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Regularity of Harmonic Maps with Prescribed Singularities

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Abstract. In this paper, we studied the regularity problem for harmonic maps into hyperbolic spaces with prescribed singularities along codimension two submanifolds. This is motivated from one of Hawking's conjectures on the uniqueness of Kerr solutions among all axially symmetric asymptotically flat stationary solutions to the vacuum Einstein equation in general relativity.

1. Introduction

In the last three decades, much progress has been made on harmonic maps between Riemannian manifolds. Among the outstanding ones, for instance, are the existence of Eells and Sampson [ES] on harmonic maps into nonpositively curved manifolds, with the generalization of R. Hamilton [Ha] to manifolds with boundary, the ones of Sacks and Uhlenbeck [SU], Lemaire [Le] and R. Schoen and S.-T. Yau on harmonic maps defined on Riemann surfaces, and regularity theories of R. Schoen and K. Uhlenbeck [SU1, SU2]. Prior to [SU1, SU2] there had been some regularity theorems due to Hildebrandt, Giusti, Giaquinta (see for example [Gi]) under various assumptions on the target manifolds. These results have brought tremendous new understandings of the geometry of manifolds.

In this paper, we consider the following problem. Let (M, ds^2) be a n-dimensional complete Riemannian manifold with or without boundary, and $N \subset M$ be a codimension two closed submanifold; let h be a smooth map from $M \backslash N$ into the naturally compactified hyperbolic space $\bar{H}^m$ such that $h(M \backslash N) \subset H^m$, where H^m is upper-half-space model of m-hyperbolic space form. Then we would like to find a harmonic map from $M \backslash N$ in H^m with "similar" asymptotic behavior to h along N. One natural approach is to perturb h to obtain the harmonic one. To make it

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precise, we let $(y_1, \ldots, y_m)(y_m > 0)$ be the global coordinates of H^m inherited from R^m, and $\partial H^m = \bar{H}^m \backslash H^m$ is just defined by $y_m = 0$. Write $h = (h_1, \ldots, h_m)$, then $h_m \geqq 0$ and $h_m > 0$ in $M \backslash N$.

We denote by $H_{1,0}(M)$ (respectively $H_{1,h_m,0}(M)$) the completion of $C_0^\infty(M)$ (resp. $C_0^\infty(M\backslash N)$) under the norm

$$\| \psi \|_{H_1}^2 = \int_M | \nabla \psi |^2 \, dV , \tag{1.1}$$

$$(\text{resp. } \| \psi \|_{H_1,h_m}^2 = \int_M h_m^{-2} | \nabla \psi |^2 \, dV) , \tag{1.2}$$

where dV is the volume form on M, and the norm on $\nabla \psi$ is taken with respect to the metric ds^2 on M.

We want to find a harmonic map into H^m of form $(\varphi_1, \ldots, \varphi_{m-1}, h_m e^{\varphi_m})$ satisfying: $\varphi_i - h_i \in H_{1,h_m,0}(M)$ for $i = 1, 2, \ldots, m-1$ and $\varphi_m \in H_{1,0}(M)$. Equivalently, we are bound to find the critical point of the functional on $(\prod_{i=1}^{m-1} (h_i + H_{1,h_m,0}(M))) \times H_{1,0}(M)$ defined as follows:

$$\begin{aligned} F(\varphi_1, \ldots, \varphi_{m-1}, \varphi_m) = \int_M \Bigg[& | \nabla \varphi_m |^2 + \frac{2 \nabla h_m \nabla \varphi_m}{h_m} \\ & + \left(\sum_{i=1}^{m-1} \frac{| \nabla \varphi_i |^2}{h_m^2} e^{-2\varphi_m} \right) \Bigg] dV . \end{aligned} \tag{1.3}$$

In the case $\log h_m$ is harmonic, i.e., $\Delta \log h_m = 0$ on $M \backslash N$, the second term in the above integration may be taken away from F. Under suitable conditions on $h_1, \ldots, h_m$, one can prove the existence of the minimizer by a standard method (see Sect. 2 or [We1]). For instance, in order to make the functional F meaningful, we have to assume that all integrals $\int_M h_m^{-2} | \nabla h_i |^2 \, dV$ are finite, where $i = 1, 2, \ldots m-1$. In fact, the solution is unique among the admissible functions. See [We1] for more details. One is then led to the problem how regular the critical point $(\varphi_1, \ldots, \varphi_m)$ could be along the submanifold N in M. This is our main concern in this paper. We demonstrate the smoothness of $(\varphi_1, \ldots, \varphi_m)$ under mild nondegeneracy conditions on h_m. Namely, we show

