Commun. Math. Phys. 149, 513–548 (1992)

Communications in Mathematical Physics

Action of Truncated Quantum Groups on Quasi-Quantum Planes and a Quasi-Associative Differential Geometry and Calculus

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Received March 13, 1992

Abstract. If q is a p^{th} root of unity there exists a quasi-coassociative truncated quantum group algebra whose indecomposable representations are the physical representations of $U_q(sl_2)$, whose coproduct yields the truncated tensor product of physical representations of $U_q(sl_2)$, and whose R-matrix satisfies quasi-Yang Baxter equations. These truncated quantum group algebras are examples of weak quasitriangular quasi-Hopf algebras ("quasi-quantum group algebras") $\mathscr{G}^*$. We describe a space $\mathscr{F}^T$ of "functions on the quasi quantum plane," i.e. of polynomials in noncommuting complex coordinate functions z_a, on which multiplication operators Z_a and the elements of $\mathscr{G}^*$ can act, so that z_a will transform according to some representation τ^f of $\mathscr{G}^*$. $\mathscr{F}^T$ can be made into a quasi-associative graded algebra $\mathscr{F}^T = \bigoplus_{n>0} \mathscr{F}^{T(n)}$ on which elements of $\mathscr{G}^*$ act as generalized derivations. In the special case of the truncated $U_q(sl_2)$ algebra we show that the subspaces $\mathscr{F}^{T(n)}$ of monomials in z_a of n^{th} degree vanish for $n \geq p-1$, and that $\mathscr{F}^{T(n)}$ carries the $2J+1$ dimensional irreducible representation of $\mathscr{G}^*$ if $n = 2J$, $J = 0, \frac{1}{2}, \ldots, \frac{1}{2}(p-2)$. Assuming that the representation τ^f of the quasi-quantum group algebra gives rise to an R-matrix with two eigenvalues, we develop a quasi-associative differential calculus on $\mathscr{F}^T$. This implies construction of an exterior differentiation, a graded algebra $\Lambda\mathscr{F}^T = \bigoplus \Lambda^n\mathscr{F}^T$ of forms and partial derivatives. A quasi-associative generalization of noncommutative differential geometry is introduced by defining a covariant exterior differentiation of forms. It is covariant under $\mathscr{G}^*$-valued gauge transformations.

0. Introduction

To explain the problem which we address, we recall the theory of the complex quantum plane [1, 2, 3].

The algebra $\mathscr{F}$ of polynomial functions on the quantum plane is a noncommutative but associative deformation of the commutative algebra $\mathscr{F}_{cl}$ of polynomial functions

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on the complex plane $\mathbf{C}^2$. It depends on a complex parameter q. The algebra $\mathscr{F}$ is generated by elements z_a ($a = 1, 2$). They are subject to relations

$$z_a z_b = z_c z_d c_R \mathscr{R}_{dc,ab} \tag{0.1}$$

with a numerical matrix $\mathscr{R}$ that is furnished by the canonical R-element for the quantum group algebra $U_q(sl_2)$, and $c_R = q^{-1/4}$. Summation over repeated indices is understood throughout.

$\mathscr{F}$ is a representation space for $U_q(sl_2)$, and the action of $\xi \in U_q(sl_2)$ on $\mathscr{F}$ is a generalized derivation [9]. This means the following.

Let $m: \mathscr{F} \otimes \mathscr{F} \to \mathscr{F}$ be the multiplication map, $m(p_1 \otimes p_2) = p_1 p_2$. Then

$$\xi(p_1 p_2) = m(\Delta_q(\xi)(p_1 \otimes p_2)), \tag{0.2}$$

where Δ_q is the coproduct for $U_q(sl_2)$. In addition to the elements of $U_q(sl_2)$, multiplication operators Z_a can act on $\mathscr{F}$ as multiplication with z_a.

When q is a root of unity, $q^p = 1$, degeneracies appear. The algebra $U_q(sl_2)$ is not semisimple in this case, and the subspaces $\mathscr{F}^n$ of homogeneous polynomials carry representations which are in general neither irreducible nor fully reducible.

However, it was shown in [7] that there exists a "truncated quantum group algebra" $\mathscr{G}^*$ associated with $U_q(sl_2)$ which is semisimple, and which carries all the structure which is necessary to interpret it as a symmetry (in quantum mechanics): a coproduct Δ, counit ε, antipode $\mathscr{S}$ and R-element $R \in \mathscr{G}^* \otimes \mathscr{G}^*$. The algebra $\mathscr{G}^*$ possesses a unit element e, but the homomorphism $\Delta: \mathscr{G}^* \to \mathscr{G}^* \otimes \mathscr{G}^*$ is not unit preserving: $\Delta(e) \neq e \otimes e$. Moreover, the bialgebra $\mathscr{G}^*$ is not coassociative but only quasi-coassociative, and R satisfies quasi-Yang Baxter equations in place of Yang Baxter equations. The indecomposable representations of $\mathscr{G}^*$ are precisely the "physical" representations of $U_q(sl_2)$ – i.e. its irreducible representations with nonzero quantum dimension – and the tensor product of representations which is furnished by the coproduct agrees with the truncated tensor product of physical representations of $U_q(sl_2)$.

These truncated quantum group algebras are examples of quasitriangular weak quasi-Hopf algebras ("quasi-quantum groups") $\mathscr{G}^*$ [7].

We will construct an algebra $\mathscr{F}^T$ of "functions on the quasi quantum plane" on which $\mathscr{G}^*$ can act as a generalized derivation. This construction works for general quasi-triangular weak quasi-Hopf algebras $\mathscr{G}^*$. The algebra $\mathscr{F}^T$ is generated by noncommuting coordinate functions z_a, as in the nontruncated case. Multiplication operators Z_a act on $\mathscr{F}^T$ by multiplication with z_a. These multiplication operators transform covariantly under $\mathscr{G}^*$. The multiplication operators satisfy braid relations

$$Z_a Z_b = Z_c Z_d c_R \tilde{\mathscr{R}}_{dc,ab}\,. \tag{0.3}$$

They substitute for commutativity of coordinates in the classical case. But $\mathscr{G}^*$-covariance of this relation turns out to require that $\tilde{\mathscr{R}}_{dc,ab}$ is not a number but an element of $\mathscr{G}^*$. The multiplication operators Z_a and elements of $\mathscr{G}^*$ generate an associative algebra $\mathscr{B}$.

In the algebra $\mathscr{F}^T$ we want relations which involve only generators z_a (and numbers). We impose relation (0.1) with a numerical matrix $\mathscr{R}$ which is furnished by the R-element for $\mathscr{G}^*$. We show that this is consistent with the braid relation (0.3) for multiplication operators, provided we do not insist on associativity of the algebra $\mathscr{F}^T$. It is still quasi-associative in the sense that homogeneous polynomials with different positions of brackets are linear combinations of each other.

In the special case of the truncated quantum group algebras associated with $U_q(sl_2)$ further properties will be established.

We show in Sect. 5 that the subspaces $\mathscr{F}^{T(n)}$ of homogeneous polynomials of degree n vanish for $n \geq p-1$. It is easy to verify (and will be explained in Sect. 5) that the nonzero subspaces $\mathscr{F}^{T(n)}$ carry irreducible representations of $\mathscr{G}^*$. If q is a p^{th} root of unity, the irreducible representations of $\mathscr{G}^*$ are labelled by $J = 0, \frac{1}{2}, 1 \ldots \frac{1}{2}(p-2)$. They have dimension $2J+1$. The spaces $\mathscr{F}^{T(n)}$ for $n = 0, 1, 2, \ldots$ carry representation $J = \frac{1}{2}n$. The quasi quantum planes associated with the truncated $U_q(sl_2)$ will turn out to be associative.

For a general weak quasi-triangular quasi-Hopf algebra we construct a graded quasi-associative algebra $\Lambda\mathscr{F}^T = \oplus \Lambda^r \mathscr{F}^T$ such that $\Lambda^0 \mathscr{F}^T = \mathscr{F}^T$. To introduce an exterior derivative d which is $\mathscr{G}^*$-invariant, satisfies $d^2 = 0$, and maps $\Lambda^r \mathscr{F}^T$ into $\Lambda^{r+1} \mathscr{F}^T$ we have to impose a restriction on $\mathscr{G}^*$. If Z_a transforms according to the representation τ^f of $\mathscr{G}^*$, we require that the R-matrix $\tau^f \otimes \tau^f(R)$ has only two eigenvalues. Under this assumption, an exterior derivative exists and elements of $\Lambda^r \mathscr{F}^T$ can be called "r-forms."

Starting from this exterior differential calculus we introduce a generalization of noncommutative differential geometry [11] to quasi-associative "algebras of functions." To this end we define covariant exterior derivatives D which act on forms (Sect. 7). They are covariant under $\mathscr{G}^*$-valued gauge transformations.

We describe (Sect. 9) how the algebra of multiplication operators Z_a and differentials Θ_a can be extended to include partial derivatives $\partial_{\bar{a}}$. They transform according to the contragredient representation $\tilde{\tau}^f$. In this construction $\tilde{\tau}^f \otimes \tilde{\tau}^f(R)$ is assumed to possess only two distinct eigenvalues.

In the quantum group case, one considers often the coaction of the quantum group [e.g. the dual bialgebra to $U_q(sl_2)$] on $\mathscr{F}^T$ [3, 10],

$$z_a \to \sum_b z_b \otimes T_{ba} \,. \tag{0.4}$$

This is also possible here. By canonical construction [7] the $\mathscr{G}^*$-module $\mathscr{F}^T$ is a comodule for the dual bialgebra $\mathscr{G}$ of $\mathscr{G}^*$, i.e. there exists a homomorphism of algebras

$$\mathscr{F}^T \to \mathscr{F}^T \otimes \mathscr{G} \,. \tag{0.5}$$

The algebra $\mathscr{G}$ is nonassociative because $\mathscr{G}^*$ is not coassociative, see [7].

1. Quasi-Quantum Group Algebras

Quantum groups $\mathscr{G}$ are noncommutative but associative generalizations of the algebra of functions on a group. To have a conventional picture of a symmetry in quantum physics, one considers the dual $\mathscr{G}^*$. It is a Hopf algebra which is coassociative but not cocommutative. In Drinfeld's quasi-triangular quasi-Hopf algebras coassociativity is weakened to quasi-coassociativity [5]. Quasi-triangularity implies that an element $R \in \mathscr{G}^* \otimes \mathscr{G}^*$ is given which furnishes a representation of the braid group (see Theorem 1.1 below).

It was shown in [7] that Drinfeld's axioms can be weakened further without loss of the physical interpretation as a symmetry, by giving up invertibility requirements. Some weak quasi-triangular quasi-Hopf algebras $\mathscr{G}^*$ of this type are canonically associated with $U_q(sl_2)$ when $q^p = 1$. All their representations are physical. We call them "truncated quantum group algebras." We proposed to regard them as the true symmetries of conformal models.

We proceed to a short review of *weak quasi-triangular quasi-Hopf algebras* $\mathcal{G}^*$. A pedagogical account is found in [7]. We will not be interested in the action of *-operations on our algebras in this paper, therefore no assumptions which involve them will be made.

$\mathcal{G}^*$ is an algebra with unit e, with additional structures as follows. There is a counit

$$\varepsilon : \mathcal{G}^* \to \mathbf{C}\,, \tag{1.1}$$

and a coproduct

$$\Delta : \mathcal{G}^* \to \mathcal{G}^* \otimes \mathcal{G}^*\,. \tag{1.2}$$

Both are homomorphisms of algebras, but Δ need not be unit preserving. The homomorphism property requires that $\Delta(\xi\eta) = \Delta(\xi)\Delta(\eta)$. Given the product Δ, there exists another one called Δ'. If $\Delta(\xi) = \sum \xi_p^1 \otimes \xi_p^2$, then Δ' is defined by

$$\Delta'(\xi) = \sum \xi_p^2 \otimes \xi_p^1\,. \tag{1.3}$$

It is demanded that

$$(\mathrm{id} \otimes \varepsilon)\Delta = (\varepsilon \otimes \mathrm{id})\Delta = \mathrm{id}\,, \tag{1.4}$$

(id = identity map). Furthermore one demands that an element $\varphi \in \mathcal{G}^* \otimes \mathcal{G}^* \otimes \mathcal{G}^*$ is given which implements (weak) quasi-coassociativity of the coproduct, Eq. (1.10) below. In contrast with Drinfeld, we admit the possibility that $\Delta(e) \neq e \otimes e$, and we do not demand invertibility of φ, but only existence of a quasiinverse, still denoted by φ^{-1}, such that

$$\varphi\varphi^{-1} = (\mathrm{id} \otimes \Delta)\Delta(e)\,, \qquad \varphi^{-1}\varphi = (\Delta \otimes \mathrm{id})\Delta(e)\,, \tag{1.5}$$

$$\begin{aligned} (\mathrm{id} \otimes \mathrm{id} \otimes \varepsilon)\,(\varphi) &= \Delta(e)\,, \\ (\mathrm{id} \otimes \varepsilon \otimes \mathrm{id})\,(\varphi) &= \Delta(e)\,, \\ (\varepsilon \otimes \mathrm{id} \otimes \mathrm{id})\,(\varphi) &= \Delta(e)\,. \end{aligned} \tag{1.6}$$

The statement that φ^{-1} is a quasiinverse of φ means that $\varphi\varphi^{-1}\varphi = \varphi$.

Finally there should exist $R \in \mathcal{G}^* \otimes \mathcal{G}^*$ such that

$$\Delta'(\eta)R = R\Delta(\eta) \quad \text{for all} \quad \eta \in \mathcal{G}^*\,. \tag{1.7}$$

We do not demand that R be invertible, instead it should have a quasiinverse R^{-1} such that

$$RR^{-1} = \Delta'(e)\,, \qquad R^{-1}R = \Delta(e)\,. \tag{1.8}$$

In addition we will impose the following requirement which is not mentioned in our article [7], but which is very natural and is needed in the construction of a differential calculus

$$(\mathrm{id} \otimes \varepsilon)\,(R) = e\,, \qquad (\varepsilon \otimes \mathrm{id})\,(R) = e\,. \tag{1.9}$$

By definition, weak quasi-coassociativity demands that

$$\varphi(\Delta \otimes \mathrm{id})\Delta(\xi) = (\mathrm{id} \otimes \Delta)\Delta(\xi)\varphi \quad \text{for all} \quad \xi \in \mathcal{G}^*\,. \tag{1.10}$$

Following Drinfeld the following relations between Δ, R, and φ are postulated:

$$(\mathrm{id}\otimes\mathrm{id}\otimes\Delta)(\varphi)(\Delta\otimes\mathrm{id}\otimes\mathrm{id})(\varphi)=(e\otimes\varphi)(\mathrm{id}\otimes\Delta\otimes\mathrm{id})(\varphi)(\varphi\otimes e), \quad (1.11)$$

$$(\mathrm{id}\otimes\Delta)(R)=\varphi_{231}^{-1}R_{13}\varphi_{213}R_{12}\varphi^{-1}, \quad (1.12)$$

$$(\Delta\otimes\mathrm{id})(R)=\varphi_{312}R_{13}\varphi_{132}^{-1}R_{23}\varphi. \quad (1.13)$$

We used the standard notation. If $R=\sum r_a^1\otimes r_a^2$, then

$$R_{13}=\sum r_a^1\otimes e\otimes r_a^2, \qquad R_{12}=\sum r_a^1\otimes r_a^2\otimes e, \quad (1.14)$$

etc.

If s is any permutation of 123 and $\varphi=\sum\varphi_\sigma^1\otimes\varphi_\sigma^2\otimes\varphi_\sigma^3$, then

$$\varphi_{s(1)s(2)s(3)}=\sum_\sigma\varphi_\sigma^{s^{-1}(1)}\otimes\varphi_\sigma^{s^{-1}(s)}\otimes\varphi_\sigma^{s^{-1}(3)}. \quad (1.15)$$

Equations (1.12), (1.13) imply validity of quasi-Yang Baxter equations,

$$R_{12}\varphi_{312}R_{13}\varphi_{132}^{-1}R_{23}\varphi=\varphi_{321}R_{23}\varphi_{231}^{-1}R_{13}\varphi_{213}R_{12}, \quad (1.16)$$

and this guarantees that R together with φ determines a representation of the braid group as will be explained below. There should also exist an antipode

$$\mathscr{S}:\mathscr{G}^*\mapsto\mathscr{G}^* \quad (1.17)$$

which is an antihomomorphism of algebras. Further properties of $\mathscr{S}$ were required in [7] in order to define covariant adjoints. We will not asume them here.

We proceed to an explanation of some consequences of the assumptions.

The counit ε provides a 1-dimensional representation which substitutes for the trivial 1-dimensional representation of groups (and their group algebras).

Given any representation τ of $\mathscr{G}^*$, one defines

$$\tilde{\tau}(\xi)={}^t\tau(\mathscr{S}^{-1}(\xi)), \quad (1.18)$$

where ^{t} denotes the transpose. This is a representation, because $\mathscr{S}$ is an antihomomorphism, and is called the contragredient representation. Later on we will assume that a particular representation τ^f has been singled out. We will refer to it as the fundamental representation. In the construction of the differential calculus it will be assumed that the tensor product of representations $\tilde{\tau}^f\bigotimes\tau^f$ contains the trivial 1-dimensional representation ε.

We use the notation $\otimes$ for the standard tensor product of matrices, algebras etc., which is associative by definition. If τ and τ' are representations of $\mathscr{G}^*$, then $\tau\otimes\tau'$ is a representation of $\mathscr{G}^*\otimes\mathscr{G}^*$. This is to be distinguished from the *tensor product* $\bigotimes$ *of representations* of $\mathscr{G}^*$ which is defined by

$$(\tau\bigotimes\tau')(\xi)=(\tau\otimes\tau')(\Delta(\xi)). \quad (1.19)$$

If $\Delta(e)\neq e\otimes e$ then this introduces "truncation." Suppose that τ is a tensor product of two or more irreducible representations. It will act on the (standard) tensor product V of the corresponding representation spaces. If $\Delta(e)\neq e\otimes e$, then $\tau(e)$ is in general not the identity map on V. But

$$\tau(\xi)\tau(e)=\tau(e)\tau(\xi)=\tau(\xi), \quad (1.20)$$

and, in particular, $\tau(e)\tau(e) = \tau(e)$. Therefore there exists a representation space, given by the range of $\tau(e)$, which is in general a proper subspace of V, and on which $\tau(e)$ will act as the identity. We call this the "true representation space of τ".

The property (1.4) tells us that tensoring with the 1-dimensional representation ε amounts to doing nothing,

$$(\tau \bigotimes \varepsilon)(\xi) = (\varepsilon \bigotimes \tau)(\xi) = \tau(\xi)\,. \tag{1.21}$$

The tensor product of representations is not associative unless φ is trivial. But weak quasi-associativity ensures that representations $(\tau \bigotimes \tau') \bigotimes \tau''$ and $\tau \bigotimes (\tau' \bigotimes \tau'')$ are equivalent. If one of the representation τ, τ', τ'' is the 1-dimensional representation ε, then the two representations are identical by (1.21). So they are related by a trivial similarity transformation. This is encoded in the relations (1.6). From the properties of φ one can also deduce that

$$\varphi = \varphi(\Delta \otimes \mathrm{id})\Delta(e) = (\mathrm{id} \otimes \Delta)\Delta(e)\varphi\,, \tag{1.22}$$

$$(\mathrm{id} \otimes \mathrm{id} \otimes \varepsilon)(\varphi^{-1}) = \Delta(e)\,. \tag{1.23}$$

The element $R \in \mathscr{G}^* \otimes \mathscr{G}^*$, which satisfies the quasi-Yang Baxter equations, furnishes a representation of the braid group (Theorem 1.1 below). Let us recall that the braid group B_n on n threads is generated by elements σ_i and σ_i^{-1} ($i = 1 \dots n-1$) which obey the Artin relations

$$\begin{gathered} \sigma_i \sigma_k = \sigma_k \sigma_i \quad \text{if} \quad |k - i| \geq 2\,, \qquad \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}\,, \\ \sigma_i \sigma_i^{-1} = \iota = \sigma_i^{-1} \sigma_i\,. \end{gathered} \tag{1.24}$$

The unit element of B_n is written as ι. We introduce some notations. Write

$$e^n = e \otimes \dots \otimes e \quad (n \text{ factors}) \tag{1.25}$$

and similarly for id^n. In addition we abbreviate $\mathscr{G}^{*\otimes n} = \mathscr{G}^* \otimes \dots \otimes \mathscr{G}^*$ (n factors), and

$$\Delta^n = (\mathrm{id}^{n-1} \otimes \Delta) \dots (\mathrm{id} \otimes \Delta)\Delta \quad \text{for} \quad n \geq 2\,, \tag{1.26}$$

$$\Delta^1 = \Delta\,, \qquad \Delta^0 = \mathrm{id}\,, \qquad \Delta^{-1} = \varepsilon\,. \tag{1.27}$$

We introduce the permutator $\mathscr{P} \in \mathscr{G}^* \otimes \mathscr{G}^*$ by (i) $\mathscr{P} \xi \otimes \eta = \eta \otimes \xi \mathscr{P}$ for all $\xi, \eta \in \mathscr{G}^*$, (ii) $\tau \otimes \tau'(\mathscr{P}) = 0$ if τ, τ' are inequivalent irreducible representations of $\mathscr{G}^*$. Such $\mathscr{P}$ exists if $\mathscr{G}^*$ is semisimple. In this case we have

Theorem 1.1 (Artin relations). *Let $\hat{R}^+ = \mathscr{P} R$ and $\hat{R}^- = R^{-1} \mathscr{P}$, and $n = r + k + 1$, $r \geq 0$. Define $\sigma_k^{n\pm} \in \mathscr{G}^{*\otimes n}$ by*

$$\sigma_k^{n\pm} = \Delta^{n-1}(e)\,(\mathrm{id}^{n-k+1} \otimes \Delta^{k-2})\,(e^{n-k-1} \otimes \varphi(\hat{R}^\pm \otimes e)\varphi^{-1})\,. \tag{1.28}$$

Then the $\sigma_k^{n\pm}$ obey Artin relations (1.24) *with $\iota = \Delta^{n-1}(e)$.*

The proof will be presented in Appendix A.

Remark. When $\mathscr{G}^*$ is not semisimple, a representation of the braid group can still be found. The formulation uses permutation maps $\mathbf{P}_k^n : \mathscr{G}^{*\otimes n} \to \mathscr{G}^{*\otimes n}$ defined by

$$\mathbf{P}_k^n(\xi_n \otimes \dots \xi_k \otimes \xi_{k-1} \dots \otimes \xi_1) = (\xi_n \otimes \dots \xi_{k-1} \otimes \xi_k \dots \otimes \xi_1)\,. \tag{1.29}$$

A representation of the braid group by maps $\varsigma_k^{n\pm} : \mathcal{G}^{*\otimes n} \to \mathcal{G}^{*\otimes n}$ is given by

$$\varsigma_k^{n\pm} = \Delta^{n-1}(e)\mathbf{P}_k^n(\mathrm{id}^{n-k+1} \otimes \Delta^{k-2})(e^{n-k-1} \otimes \varphi_{213}(R^\pm \otimes e)\varphi^{-1}), \tag{1.30}$$

where $R^+ = R$ and $R^- = R^{-1'}$ and $'$ interchanges factors in $\mathcal{G}^* \otimes \mathcal{G}^*$. The proof of this more general result parallels the one given in Appendix A but is slightly more complicated.

A weak quasi-triangular quasi-Hopf algebra $\mathcal{G}^*$ is canonically associated with $U_q(sl_2)$ with q a root of unity. As an algebra $\mathcal{G}^* = U_q(sl_2)/\mathcal{T}$, where $\mathcal{T}$ is the ideal which is annihilated by all the physical representations τ^I, $2I = 0 \dots p-2$, of $U_q(sl_2)$. $\mathcal{G}^*$ is semisimple, its representations are fully reducible, and the irreducible ones are precisely the physical representations of $U_q(sl_2)$. Let

$$u(I,J) = \min\{|I+J|, p-2-I-J\} \tag{1.31}$$

and let P_{IJ} be the projector on the physical subrepresentations K, $|I-J| \le K \le u(I,J)$ of the tensor product $\tau^I \bigotimes_q \tau^J$ of $U_q(sl_2)$ representations. There exists $P \in \mathcal{G}^*$ such that $P_{IJ} = (\tau^I \otimes \tau^J)(P)$. The coproduct in $\mathcal{G}^*$ is determined in terms of the coproduct Δ_q in $U_q(sl_2)$ as

$$\Delta(\xi) = P\Delta_q(\xi), \tag{1.32}$$

hence $\Delta(e) = P \neq e \otimes e$. This coproduct specifies a tensor product $\bigotimes$ which is equal to the truncated tensor product of physical $U_q(sl_2)$ representations. Thus

$$\tau^I \bigotimes \tau^J = \bigoplus_{|I-J| \le K \le u(IJ)} \tau^K. \tag{1.33}$$

There exists an element $\varphi \in \mathcal{G}^* \otimes \mathcal{G}^* \otimes \mathcal{G}^*$ such that $\varphi_{IJK} = (\tau^I \otimes \tau^J \otimes \tau^K)(\varphi)$ implements the well known unitary equivalence of the truncated tensor products $\tau^I \bigotimes (\tau^J \bigotimes \tau^K)$ and $(\tau^I \bigotimes \tau^J) \bigotimes \tau^K$. A truncated tensor product $\bigotimes$ is defined also for basis vectors $\hat{e}_i^I$ in the dual representation spaces $\hat{V}^I$ on which $\mathcal{G}^*$ acts from the right, viz. $\hat{e}_i^I \bigotimes \hat{e}_j^J = \hat{e}_i^I \otimes \hat{e}_j^J P_{IJ}$. The map φ_{IJK} can be specified by its action on triple truncated products of basis vectors, together with the condition $\varphi = (\mathrm{id} \otimes \Delta)\Delta(e)\varphi$, viz.

$$\begin{aligned} &\sum_{ijkp} \begin{bmatrix} I & P & L \\ i & p & l \end{bmatrix}_q \begin{bmatrix} J & K & P \\ j & k & p \end{bmatrix}_q \hat{e}_i^I \otimes \hat{e}_j^J \otimes \hat{e}_k^K \varphi \\ &= \sum_{Q,ijkq} F_{PQ} \begin{bmatrix} J & I \\ K & L \end{bmatrix} \begin{bmatrix} I & J & Q \\ i & j & q \end{bmatrix}_q \begin{bmatrix} Q & K & L \\ q & k & l \end{bmatrix}_q \hat{e}_i^I \otimes \hat{e}_j^J \otimes \hat{e}_k^K \end{aligned} \tag{1.34}$$

with fusion matrices given by $6j$-symbols, $F_{PQ} \begin{bmatrix} J & I \\ K & L \end{bmatrix} = \begin{Bmatrix} K & J & P \\ I & L & Q \end{Bmatrix}_q$. $\begin{bmatrix} \cdots \\ \cdots \end{bmatrix}_q$ are Clebsch Gordan coefficients for $U_q(sl_2)$ (and at the same time for $\mathcal{G}^*$).

The R-element of $\mathcal{G}^* \otimes \mathcal{G}^*$ is given in terms of the R-element R_q for $U_q(sl_2)$ by

$$R = R_q \Delta(e) = \Delta'(e) R_q, \tag{1.35}$$

while antipode and counit are the same as in $U_q(sl_2)$. It is shown in [7] that the defining properties of a weak quasi-triangular quasi-Hopf algebra are satisfied.

2. The Associative Algebra $\mathscr{B}$ Generated by Multiplication Operators Z_a and elements of $\mathscr{G}^*$

In order not to clutter the presentation, the proofs of the results in this section are collected in Appendix B, with few exceptions.

We assume that some particular irreducible representation τ^f of $\mathscr{G}^*$ has been singled out. We will refer to it sometimes as "the fundamental representation." Let its dimension be N. In the case of the truncated quantum group algebra $\mathscr{G}^*$ which is associated with $U_q(sl_2)$ when q is a p^{th} root of unity, $p \geq 3$, we select the 2-dimensional representation, so that $N = 2$ in this case.

Let $\Delta, R, \varphi, \varphi^{-1}$ be as in the last section. Given

$$\varphi = \sum_\sigma \varphi^1_\sigma \otimes \varphi^2_\sigma \otimes \varphi^3_\sigma \tag{2.1}$$

one introduces

$$\varphi_{213} = \sum_\sigma \varphi^2_\sigma \otimes \varphi^1_\sigma \otimes \varphi^3_\sigma . \tag{2.2}$$

To state the relations in $\mathscr{B}$ we introduce the matrix $\tilde{\mathscr{R}}$ with entries in $\mathscr{G}^*$,

$$\tilde{\mathscr{R}}_{dc,ab} = (\tau^f_{da} \otimes \tau^f_{cb} \otimes \mathrm{id})\,(\varphi_{213}(R \otimes e)\varphi^{-1}) \in \mathscr{G}^* . \tag{2.3}$$

Definition 2.1 (Algebra $\mathscr{B}$). *Choose $c_R \in \mathbf{C}$ such that $c_R(\tau^f \otimes \tau^f)(R)$ possesses an eigenvalue* 1. *The associative algebra $\mathscr{B}$ is generated by elements Z_a, $(a = 1 \dots N)$, and the elements of $\mathscr{G}^*$. The unit element e of $\mathscr{G}^*$ acts as a unit element of $\mathscr{B}$ so that*
(1) $Z_a e = Z_a = e Z_a$,
and the following further relations are imposed.
(2) *($\mathscr{G}^*$-covariance)* $\xi Z_a = Z_b(\tau^f_{ba} \otimes \mathrm{id})\,(\Delta(\xi))$ *for* $\xi \in \mathscr{G}^*$,
(3) *(braid relations)* $Z_a Z_b = c_R Z_c Z_d \tilde{\mathscr{R}}_{dc,ab}$.

Summations over repeated indices are understood throughout.

If $\Delta(\xi) = \sum_\sigma \xi^1_\sigma \otimes \xi^2_\sigma$ then relation (2) reads explicitly

$$\xi Z_a = \sum_\sigma Z_b \tau^f_{ba}(\xi^1_\sigma)\xi^2_\sigma . \tag{2.4}$$

This tells us how to shift elements ξ of $\mathscr{G}^*$ through factors Z_a from left to right. It follows that we have

Proposition 2.2. *Every element of $\mathscr{B}$ is a complex linear combination of elements of the form*

$$Z_{a_n} \dots Z_{a_1}\eta \quad \text{with} \quad n \geq 0,\ \eta \in \mathscr{G}^* .$$

Relation (1) in Definition 1 takes the form

$$Z_a e = Z_a = Z_b(\tau^f_{ba} \otimes \mathrm{id})\,(\Delta(e)) . \tag{2.5}$$

This is a nontrivial linear relation between Z_a's with coefficients in $\mathscr{G}^*$ if $\Delta(e) \neq e \otimes e$.

Definition 2.3 ($\mathscr{G}^*$-covariance). *Let $\tau = (\tau_{\alpha\beta})_{\alpha,\beta \in I}$ be the representation matrix of a n-dimensional representation of $\mathscr{G}^*$. An n-tupel $F = (F_\alpha)_{\alpha \in I}$, $F_\alpha \in \mathscr{B}$, is said to transform covariantly according to the representation τ of $\mathscr{G}^*$ if*

$$\xi F_\alpha = F_\beta(\tau_{\beta\alpha} \otimes \mathrm{id})\,(\Delta(\xi)) \tag{2.6}$$

for all $\xi \in \mathscr{G}^*$. $F \in \mathscr{B}$ *is called* $\mathscr{G}^*$-invariant *if it transforms according to the* 1*-dimensional representation* ε *of* $\mathscr{G}^*$ *or, equivalently, if*

$$\xi F = F\xi \tag{2.7}$$

for all $\xi \in \mathscr{G}^*$.

Relation (2) in Definition 2.1 says that $(Z_a)_{a=1\dots N}$ transforms covariantly according to the fundamental representation of $\mathscr{G}^*$. We show now how to construct composite objects which transform covariantly. Because of lack of coassociativity in $\mathscr{G}^*$, products $Z_a Z_b$ do not transform convariantly. But we have

Theorem 2.4 (Covariant products). *Suppose that* $(F_\alpha)_{\alpha\in I}$ *and* $(F'_\beta)_{\beta\in I'}$ *transform covariantly according to representations* τ *and* τ' *of* $\mathscr{G}^*$ *with dimensions* n *and* n'. *Define the* nn'*-tuple* $F\times F'$ *by*

$$(F\times F')_{\alpha\beta} = \sum_{\gamma\in I}\sum_{\delta\in I'} F_\gamma F'_\delta(\tau_{\gamma\alpha}\otimes\tau'_{\delta\beta}\otimes \mathrm{id})(\varphi) \in \mathscr{B}\,. \tag{2.8}$$

Then $F\times F'$ *transforms covariantly according to the tensor product representation* $\tau \bigotimes \tau'$ *of* $\mathscr{G}^*$.

Moreover, Eq. (2.8) *can be inverted to recover ordinary products from covariant one, viz.*

$$F_\alpha F'_\beta = \sum_{\gamma\in I}\sum_{\delta\in I'} (F\times F')_{\gamma\delta}(\tau_{\gamma\alpha}\otimes\tau'_{\delta\beta}\otimes \mathrm{id})(\varphi^{-1})\,. \tag{2.9}$$

If $G\in\mathscr{B}$ *is* $\mathscr{G}^*$*-invariant then*

$$(G\times F)_\alpha = GF_\alpha\,,\qquad (F\times G)_\alpha = F_\alpha G\,. \tag{2.10}$$

Note that $\times$ is in general a product of vectors whose entries are elements of $\mathscr{B}$, and not a product of individual elements of $\mathscr{B}$.

Using the notation (2.1), the defining Eq. (2.8) takes the form

$$(F\times F')_{\alpha\beta} = \sum_\sigma F_\gamma F'_\delta \tau_{\gamma\alpha}(\varphi^1_0)\tau'_{\delta\beta}(\varphi^2_\sigma)\varphi^3_\sigma\,. \tag{2.11}$$

This exhibits the fact that the $(F\times F')_{\alpha\beta}$ are complex linear combinations of terms $F_\gamma F'_\delta \varphi^3_\sigma$ with coefficients $\varphi^3_\sigma \in \mathscr{G}^*$.

We will show the proof of covariance, although it is straightforward, in order to get the reader accustomed to such computations. The rest of the proof of Theorem 2.4 is in Appendix B.

Proof of Theorem 2.4 (Covariance). We adopt the summation convention for all repeated indices. The range of summation will be clear from the context.

Let $\xi = \sum \xi^1_\sigma \otimes \xi^2_\sigma$. The hypotheses of Theorem 2.4 and Definition 2.3 imply

$$\begin{aligned}
\xi(F\times F')_{\alpha\beta} &= F_\varepsilon(\tau_{\varepsilon\gamma}\otimes\mathrm{id})(\Delta(\xi))F'_\delta(\tau_{\gamma\alpha}\otimes\tau'_{\delta\beta}\otimes\mathrm{id})(\varphi)\\
&= \sum_\sigma F_\varepsilon \tau_{\varepsilon\gamma}(\xi^1_\sigma)\xi^2_\sigma F'_\delta(\tau_{\gamma\alpha}\otimes\tau'_{\delta\beta}\otimes\mathrm{id})(\varphi)\\
&= \sum_\sigma F_\varepsilon F'_\omega \tau_{\varepsilon\gamma}(\xi^1_\sigma)\,(\tau'_{\omega\delta}\otimes\mathrm{id})\,(\Delta(\xi^2_\sigma))\,(\tau_{\gamma\alpha}\otimes\tau'_{\delta\beta}\otimes\mathrm{id})\,(\varphi)\\
&= F_\varepsilon F'_\omega(\tau_{\varepsilon\alpha}\otimes\tau'_{\omega\beta}\otimes\mathrm{id})\,((\mathrm{id}\otimes\Delta)\Delta(\xi)\varphi)\\
&= F_\varepsilon F'_\omega(\tau_{\varepsilon\alpha}\otimes\tau'_{\omega\beta}\otimes\mathrm{id})\,(\varphi(\Delta\otimes\mathrm{id})\Delta(\xi))
\end{aligned}$$

by the intertwining property (1.10) of φ. Thus

$$\begin{aligned}\xi(F\times F')_{\alpha\beta} &= F_\varepsilon F'_\omega(\tau_{\varepsilon\gamma}\otimes\tau'_{\omega\delta}\otimes\mathrm{id})(\varphi)(\tau_{\gamma\alpha}\otimes\tau'_{\delta\beta}\otimes\mathrm{id})((\Delta\otimes\mathrm{id})\Delta(\xi))\\ &= (F\times F')_{\gamma\delta}(\hat\tau\otimes\mathrm{id})(\Delta(\xi)),\end{aligned}$$

where $\hat\tau\equiv\tau\bigotimes\tau'$ is the tensor product of representations which is defined by

$$\hat\tau(\eta) = (\tau\otimes\tau')(\Delta(\eta)) = (\tau\bigotimes\tau')(\eta). \tag{2.12}$$

This is the desired result. q.e.d.

With the help of the $\times$-product, the braid relations (3) in Definition 2.1 can be written in another way which involves a braid matrix whose entries are c-numbers.

Theorem 2.5 (Braid relations). (i) *The braid relations* (3) *of Definition 2.1 are equivalent to*

$$(Z\times Z)_{ab} = c_R(Z\times Z)_{cd}\mathscr{R}_{dc,ab} \tag{2.13}$$

with

$$\mathscr{R}_{dc,ab} = (\tau^f_{da}\otimes\tau^f_{cb})(R)\in\mathbf{C}. \tag{2.14}$$

(ii) *Both sides of Eq.* (2.13) *transform covariantly according to the representation* $\tau^f\bigotimes\tau^f$ *of* $\mathscr{G}^*$.

This theorem explains why the braid relations Definition 2.1 (3) are consistent with $\mathscr{G}^*$-covariance.

The $\times$-product is not associative. But it is quasi-associative in the following sense.