Theorem 1.1. *Let (M, g) be a smooth n-dimensional Riemannian manifold without boundary, $N \subset M$ be a smooth $(n-2)$-dimensional closed submanifold, $h_1, \ldots, h_{m-1}$ be smooth functions on M, $h_m > 0$ be smooth in $M \backslash N$ and*

$$\Delta \log h_m(x) = 0, \quad x \in M \backslash N , \tag{1.4}$$

$$\lim_{\rho(x) \to 0} \frac{\log h_m(x)}{\alpha \log \rho(x)} = 1 , \tag{1.5}$$

where $\rho(x) = \operatorname{dist}(x, N)$ is the distance between $x \in M$ and N, $\alpha > 0$ is some positive constant.

Let $(\varphi_1, \ldots, \varphi_m)$ be the minimizer of F defined in (1.3) in the space $(\prod_{i=1}^{m-1} (h_i + H_{1,h_m,0}(M))) \times (H_{1,0}(M) \cap L^\infty(M))$. Then for any $\varepsilon > 0$, $\varepsilon < 2\alpha$, $(\varphi_1, \ldots, \varphi_m) \in C^{k_\alpha, \lambda_\alpha}$ with $k_\alpha = [2\alpha - 2\varepsilon]$ and $\lambda_\alpha = \min\{2\alpha - 2\varepsilon - k_\alpha, 1 - \varepsilon\}$.

The conclusion of the above theorem still holds if, instead of (1.4), we assume that $\Delta \log h_m(x)$ can be extended to N as a smooth function. More generally, if

N is a submainfold with boundary, this theorem implies that the minimizer $(\varphi_1, \ldots, \varphi_m)$ is Hölder continuous in the interior of N. But our method also yields Hölder continuity of the minimizer on the boundary of N. This is discussed in [LT1].

The existence of such h_m is elementary and has been explicitly written down in terms of the Green's function. At those points where $h_m > 0$, the regularity of $(\varphi_1, \ldots, \varphi_m)$ is just the same as that of the harmonic map $(h_1 + \varphi_1, \ldots, h_{m-1} + \varphi_{m-1}, h_m e^{\varphi_m})$ into H^m, and it then follows from the well-known regularity theorem for harmonic maps (cf. [SU1]). Therefore, in order to prove our main theorem, we only need to give the regularity of $(\varphi_1, \ldots, \varphi_m)$ at those points where $h_m = 0$. Some easy computations show that the Euler–Lagrange equation of (1.3) is degenerate at these points. This is the essential difficulty to the proof of Theorem 1.1. The proof we have here is in the spirit of [SU1].

For harmonic maps, it is well known that C^α-regularity ($\alpha > 0$) automatically implies higher order ones (cf. [Sc]). However it is not clear in our case because of the degeneracy of the Euler–Lagrange equations. Therefore, we also need to derive the regularity estimates of higher order for $(\varphi_1, \ldots, \varphi_m)$ from C^α-estimate ($\alpha > 0$).

We also give an existence theorem of such a harmonic map in case h_i ($i = 1, \ldots, m-1$) are constants along the connected components of N.

One of our motivations towards such a problem is from one conjecture of Hawking in the formulation of G. Weinstein [We1]. Hawking's conjecture asserts that Kerr solutions are the only asymptotically flat, axially symmetric, stationary ones of Einstein Vacuum Equation in general relativity with certain nondegenerate conditions on event horizons. In case the event horizon is connected, it was settled down by Robinson [Ro] in the 70's. Robinson's proof is based on the uniqueness of harmonic maps into hyperbolic space and earlier work by Ernst and Carter ([Er, Ca]). But this conjecture is still open in general. As an application of our theorem, we will prove that there are no asymptotically flat, axially symmetric, stationary solutions of *EVE* with disconnected event horizons of small angular momentum.

The organization of this paper is as follows. In Sect. 2, we prove the existence and boundedness of the minimizer of the functional F in (1.3). In Sect. 3, we discuss some total energy estimate. Section 4 contains a modified monotonicity formula for our solutions. Then the Hölder regularity follows from the standard De Giorgi estimate. In Sect. 5, we discuss the regularity of higher order. The application of our theorem is given in the last section (Sect. 6).