Theorem 2.6 (Quasi-associativity of the $\times$-product). *Suppose that* $F=(F_\alpha)$, $F'=(F_\beta)$, *and* $F''=(F''_\gamma)$ *transform covariantly according to representations* τ, τ', *and* τ'' *of* $\mathscr{G}^*$. *Then*

(i) $$((F\times F')\times F'')_{\alpha\beta\gamma} = (F\times(F'\times F''))_{\delta\varepsilon\kappa}(\tau_{\delta\alpha}\otimes\tau'_{\varepsilon\beta}\otimes\tau''_{\kappa\gamma})(\varphi), \tag{2.15}$$

(ii) $$(F\times(F'\times F''))_{\alpha\beta\gamma} = ((F\times F')\times F'')_{\delta\varepsilon\kappa}(\tau_{\delta\alpha}\otimes\tau'_{\varepsilon\beta}\otimes\tau''_{\kappa\gamma})(\varphi^{-1}). \tag{2.16}$$

Corollary 2.7 (Complex linear relations). *If* $F=(F_\alpha)$ *transforms according to the representation* τ *of* $\mathscr{G}^*$ *then*

$$F_\alpha = F_\beta\tau_{\beta\alpha}(e). \tag{2.17}$$

If $F_a = Z_a$ then Eq. (2.17) is trivial because $\tau^f(e)$ is the identity matrix. Other covariants are built up as $\times$-products of Z's. The case of practical interest is therefore when $F=(F'\times F'')$. This transforms according to $\tau=\tau'\bigotimes\tau''$ and $\tau(e)=(\tau'\otimes\tau'')(\Delta(e))$ is not in general the identity matrix if $\Delta(e)\neq e\otimes e$. This reflects the fact that the "true representation space" of a tensor product representation may in general be a proper subspace of the tensor product of representation spaces. In this case Eq. (2.17) is a nontrivial complex linear relation among the F_α's. It tells us that those components which are not in the true representation space are in fact zero.

We are now ready to describe the elements of $\mathscr{B}$ in another way.

Theorem 2.8 (Covariant products span $\mathscr{B}$). *Every element of* $\mathscr{B}$ *is a complex linear combination of elements of the form*

$$(Z\times(Z\times\ldots\times(Z\times Z)\ldots))_{a_n\ldots a_1}\eta \tag{2.18}$$

with $n\geq 0$ *and* $\eta\in\mathscr{G}^*$.

Theorem 2.8 follows from Proposition 2.2 and the inversion formula in Theorem 2.4.

Theorem 2.9 (Braid relations for composite operators). *Suppose that* $F = (F_\alpha)$, $F' = (F'_\beta)$, *and* $F'' = (F''_\gamma)$ *transform covariantly according to representations* τ, τ', *and* τ'' *of* $\mathscr{G}^*$.

(i) *Suppose that the braid relations*

$$(F \times F')_{\alpha\beta} = c'_R (F' \times F)_{\mu\nu} (\tau_{\nu\alpha} \otimes \tau'_{\mu\beta})(R)\,,$$
$$(F \times F'')_{\alpha\beta} = c''_R (F'' \times F)_{\mu\varrho} (\tau_{\varrho\alpha} \otimes \tau''_{\mu\beta})(R)$$

hold true. Then F *and* $F' \times F''$ *satisfy braid relations*

$$(F \times (F' \times F''))_{\alpha\beta\gamma} = c'_R c''_R ((F' \times F'') \times F)_{\mu\nu\varrho} (\tau_{\varrho\alpha} \otimes (\tau' \bigotimes \tau'')_{\mu\nu,\beta\gamma})(R)\,,$$
$$((F' \times F'') \times F)_{\beta\gamma\alpha} = c'_R c''_R (F \times (F' \times F''))_{\varrho\mu\nu} ((\tau' \bigotimes \tau'')_{\mu\nu,\beta\gamma} \otimes \tau_{\varrho\alpha})(R)\,.$$

(ii) *If* F_α, F'_β *are complex linear combinations of* $\times$*-products of* Z_a*'s (with brackets in arbitrary positions), and are homogeneous in the* Z*'s of degree* n *and* m*, respectively, then*

$$(F \times F')_{\alpha\beta} = (F' \times F)_{\delta\gamma} c_R^{nm} (\tau_{\gamma\alpha} \otimes \tau'_{\delta\beta})(R)\,. \tag{2.19}$$

In particular, if $F' \in \mathscr{B}$ *is* $\mathscr{G}^*$*-invariant then*

$$F_\alpha F' = F' F_\alpha c_R^{nm}\,. \tag{2.20}$$

The proof of this and other results of this section is in Appendix B.

3. The Coset Space $\mathscr{F}^T = \mathscr{B}/\mathscr{G}^*$

We will construct a $\mathscr{B}$-module $\mathscr{F}^T$ – i.e. a space which is a representation space for $\mathscr{G}^*$ and on which Z_a can act. It is obtained as a coset space $\mathscr{F}^T = \mathscr{B}/\mathscr{G}^*$. Its elements are equivalence classes of elements of $\mathscr{B}$. Since $\mathscr{G}^*$ is an algebra and not a group, we need a homomorphism

$$\varepsilon : \mathscr{G}^* \to \mathbf{C}$$

to define the equivalence relation. This homomorphism is given by the counit. Recalling Proposition 2.2 we define the equivalence relation by equating

$$Z_{a_1} \dots Z_{a_n} \xi \sim Z_{b_1} \dots Z_{b_n} \eta$$

if

$$m = n\,, \qquad a_i = b_i \quad (i = 1, \dots n)\,, \qquad \varepsilon(\xi) = \varepsilon(\eta)\,. \tag{3.1}$$

We denote the maps into cosets by

$$\varepsilon : \mathscr{B} \to \mathscr{F}^T = \mathscr{B}/\mathscr{G}^*\,. \tag{3.2}$$

$\mathscr{F}^T$ becomes a $\mathscr{B}$-module by setting

$$F\varepsilon(G) = \varepsilon(FG) \quad \text{for} \quad F, G \in \mathscr{B}\,. \tag{3.3}$$

We introduce a special notation for the image of the unit element e of $\mathscr{B}$,

$$\Omega = \varepsilon(e)\,. \tag{3.4}$$

Theorem 3.1 ($\mathscr{G}^*$-invariance of 0th order monomials). *Ω is $\mathscr{G}^*$-invariant in the sense that*

$$\xi\Omega = \Omega\varepsilon(\xi) \quad \textit{for all} \quad \xi \in \mathscr{G}^* . \tag{3.5}$$

If (F_α) transforms according to representation τ of $\mathscr{G}^*$, then $F_\alpha\Omega$ does so, too. That is

$$\xi F_\alpha \Omega = F_\beta \Omega \tau_{\beta\alpha}(\xi) . \tag{3.6}$$

Proof. $\xi\Omega = \varepsilon(\xi) = \Omega\varepsilon(\xi)$ because $\xi \sim e\varepsilon(\xi)$. The second assertion follows from the first because $(\mathrm{id} \otimes \varepsilon)\Delta(\xi) = \xi$. q.e.d.

Theorem 3.2 (Equivalent representations of elements of $\mathscr{F}^T$).

$$\mathscr{F}^T = \bigoplus_{n \geq 0} \mathscr{F}^{T(n)}$$

with $\mathscr{F}^{T(n)}$ spanned by elements of the form

$$z^n_{a_n \dots a_1} = Z_{a_n} \dots Z_{a_1} \Omega \quad (n \geq 0) , \tag{3.7}$$

$z^0 = \Omega$. These elements can be written in the equivalent form

$$z^n_{a_n \dots a_1} = (Z \times (Z \times \dots \times (Z \times Z) \dots))_{a_n \dots a_1} \Omega . \tag{3.8}$$

Proof of Theorem 3.1. Since $\varepsilon(\xi) = \Omega\varepsilon(\xi)$, the first part follows from Proposition 2.2. The equivalent representation (3.8) follows from the following lemma by induction on m with $F_a = Z_a$.

Lemma 3.3. *Suppose that $F = (F_\alpha)$ and $F' = (F'_\beta)$ transform covariantly according to some representation τ and τ'. Then*

$$(F \times F')_{\alpha\beta}\Omega = F_\alpha F'_\beta \Omega . \tag{3.9}$$

Proof of Lemma 3.3.

$$\begin{aligned} F_\alpha F'_\beta \Omega &= (F \times F')_{\gamma\delta} (\tau_{\gamma\alpha} \otimes \tau'_{\delta\beta} \otimes \mathrm{id}) (\varphi^{-1}) \Omega \\ &= (F \times F')_{\gamma\delta} \Omega (\tau_{\gamma\alpha} \otimes \tau'_{\delta\beta}) (\Delta(e)) . \end{aligned}$$

This follows from Eq. (1.23), i.e. $(\mathrm{id} \otimes \mathrm{id} \otimes \varepsilon)(\varphi^{-1}) = \Delta(e)$. The assertion of Lemma 3.3 follows now from covariance of $(F \times F')$ and Corollary 2.7. q.e.d.

We will use multiindices $\alpha = (a_n \dots a_1)$ $(a_i = 1 \dots N)$ said to be of length $|\alpha| = n$. If $\alpha = (a_n \dots a_1)$ and $\beta = (b_m \dots b_1)$ we write

$$\alpha \vee \beta = (a_n \dots a_1 b_m \dots b_1) .$$

We introduce a shorthand notation for n-fold tensor products of fundamental N-dimensional representations

$$\tau^{(n)} = (\tau^f \bigotimes (\tau^f \bigotimes \dots \bigotimes (\tau^f \bigotimes \tau^f) \dots)) \quad (n \text{ factors}) . \tag{3.10}$$

$\tau^{(n)}_{\alpha\beta}$ is defined for multiindices α, β of length n, and $\tau^{(0)} = \varepsilon$.

Lemma 3.4. $\tau^{(n)}_{\delta\vee\alpha\,\varepsilon\vee\beta}(\xi)\tau^{(m)}_{\beta\gamma}(e) = \tau^{(n)}_{\delta\vee\alpha\,\varepsilon\vee\gamma}(\xi)$ *for* $n \geq m$.

Proof. Lemma 3.4 follows straight from the definitions and the homomorphism property of Δ. For instance

$$\begin{aligned}\tau^{(3)}_{d\vee\alpha\, e\vee\beta}(\xi)\tau^{(2)}_{\beta\gamma}(e) &= (\tau^f \otimes \tau^f \otimes \tau^f)_{d\vee\alpha\, e\vee\gamma}((\mathrm{id}\otimes\Delta)\Delta(\xi)\,(e\otimes\Delta(e))) \\ &= (\tau^f \otimes \tau^f \otimes \tau^f)_{d\vee\alpha\, e\vee\gamma}((\mathrm{id}\otimes\Delta)\Delta(\xi)) \\ &= \tau^{(3)}_{d\vee\alpha\, e\vee\gamma}(\xi)\end{aligned}$$

since

$$(\mathrm{id}\otimes\Delta)\Delta(\xi)\,(e\otimes\Delta(e)) = (\mathrm{id}\otimes\Delta)\,(\Delta(\xi)e\otimes e) = (\mathrm{id}\otimes\Delta)\Delta(\xi)\,. \tag{3.11}$$

This generalizes to arbitrary $n \geq m$ in an obvious way. q.e.d.

For multiindices $\alpha = (a_n \dots a_1)$, $\beta = (b_n \dots b_1)$ of length $n \geq k+1$ we define the permutation matrices

$$P^k_{\alpha\beta} = \delta_{a_k b_{k+1}}\,\delta_{a_{k+1}\, b_k} \prod_{i\neq k,\, k+1} \delta_{a_i b_i}\,. \tag{3.12}$$

Theorem 3.5. *Define matrices $T^{(n)}(\sigma^\pm_k)$ ($n \geq 2$, $k = 1 \dots n-1$) with multiindices α, β of length n as follows. Set $R^+ = R$ and $R^- = R^{-1'}$, where $'$ stands for interchange of factors in $\mathscr{G}^* \otimes \mathscr{G}^*$. Define*

$$T^{(n)}_{\alpha\beta}(\sigma^\pm_{n-1}) = P^{n-1}_{\alpha\gamma}(\tau^f \otimes \tau^f \otimes \tau^{(n-2)})_{\gamma\beta}(\varphi_{213}(R^\pm \otimes e)\varphi^{-1})\,, \tag{3.13}$$

$$T^{(n)}_{\delta\vee\alpha\,\varepsilon\vee\beta}(\sigma^\pm_k) = \tau^{(n)}_{\delta\vee\alpha\,\varepsilon\vee\gamma}(e)T^{(k+1)}_{\gamma\beta}(\sigma^\pm_k) \quad \text{for} \quad k \leq n-2\,. \tag{3.14}$$

This defines a representation of the braid group B_n with $T^{(n)}(\iota) = \tau^{(n)}(e)$.

For $n = 2$ the expression for $T^{(n)}$ simplifies, because $\tau^{(0)} = \varepsilon$,

$$T^{(2)}_{\alpha\beta}(\sigma^{\pm 1}_1) = P^1_{\alpha\gamma}(\tau^f \otimes \tau^f)_{\gamma\beta}(R^\pm)\,. \tag{3.15}$$

The unit element ι is not in general represented by the unit matrix. But this unusual feature disappears when the action is restricted to the "true representation space" of $\mathscr{G}^*$ on which $\tau^{(n)}(e)$ acts as the identity.

Proof of Theorem 3.5.. This is a corollary of Theorem 1.1. Using the notation of Theorem 1.1 and the equality $P^k_{\alpha\beta} = (\otimes^n_1 \tau^f(e^{n-k-1} \otimes \mathscr{P} \otimes \varepsilon^{k-1}))_{\alpha\beta}$,

$$T^{(n)}(\sigma^\pm_k) = \otimes^n_1 \tau^f(\sigma^{n\pm}_k)\,. \tag{3.16}$$

Since the $T^{(n)}(\sigma^\pm_k)$ appear as a homomorphic image of $\sigma^{n\pm}_k$, Artin relations for $T^{(n)}(\sigma^\pm_k)$ follow from those of $\sigma^{n\pm}_k$. Even if $\mathscr{G}^*$ is not semisimple, the assertion of Theorem 3.5 holds. In this case we can construct $T^{(n)}(\sigma^\pm_k)$ from maps $\varsigma^{n\pm}_k$ defined in the remark after Theorem 1.1,

$$T^{(n)}(\sigma^\pm_k)_{\alpha\beta} = P^k_{\alpha\gamma}(\otimes^n_1 \tau^f(\mathbf{P}^n_k \varsigma^{n\pm}_k)\,. \tag{3.17}$$

This completes the proof of Theorem 3.5. q.e.d.

Theorem 3.6. *The $\mathscr{B}$-module $\mathscr{F}^T$ admits an alternative description as follows. $\mathscr{F}^T = \bigoplus_{n\geq 0} \mathscr{F}^{T(n)}$ and $\mathscr{F}^{T(n)}$ is spanned by elements z^n_α, $\alpha = (a_n \dots a_1)$, with complex linear dependencies among them as follows:*

(i) $z^n_\alpha = z^n_\beta \tau^{(n)}_{\beta\alpha}(e)$, *truncation*,
(ii) $z^n_\alpha = z^n_\beta c_R T^{(n)}_{\beta\alpha}(\sigma_k)$, $k = 1 \dots n-1$, *braid invariance*.
Elements $\xi \in \mathscr{G}^*$ *act according to*

$$\xi z^n_\alpha = z^n_\beta \tau^{(n)}_{\beta\alpha}(\xi)\,, \tag{3.18}$$

where $\tau^{(n)}$ *is the n-fold tensor product* (3.10) *of fundamental representations. Generators* $Z_a \in \mathscr{B}$ *act according to*

$$Z_a z^n_\alpha = z^{n+1}_{a\vee\alpha}\,.$$

We remark that the truncation identity (i) follows from braid invariance (ii) if the latter is postulated also for σ_k^{-1}.

Proof of Theorem 3.6. The transformation law follows from Theorem 2.4, 3.1, and 3.2.

$$\begin{aligned}\xi z^n_\alpha &= \xi(Z\times(\dots(Z\times Z)\dots))_\alpha\Omega = (Z\times(\dots(Z\times Z)\dots))_\beta(\tau^{(n)}\otimes\mathrm{id})_{\beta\alpha}(\Delta(\xi))\Omega\\ &= (Z\times(\dots(Z\times Z)\dots))_\beta\Omega\tau^{(n)}_{\beta\alpha}((\mathrm{id}\otimes\varepsilon)\Delta(\xi)) = z^n_\beta\tau^{(n)}_{\beta\alpha}(\xi)\end{aligned}$$

since $(\mathrm{id}\otimes\varepsilon)\Delta(\xi) = \xi$.

The action of Z_a follows from the original definition of Z^n_α, with $z^0 = \Omega$. It only remains to show validity of the linear relations and that they are in fact the only ones.

Since $\varepsilon(e) = 1$ by the homomorphism property, $e\Omega = \Omega$ follows from Theorem 3.1. Since $eZ_a = Z_a$ it follows that $ez^n_\alpha = z^n_\alpha$ for all α, n. This shows that (i) holds. Acting on this equation with $Z_{a_m}\dots Z_{a_{n+1}}$ $(m > n)$ gives no new relation because of Lemma 3.4 with $\xi = e$. The consequences of $Z_a e = Z_a$ are implied by $e\Omega = \Omega$ and $Z_a e Z_b = Z_a Z_b$, which follows from $eZ_b = Z_b$. Therefore this gives no new relations either. Therefore, the implication of the relation (i) of Definition 2.1 are embodied in (i) of Theorem 3.5.

Now we turn to the consequences of the braid relations (3) of Definition 2.1. Let $\alpha = (a_n \dots a_1) = (a_n a_{n-1}\alpha')$. Using covariance property (3.18) we compute

$$\begin{aligned} z^n_\alpha &= Z_{a_n} Z_{a_{n-1}} z^{n-2}_{\alpha'} = Z_{b_{n-1}} Z_{b_n} c_R \mathscr{R}_{b_{n-1}b_n a_n a_{n-1}} z^{n-2}_{\alpha'}\\ &= Z_{b_n} Z_{b_{n-1}} z^{n-2}_{\beta'} c_R (\tau^f_{b_{n-1}a_n} \otimes \tau^f_{b_n a_{n-1}} \otimes \tau^{(n-2)}_{\beta'\alpha'})(\varphi_{213}(R\otimes e)\varphi^{-1})\\ &= z^n_\beta c_R T^{(n)}_{\beta\alpha}(\sigma_{n-1})\,.\end{aligned}$$

Applying $Z_{a_m}\dots Z_{a_{n+1}}$ to both sides we obtain

$$z^m_\gamma = z^m_\delta c_R T^{(m)}_{\delta\gamma}(\sigma_{n-1}) \quad \text{for} \quad m \geq n\,. \tag{3.19}$$

The factor $\tau^{(m)}(e)$ in the definition of $T^{(m)}(\sigma_{n-1})$ is irrelevant because of relation (i) which was already established. This proves (ii). Conversely it is clear that all the implications of the braid relations (3) of Definition 2.1 were found, since we examined the effect of the interchange of an arbitrary pair $Z_{a_n}Z_{a_{n-1}}$ in $Z_{a_m}\dots Z_{a_n}Z_{a_{n-1}}\dots Z_{a_1}\Omega$. q.e.d.

4. The Algebra $\mathscr{F}^T$

We define a product $\cdot$ in $\mathscr{F}^T$ which makes $\mathscr{F}^T$ into a not necessarily associative algebra.

Definition 4.1 (Product in $\mathscr{F}^T$). *Let $Z^0 = e$ and*

$$Z^n_{a_n \ldots a_1} = (Z \times (Z \times \ldots \times (Z \times Z) \ldots))_{a_n \ldots a_1} \quad \textit{for} \quad n \geq 1\,, \tag{4.1}$$

so that $z^n_\alpha = Z^n_\alpha \Omega$. Define

$$(z^n_\alpha \cdot z^m_\beta) = (Z^n \times Z^m)_{\alpha\beta} \Omega \tag{4.2}$$

for $n, m \geq 0$. In particular, $\Omega = z^0$ is the identity of the algebra $\mathscr{F}^T$.

We show below that Definition 4.1 is meaningful. Before we turn to the proof, we list some properties of the algebra $\mathscr{F}^T$.

Theorem 4.2. (i) *Generators $Z_a \in \mathscr{B}$ act on $\mathscr{F}^T$ by multiplication with $z_a = z^1_a$*

$$Z_a f = (z_a \cdot f)\,.$$

(ii) *$\xi \in \mathscr{G}^*$ acts on the algebra $\mathscr{F}^T$ as a generalized derivation*

$$\xi z^n_\alpha = z^n_\beta \tau^{(n)}_{\beta\alpha}(\xi)\,, \tag{4.3}$$

$$\xi(f^1 \cdot f^2) = m(\Delta(\xi)\,[f^1 \otimes f^2])\,, \tag{4.4}$$

where $\tau^{(n)}$ is the n-fold tensor product $\tau^f \bigotimes(\tau^f \bigotimes \ldots \bigotimes(\tau^f \bigotimes \tau^f)\ldots))$ of fundamental representations τ^f of $\mathscr{G}^*$ and $m : \mathscr{F}^T \otimes \mathscr{F}^T \to \mathscr{F}^T$ is multiplication in $\mathscr{F}^T$.
(iii) *The algebra $\mathscr{F}^T$ is quasi-associative in the sense that the product $z_{a_n} \ldots z_{a_1}$ with arbitrary specification of the position of brackets can be written as a complex linear combination of products $z_{b_n} \ldots z_{b_1}$ with any other specification of brackets. Reassociation is performed with the help of the formulae*

$$((f_\alpha \cdot f'_\beta) \cdot f''_\gamma) = (f_\delta \cdot (f'_\varepsilon \cdot f''_\kappa))\,(\tau_{\alpha\delta} \otimes \tau'_{\beta\varepsilon} \otimes \tau''_{\gamma\kappa})\,(\varphi)\,, \tag{4.5}$$

$$(f_\delta \cdot (f'_\varepsilon \cdot f''_\kappa)) = ((f_\alpha \cdot f'_\beta) \cdot f''_\gamma)\,(\tau_{\delta\alpha} \otimes \tau'_{\varepsilon\beta} \otimes \tau''_{\kappa\gamma})\,(\varphi^{-1})\,. \tag{4.6}$$

They are valid if $f, f', f'' \in \mathscr{F}^T$ transform according to representations τ, τ', τ'' of $\mathscr{G}^*$.
(iv) *The product $\cdot$ is braid-commutative in the sense that*

$$(z^n_\alpha \cdot z^m_\beta) = (z^m_\gamma \cdot z^n_\delta) c^{nm}_R (\tau^{(n)}_{\delta\alpha} \otimes \tau^{(m)}_{\gamma\beta})\,(R)\,. \tag{4.7}$$

The generalized derivation property can be written as follows. Let $\Delta(\xi) = \sum_\sigma \xi^1_\sigma \otimes \xi^2_\sigma$. Then

$$\xi(f^1 \cdot f^2) = \sum_\sigma (\xi^1_\sigma f^1 \cdot \xi^2_\sigma f^2)\,. \tag{4.8}$$

Remark. We may convert Definition 4.1 into an explicit formula which exhibits $z^n_\alpha \cdot z^n_\beta$ as a sum of terms z^{n+m}_γ with complex coefficients. Straightforward computation leads to the result

$$(z^n_\alpha \cdot z^m_\beta) = z^{n+m}_\mu M^{n,m}_{\mu\alpha\beta}\,, \tag{4.9}$$

with

$$M^{n,m}_{\mu\,\alpha\beta} = \left(\bigotimes_1^{n+m} \tau^f\right)_{\mu\,\alpha\vee\beta} (\Phi_{n-2} \ldots \Phi_1 \Phi_0)\,, \tag{4.10}$$

$$\Phi_k = e^k \otimes ((\mathrm{id} \otimes \Delta^{n-2-k} \otimes \Delta^{m-1})\,(\varphi))\,, \qquad (0 \leq k \leq n-2) \tag{4.11}$$

in the notation (1.25)f.

Proof that Definition 4.1 is meaningful. It must be verified that the definition of the product is consistent with the linear dependencies among the elements z^n_α. We must therefore show that

$$(z^n_\alpha \cdot z^m_\gamma)\tau^{(m)}_{\gamma\beta}(e) = (z^n_\alpha \cdot z^m_\beta)\,, \tag{4.12}$$

$$(z^n_\delta \cdot z^m_\beta)\tau^{(n)}_{\delta\alpha}(e) = (z^n_\alpha \cdot z^m_\beta)\,, \tag{4.13}$$

and

$$(z^n_\alpha \cdot z^m_\gamma)c_R T^{(m)}_{\gamma\beta}(\sigma_k) = (z^n_\alpha \cdot z^m_\beta)\,, \tag{4.14}$$

$$(z^n_\delta \cdot z^m_\beta)c_R T^{(n)}_{\delta\alpha}(\sigma_k) = (z^n_\alpha \cdot z^m_\beta) \tag{4.15}$$

for $1 \le k \le m-1$ and for $1 \le k \le n-1$, respectively.

We wrote the definition of the product $\cdot$ in manifestly covariant form, but Lemma 3.3 tells us that it can also be written in the equivalent form

$$(z^n_\alpha \cdot z^m_\beta) = Z^n_\alpha Z^m_\beta \Omega\,. \tag{4.16}$$

Validity of Eqs. (4.12 ... 4.15) will therefore follow from the identities

$$Z^r_\beta \tau^{(r)}_{\beta\alpha}(e) = Z^r_\alpha\,, \tag{4.17}$$

and from

Proposition 4.3. *$Z^r_\beta c_R T^{(r)}_{\beta\alpha}(\sigma_k) = Z^r_\alpha$ for $1 \le k \le n-1$.*

Equation (4.17) follows from Corollary 2.7, and Proposition 4.3 is proven in Appendix C. This shows that Definition 4.1 is meaningful. q.e.d.

Proof of Theorem 4.2. (i) follows from Eq. (4.16). (ii) follows from Theorem 3.1, the covariance of $Z^n \times Z^m$ (Theorem 2.4), and the definition of the tensor product of representations. (iii) is a corollary of the reassociation identities, Theorem 2.6. (iv) follows from the braid relations for composites, Theorem 2.9. q.e.d.

5. The Structure of the $\mathscr{G}^*$-Module $\mathscr{F}^T$

In this section we specialize to the truncated quantum group algebra $\mathscr{G}^*$ associated with $U_q(sl_2)$. The irreducible representations τ^J of $\mathscr{G}^*$ are the "physical" representations of $U_q(sl_2)$, i.e. the irreducible representations of $U_q(sl_2)$ with nonzero quantum dimension. They are labelled by

$$J = 0,\ \tfrac{1}{2},\ \ldots\ ,\ \tfrac{1}{2}(p-2)\,,$$

and have dimension $2J+1$, if q is a p^{th} root of unity. We assume $p \ge 4$ for simplicity.

Theorem 5.1 (Structure of $\mathscr{F}^T$). *Let $\mathscr{G}^*$ be the truncated quantum group algebra associated with $U_q(sl_2)$, q a primitive p^{th} root of unity, $p \ge 4$, and let τ^f be its fundamental 2-dimensional representation. Then*
(i) $\mathscr{F}^{T(n)} = 0$ *for all* $n \ge p-1$.
(ii) $\mathscr{F}^{T(n)}$ *carries the* $n+1$*-dimensional irreducible representation of* $\mathscr{G}^*$ *if* $n \le p-2$.
(iii) $\mathscr{F}^T$ is associative.

The main ingredients of the proof are in the following two lemmas.

Lemma 5.2. $\dim \mathscr{F}^{T(n)} = n+1$ *if* $n \le p-2$.

Lemma 5.3. *Suppose that $\mathscr{F}^{T(r)}$ carries the irreducible representation τ and $\mathscr{F}^{T(s)}$ carries the irreducible representation τ'. Then $\mathscr{F}^{T(r+s)}$ is either zero or it carries a subrepresentation of $\tau \bigotimes \tau'$.*

Proof of Lemma 5.2. We recall that the tensor product of representations of $\mathscr{G}^*$ is the truncated tensor product of physical representations of $U_q(sl_2)$, see Sect. 1. No truncation appears in the tensor product $\tau^I \bigotimes \tau^J$ if $I + J \leq p - 2 - I - J$, i.e. if $I+J \leq \frac{1}{2}(p-2)$. Therefore the tensor product $\tau^{(n)}$ of the n fundamental 2-dimensional representations τ^f is untruncated if $n \leq p-2$. It follows that $\tau^{(n)}(e) = 1$ in this case, so that the linear relations in Theorem 3.6 come from braid symmetrization only, and are the same as if we used the invertible $U_q(sl_2)$ R-element R_q in place of R. But in the nontruncated quantum plane, the braid relations reduce to [1, 2, 3]

$$z_1 z_2 = q^{-1/2} z_2 z_1 \,. \tag{5.1}$$

This can be used to shift all factors z_1 to the right of all factors z_2. There can be $0 \ldots n$ factors z_1. It follows that the number of linearly independent vectors in $\mathscr{F}^{T(n)}$ is $n+1$. q.e.d.

Proof of Lemma 5.3. Because of quasi-associativity of the product in $\mathscr{F}^T$, $\mathscr{F}^{T(r+s)}$ is spanned by elements of the form $(w \cdot v)$, with $w \in \mathscr{F}^{T(r)}$ and $v \in \mathscr{F}^{T(s)}$. By hypothesis $\mathscr{F}^{T(r)}$ carries representation τ and $\mathscr{F}^{T(s)}$ carries representation τ'. Since $\xi \in \mathscr{G}^*$ act on $\mathscr{F}^T$ as generalized derivations by Theorem 4.2((ii), it follows that $\mathscr{F}^{T(r+s)}$ carries a subrepresentation of $\tau \bigotimes \tau'$ if it is not zero. q.e.d.

Proof of Theorem 5.1. Validity of Theorem 5.1(ii) for $0 \leq n \leq (p-2)$ follows from the two lemmas and the tensor product decomposition (1.33, 1.31) applied to $\tau^J \otimes \tau^{1/2}$.

To prove the first part of Theorem 5.1 it suffices to show that $\mathscr{F}^{T(p-1)} = 0$ since all higher order polynomials contain factors of order $p-1$. In the following $P^J \in \mathscr{G}^*$ should denote the minimal central element of $\mathscr{G}^*$ that is associated with the irreducible $2J+1$-dimensional representation of $\mathscr{G}^*$. To prove $\mathscr{F}^{T(p-1)} = 0$ we use that

$$(Z \times Z)_{ab} (\tau^f_{ac} \otimes \tau^f_{bd}) (\Delta(P^0)) = 0 \,. \tag{5.2}$$

This expresses the fact that there are no homogeneous polynomials of degree two which transform according to the trivial representation. A formal proof can be found in Appendix D. Moreover we know that all polynomials of degree $p-2$ transform according to the $p-1$-dimensional representation. In mathematical terms this means that

$$(Z^{p-3} \times Z)_{\alpha a} = (Z^{p-3} \times Z)_{\beta b} (\tau^{(p-3)}_{\beta\alpha} \otimes \tau^f_{ba}) (\Delta(P^{\frac{1}{2}(p-2)})) \,. \tag{5.3}$$

We wish to show that

$$(Z^{p-3} \times (Z \times Z))_{\alpha ab} = 0 \,.$$

In order to establish this we will show that all the polynomials $((Z^{p-3} \times Z) \times Z)_{\beta cd}$ are linear combinations of the polynomials

$$(Z^{p-3} \times (Z \times Z))_{\alpha ab} (\tau^{\frac{1}{2}(p-3)}_{\alpha\beta} \otimes \tau^f_{ac} \otimes \tau^f_{bd}) (e \otimes \Delta(P^0))$$

which vanish due to (5.2). More precisely we show that

$$\begin{aligned} &((Z^{p-3} \times Z) \times Z)_{\beta cd} a \\ &\quad = (Z^{p-3} \times (Z \times Z))_{\alpha ab} (\tau^{\frac{1}{2}(p-3)}_{\alpha\beta} \otimes \tau^f_{ac} \otimes \tau^f_{bd}) ((e \otimes \Delta(P^0))A) \,, \end{aligned} \tag{5.4}$$

where $a \neq 0$ is a real number and $A \in \mathcal{G}^* \otimes \mathcal{G}^* \otimes \mathcal{G}^*$ is given by $A \equiv \varphi(\Delta(P^{\frac{1}{2}(p-2)}) \otimes e)$. It follows from eq. (5.3) by reassociation that the right-hand side of Eq. (5.4) equals

$$((Z^{p-3} \times Z) \times Z)_{\alpha ab}(\tau_{\alpha\beta}^{\frac{1}{2}(p-3)} \otimes \tau_{ac}^f \otimes \tau_{bd}^f)((\Delta(P^{\frac{1}{2}(p-2)}) \otimes e)\varphi^{-1}(e \otimes \Delta(P^0))A) . \quad (5.5)$$

Using the explicit expression (1.34) for φ it is easy to see that

$$(\Delta(P^{\frac{1}{2}(p-2)}) \otimes e)\varphi^{-1}(e \otimes \Delta(P^0))A = (\Delta(P^{\frac{1}{2}(p-2)}) \otimes e)a ,$$

where a is some nonzero real number. So we get finally that the right-hand side of Eq. (5.4) equals

$$\begin{aligned}&((Z^{p-3} \times Z) \times Z)_{\alpha ab}(\tau_{\alpha\beta}^{\frac{1}{2}(p-3)} \otimes \tau_{ac}^f \otimes \tau_{bd}^f)(\Delta(P^{\frac{1}{2}(p-2)}) \otimes e)a \\ &= ((Z^{p-3} \times Z) \times Z)_{\beta cd}a \end{aligned} \quad (5.6)$$

by (5.3). This completes the proof that all products of the type $(Z^{p-3} \times Z) \times Z$ vanish. But since all other products of $p-1$ generators Z can be obtained out of these by reassociation, we established that $\mathcal{F}^{T(p-1)} = 0$. This concludes the proof of Theorem 5.1(i).

The associativity of $\mathcal{F}^T$ (Theorem 5.1(iii)) is by now a simple consequence of some arguments displayed above. In the proof of Lemma 5.2 we have seen that for polynomials of total degree $n \leq p-2$ there are no relations from truncation. This means that in such polynomials brackets can be moved (and removed) as in the untruncated (associative) quantum plane. Theorem 5.1(i) states that nonvanishing polynomials of degree $n > p-2$ do not exist.

6. Quasi-Associative Exterior Differential Calculus

We return to the consideration of general quasi-quantum group algebras as described in Sect. 1.

In this section we extend the algebra $\mathcal{F}^T$ to a quasi-associative algebra

$$\Lambda\mathcal{F}^T = \bigoplus_{n\geq 0} \Lambda^n\mathcal{F}^T \quad (6.1)$$

by adjoining θ_b to $\mathcal{F}^T \cdot \Lambda^0\mathcal{F}^T = \mathcal{F}^T$. The space $\Lambda^n\mathcal{F}^T$ will be spanned by elements of the form

$$x_{\alpha\beta}^n = (z_{a_m} \cdot (z_{a_{m-1}} \cdot \ldots (z_{a_1} \cdot (\theta_{b_n} \cdot (\theta_{b_{n-1}} \cdot \ldots (\theta_{b_2}\theta_{b_1}) \ldots))) \ldots)) , \quad (6.2)$$

with $\alpha = (a_m \ldots a_1)$, $\beta = (b_n \ldots b_1)$, $m = 0, 1, \ldots$. The quasi-associative product in $\Lambda\mathcal{F}^T$ is written as $\cdot$ in what follows.

Under restrictive assumptions on the representation τ^f an exterior derivative d will be defined which acts on $\Lambda\mathcal{F}^T$ and enjoys the standard properties

$$d: \Lambda^n\mathcal{F}^T \mapsto \Lambda^{n+1}\mathcal{F}^T , \quad (6.3)$$

$$dz_a = \theta^a , \quad (6.4)$$

$$d^2 = 0 , \quad (6.5)$$

$$\begin{aligned}&d(x \cdot y) = (dx \cdot y) + (-1)^n(x \cdot dy) \\ &\qquad \text{if } x \in \Lambda^n\mathcal{F}^T,\ y \in \Lambda\mathcal{F}^T \text{ (Leibniz rule)} . \end{aligned} \quad (6.6)$$

The action of $\mathscr{G}^*$ on $\mathscr{F}^T$ extends to an action on $\Lambda\mathscr{F}^T$ by generalized derivations.

We will obtain the quasi-associative algebra $\Lambda\mathscr{F}^T$ as a quotient from an associative algebra

$$\Lambda\mathscr{B} = \bigoplus_{n\geq 0} \Lambda^n\mathscr{B} \tag{6.7}$$

which is generated by $\mathscr{G}^*$ and Z_a, Θ_a subject to the relations (e = unit element of $\mathscr{G}^*$)

$$Z_a e = eZ_a = Z_a\,, \tag{6.8}$$
$$\Theta_a e = e\Theta_a = \Theta_a\,, \tag{6.9}$$
$$\xi Z_a = Z_b(\tau^f_{ba}\otimes \mathrm{id})(\Delta(\xi)) \quad \text{and} \tag{6.10}$$
$$\xi\Theta_a = \Theta_b(\tau^f_{ba}\otimes \mathrm{id})(\Delta(\xi)) \quad \text{for} \quad \xi\in\mathscr{G}^*\,, \tag{6.11}$$
$$(Z\times Z)_{ab} = (Z\times Z)_{dc}c_B^{-1}(\tau^f_{ca}\otimes\tau^f_{db})(R)\,, \tag{6.12}$$
$$(\Theta\times\Theta)_{ab} = (\Theta\times\Theta)_{dc}c_A^{-1}(\tau^f_{ca}\otimes\tau^f_{db})(R)\,, \tag{6.13}$$
$$(\Theta\times Z)_{ab} = -(Z\times\Theta)_{dc}c_A^{-1}(\tau^f_{ca}\otimes\tau^f_{db})(R)\,. \tag{6.14}$$

The phase factor $c_B = c_R^{-1}$ in our previous notation, and $c_A \neq c_B$ is another phase factor which remains arbitrary for now. Using these relations, a general element of $\Lambda^n\mathscr{B}$ can be exhibited as a c-linear combination of elements of the form

$$Z_{a_n}\dots Z_{a_1}\Theta_{b_m}\dots\Theta_{b_1}\xi\,. \tag{6.15}$$

The homomorphism (counit) $\varepsilon:\mathscr{G}^*\mapsto \mathbf{C}$ induces a map ε to cosets similarly as for $\mathscr{B}$,

$$\varepsilon:\Lambda\mathscr{B}\mapsto\Lambda\mathscr{F}^T = \Lambda\mathscr{B}/\mathscr{G}^*\,. \tag{6.16}$$

We use again a special notation for the image of the unit element.

$$\Omega = \varepsilon(e)\,.$$

It is $\mathscr{G}^*$-invariant,

$$\xi\Omega = \Omega\varepsilon(\xi)\,\text{for} \quad \xi\in\mathscr{G}^*\,, \tag{6.17}$$

Theorem 6.1. *$\Lambda^n\mathscr{F}^T = \bigoplus_{n\geq 0}\Lambda\mathscr{F}^T$ and the space $\Lambda^n\mathscr{F}^T$ of n-forms is spanned by elements*

$$x^n_{\alpha\beta} = Z_{a_m}\dots Z_{a_1}\Theta_{b_n}\dots\Theta_{b_1}\Omega \tag{6.18}$$

with $\alpha = (a_m\dots a_1)$, $\beta = (b_n\dots b_1)$, $m = 0,1,\dots$.