Finally, we would like to remark that two possible generalizations can be made in the future. The simpler one is to remove the smoothness condition of N; instead we assume that N is a union of submanifolds intersecting with each other transversally. The more interesting one is to lift the assumption on the hyperbolicity of H^m. In general, when the target may not have nonpositive curvature, one expects the regularity of the minimizer of (1.3) outside a subset of M of codimension 3 as R. Schoen and K. Uhlenbeck found for harmonic maps. All these generalizations will be discussed in the future.

After we finished this work, we learned from G. Weinstein that in the special case where $M = R^3$, N is the z-axis, $m = 2$ and (φ_1, φ_2) is the minimizer among axisymmetric functions; our theorem is also proved independently by G. Weinstein ([We2]). His method is completely different from ours and seems to be unlikely to be generalized to higher dimensional or nonaxially symmetric cases.

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2. Existence

The existence results in this section are essentially due to G. Weinstein [We1]. We present them here for the sake of completeness.

Let (M^n, g) be a n-dimensional ($n \geqq 3$) smooth compact Riemannian manifold with smooth boundary, $N^{n-2} \subset M^n$ be a $(n-2)$-dimensional smooth closed submanifold. Let $\alpha > 0$ be some positive number, $\rho(x)$ be a function defined on M, $\rho(x) = \operatorname{dist}(x, N)$ for x near N and smooth, strictly positive elsewhere. Let u be the solution of

$$- \Delta u(x) = \Delta(\alpha \log \rho(x)), \quad x \in M ,$$

$$u|_{\partial M} = 0 .$$

Here Δ denotes the Laplace–Beltrami operator with respect to the metric of M.

It is obvious that u is smooth away from N. The question is how smooth it is across N. After some essentially elementary calculations, we see that w is at least Hölder continuous. In fact, for any $\varepsilon \in (0, 1)$, there exists some positive constant $C(\varepsilon) > 0$, such that,

$$|\nabla u(x)| \leqq C(\varepsilon)\rho(x)^{\varepsilon-1} \quad \forall x \in M \backslash N .$$

Let $h_m(x) = \rho(x)^\alpha e^{u(x)}$, $x \in M$, clearly $\Delta(\log h_m) = 0$ on $M \backslash N$.

Let f_i: $M \to \mathbb{R}$ $(1 \leqq i \leqq m)$ be $H^1(M)$ function and bounded on ∂M. For $i = 1, 2, \ldots, m-1$, $f_i = 0$ on N. We look for harmonic maps

$$(\varphi_1, \ldots, \varphi_{m-1}, h_m e^{\varphi_m}) \colon M \backslash N \to H^m$$

with prescribed boundary data:

$$\varphi_i|_{\partial M} = f_i, \quad 1 \leqq i \leqq m ,$$

$$\varphi_i|_N = f_i = 0, \quad 1 \leqq i \leqq m-1,$$

and the prescribed singularity on N in the sense that $|\varphi_m| \leqq$ *constant.*

Let us set up the problem rigorously, consider

$$H(\varphi_1, \ldots, \varphi_{m-1}, \varphi_m) = \int_M \left\{ |\nabla \varphi_m|^2 + h_m^{-2} e^{-2\varphi_m} \sum_{i=1}^{m-1} |\nabla \varphi_i|^2 \right\} dV , \tag{2.1}$$

where

$$\varphi_i - f_i \in H_{1, h_m, 0}(M) \quad \text{for } i = 1, 2, \ldots, m-1 , \tag{2.2}$$

$$\varphi_m - f_m \in H_0^1(M) . \tag{2.3}$$

It has been explained that if we can obtain a minimizer of $H(\varphi_1, \ldots, \varphi_m)$ among the admissible functions with $|\varphi_m| \leqq$ *constant*, we will have the harmonic map with prescribed singularity.

Theorem 2.1. *The minimum of* $H(\varphi_1, \ldots, \varphi_m)$ *can be attained among those admissible functions* $\varphi_1, \ldots, \varphi_m$ *which satisfy* (2.2) *and* (2.3). *Furthermore the minimizer satisfies* $|\varphi_m| \leqq$ *constant.*

Remark 2.1. The condition on the boundary data $f_i = 0$ on N $(i = 1, 2, \ldots, m-1)$ can be replaced by $f_i = c_i$ on N $(i = 1, 2, \ldots, m-1)$ with $c_i \in \mathbb{R}$ being constants. One only needs to make a translation in $(\varphi_1, \ldots, \varphi_{m-1})$ to achieve this.