Proof of Theorem 6.1. Theorem 6.1 follows from the observation (6.15). q.e.d.

The elements $x^n_{\alpha\beta}$ can be rewritten in covariant form. Set

$$X^n_{\alpha\beta} = (Z\times(Z\times\dots(Z\times(\Theta\times\dots\times(\Theta\times\Theta)\dots))\dots))_{\alpha\beta}\Omega\,. \tag{6.19}$$

Then

$$x^n_{\alpha\beta} = X^n_{\alpha\beta}\Omega\,. \tag{6.20}$$

This follows from familiar arguments, using $(\mathrm{id}\otimes\mathrm{id}\otimes\varepsilon)(\varphi) = \Delta(e)$.

Finally we make $\Lambda\mathscr{F}^T$ into a quasi-associative algebra.

Definition 6.2a (Product in $\Lambda\mathscr{F}^T$). *Using the notation* (6.19, 6.20), *a not necessarily associative product* $\cdot$ *can be defined in* $\Lambda\mathscr{F}^T$ *by*

$$(x^n_{\alpha\beta} \cdot x^m_{\gamma\delta}) = (X^n \times X^m)_{\alpha\beta,\gamma\delta}\Omega\,, \tag{6.21}$$

or, equivalently, by $(x^n_{\alpha\beta} \cdot x^m_{\gamma\delta}) = X^n_{\alpha\beta}X^m_{\gamma\delta}\Omega$. Ω *acts as unit element in* $\Lambda\mathscr{F}^T$.

Theorem 6.2b. *The product* (6.21) *in* $\Lambda\mathscr{F}^T$ *is well defined and quasi-associative. It makes* $\mathscr{F}^T$ *into a graded algebra.* $\xi \in \mathscr{G}^*$ *acts on* $\Lambda\mathscr{F}^T$ *as a generalized derivation.*

It is obvious that products of elements in $\Lambda^n\mathscr{F}^T$ and $\Lambda^p\mathscr{F}^T$ are in $\Lambda^{n+p}\mathscr{F}^T$. The rest of the theorem is proven in the same way as it was proven for $\mathscr{F}^T$ (Theorem 4.2).

If we set

$$z_a = \varepsilon(Z_a)\,; \qquad \theta_a = \varepsilon(\Theta_a)\,, \tag{6.22}$$

then

$$x^n_{\alpha\beta} = (z_{a_m} \cdot (z_{a_{m-1}} \cdot \ldots (z_{a_1} \cdot (\theta_{b_n} \cdot (\theta_{b_{n-1}} \cdot \ldots (\theta_{b_2}\theta_{b_1})\ldots)))\ldots))\,. \tag{6.23}$$

Thus, Z_a acts as multiplication by z_a, and Θ_a acts as multiplication by θ_a. The algebra has properties very much like the properties of the algebra $\mathscr{F}^T$ stated in Theorem 4.2, i.e. it is quasi-associative and (graded) braid commutative. Braid-commuting two factors θ gives an extra $c_B c_A^{-1}$-factor compared to the braid-commutation of z. This generalizes the anticommutation of differential forms.

Let us finally investigate under which assumptions an exterior derivative d can be adjoined to $\Lambda\mathscr{B}$. It is required to be $\mathscr{G}^*$-invariant,

$$d\xi = \xi d \quad \text{for} \quad \xi \in \mathscr{G}^*\,, \tag{6.24}$$

and subject to the relations

$$\begin{aligned} d^2 &= 0\,, \\ dZ_a &= \Theta_a + Z_a d \\ d\theta_a &= -\Theta_a d\,. \end{aligned} \tag{6.25}$$

These relations are actually not independent. The last relation follows from the first two relations, as is seen by multiplying the second one with d from left or right. However, consistency of (6.25) with braid relations (6.12, 6.13, 6.14) does not hold in general. As an example for the checks to be done, let us multiply (6.12) by d from the left. After shifting d to the right by application of (6.25) we obtain

$$(Z \times \Theta)_{ab} + (\Theta \times Z)_{ab} = ((Z \times \Theta)_{dc} + (\Theta \times Z)_{dc})c_B^{-1}(\tau^f_{ca} \otimes \tau^f_{db})(R)\,. \tag{6.26}$$

In this expression we already subtracted terms of the type $(Z \times Z)d$. Now we use (6.14) to deduce

$$(Z \times \Theta)_{ab}(\tau^f_{be} \otimes \tau^f_{af})(e \otimes e - c_A^{-1}R)(\tau^f_{fc} \otimes \tau^f_{ed})(e \otimes e - c_B^{-1}R) = 0\,. \tag{6.27}$$

When applied to an element in $\Lambda^r\mathscr{F}^T$ this would give rise to a new linear relation in $\Lambda^{r+1}\mathscr{F}^T$, if the coefficients of the $Z \times \Theta$-terms are nonzero. In conclusion, for an exterior derivative d with properties (6.25) to exist, we have to assume

$$(\tau^f_{be} \otimes \tau^f_{af})(e \otimes e c_A - R)(\tau^f_{fc} \otimes \tau^f_{ed})(e \otimes e c_B - R) = 0\,. \tag{6.28}$$

This means that the matrix $(\tau^f \otimes \tau^f)(R)$ should have only two distinct eigenvalues. We will prove in Sect. 8 and Appendix D that this is also sufficient.

Theorem 6.3 (Leibniz rule). *Assume that* (6.25) *are consistent with the braid relations. Defining* $d\Omega = 0$ *we have*

$$dz_a = \theta_a \,, \tag{6.29}$$

$$d(x \cdot y) = (dx \cdot y) + (-1)^n (x \cdot dy) \quad \text{if} \quad x \in \Lambda^n \mathscr{F}^T,\ y \in \Lambda \mathscr{F}^T. \tag{6.30}$$

To prove this we will need

Lemma 6.4. *Let* $X'^{n}_{\alpha\beta} = dX^n_{\alpha\beta} - (-1)^n X^n_{\alpha\beta} d$. *Then* $X'^{n}_{\alpha\beta}$ *is a linear combination of elements* $X_{\mu\nu}$ *with complex coefficients, and* $X'^{n}_{\alpha\beta}$ *transforms like* $X^n_{\alpha\beta}$ *under* $\mathscr{G}^*$.

Proof of Lemma 6.4. Since d is $\mathscr{G}^*$-invariant we have $dF = (d \times F)$ and $(d \times Z)_a = (Z \times d)_a + \Theta_a$, $(d \times \Theta)_b = -(\Theta \times d)_b$. We push d through the factors Z_a, Θ_b in $X^n_{\alpha\beta}$. To apply the formulae just mentioned one needs to do various reassociations. Further reassociations are needed in order to use the covariant $\Theta \times Z$ braid relation to push emerging factors Θ to the right. But all the reassociations involve only taking complex linear combinations. This shows that $X^{n'}_{\alpha\beta}$ is a linear combination of elements $X^r_{\mu\nu}$ with complex coefficients. Covariance follows because d is $\mathscr{G}^*$-invariant, and $r = n + 1$ by counting factors Θ. This proves Lemma 6.4. q.e.d.

Proof of Theorem 6.3. By definition

$$d(x \cdot y) = d(X^n \times X^m)_{\alpha\beta\,\gamma\delta}\Omega\,.$$

Since d transforms according to the trivial 1-dimensional representation ε of $\mathscr{G}^*$ we have that

$$\begin{aligned} d(X^n \times X^m)_{\alpha\beta\,\gamma\delta} &= (d \times (X^n \times X^m))_{\alpha\beta\,\gamma\delta} \\ &= ((d \times X^n) \times X^n)_{\alpha\beta\,\gamma\delta} \\ &= (-1)^n((X^n \times d) \times X^m)_{\alpha\beta\,\gamma\delta} + (X^{n'} \times X^m)_{\alpha\beta\,\gamma\delta} \end{aligned}$$

because of the definition and covariance property of X', Lemma 6.4. Thus

$$\begin{aligned} d(X^n \times X^m) &= (-1)^n (X^n \times (d \times X^m))_{\alpha\beta\,\gamma\delta} + (X^{n'} \times X^m)_{\alpha\beta\,\gamma\delta} \\ &= (-1)^{n+m} (X^n \times (X^m \times d))_{\alpha\beta\,\gamma\delta} + (-1)^n (X^n \times X^{m'})_{\alpha\beta\,\gamma\delta} \\ &\quad + (X^{n'} \times X^m)_{\alpha\beta\,\gamma\delta}\,. \end{aligned}$$

We will apply both sides to Ω. The first term equals

$$(-1)^{n+m}((X^n \times X^m) \times d)_{\alpha\beta\,\gamma\delta} = (-1)^{n+m}(X^n \times X^m)_{\alpha\beta\,\gamma\delta} d$$

and vanishes when applied to Ω. Since $d\Omega = 0$, $\varepsilon(X^{n'}_{\alpha\beta}) = dx^n_{\alpha\beta} = dx$, and similarly for dy. Thus we get

$$\begin{aligned} d(x \cdot y) &= X^{n'}_{\alpha\beta} X^m_{\gamma\delta}\Omega + (-1)^n X^n_{\alpha\beta} X^{m'}_{\gamma\delta}\Omega \\ &= (dx \cdot y) + (-1)^n (x \cdot dy) \end{aligned}$$

by Lemma 6.4 and Definition 6.2a of the product in $\Lambda\mathscr{F}^T$. q.e.d.

7. A Quasi-Associative Generalization of Noncommutative Differential Geometry

The spaces $\Lambda^n\mathcal{F}^T \supset \mathcal{F}^T$ are $\mathcal{F}^T$- bimodules. In the algebraic setting, such modules substitute for spaces of sections in vectorbundles. Elements of $\Lambda^n\mathcal{F}^T$ substitute for n^{th} rank antisymmetric tensor fields on the plane. We wish to introduce covariant exterior derivatives

$$D: \Lambda^n\mathcal{F}^T \mapsto \Lambda^{n+1}\mathcal{F}^T. \tag{7.1}$$

They should be covariant under $\mathcal{G}^*$-valued gauge transformations.

Every element z_α in $\mathcal{F}^T = \mathcal{B}/\mathcal{G}^*$ has a preferred representative Z_α in $\mathcal{B}$, and $\mathcal{B}$ is spanned by elements $Z_\alpha\xi$ with $\xi \in \mathcal{G}^*$. Therefore $\mathcal{B}$ can be regarded as an algebra of functions on the quantum plane with values in $\mathcal{G}^*$. The invertible elements of $\mathcal{B}$ will be considered as gauge transformations.

Similarly, the elements of $\Lambda^1\mathcal{B}$ may be regarded as $\mathcal{G}^*$-valued 1-forms.

We introduce covariant exterior derivatives

$$D = d + A, \quad A \in \Lambda^1\mathcal{B}. \tag{7.2}$$

They can act on elements of $\Lambda\mathcal{F}^T$; $D\omega = d\omega + A\omega$, where the second term involves the action of elements of $\Lambda\mathcal{B}$ on $\Lambda\mathcal{F}^T = \Lambda\mathcal{B}/\mathcal{G}^*$.

If Ξ is a gauge transformation, $\Xi \in \mathcal{B}$, *invertible*, then

$$D\Xi\omega = \Xi D'\omega \quad \text{for} \quad \omega \in \Lambda\mathcal{F}^T, \tag{7.3}$$

$$D' = d + A', \tag{7.4}$$

$$A' = \Xi^{-1}A\Xi + \Xi^{-1}[d, \Xi]. \tag{7.5}$$

The field strength tensor $F \in \Lambda^2$, is defined by

$$F = D^2 = \{d, A\} + AA. \tag{7.6}$$

$A \in \Lambda^1\mathcal{B}$ contains a single factor Θ. It follows from this and from the relations (6.25) between d, Z_a, and Θ_a that indeed $\{d, A\} \in \Lambda^2\mathcal{F}^T$.

Consider the special case that $\mathcal{G}^*$ is the truncated quantum group associated with $U_q(sl_2)$. Then $\mathcal{G}^*$ is a finite sum of full matrix algebras. Therefore it has plenty of invertible elements. The elements of $\mathcal{B}$ are polynomials in Z_a's up to some maximal degree which is determined by q, by the result of Sect. 5. All the monomials except those of degree 0 are nilpotent elements of $\mathcal{B}$. It follows that $\mathcal{B}$ also has plenty of invertible elements.

8. Partial Derivatives

In this section we wish to extend $\Lambda\mathcal{F}^T$ to an algebra which contains also partial derivatives $\partial_{\dot{a}}$ as generators. We will be able to do this under more restrictive assumptions on the symmetry algebra. They are satisfied for the truncated quantum group algebras $\mathcal{G}^*$ which are associated with $U_q(sl_2)$ if τ^f is the 2-dimensional fundamental representation.

We recall that the contragredient representation $\tilde{\tau}$ of a representation τ of $\mathcal{G}^*$ is defined with the help of the antipode $\mathcal{S}$

$$\tilde{\tau}(\xi) = {}^t\tau(\mathcal{S}^{-1}(\xi)). \tag{8.1}$$

We assume that $\mathscr{G}^*$ is semisimple, that $\tilde{\tau}^f \bigotimes \tau^f$ contains the 1-dimensional representation ε as a subrepresentation, and that the representations $\tau^f \bigotimes \tau^f$ and $\tilde{\tau}^f \bigotimes \tilde{\tau}^f$ decompose into a sum of two representations each.

Explicitly we have $\tau^f \bigotimes \tau^f \sim \tau^A \oplus \tau^B$ and $\tilde{\tau}^f \bigotimes \tilde{\tau}^f \sim \tau^{\tilde{A}} \oplus \tau^{\tilde{B}}$ with irreducible representations $\tau^A, \tau^B, \tau^{\tilde{A}}, \tau^{\tilde{B}}$. We recall that semisimplicity implies the existence of the element $\mathscr{P} \in \mathscr{G}^* \otimes \mathscr{G}^*$ which was defined before Theorem 1.1 and which interchanges factors in $\mathscr{G}^* \otimes \mathscr{G}^*$, $\mathscr{P}(\xi \otimes \eta) = (\eta \otimes \xi)\mathscr{P}$. Let $\hat{R}^+ = \mathscr{P}R$ and $\hat{R}^- = R^{-1}\mathscr{P}$ as before. It follows from the intertwining properties of R and $\mathscr{P}$ that

$$\hat{R}^+ \Delta(\xi) = \Delta(\xi)\hat{R}^+ .$$

As a consequence

$$(\tau^f \otimes \tau^f)(\hat{R}^+) = (\tau^f \otimes \tau^f)(c_A \Delta(P^A) + c_B \Delta(P^B)) , \tag{8.2}$$

$$(\tilde{\tau}^f \otimes \tilde{\tau}^f)(\hat{R}^+) = (\tilde{\tau}^f \otimes \tilde{\tau}^f)(c_{\tilde{A}} \Delta(P^{\tilde{A}}) + c_{\tilde{B}}(P^{\tilde{B}})) , \tag{8.3}$$

so that these R-matrices have only two eigenvalues each. We assume that

$$c_{\tilde{A}} = c_A , \tag{8.4}$$

$$c_B \neq c_A , \tag{8.5}$$

$$c_{\tilde{B}} \neq c_A . \tag{8.6}$$

In these formulas, $P^c \in \mathscr{G}^*$ denotes the minimal central projector which belongs to the irreducible representation τ^c and the c_c are complex constants. Let us choose $c_R = c_B^{-1}$ as in Definition 1.1. We are going to introduce differential operators $\partial_{\dot{a}}$ which transform according to the contragredient $\tilde{\tau}^f$ of the fundamental representation τ^f. The differentials Θ_a of the last section transform like Z_a. Latin indices $a, \dot{a}$ etc. will take values $1 \ldots N$ if τ^f has dimension N.

Because of covariance properties, $\times$-products can be defined as in Sect. 3. For instance

$$(\Theta \times \partial)_{a\dot{b}} = \Theta_c \partial_{\dot{d}} (\tau^f_{ca} \otimes \tilde{\tau}^f_{\dot{d}\dot{b}} \otimes \mathrm{id})(\varphi) . \tag{8.7}$$

Definition 8.1 (Metric tensor). *We write τ and $\tilde{\tau}$ in this section in place of τ^f and $\tilde{\tau}^f$. The entries $g_{\dot{a}b}$ of a metric tensor and $g^{a\dot{b}}$ of a twisted inverse metric tensor are complex numbers with the following defining properties.*
(i) *Covariance: For all $\xi \in \mathscr{G}^*$*

$$g_{\dot{c}d}(\tilde{\tau} \bigotimes \tau)_{\dot{c}d\dot{a}b}(\xi) = g_{\dot{a}b}\varepsilon(\xi) , \tag{8.8}$$

$$(\tau \bigotimes \tilde{\tau})_{c\dot{d}a\dot{b}} g^{a\dot{b}} = g^{c\dot{d}}\varepsilon(\xi) . \tag{8.9}$$

(ii) *Normalization*

$$g_{\dot{f}g}(\tau_{ha} \otimes \tilde{\tau}_{\dot{f}\dot{b}} \otimes \tau_{gc})(\varphi) g^{a\dot{b}} = \delta_{hx} . \tag{8.10}$$

It will become apparent later on that left-hand side and right-hand side of Eq. (8.10) are proportional as a consequence of covariance, therefore Eq. (8.10) can be ensured by suitable normalization, if the left-hand side of Eq. (8.10) is not 0. Suitable tensors $g_{\dot{a}b}$ and $g^{a\dot{b}}$ exist then and are provided by Clebsch Gordon coefficients for $\mathscr{G}^*$. The normalization convention leaves the freedom

$$g_{\dot{a}b} \to \lambda g_{\dot{a}b} , \qquad g^{a\dot{b}} \to \lambda^{-1} g^{a\dot{b}} , \qquad (\lambda \in \mathbf{C}) .$$

Definition 8.2 (Algebra of multiplication and differentiation operators and differentials). *Given metric tensors with properties as in Definition 8.1, the associative algebra $\mathscr{D}$ is generated by elements $\xi \in \mathscr{G}^*$, Z_a, Θ_a, and $\partial_{\dot a}$ subject to the following relations:*
(i) *The unit element $e \in \mathscr{G}^*$ is also unit element of $\mathscr{D}$.*
(ii) *Covariance: For $\xi \in \mathscr{G}^*$,*

$$\xi Z_a = Z_b(\tau_{ba} \otimes \mathrm{id})(\Delta(\xi)), \tag{8.11}$$
$$\xi \Theta_a = \Theta_b(\tau_{ba} \otimes \mathrm{id})(\Delta(\xi)), \tag{8.12}$$
$$\xi \partial_{\dot a} = \partial_{\dot b}(\tilde\tau_{\dot b\dot a} \otimes \mathrm{id})(\Delta(\xi)). \tag{8.13}$$

(iii) *Braid relations*

$$(Z \times Z)_{ab} = (Z \times Z)_{dc} c_B^{-1}(\tau_{ca} \otimes \tau_{db})(R), \tag{8.14}$$
$$(\Theta \times \Theta)_{ab} = (\Theta \times \Theta)_{dc} c_A^{-1}(\tau_{ca} \otimes \tau_{db})(R), \tag{8.15}$$
$$(\partial \times \partial)_{\dot a\dot b} = (\partial \times \partial)_{\dot d\dot c} c_{\tilde B}^{-1}(\tilde\tau_{\dot c\dot a} \otimes \tilde\tau_{\dot d\dot b})(R), \tag{8.16}$$
$$(\Theta \times Z)_{ab} = -(Z \times \Theta)_{dc} c_A^{-1}(\tau_{ca} \otimes \tau_{db})(R), \tag{8.17}$$
$$(\Theta \times \partial)_{a\dot b} = -(\partial \times \Theta)_{\dot d c} c_A(\tau_{ca} \otimes \tilde\tau_{\dot d\dot b})(R), \tag{8.18}$$
$$(\partial \times Z)_{\dot a b} = g_{\dot a b} e - (Z \times \partial)_{d\dot c} c_A(\tilde\tau_{\dot c\dot c} \otimes \tau_{db})(R). \tag{8.19}$$

Inverse braid relations involving R^{-1} can be derived from those stated here. The braid relations are here written in $\mathscr{G}^*$-covariant form, but they can be transformed into relations involving ordinary products, as in Definition 2.1.

It follows from the definitions that elements of $\mathscr{D}$ can be written as a linear combination of terms

$$Z_{a_1} \dots Z_{a_n} \Theta_{b_1} \dots \Theta_{b_m} \partial_{\dot c_1} \dots \partial_{\dot c_p} \xi \tag{8.20}$$

with $\xi \in \mathscr{G}^*$. One verifies that all braid relations, including (8.19), are $\mathscr{G}^*$-covariant.

Theorem 8.3. $\mathscr{D} \supset \mathscr{B}$.

The proof of this theorem is given in Appendix D. Let us emphasize that this result is not trivial. The inhomogeneous ones among the relations in Definition 8.2 could in principle induce new relations among the Z's. In this case, the subalgebra of $\mathscr{D}$ which is generated by the Z's would not be $\mathscr{B}$, but a factor $\mathscr{B}/\mathscr{I}$ where the ideal $\mathscr{I}$ is furnished by the new relations. The restrictive assumptions stated at the beginning of this section are imposed in order to ensure that this does not happen.

Next we show that the exterior derivative d with the properties (6.3)ff. can be constructed from differentials Θ_b and partial derivatives. Note that the necessary condition (6.28) for existence of d is satisfied by the decomposition (8.2) of the R-matrix.

Theorem 8.4 (exterior derivative). *Define $d = g^{a\dot b}(\Theta \times \partial)_{a\dot b}$. Then d is $\mathscr{G}^*$-invariant and*

$$d^2 = 0, \tag{8.21}$$
$$dZ_a = \Theta_a + Z_a d. \tag{8.22}$$

Proof of Theorem 8.4. First we prove $\mathscr{G}^*$-invariance, $d\xi = \xi d$. By covariance of the $\times$-product,

$$\begin{aligned}\xi(\Theta \times \partial)_{a\dot b} g^{a\dot b} &= (\Theta \times \partial)_{c\dot d}((\tau \bigotimes \tilde\tau)_{c\dot d a\dot b} \otimes \mathrm{id})(\Delta(\xi)) g^{a\dot b} \\ &= (\Theta \times \partial)_{c\dot d} g^{c\dot d}(\varepsilon \otimes \mathrm{id})(\Delta(\xi)) = (\Theta \times \partial)_{c\dot d} g^{c\dot d} \xi\,'\end{aligned}$$

as claimed. Next we prove $d^2 = 0$. Since d transforms according to the 1-dimensional representation, it follows from Theorem 2.4 and the braid relation for composites, Theorem 2.9, that

$$d^2 = d \times d = \frac{c_A}{c_{\tilde{B}}}(d \times d)(\varepsilon \otimes \varepsilon)(R) = \frac{c_A}{c_{\tilde{B}}} d \times d = \frac{c_A}{c_{\tilde{B}}} d^2 . \tag{8.23}$$

Property (1.9) was used to conclude that $(\varepsilon \otimes \varepsilon)(R) = e$. The factor is a consequence of Theorem 2.9. This proves $d^2 = 0$ since $c_\alpha \neq c_{\tilde{B}}$ was assumed. Finally we show that $dZ_a = \Theta_a + Z_a d$. Since d is $\mathscr{G}^*$-invariant, $dZ_c = (d \times Z)_c$ by Theorem 2.4. If the first term in the $(\partial \times Z)$ braid relations were absent, the result would be given by the braid relation for composites again. Thus

$$\begin{aligned}(d \times Z)_c &= 1^{\text{st}} \text{ term} + (Z \times d)_e(\varepsilon \otimes \tau_{ec})(R) \\ &= 1^{\text{st}} \text{ term} + Z_c d ,\end{aligned} \tag{8.24}$$

because $(\varepsilon \otimes \mathrm{id})(R) = e$ by Eq. (8.9). To find the first term we write

$$\begin{aligned}(d \times Z)_c &= g^{a\dot{b}}((\Theta \times \partial) \times Z)_{a\dot{b}c} \\ &= g^{a\dot{b}}(\Theta \times (\partial \times Z))_{e\dot{f}g}(\tau_{ea} \otimes \tilde{\tau}_{\dot{f}\dot{b}} \otimes \tau_{gc})(\varphi) \\ &= g^{a\dot{b}}\Theta_h(\partial \times Z)_{\dot{i}k}(\tau_{he} \otimes (\tilde{\tau} \bigotimes \tau)_{\dot{i}k\dot{f}g} \otimes \mathrm{id})(\varphi)(\tau_{ea} \otimes \tilde{\tau}_{\dot{f}\dot{b}} \otimes \tau_{gc})(\varphi) \\ &= g^{a\dot{b}}g_{\dot{i}k}\Theta_h(\tau_{he} \otimes (\tilde{\tau} \bigotimes \tau)_{\dot{i}k\dot{f}g} \otimes \mathrm{id})(\varphi)(\tau_{ea} \otimes \tilde{\tau}_{\dot{f}\dot{b}} \otimes \tau_{gc})(\varphi) + 2^{\text{nd}} \text{ term} .\end{aligned}$$

We can use the covariance property of $g_{\dot{i}k}$ to write the explicit term as

$$\begin{aligned}1^{\text{st}} \text{ term} &= g^{a\dot{b}}g_{\dot{f}g}\Theta_h(\tau_{he} \otimes \mathrm{id})((\mathrm{id} \otimes \varepsilon \otimes \mathrm{id})(\varphi))(\tau_{ea} \otimes \tilde{\tau}_{\dot{f}\dot{b}} \otimes \tau_{gc})(\varphi) \\ &= \Theta_h \alpha_{hc} .\end{aligned}$$

with

$$\alpha_{hc} = g^{a\dot{b}}(\tau_{ha} \otimes \tilde{\tau}_{\dot{f}d\dot{b}} \otimes \tau_{gc})(\varphi)g_{\dot{f}g} . \tag{8.25}$$

Since d is $\mathscr{G}^*$-invariant, $(d \times Z)_c$ and $(Z \times d)_d$ transform like Z_c. Since τ is irreducible, the same must be true of the term $\Theta_h \alpha_{hc}$. It follows that $\alpha_{hc} \propto \delta_{hc}$, as was mentioned after Definition 8.1. By hypothesis, the metric tensors have the normalization property of Definition 8.1. Therefore the 1^{st} term equals Θ_c. This proves that $dZ_a = \Theta_a + Z_a d$, and completes the proof of Theorem 8.4. q.e.d.

Next we reconsider the space $\Lambda\mathscr{F}^T$ of forms. We reconstruct this linear space as a factor space of $\mathscr{D}$. In this way it becomes a $\mathscr{D}$-module. The elements of $\mathscr{D}$ can act on it as multiplication and differentiation operators. A product in $\Lambda\mathscr{F}^T$ was constructed in Sect. 6. $\Lambda\mathscr{F}^T$ is constructed as a coset space $\mathscr{D}/\mathscr{J}$ with the help of a homomorphism $\varepsilon : \mathscr{J} \to \mathbf{C}$. The coset space consists of equivalence classes, $X\xi \sim X\varepsilon(\xi)$ if $\xi \in \mathscr{J}$.

Definition 8.5 ($\mathscr{D}$-module $\Lambda\mathscr{F}^T$). *Let $\mathscr{J} \subset \mathscr{D}$ be generated by elements $\xi \in \mathscr{G}^*$ and by $\partial_{\dot{a}}$ ($\dot{a} = 1 \dots N$). Extend the counit of $\mathscr{G}^*$ to a homomorphism $\varepsilon : \mathscr{J} \to \mathbf{C}$ by setting $\varepsilon(\partial_{\dot{a}}) = 0$. Define*

$$\Lambda\mathscr{F}^T = \mathscr{D}/\mathscr{J} , \tag{8.26}$$

where cosets are formed with the help of ε as explained above. Writing $\varepsilon : \mathscr{D} \to \Lambda\mathscr{F}^T$ for the map to cosets, $E \in \mathscr{D}$ acts on $\Lambda\mathscr{F}^T$ in the obvious way

$$E\varepsilon(X) = \varepsilon(EX) . \tag{8.27}$$

The special element $\Omega = \varepsilon(e)$ is annihilated by differential operators

$$\partial_{\dot{a}}\Omega = 0 .$$

Since d is an element of $\mathscr{D}$, it can act on $\Lambda\mathscr{F}^T$. We established the relations stated in Theorem 8.4. It was proven before that Leibniz rule follows.

8.1. Example (Truncated Quantum Group Algebras $U_q(sl_2)$). Let q be a primitive p^{th} root of unity. For simplicity we restrict attention to $p \geq 5$ and representation $\tau^f = \tau^{1/2}$. The representation $\tilde{\tau}^f$ is then equivalent to $\tau^{1/2}$. In tensor products of up to three of these two-dimensional representations no truncation appears if $p \geq 5$. Therefore

$$(\tau^f \otimes \tilde{\tau}^f \otimes \tau^f)(\varphi) = 1 \quad \text{if} \quad p \geq 5 , \tag{8.28}$$

and $(\tau^f \otimes \tau^f)(R)$, $(\tilde{\tau}^f \otimes \tau^f)(R)$, and $(\tilde{\tau}^f \otimes \tilde{\tau}^f)(R)$ are the same as in the nontruncated case. As a result, the fundamental braid relations are the same as in the ordinary quantum plane, except that ordinary products must be replaced by covariant ones. Thus the $(Z \times Z)$ and $(\partial \times \partial)$ braid relations reduce to

$$(Z \times Z)_{12} = q^{-1/2}(Z \times Z)_{21} , \tag{8.29}$$

$$(\partial \times \partial)_{12} = q^{1/2}(\partial \times \partial)_{21} , \tag{8.30}$$

while the $(\Theta \times \Theta)$ braid relations give

$$\begin{aligned}(\Theta \times \Theta)_{12} &= -q^{1/2}(\Theta \times \Theta)_{21} ,\\ (\Theta \times \Theta)_{11} &= 0 = (\Theta \times \Theta)_{22} .\end{aligned}$$

The $(\partial \times Z)$ braid relations give

$$\begin{aligned}(\partial \times Z)_{12} &= q^{-1/2}(Z \times \partial)_{21} ,\\ (\partial \times Z)_{21} &= q^{-1/2}(Z \times \partial)_{12} ,\\ (\partial \times Z)_{11} &= e + q^{-1}(Z \times \partial)_{11} + (q^{-1} - 1)(Z \times \partial)_{22} ,\\ (\partial \times Z)_{22} &= e + q^{-1}(Z \times \partial)_{22} .\end{aligned} \tag{8.31}$$

The $(\partial \times \Theta)$ braid relations are of the same form, except that the inhomogeneous terms e are absent.

The exterior derivative is given by

$$d = (\Theta \times \partial)_{11} + (\Theta \times \partial)_{22} . \tag{8.32}$$

One reads off that

$$g_{\dot{a}b} = \delta_{\dot{a}b} , \qquad g^{a\dot{b}} = \delta^{a\dot{b}} . \tag{8.33}$$

9. Appendix A: Proof of Theorem 1.1

Lemma A.1. *$\Delta^{n-1}(\xi)$ commutes with $\sigma_k^{n\pm}$ for all $\xi \in \mathscr{G}^*$, n, $1 \leq k \leq n-1$.*

Proof. We will give the proof for $n = 3$, $k = 2$. The general case uses essentially the same argument:

$$\begin{aligned}\Delta^2(\xi)\varphi(\hat{R}^+ \otimes e)\varphi^{-1} &= \varphi(\Delta \otimes \mathrm{id})\Delta(\xi)\,(\hat{R}^+ \otimes e)\varphi^{-1} \\ &= \varphi(\mathscr{P} \otimes e)\,(\Delta' \otimes \mathrm{id})\Delta(\xi)\,(R \otimes e)\varphi^{-1} \\ &= \varphi(\hat{R}^+ \otimes e)\,(\Delta \otimes \mathrm{id})\Delta(\xi)\varphi^{-1} = \varphi(\hat{R}^+ \otimes e)\varphi^{-1}\Delta^2(\xi)\,.\end{aligned}$$

Lemma A.2. *Let $B_n[B'_n]$ denote the subalgebra generated by σ_k^n $(k = 1, \ldots, n-1)$ and $\Delta^{n-1}(e)$ $[\sigma_k^n$ $(k = 2, \ldots, n-1)$ and $\Delta^{n-1}(e)]$. Then the following maps:*
(i) $I_n : B_n \to B_{n+1}$ *defined on generators of B_n by*

$$I_n(\sigma_k^n) = \Delta^n(e)\,(e \otimes \sigma_k^n) = \sigma_k^{n+1} \quad \text{for all} \quad 1 \le k \le n-1\,, \tag{9.1}$$

$$I_n(\Delta^{n-1}(e)) = \Delta^n(e)\,, \tag{9.2}$$

(ii) $C_n^{\pm} : B'_n \to B_{n\pm 1}$ *defined on generators of B'_n by*

$$C_n^{\pm}(\sigma_k^n) = \Delta^{n\pm 1}(e)\,(\mathrm{id}^{n-1} \otimes \Delta^{\pm 1})\,(\sigma_k^n) = \sigma_{k\pm 1}^{n\pm 1} \quad \text{for all} \quad 2 \le k \le n-1\,, \tag{9.3}$$

$$C_n^{\pm}(\Delta^{n-1}(e)) = (\Delta^{n-1\pm 1})\,(e) \tag{9.4}$$

extend to homomorphisms of algebras.

Proof. The lemma is a simple consequence of Lemma A.1 with the special choice $\xi = e$ and the relation $\Delta^j(e)\Delta^j(e) = \Delta^j(e)$.

Lemma A.3. $\sigma_2^{3-}\sigma_2^3 = \Delta^2(e) = \sigma_2^3\sigma_2^{3-}$.

Proof. Since $\hat{R}^- = R^{-1}\mathscr{P}$, $\sigma_2^{3-} = \varphi(\hat{R}^- \otimes e)\varphi^{-1}$ is the quasi-inverse of $\sigma_2^3 = \varphi(\hat{R}^+ \otimes e)\varphi^{-1}$. This proves the lemma.

Lemma A.4. $\sigma_{n-1}^n\sigma_2^n = \sigma_2^n\sigma_{n-1}^n$ *for all* $n \ge 5$.

Proof. Since $\sigma_{n-1}^n = (\mathrm{id}^2 \otimes \Delta^{n-3})\,(\varphi(\hat{R}^+ \otimes e)\varphi^{-1})$ and $\sigma_2^n = \Delta^{n-1}(e)\,(e^2 \otimes \sigma_2^{n-2})$, the lemma follows directly from Lemma A.1.

Lemma A.5. *Let Φ be defined by $\Phi \equiv (\mathrm{id} \otimes \Delta \otimes \mathrm{id})\,(\varphi^{-1})\,(e \otimes \varphi^{-1})$. Then the following two relations hold:*
(i) $\Phi\sigma_3^4 = (\varphi(\hat{R}^+ \otimes e)\varphi^{-1} \otimes e)\Phi$,
(ii) $\Phi\sigma_2^4 = (e \otimes \hat{R}^+ \otimes e)\Phi$.

Proof. (i) Using (1.11) and the intertwining properties of $\mathscr{P}, R$, we get

$$\begin{aligned}\Phi\sigma_3^4 &= (\mathrm{id} \otimes \Delta \otimes \mathrm{id})\,(\varphi^{-1})\,(e \otimes \varphi^{-1})\,(\mathrm{id} \otimes \mathrm{id} \otimes \Delta)\,(\varphi(\hat{R}^+ \otimes e)\varphi^{-1}) \\ &= (\varphi \otimes e)\,(\Delta \otimes \mathrm{id} \otimes \mathrm{id})\,(\varphi^{-1})\,(\hat{R}^+ \otimes e^2)\,(\mathrm{id} \otimes \mathrm{id} \otimes \Delta)\,(\varphi^{-1}) \\ &= (\varphi(\hat{R}^+ \otimes e) \otimes e)\,(\varphi^{-1} \otimes e)\,(\mathrm{id} \otimes \Delta \otimes \mathrm{id})\,(\varphi^{-1})\,(e \otimes \varphi^{-1}) \\ &= (\varphi(\hat{R}^+ \otimes e)\varphi^{-1} \otimes e)\Phi\,.\end{aligned}$$

The proof of (ii) is similar. For detailed proofs of the used reformulations of (1.11) cp. Appendix B.

Lemma A.6. $\sigma_3^4\sigma_2^4\sigma_3^4 = \sigma_2^4\sigma_3^4\sigma_2^4$.

Proof. Lemma 6 follows by recurrent use of Lemma 5 and the quasi-Yang Baxter equation.

Proof of Theorem 1.1. In Lemma A.3, 4, 6, the Artin relations have been established for special values of n, k. But under the action of the homomorphisms defined in Lemma A.2, these relations map to all the Artin relations for arbitrary values of n, k. Thus the proof of Theorem 1.1 is complete.

10. Appendix B: Proofs for Sect. 2

Proof of Theorem 2.4 (Inversion formula). The proof of the inversion formula (2.9) is not an entirely trivial consequence of the definition of the $\times$-product because φ^{-1} is not a true inverse of φ. It will be necessary to use the information that e acts as the identity in $\mathscr{B}$.