Remark 2.2. The assumption of $N^{n-2} \subset M^n$ being a closed submanifold can be relaxed to $N^{n-2} \subset M^n$ being a submanifold with $\partial N \subset \partial M$, then we need the boundary conditions $\{f_i\}_{i=1}^{m-1}$ to be compatible on ∂N and on ∂M. The proof is exactly the same.

Sketch of the Proof of Theorem 2.1. Let $\varphi^{(k)} = (\varphi_1^{(k)}, \ldots, \varphi_{m-1}^{(k)}, \varphi_m^{(k)})$ be a minimizing sequence, with the boundary conditions (2.2) and (2.3). Since f_m is bounded on ∂M, we can replace $\varphi_m^{(k)}$ by $\tilde{\varphi}_m^{(k)} = \max\{\varphi_m, -C\}$. The new sequence $\varphi^{(k)} = (\varphi_1^{(k)}, \ldots, \varphi_{m-1}^{(k)}, \tilde{\varphi}_m^{(k)})$ will have energy no more than $\varphi^{(k)}$, and the same boundary value as $\varphi^{(k)}$. Therefore $\tilde{\varphi}^{(k)}$ is also a minimizing sequence. We replace $\varphi^{(k)}$ by $\tilde{\varphi}^{(k)}$, but for simplicity, we still denote it as $\varphi^{(k)}$. After this truncation, the minimizing sequence $\varphi^{(k)}$ satisfies

$$\varphi_m^{(k)} \geqq -C\,. \tag{2.4}$$

Once we have (2.4), we can use a simple density argument to replace $\varphi_i^{(k)}$ $(i = 1, 2, \ldots, m-1)$ by $\tilde{\varphi}_i^{(k)}$ $(i = 1, 2, \ldots, m-1)$ with $\tilde{\varphi}_i^{(k)} - f_i \in C_0^\infty(M \backslash N)$. Once again we replace $\varphi_i^{(k)}$ by $\tilde{\varphi}_i^{(k)}$, but still keep the old notation.

In order to get an upper bound on $\varphi_m^{(k)}$, we explore the isometry group of H^m. Let us denote

$$\Phi = (\Phi_1, \ldots, \Phi_{m-1}, \Phi_m) = (\varphi_1, \ldots, \varphi_{m-1}, h_m e^{\varphi_m}) \in H^m. \tag{2.5}$$

We know that

$$\begin{cases} \bar{\Phi}_i = -\dfrac{\Phi_i}{\Phi_m^2 + \sum\limits_{i=1}^{m-1} \Phi_i^2} & 1 \leqq i \leqq m-1 \\[2ex] \bar{\Phi}_m = \dfrac{\Phi_m}{\Phi_m^2 + \sum\limits_{i=1}^{m-1} \Phi_i^2} & \end{cases} \tag{2.6}$$

is an isometry of H^m, namely,

$$\frac{\sum\limits_{i=1}^{m-1} d\Phi_i^2 + d\Phi_m^2}{\Phi_m^2} = \frac{\sum\limits_{i=1}^{m-1} d\bar{\Phi}_i^2 + d\bar{\Phi}_m^2}{\bar{\Phi}_m^2}\,. \tag{2.7}$$

Let us write $\bar{\Phi}$ as

$$\bar{\Phi} = (\bar{\Phi}_1, \ldots, \bar{\Phi}_{m-1}, \bar{\Phi}_m) = (\bar{\varphi}_1, \ldots, \bar{\varphi}_{m-1}, h_m^{-1} e^{\bar{\varphi}_m})\,. \tag{2.8}$$

It follows easily from (2.7) that

$$|\nabla\varphi_m|^2 + \sum_{i=1}^{m-1} |\nabla\varphi_i|^2 e^{-2\varphi_m} h_m^{-2} = |\nabla\bar{\varphi}_m|^2 + \sum_{i=1}^{m-1} |\nabla\bar{\varphi}_i|^2 e^{-2\bar{\varphi}_m} h_m^2$$
$$- 2\nabla(\log h_m)\,\nabla(\varphi_m + \bar{\varphi}_m)\,.$$