Using the formula $\varphi\varphi^{-1} = (\mathrm{id}\otimes\Delta)\Delta(e)$, we get from the definition (2.8) of the $\times$-product

$$(F \times F')_{\alpha\beta}(\tau_{\alpha\delta} \otimes \tau'_{\beta\gamma} \otimes \mathrm{id})\,(\varphi^{-1}) = F_\varepsilon F'_\kappa(\tau_{\varepsilon\delta} \otimes \tau'_{\kappa\gamma} \otimes \mathrm{id})\,((\mathrm{id}\otimes\Delta)\Delta(e))\,. \quad (10.1)$$

Write $\Delta(e) = \sum_\sigma e^1_\sigma \otimes e^2_\sigma$. Because e is the unit element of $\mathscr{B}$ according to Definition 2.1, we have

$$\begin{aligned} F_\gamma F'_\delta &= eF_\gamma F'_\delta = \sum_\sigma F_\varepsilon \tau_{\varepsilon\gamma}(e^1_\sigma) e^2_\sigma F_\delta \\ &= \sum_\sigma F_\varepsilon F'_\kappa \tau_{\varepsilon\gamma}(e^1_\sigma)\,(\tau'_{\kappa\delta} \otimes \mathrm{id})\,(\Delta(e^2_\sigma))\,. \end{aligned}$$

That is

$$F_\gamma F'_\delta = F_\varepsilon F'_\kappa(\tau_{\varepsilon\gamma} \otimes \tau'_{\kappa\delta} \otimes \mathrm{id})\,((\mathrm{id}\otimes\Delta)\Delta(e))\,. \quad (10.2)$$

Inserting this into Eq. (10.1) proves the inversion formula (2.9). q.e.d.

Proof of Theorem 2.4 (Invariants). If one of the factors is $\mathscr{G}^*$-invariant, the definition of the $\times$-product simplifies to (2.10) because of Eqs. (1.6). q.e.d.

Proof of Theorem 2.5 (Covariance). The left-hand side of Eq. (2.13) transforms according to the representation $\hat{\tau} = \tau^f \bigotimes \tau^f$ by Theorem 2.4. [This representation is described explicitly in Eq. (2.12).] For the same reason, the right-hand side of Eq. (2.13) transforms according to

$$\begin{aligned} \xi(Z \times Z)_{cd}\mathscr{R}_{dc,ab} &= (Z \times Z)_{ef}(\hat{\tau}_{ef,cd} \otimes \mathrm{id})\,(\Delta(\xi))\mathscr{R}_{dc,ab} \\ &= (Z \times Z)_{ef}(\tau^f_{ec} \otimes \tau^f_{fd} \otimes \mathrm{id})\,((\Delta \otimes \mathrm{id})\Delta(\xi))\mathscr{R}_{dc,ab} \\ &= \sum_\sigma (Z \times Z)_{ef}(\tau^f_{fd} \otimes \tau^f_{ec} \otimes \mathrm{id})\,(\Delta'(\xi^1_\sigma) \otimes \xi^2_\sigma)\mathscr{R}_{dc,ab} \\ &= \sum_\sigma (Z \times Z)_{ef}(\tau^f_{fa} \otimes \tau^f_{eb} \otimes \mathrm{id})\,((\Delta'(\xi^1_\sigma) \otimes \xi^2_\sigma)\,(R \otimes e))\,. \end{aligned}$$

The principal property (1.7), $R\Delta(\xi) = \Delta'(\xi)R$ of R tells us that this equals

$$\begin{aligned} \xi(Z \times Z)_{cd}\mathscr{R}_{dc,ab} &= \sum_\sigma (Z \times Z)_{ef}(\tau^f_{fa} \otimes \tau^f_{eb} \otimes \mathrm{id})\,((R \otimes e)\,(\Delta(\xi^1_\sigma) \otimes \xi^2_\sigma)) \\ &= (Z \times Z)_{ef}\mathscr{R}_{fe,dc}(\tau^f_{da} \otimes \tau^f_{cb} \otimes \mathrm{id})\,((\Delta \otimes \mathrm{id})\Delta(\xi)) \\ &= (Z \times Z)_{ef}\mathscr{R}_{fe,dc}(\hat{\tau}_{dc,ab} \otimes \mathrm{id})\,(\Delta(\xi))\,. \end{aligned}$$

This is the desired transformation law. q.e.d.

Proof of Theorem 2.5 (Equivalence of Braid Relations). (i) The braid relations (3) of Definition 2.1 imply in view of the definition (2.8) of the $\times$-product

$$\begin{aligned}(Z \times Z)_{ef} &= c_R Z_c Z_d \tilde{\mathscr{R}}_{dc,ab}(\tau^f_{ae} \otimes \tau^f_{bf} \otimes \mathrm{id})(\varphi) \\ &= c_R Z_c Z_d(\tau^f_{de} \otimes \tau^f_{cf} \otimes \mathrm{id})(\varphi_{213}(R \otimes e)\varphi^{-1}\varphi)\,. \qquad (10.3)\end{aligned}$$

φ^{-1} is not a true inverse of φ. Therefore $\varphi^{-1}\varphi$ is not the identity. But $\varphi^{-1}\varphi = (\Delta \otimes \mathrm{id})\Delta(e)$ according to Eq. (1.5) of sect. 1. Thus

$$\varphi_{213}(R \otimes e)\varphi^{-1}\varphi = \varphi_{213}(R \otimes e)\,(\Delta \otimes \mathrm{id})\Delta(e) = \varphi_{213}(\Delta' \otimes \mathrm{id})\Delta(e)\,(R \otimes e)$$

by the principal property (1.7) of R. Introduce the permutation operator P_{12} on $\mathscr{G}^* \otimes \mathscr{G}^* \otimes \mathscr{G}^*$ by

$$P_{12}[\xi \otimes \eta \otimes \zeta] = \eta \otimes \xi \otimes \zeta\,.$$

In this notation

$$\varphi_{213}(R \otimes e)\varphi^{-1}\varphi = P_{12}[\varphi(\Delta \otimes \mathrm{id})\Delta(e)]\,(R \otimes e)\,.$$

Inserting this into Eq. (10.3) yields

$$\begin{aligned}&(Z \times Z)_{ef} \\ &\quad = c_R Z_c Z_d(\tau^f_{de'} \otimes \tau^f_{cf'} \otimes \mathrm{id})\,(P_{12}[\varphi(\Delta \otimes \mathrm{id})\Delta(e)])\,(\tau^f_{e'e} \otimes \tau^f_{f'f} \otimes \mathrm{id})\,(R \otimes e) \\ &\quad = c_R Z_c Z_d(\tau^f_{cf'} \otimes \tau^f_{de'} \otimes \mathrm{id})\,(\varphi(\Delta \otimes \mathrm{id})\Delta(e))\,(\tau^f_{e'e} \otimes \tau^f_{f'f} \otimes \mathrm{id})\,(R \otimes e)\,. \quad (10.4)\end{aligned}$$

Now we use Eq. (1.22) of Sect. 1 which tells us that $\varphi(\Delta \otimes \mathrm{id})\Delta(e) = \varphi$.

Inserting this in expression (10.4) and using the definition of the $\times$-product again yields

$$(Z \times Z)_{ef} = c_R (Z \times Z)_{f'e'}(\tau^f_{e'e} \otimes \tau^f_{f'f})\,(R)\,.$$

This is the desired result in view of the definition of $\mathscr{R}$. This shows that the braid relations (2.13) are implied by Definition 1.

(ii) Now we start from the braid relations (2.13) in Theorem 2.5. We contract them with $(\tau^f \otimes \tau^f \otimes \mathrm{id})\,(\varphi^{-1})$ to get

$$(Z \times Z)_{ab}(\tau^f_{af} \otimes \tau^f_{bg} \otimes \mathrm{id})\,(\varphi^{-1}) = Z_k Z_l(\tau^f_{kf} \otimes \tau^f_{lg} \otimes \mathrm{id})\,(\varphi\varphi^{-1}) = Z_f Z_g\,. \quad (10.5)$$

The second of these equalities obtains from $\varphi\varphi^{-1} = (\mathrm{id} \otimes \Delta)\Delta(e)$ using the following special case of Eq. (10.2):

$$Z_a Z_c = Z_b Z_d(\tau^f_{ba} \otimes \tau^f_{dc} \otimes \mathrm{id})\,((\mathrm{id} \otimes \Delta)\Delta(e))\,. \qquad (10.6)$$

Equation (2.13) of Theorem 2.5 implies therefore that

$$\begin{aligned}Z_f Z_g &= (Z \times Z)_{cd} c_R \tilde{\mathscr{R}}_{dc,ab}(\tau^f_{af} \otimes \tau^f_{bg} \otimes \mathrm{id})\,(\varphi^{-1}) \\ &= Z_k Z_l c_R(\tau^f_{kf} \otimes \tau^f_{lg} \otimes \mathrm{id})\,(\varphi_{213}(R \otimes e)\varphi^{-1})\,.\end{aligned}$$

This is the desired braid relation of Definition 2.1 in view of the Definition Eq. (2.3) of $\tilde{\mathscr{R}}$. q.e.d.

Proof of Theorem 2.6 (i). From the definition of the $\times$-product, its covariance property stated in Theorem 2.4, and the definition of the tensor product $\bigotimes$ of representations one computes

$$\begin{aligned}((F \times F') \times F'')_{\alpha\beta\gamma} \\ = F_\kappa F'_\mu F''_\nu (\tau_{\kappa\alpha} \otimes \tau'_{\mu\beta} \otimes \tau''_{\nu\gamma} \otimes \mathrm{id})([\mathrm{id} \otimes \mathrm{id} \otimes \Delta](\varphi)\,[\Delta \otimes \mathrm{id} \otimes \mathrm{id}](\varphi)), \quad (10.7)\end{aligned}$$

and

$$(F \times (F' \times F''))_{\alpha\beta\gamma} = F_\kappa F'_\mu F''_\nu (\tau_{\kappa\alpha} \otimes \tau'_{\mu\beta} \otimes \tau''_{\nu\gamma} \otimes \mathrm{id})((e \otimes \varphi)\,[\mathrm{id} \otimes \Delta \otimes \mathrm{id}](\varphi)) . \quad (10.8)$$

Theorem 2.6 (i) follows now from Drinfelds relation (1.11), viz.

$$(\mathrm{id} \otimes \mathrm{id} \otimes \Delta)(\varphi)(\Delta \otimes \mathrm{id} \otimes \mathrm{id})(\varphi) = (e \otimes \varphi)(\mathrm{id} \otimes \Delta \otimes \mathrm{id})(\varphi)(\varphi \otimes e) . \quad (10.9)$$

Similarly, Theorem 2.6 (ii) follows from the following relation:

$$(\mathrm{id} \otimes \mathrm{id} \otimes \Delta)(\varphi)(\Delta \otimes \mathrm{id} \otimes \mathrm{id})(\varphi)(\varphi^{-1} \otimes e) = (e \otimes \varphi)(\mathrm{id} \otimes \Delta \otimes \mathrm{id})(\varphi) . \quad (10.10)$$

This proves Theorem 2.6 assuming relation (10.10).

Proof of Relation (10.10). We deduce (10.10) from (10.9). Multiplying (10.9) with $(\varphi^{-1} \otimes e)$ from the right we obtain Eq. (10.10) except that the right-hand side is replaced by

$$\begin{aligned}&(e \otimes \varphi)(\mathrm{id} \otimes \Delta \otimes \mathrm{id})(\varphi)(\varphi\varphi^{-1} \otimes e) \\ &= (e \otimes \varphi)(\mathrm{id} \otimes \Delta \otimes \mathrm{id})(\varphi)((\mathrm{id} \otimes \Delta)\Delta(e) \otimes e) \\ &= (e \otimes \varphi)(\mathrm{id} \otimes \Delta \otimes \mathrm{id})(\varphi)(\mathrm{id} \otimes \Delta \otimes \mathrm{id})(\Delta(e) \otimes e)) \\ &= (e \otimes \varphi)(\mathrm{id} \otimes \Delta \otimes \mathrm{id})(\varphi\Delta(e) \otimes e) .\end{aligned}$$

From (1.22) it follows that

$$\varphi(\Delta(e) \otimes e) = \varphi . \quad (10.11)$$

Inserting this reproduces the right-hand side of Eq. (10.10). This completes the proof of Eq. (10.10) and of Theorem 2.6. q.e.d.

Proof of Corollary 2.7. Inserting Theorem 2.6 (i) into Theorem 2.6 (ii) we obtain with $\varphi\varphi^{-1} = (\mathrm{id} \otimes \Delta)\Delta(e)$,

$$\begin{aligned}(F \times (F' \times F''))_{\alpha\beta\gamma} &= (F \times (F' \times F''))_{\kappa\mu\nu}(\tau_{\kappa\alpha} \otimes \tau'_{\mu\beta} \otimes \tau''_{\nu\gamma})((\Delta \otimes \mathrm{id})\Delta(e)) \\ &= (F \times (F' \times F''))_{\kappa\mu\nu}(\tau \bigotimes (\tau' \bigotimes \tau''))_{\kappa\mu\nu,\alpha\beta\gamma}(e) . \quad (10.12)\end{aligned}$$

Now we specialize to singlets $F' = e$, $F'' = e$. They transform according to the trivial 1-dimensional representation $\tau' = \tau'' = \varepsilon$ which is given by the counit. $\tau \bigotimes \varepsilon = \tau$ according to Eq. (1.21), for any representation τ. Furthermore $(e \times F)_\alpha = F_\alpha$ by Eq. (2.10) of Theorem 2.4. In the special case considered, Eq. (10.12) reduces therefore to the assertion of Corollary 2.7. q.e.d.

Proof of Theorem 2.8. One converts ordinary products into covariant ones by use of Theorem 2.4, starting to the right. Writing $\varphi^{-1} = \sum_\sigma \xi^1_\sigma \otimes \xi^2_\sigma \otimes^3_\sigma$, factors ξ^3_σ will appear which can be multiplied with the factors $\eta \in \mathscr{G}^*$ standing to the right of them. After all ordinary products are converted in this way, one ends up with an expression of the form (2.18). q.e.d.

Proof of Theorem 2.9 (Braid Relations for Composites). (i): The result of the exchange of F with $(F' \times F'')$ can be determined from the braid relations in the hypothesis of the theorem and quasi-associativity (Theorem 2.6) by the sequence of steps shown in the following diagram.

$$\begin{array}{ccc} (F \times F') \times F'' & \mapsto & F'' \times (F \times F') \\ \downarrow & & \uparrow \\ F \times (F' \times F'') \mapsto F \times (F'' \times F') & \mapsto (F \times F'') \times F' \mapsto & (F'' \times F) \times F' . \end{array} \tag{10.13}$$

This leads to the same result as claimed in Theorem 2.9 if

$$(\mathrm{id} \otimes \Delta)(R) = \varphi_{231}^{-1} R_{13} \varphi_{213} R_{12} \varphi^{-1} . \tag{10.14}$$

Similarly, the braid relation which effects the inverse exchange follows from the identity

$$(\Delta \otimes \mathrm{id})(R) = \varphi_{312} R_{13} \varphi_{132}^{-1} R_{23} \varphi . \tag{10.15}$$

These two relations are true by definition in a weak quasitriangular quasi Hopf algebra, see Eqs. (10.12, 10.13) in Sect. 1. This proves Theorem 2.9 (i).
(ii): Theorem 2.9 (ii) follows from part (i) by induction, with the assertion of Theorem 2.5 as the starting point. In the special case when one of the factors is $\mathcal{G}^*$ invariant, the result can be simplified as indicated with the help of Eq. (1.9). q.e.d.

11. Appendix C: Proof of Proposition 4.3

We wish to prove that

$$Z_\beta^n c_R T_{\beta\alpha}^{(n)}(\sigma_{k-1}) = Z_\alpha^n \tag{11.1}$$

for $k = 2 \dots n$.

To prepare for the proof, two lemmas will be stated and proven. We take it as a standing hypothesis that vectors $F = (F_\alpha)$, $F' = (F'_\beta)$, $F'' = (F''_\gamma)$ transform covariantly according to some representations τ, τ', τ'' of $\mathcal{G}^*$.

Lemma C.1. *Suppose that*

$$(F \times F')_{\alpha\beta} = (F' \times F)_{\delta\gamma} c(\tau_{\gamma\alpha} \otimes \tau'_{\delta\beta})(R) \tag{11.2}$$

with some factor $c \in \mathbf{C}$. This implies

$$((F \times F') \times F'')_{\alpha\beta\mu} = ((F' \times F) \times F''))_{\delta\gamma\mu} c(\tau_{\gamma\alpha} \otimes \tau'_{\delta\beta})(R) , \tag{11.3}$$

$$(F'' \times (F \times F'))_{\mu\alpha\beta} = (F'' \times (F' \times F))_{\mu\delta\gamma} c(\tau_{\gamma\alpha} \otimes \tau'_{\delta\beta})(R) . \tag{11.4}$$

Proof of Lemma C.1. From the definition of the $\times$-product and its covariance property, the definition of the tensor product of representations and hypothesis (11.2) one computes

$$\begin{aligned} \text{l.h.s.} &\equiv ((F \times F') \times F'')_{\alpha\beta\mu} \\ &= ((F' \times F) \times F'')_{\delta\gamma\nu} c(\tau_{\gamma\alpha} \otimes \tau'_{\delta\beta} \otimes \tau''_{\nu\mu} \otimes \mathrm{id})(A) , \\ A &= (\Delta' \otimes \mathrm{id} \otimes \mathrm{id})(\varphi^{-1})(R \otimes e \otimes e)(\Delta \otimes \mathrm{id} \otimes \mathrm{id})(\varphi) . \end{aligned}$$

By the fundamental property of R

$$\begin{aligned} A &= (\Delta' \otimes \mathrm{id} \otimes \mathrm{id})(\varphi^{-1}\varphi)(R \otimes e \otimes e) \\ &= (\Delta' \otimes \mathrm{id} \otimes \mathrm{id})(\Delta \otimes \mathrm{id})\Delta(e)(R \otimes e \otimes e) . \end{aligned}$$

But

$$(\tau_{\gamma\alpha} \otimes \tau'_{\delta\beta} \otimes \tau''_{\nu\mu} \otimes \mathrm{id})(\Delta' \otimes \mathrm{id} \otimes \mathrm{id})(\Delta \otimes \mathrm{id})\Delta(e))$$
$$= (((\tau' \bigotimes \tau) \bigotimes \tau'')_{\delta\gamma\nu,\beta\alpha\mu} \otimes \mathrm{id})(e) . \tag{11.5}$$

Because of the covariance property of $((F \times F') \times F'')$ we get from this

$$\text{l.h.s.} = e((F \times F') \times F'')_{\delta\gamma\mu} c(\tau_{\gamma\alpha} \otimes \tau'_{\delta\beta})(R) . \tag{11.6}$$

This is the desired result (11.3) because e acts as a unit element in $\mathscr{B}$. Relation (11.4) is proven in the same way. q.e.d.

Lemma C.2. *Validity of the relation*

$$Z^n_\beta c_R T^{(n)}_{\beta\alpha}(\sigma_{k-1}) = Z^n_\alpha \tag{11.7}$$

for some given n, k with $n \geq k$ implies that

$$(F \times Z^n)_{\mu\beta} c_R T^{(n)}_{\beta\alpha}(\sigma_{k-1}) = (F \times Z^n)_{\mu\alpha} . \tag{11.8}$$

Proof of Lemma C.2 for $k = n$. By definition

$$T^{(n)}_{\beta\alpha}(\sigma_{n-1}) = (\tau^f \otimes \tau^f \otimes \tau^{(n-2)}_{\bar\beta\alpha})(\varphi_{213}(R \otimes e)\varphi) , \tag{11.9}$$

where $\bar\beta$ differs from $\beta = (b_n, b_{n-1}, \ldots , b_1)$ by interchange of b_n and b_{n-1}.