Integrating the above over M,

$$\int_M |\nabla\varphi_m|^2 + \sum_{i=1}^{m-1} |\nabla\varphi_i|^2 e^{-2\varphi_m} h_m^{-2} = \int_M |\nabla\bar{\varphi}_m|^2 + \sum_{i=1}^{m-1} |\nabla\bar{\varphi}_i|^2 e^{-2\bar{\varphi}_m} h_m^2$$
$$- 2\int_M \nabla(\log h_m)\,\nabla(\varphi_m + \bar{\varphi}_m)\,.$$

Since $\varphi_i^{(k)} = 0$ $(i = 1, 2, \ldots, m-1)$ near N, it follows from (2.6) that $\varphi_m + \bar{\varphi}_m = 0$ near N. Using Stokes theorem and the harmonicity of $\log h_m$, we have

$$- 2\int_M \nabla(\log h_m)\,\nabla(\varphi_m + \bar{\varphi}_m) = 2\int_{\partial M} (\varphi_m + \bar{\varphi}_m)\frac{\partial}{\partial\nu}(\log h_m)$$

which depends only on the boundary data $\{f_i\}$. Hence

$$\int_M |\nabla\varphi_m|^2 + \sum_{i=1}^{m-1} |\nabla\varphi_i|^2 e^{-2\varphi_m} h_m^{-2} = \int_M |\nabla\bar{\varphi}_m|^2 + \sum_{i=1}^{m-1} |\nabla\bar{\varphi}_i|^2 e^{-2\bar{\varphi}_m} h_m^2$$
$$+ \textit{constant}$$

clearly, $\bar{\varphi}_m \geqq -C$ on ∂M, therefore we can chop it off from below as before without increasing the energy. Therefore we can assume that our minimizing sequence $\varphi^{(k)}$ has the property that $\bar{\varphi}_m \geqq -C$, which, according to (2.6), implies that

$$\varphi_m^{(k)} \leqq C\,. \tag{2.9}$$

The new minimizing sequence might lose the lower bound (2.4), but we can chop it off from below and gain back this property easily.

Putting the above together, we have obtained a minimizing sequence $\varphi^{(k)}$ with bounds on $\varphi_m^{(k)}$ from above and below ((2.4) and (2.8)). With these bounds, one can easily obtain a minimum by using some standard functional analysis argument.

Remark 2.3. In Theorem 2.1, if there are finite disjoint $n-2$ closed dimensional submanifolds $N_1, \ldots, N_l \subset M$, one can prove the same result by letting $\rho(x) = \mathrm{dist}(x, N_j)$ $(j = 1, 2, \ldots, l)$ near $\bigcup_{j=1}^{l} N_j$ and smooth, positive elsewhere, also $f_i = C_i^j$ on N_j $(i \leqq j \leqq l, 1 \leqq i \leqq m-1)$. The proof is very similar, just perform the chopping off procedure one by one.

3. Energy Estimates

Let $(\varphi_1, \ldots, \varphi_m)$ be the solution of the Euler–Lagrange equation of (1.3), i.e.

$$\begin{cases} \Delta\varphi_m = -\dfrac{1}{h_m^2} e^{-2\varphi_m} \displaystyle\sum_{i=1}^{m-1} |\nabla\varphi_i|^2 \\ \mathrm{div}\left(\dfrac{e^{-2\varphi_m}}{h_m^2}\nabla\varphi_i\right) = 0 \quad \text{for } i = 1, 2, \ldots, m-1 \end{cases} \quad \text{on } M\backslash N \tag{3.1}$$

with $|\varphi_m| \leqq C$. Note that here we need to use the assumption in Theorem 1 that $\Delta \log h_m = 0$ outside N. In our proof below and the next two sections, we always use C to denote a universal constant, although its actual value may vary in different places. From now on, we fix a point x_0 in $N \subset M$.

Write ρ to be the distance function from the subspace N with respect to the metric ds^2. By our assumptions in Theorem 1.1, we have

$$h_m = \rho^\alpha e^u \quad (\alpha > 0) \tag{3.2}$$

in a neighborhood of x_0, say the unit geodesic ball $B_1(x_0)$, where u is smooth in $B_1(x_0)\backslash N$ such that for any $0 < \varepsilon < 1$, there exists some positive constant $C(\varepsilon) > 0, |\nabla u(x)| \leqq C(\varepsilon)\rho(x)^{\varepsilon - 1}$, $\forall x \in N$.

We denote by $r_x(\cdot)$ the distance function on M from x_0.

Lemma 3.1. *The solution* $(\varphi_1, \ldots, \varphi_m)$ *can be extended across* $M \cap B_1(x_0)$ *to be the weak solutions of* (3.1) *in the sense: for any smooth functions* $\psi_1, \ldots, \psi_n$ *with compact support in* $B_1(x_0)$, *we have*

$$\int_{B_1(0)} \left\{ \nabla\varphi_m \nabla\psi_m - \frac{1}{h_m^2} e^{-2\varphi_m} \sum_{i=1}^{m-1} |\nabla\varphi_i|^2 \psi_m \right\} dV = 0\,, \tag{3.3}$$

$$\int_{B_1} \left\{ \nabla\psi_i \nabla\varphi_i + 2\nabla\varphi_i \left(\frac{\nabla h_m}{h_m} - \nabla\varphi_m \right) \psi_i \right\} dV = 0 \quad i = 1, 2, \ldots, m-1\,. \tag{3.4}$$

Proof. Outside N, the second equation in (3.1) is equivalent to

$$\Delta\varphi_i - 2\nabla\varphi_i \left(\frac{\nabla h_m}{h_m} - \nabla\varphi_m \right) = 0\,. \tag{3.5}$$

Let η be a cut-off function from R^1 into R^1 satisfying $\eta(t) \equiv 0$ for $t \leqq 1$, $\eta(t) \equiv 1$ for $t \geqq 2$. $\eta(t) \geqq 0$, $|\eta'(t)| \leqq 1$. Then the lemma is proved by multiplying $\eta\left(\dfrac{\log(-\log\rho)}{\log(-\log\varepsilon)}\right)\psi_j$ $(1 \leqq j \leqq m)$ to the first equation of (3.1) and (3.4) above, respectively, integrating by parts, and then taking the limit as ε goes to zero. Note here that we need to use the fact

$$\int_0^1 \frac{d\rho}{\rho(\log\rho)^2} < \infty\,.$$

Lemma 3.2. *There is a uniform constant* C *such that for any* x *in* $B_{\frac{1}{2}}(x_0)$,

$$\int_{B_{\frac{1}{4}}(x)} r_x(z)^{-n+2} \left(|\nabla\varphi_m|^2 + \sum_{i=1}^{m-1} \frac{|\nabla\varphi_i|^2}{h_m^2} \right)(z)\,dV(z) \leqq C < \infty\,. \tag{3.6}$$

Proof. Let $G_x(z)$ be the Green function on $B_1(x_0)$ with singularity at x and vanishing on $\partial B_1(x_0)$, η be the cut-off function defined as above. Put $\lambda = -1 + \inf_M \varphi_m$, then $-\varphi_m - \lambda \geqq 1$. For any $\varepsilon > 0$, we smooth the Green function as follows:

$$G_x^\varepsilon(z) = \frac{1}{\mathrm{Vol}(B_\varepsilon(z)) \int_{B_\varepsilon(z)} G_x(z')dz'}\,. \tag{3.7}$$

Substituting ψ_m in (3.3) by $(1 - \eta(4r_x(z)))^2 G_x^\varepsilon(z)(-\varphi_m(z) - \lambda)$ and using the boundedness of φ_m, we immediately obtain

$$\int_{B_1(0)} \left\{ h_m^{-2} \sum_{i=1}^{m-1} |\nabla \varphi_i|^2 + |\nabla \varphi_m|^2 \right\} (1 - \eta(4r_x))^2 G_x^\varepsilon(z) dV(z)$$

$$\leqq C\left\{1 - \int_{B_1(0)} \nabla\varphi_m \cdot (\varphi_m - \lambda) \nabla((1-\eta)^2 G_x^\varepsilon) dV\right\}$$

$$\leqq C\left\{1 + \frac{1}{2}\int_{B_1(0)} (\varphi_m - \lambda)^2 \Delta((1-\eta)^2 G_x^\varepsilon)(z) dV(z)\right\}$$

$$\leqq C\left\{1 + \frac{1}{2}\int_{B_1(0)} (\varphi_m - \lambda)^2 (1-\eta)^2 \Delta G_x^\varepsilon dV\right\}. \tag{3.8}$$

(Note that C is always a uniform constant.)