From the definition of the $\times$-product and Definition (3.10) of the multiple tensor product $\tau^{(n-2)}$ of representations we deduce

$$\begin{aligned}(F \times Z^n)_{\mu\alpha} &= F_\nu Z^n_\beta(\tau_{\nu\mu} \otimes \tau^{(n)}_{\beta\alpha} \otimes \mathrm{id})(\varphi) \\ &= F_\nu Z^n_\beta(\tau_{\nu\mu} \otimes (\tau^f \otimes \tau^f \otimes \tau^{(n-2)})_{\beta\alpha} \otimes \mathrm{id})((\mathrm{id} \otimes (\mathrm{id} \otimes \Delta)\Delta \otimes \mathrm{id})(\varphi)) .\end{aligned}$$

Now the hypothesis of the lemma and the definition of $T^{(n)}(\sigma_{n-1})$ are inserted, and the ordinary product $F_\varrho Z^n_\gamma$ is converted back into a covariant product. As a result one obtains

$$(F \times Z^n)_{\mu\alpha} = (F \times Z^n)_{\nu\beta} c_R(\tau_{\nu\mu} \otimes (\tau^f \otimes \tau^f \otimes \tau^{(n-2)})_{\beta\alpha} \otimes \mathrm{id})(A) , \tag{11.10}$$

with

$$\begin{aligned}A = {}& P_{2,3}(\mathrm{id} \otimes (\mathrm{id} \otimes \Delta)\Delta \otimes \mathrm{id})(\varphi^{-1}))(e \otimes \varphi_{213}(R \otimes e)\varphi^{-1} \otimes e) \\ & \cdot (\mathrm{id} \otimes (\mathrm{id} \otimes \Delta)\Delta \otimes \mathrm{id})(\varphi) .\end{aligned} \tag{11.11}$$

$P_{2,3}$ interchanges ξ^2 and ξ^3 in products $\xi^1 \otimes \xi^2 \otimes \xi^3 \otimes \ldots \in \mathscr{G}^* \otimes \mathscr{G}^* \otimes \mathscr{G}^* \otimes \ldots$. Inserting the decomposition $\varphi = \sum_\sigma \varphi^1_\sigma \otimes \varphi^2_\sigma \otimes \varphi^3_\sigma$ and using the intertwining properties of φ and R, one verifies that

$$\begin{aligned}&(e \otimes R \otimes e \otimes e)((e \otimes \varphi^{-1}) \otimes e)(\mathrm{id} \otimes (\mathrm{id} \otimes \Delta)\Delta \otimes \mathrm{id})(\varphi) \\ &= (e \otimes R \otimes e \otimes e)(\mathrm{id} \otimes (\Delta \otimes \mathrm{id})\Delta \otimes \mathrm{id})(\varphi)(e \otimes \varphi^{-1} \otimes e) \\ &= (\mathrm{id} \otimes (\Delta' \otimes \mathrm{id})\Delta \otimes \mathrm{id})(\varphi)(e \otimes R \otimes e \otimes e)(e \otimes \varphi^{-1} \otimes e) .\end{aligned}$$

Similarly

$$\begin{aligned}
&P_{2,3}[(\mathrm{id}\otimes(\mathrm{id}\otimes\Delta)\Delta\otimes\mathrm{id})(\varphi^{-1})](e\otimes\varphi_{213}\otimes e)\\
&\quad=P_{2,3}[(\mathrm{id}\otimes(\mathrm{id}\otimes\Delta)\Delta\otimes\mathrm{id})(\varphi^{-1})(e\otimes\varphi\otimes e)]\\
&\quad=P_{2,3}[(e\otimes\varphi\otimes e)(\mathrm{id}\otimes(\Delta\otimes\mathrm{id})\Delta\otimes\mathrm{id})(\varphi^{-1})]\\
&\quad=(e\otimes\varphi_{213}\otimes e)(\mathrm{id}\otimes(\Delta'\otimes\mathrm{id})\Delta\otimes\mathrm{id})(\varphi^{-1})\,.
\end{aligned}$$

Using the formula for $\varphi^{-1}\varphi$ one finds from this that

$$A=(e\otimes\varphi_{213}\otimes e)(\mathrm{id}\otimes(\Delta'\otimes\mathrm{id})\Delta\otimes\mathrm{id})(\Delta\otimes\mathrm{id})\Delta(e)(e\otimes R\otimes e\otimes e)(e\otimes\varphi^{-1}\otimes e)\,. \tag{11.12}$$

Finally we use the intertwining property of φ once again to write this as

$$A=(\mathrm{id}\otimes(\mathrm{id}\otimes\Delta')\Delta\otimes\mathrm{id})(\Delta\otimes\mathrm{id})\Delta(e)(e\otimes\varphi_{213}\otimes e)(e\otimes R\otimes e\otimes e)(e\otimes\varphi^{-1}\otimes e)\,. \tag{11.13}$$

By definition,

$$\begin{aligned}
&((\tau\bigotimes\tau^{(n)})\otimes\mathrm{id})(\Delta(e))=((\tau\otimes\tau^{(n)})\otimes\mathrm{id})((\Delta\otimes\mathrm{id})\Delta(e))\\
&\quad=(\tau\otimes\tau^{f}\otimes\tau^{f}\otimes\tau^{(n-2)}\otimes\mathrm{id})(\mathrm{id}\otimes(\mathrm{id}\otimes\Delta)\Delta\otimes\mathrm{id})((\Delta\otimes\mathrm{id})\Delta(e))\,.
\end{aligned}$$

The $'$ on Δ' in the expression for A is absorbed by an interchange of matrix indices, and the factor $((\tau\bigotimes\tau^{(n)})\otimes\mathrm{id})(\Delta(e))$ is absorbed by using that

$$(F\times Z^{n})((\tau\bigotimes\tau^{(n)})\otimes\mathrm{id})(\Delta(e))=e(F\times Z^{n})=(F\times Z^{n})\,. \tag{11.14}$$

Reinserting the definition of $T^{(n)}(\sigma_{n-1})$ one obtains the desired result

$$\text{l.h.s.}=(F\times Z^{n})_{\mu\beta}c_{R}T^{(n)}_{\beta\alpha}(\sigma_{n-1})\,.\quad\text{q.e.d.} \tag{11.15}$$

Proof of Lemma C.2 for general k. Let $n=r+k$. The factor $\tau^{(n)}(e)$ in the definition of $T^{(n)}(\sigma_{k-1})$ is irrelevant because of Corollary 2.7. Therefore we may drop it so that

$$T^{(n)}_{\beta\alpha}(\sigma_{k-1})=((\otimes_{1}^{r+2}\tau^{f})\otimes\tau^{(k-2)})_{\bar\beta\alpha}(e^{r}\otimes\varphi_{213}(R\otimes e)\varphi^{-1})\,, \tag{11.16}$$

where $\bar\beta$ differs from $\beta=(b_n,b_{n-1},\ldots,b_1)$ by interchange of b_k and b_{k-1}. Here and below we write again e^{r} for $e\otimes\ldots\otimes e$, and similarly for id^{r}. By its definition

$$\tau^{(n)}(\xi)=(\otimes_{1}^{r+2}\tau^{f})\otimes\tau^{(k-2)}((\mathrm{id}^{r+1}\otimes\Delta)\ldots(\mathrm{id}\otimes\Delta)\Delta(\xi))\,. \tag{11.17}$$

Using these formulas, the calculation proceeds as in the case $k=n$ and one is led to evaluating

$$\begin{aligned}
A=P_{r+2,r+3}\;&[(\mathrm{id}\otimes(\mathrm{id}^{r+1}\otimes\Delta)\ldots(\mathrm{id}\otimes\Delta)\Delta\otimes\mathrm{id})(\varphi^{-1})]\\
&\cdot(e^{r+1}\otimes\varphi_{231}(R\otimes e)\varphi^{-1}\otimes e)(\mathrm{id}\otimes(\mathrm{id}^{r+1}\otimes\Delta)\ldots(\mathrm{id}\otimes\Delta)\Delta\otimes\mathrm{id})(\varphi)\,.
\end{aligned}$$

Using the intertwining properties of φ^{-1} and of R one deduces that

$$(e^{r}\otimes R\otimes e)(e^{r}\otimes\varphi^{-1})(\mathrm{id}^{r+1}\otimes\Delta)\ldots(\mathrm{id}\otimes\Delta)\Delta(\varphi^{3}_{\sigma}) \tag{11.18}$$

$$=(e^{r}\otimes R\otimes e)((\mathrm{id}^{r}\otimes\Delta\otimes\mathrm{id})(\mathrm{id}^{r}\otimes\Delta)\ldots\Delta)(\varphi^{3}_{\sigma})(e^{r}\otimes\varphi^{-1}) \tag{11.19}$$

$$=((\mathrm{id}^{r}\otimes\Delta'\otimes\mathrm{id})(\mathrm{id}^{r}\otimes\Delta)\ldots\Delta)(\varphi^{3}_{\sigma})(e^{r}\otimes R\otimes e)(e^{r}\otimes\varphi^{-1})\,. \tag{11.20}$$

The factor to the left of $e^{r+1} \otimes R \otimes e \otimes e$ is similarly transformed. The calculation proceeds further in the same manner as in the case $k = n$ and leads finally to the desired result. This completes the proof of Lemma C.2.

We turn to the proof of Proposition 4.3. It proceeds in two steps.

Proof of Eq. (11.1) *for* $n = k$. Write $\gamma = (c_1 c_2 \gamma')$, etc. By its definition

$$Z^n_\alpha = (Z \times (Z \times Z^{(n-2)}))_\alpha = ((Z \times Z) \times Z^{(n-2)})_\beta (\tau^f \otimes \tau^f \otimes \tau^{(n-2)})_{\beta\alpha}(\varphi^{-1}) . \quad (11.21)$$

Now we use the braid relations of Theorem 2.5 for $(Z \times Z)$ and Lemma C.1 to rewrite this as

$$= ((Z \times Z) \times Z^{(n-2)})_{c_2 c_1 \gamma'} c_R (\tau^f \otimes \tau^f \otimes \tau^{(n-2)})_{\gamma\alpha}((R \otimes e)\varphi^{-1}) . \quad (11.22)$$

Reassociate to get

$$\begin{aligned} c_R^{-1} Z^n_\alpha &= ((Z \times (Z \times Z^{n-2}))_{d_2 d_1 \delta'} (\tau^f_{d_2 c_2} \otimes \tau^f_{d_1 c_1} \otimes \tau^{n-2}_{\delta'\gamma'}) (\varphi) \\ &\quad \times (\tau^f \otimes \tau^f \otimes \tau^{(n-2)})_{\gamma\alpha}((R \otimes e)\varphi^{-1}) \\ &= ((Z \times (Z \times Z^{n-2}))_{d_2 d_1 \delta'} (\tau^f \otimes \tau^f \otimes \tau^{n-2})_{\delta\alpha} (\varphi_{213}(R \otimes e)\varphi^{-1}) \\ &= Z^n_\delta T^{(n)}_{\delta\alpha}(\sigma_{n-1}) . \quad \text{q.e.d.} \end{aligned}$$

Proof of Eq. (11.1) *for* $n > k$. Having established Eq. (11.1) for $n = k$ we proceed by induction in n. The factor $\tau^{(n)}(e)$ in the definition of $T^{(n)}(\sigma_l)$ is again irrelevant because of Corollary 2.7. Therefore we may suppose that

$$T^{(n+1)}_{\beta\alpha}(\sigma_{k-1}) = \delta_{ba} T^{(n)}_{\beta'\alpha'}(\sigma_{k-1}) \quad (11.23)$$

with $\beta = (b\beta')$ etc. By definition $Z^{(n+1)} = Z \times Z^n$. Lemma C.2 with $F = Z$ provides therefore the induction step from n to $n + 1$. This completes the proof of Proposition 4.3.

12. Appendix D: Proof that $\mathscr{B} \subset \mathscr{D}$

Due to the appearance of the $g_{\dot a \dot b}$-term in the $\partial \times Z$-braid relations (12.19), $\mathscr{B} \subset \mathscr{D}$ is not obvious. Indeed for different choices of the phases in Definition 8.2 this term would lead in general to new relations among the generators $Z_a \in \mathscr{D}$ which do not hold in $\mathscr{B}$. Our aim here is to show that our choice of phases in Definition 8.2 does not lead to such new relations.

Under the assumptions on the tensor product decomposition spelled out at the beginning of Sect. 8 (namely that tensor products $\tau^f \bigotimes \tau^f$ and $\tilde\tau^f \bigotimes \tilde\tau^f$ decompose into a direct sum of exactly two irreducible representations) we can reformulate the $Z \times Z$- and $\partial \times \partial$-braid relations in a way which is more convenient for the proof.

Lemma D.1. *With the notations and assumptions described in Sect. 6 we have:*
(i) *The relations* (12.14) *are equivalent to*

$$(Z \times Z)_{ab} (\tau^f_{ac} \otimes \tau^f_{bd}) (\Delta(P^A)) = 0 . \quad (12.1)$$

(ii) *The relations* (12.16) *are equivalent to*

$$(\partial \times \partial)_{\dot a \dot b} (\tilde\tau^f_{\dot a \dot c} \otimes (\tilde\tau^f_{\dot b \dot d}) (\Delta(P^{\tilde A})) = 0 . \quad (12.2)$$

Proof. Using $c_R = c_B^{-1}$ and the formula (8.2) for $(\tau^f \otimes \tau^f)(\hat{R}^+)$ we easily calculate

$$
\begin{aligned}
c_R(Z \times Z)_{cd}(\tau^f_{da} \otimes \tau^f_{cb})(R) &= c_R(Z \times Z)_{cd}(\tau^f_{ca} \otimes \tau^f_{db})(\hat{R}^+) \\
&= (Z \times Z)_{cd}(\tau^f_{ca} \otimes \tau^f_{db})\left(\left(\frac{c_A}{c_B} - 1\right)\Delta(P^A) + e \otimes e\right) \\
&= (Z \times Z)_{cd}(\tau^f_{ca} \otimes \tau^f_{db})\left(\left(\frac{c_A}{c_B} - 1\right)\Delta(P^A)\right) + c_R(Z \times Z)_{cd}(\tau^f_{da} \otimes \tau^f_{cb})(R)\,.
\end{aligned}
$$

This proves (i). The proof of (ii) is the same.

Lemma D.2. *From the $\partial \times Z$-braid relation* (8.19) *it follows that*
(i) $(\partial \times (Z \times Z))_{def}(\tilde{\tau}^f_{\dot{d}\dot{a}} \otimes \tau^f_{eb} \otimes \tau^f_{fc})(e \otimes \Delta(P^A)) = 0$,
(ii) $((\partial \times \partial) \times Z)_{\dot{d}\dot{e}f}(\tilde{\tau}^f_{\dot{d}\dot{a}} \otimes \tilde{\tau}^f_{\dot{e}\dot{b}} \otimes \tau^f_{fc})(\Delta(P^{\tilde{A}}) \otimes e) = 0$.

Proof. Again we will prove only (i). The proof of (ii) uses exactly the same ideas.

$$
\begin{aligned}
&(\partial \times (Z \times Z))_{def}(\tilde{\tau}^f_{\dot{d}\dot{a}} \otimes \tau^f_{eb} \otimes \tau^f_{fc})(e \otimes \Delta(P^A)) \\
&\quad = g_{\dot{d}e} Z_f(\tilde{\tau}^f_{\dot{d}\dot{a}} \otimes \tau^f_{eb} \otimes \tau^f_{fc})(\varphi^{-1}(e \otimes \Delta(P^A))) \\
&\qquad - g_{\dot{d}e} Z_f(\tilde{\tau}^f_{\dot{d}\dot{a}} \otimes \tau^f_{fb} \otimes \tau^f_{ec})(\varphi_{213}(R_{12})\varphi^{-1}(e \otimes \Delta(P^A)))c_A \\
&\quad = g_{\dot{d}e} Z_f(\tilde{\tau}^f_{\dot{d}\dot{a}} \otimes \tau^f_{eb} \otimes \tau^f_{fc})(\varphi^{-1}(e \otimes \Delta(P^A))) \\
&\qquad - g_{\dot{d}e} Z_f(\tilde{\tau}^f_{\dot{d}\dot{a}} \otimes \tau^f_{ec} \otimes \tau^f_{fb})(\varphi_{312}(R_{13})\varphi^{-1}_{132}(e \otimes \Delta'(P^A)))c_A \\
&\quad = g_{\dot{d}e} Z_f(\tilde{\tau}^f_{\dot{d}\dot{a}} \otimes \tau^f_{eb} \otimes \tau^f_{fc})(\varphi^{-1}(e \otimes \Delta(P^A))) \\
&\qquad - g_{\dot{d}e} Z_f(\tilde{\tau}^f_{\dot{d}\dot{a}} \otimes \tau^f_{ec} \otimes \tau^f_{fb})(\varphi^{-1}(R^{-1}_{23})(e \otimes \Delta'(P^A)))c_A \\
&\quad = g_{\dot{d}e} Z_f(\tilde{\tau}^f_{\dot{d}\dot{a}} \otimes \tau^f_{eb} \otimes \tau^f_{fc})(\varphi^{-1}(e \otimes (\Delta(P^A) - c_A(P^A)\hat{R}^-))) = 0\,.
\end{aligned}
$$

We used the $\partial \times Z$-braid relations (8.19), exchanged the third and second component of the tensor product, rewrote the expression with help of (8.12) and dropped the factor $(\mathrm{id} \otimes \Delta)(R)$ by (8.8). Finally we used the definition and expansion (8.3) of $\hat{R}^-$ to get the result. Terms of the type $(Z \times Z) \times \partial$ are eliminated by application of Lemma D.1. This completes the proof of the lemma.

Proof that $\mathscr{B} \subset \mathscr{D}$. Let us first discuss how the presence of the $g_{\dot{a}b}$-term could lead to new relations among coordinates $Z_a \in \mathscr{D}$. By definition all relations in $\mathscr{B}$ are $\mathscr{G}^*$-linear combinations of the following fundamental relations:

$$((Z^n \times (Z \times Z)) \times Z^m)_{\alpha ab\beta}(\tau^n_{\alpha\gamma} \otimes \tau^f_{ac} \otimes \tau^f_{bd} \otimes \tau^m_{\beta\delta})(e \otimes \Delta(P^0) \otimes e) = 0\,. \quad (12.3)$$

After multiplication with $\partial_{\dot{b}}$ from the left we bring the product into the standard form (8.20). This is done by recurrent use of (8.19). Each time we interchange the order of ∂ and Z we create an additional term which is purely generated by Z and $\xi \in \mathscr{G}^*$. The sum of all these terms has to vanish since the whole product was zero. We have to show that this sum is again a linear combination of products (12.3). The simplest case of this calculation was done in the proof of Lemma D.2. The general case is a simple consequence of the lemma. This follows by suitable reassociation.

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Communicated by A. Jaffe