But some direct computations show that the last integral is nonpositive when ε is sufficiently small. Thus our lemma follows from (3.8) by taking the limit as ε goes to zero and the fact that $G_x(z)$ is equivalent to $r_x(z)^{-n+2}$ in $B_{\frac{1}{2}}(x)$.

Before we go further, we would like to make a few remarks. By the well-known regularity theory of harmonic maps (cf. [SU1]), we know that $(\varphi_1, \ldots, \varphi_m)$ are as smooth as $(h_1, \ldots, h_m)$ outside $N \subset M$, or more precisely, the set where $h_m > 0$.

Lemma 3.3. *For any point x in $B_1(x_0)\backslash N$, we have*

$$|\nabla \varphi_m|(x) \leqq \frac{C}{\rho(x)}, \quad |\nabla \varphi_i|(x) \leqq C_\rho(x)^{-1+\alpha}, \tag{3.9}$$

where C is a uniform constant independent of $(\varphi_1, \ldots, \varphi_m)$.

Proof. Recall that $(\varphi_1, \ldots, \varphi_{m-1}, h_m e^{\varphi_m})$ define a harmonic map Φ from $M\backslash N$ into H^m. Denote by $e = e(\Phi)$ the energy density of this harmonic map. Then by the standard Bochner formula, one finds by using the hyperbolicity of H^m,

$$-\Delta\sqrt{e(\Phi)} \leqq \mu\sqrt{e(\Phi)}, \tag{3.10}$$

where μ is a constant depending only on the upper bound of Ricci curvature on M.

Then by applying mean-value inequality or direct Moser iteration to (3.10), we obtain $(\bar{\rho} = \rho(x))$

$$e(\Phi)(x) \leqq \frac{C}{\mathrm{Vol}(B_{\frac{\bar\rho}{2}}(x))} \int_{B_{\frac{\bar\rho}{2}}(x)} e(\Phi)(z) dV(z). \tag{3.11}$$

On the other hand, according to the definition, we have

$$e(\Phi) = \left(\sum_{i=1}^{m-1} |\nabla\varphi_i|^2\right) h_m^{-2} e^{-2\varphi_m} + |\nabla \log h_m + \nabla\varphi_m|^2$$

$$= h_m^{-2} e^{-2\varphi_m} \left(\sum_{i=1}^{m-1} |\nabla\varphi_i|^2\right) + \left|\frac{\alpha\nabla\rho}{\rho} + \nabla u + \nabla\varphi_m\right|^2. \tag{3.12}$$

Thus by the previous lemma and the assumption of ∇u near N (cf. (3.2)),

$$\int_{B_{\frac{\bar\rho}{2}}(x)} e(\Phi)(z)\,dV(z) \leqq C\bar\rho^{n-2} = C\rho(x)^{n-2}. \tag{3.13}$$

It follows

$$e(\Phi)(x) \leqq C\rho(x)^{-2}. \tag{3.14}$$

Note that this C may be different from the previous one, but still independent of x in $B_1(x_0)$ and $(\varphi_1, \ldots, \varphi_m)$. Then (3.9) follows from (3.14) and (3.12).

Without loss of generality, we may assume that $B_4(x_0)$ is geodesically convex.

Corollary 3.1. *Let $(\varphi_1, \ldots, \varphi_m)$ be the solution of* (3.1) *with $|\varphi_m|$ bounded. Then*

$$|\varphi_i(x) - \varphi_i(\pi(x))| \leqq C\rho^\alpha \quad i = 1, 2, \ldots, m-1, \tag{3.15}$$

where π is the map from $B_1(x_0)$ to N defined as follows: for any x, there is a unique geodesic γ such that $\gamma(0) \in N$, $\gamma(\rho(x)) = x$, and $\gamma(0)$ is perpendicular to N at $\gamma(0)$, then define $\pi(x) = \gamma(0)$.

Proof. Let γ be the unique geodesic joining $\pi(x)$ to x with length $\rho(x)$. Then $|\gamma'| \equiv 1$. By the fundamental theorem of calculus,

$$\begin{aligned}\varphi_i(x) - \varphi_i(\pi(x)) &= \int_0^{\rho(x)} \left(\frac{d}{dt}\varphi_i(\gamma(t))\right)dt = \int_0^{\rho(x)} \nabla\varphi_i(\gamma(t))\cdot(\gamma'(t))\,dt \\ &\leqq C\int_0^{\rho(x)} t^{\alpha-1}\,dt = \frac{C}{\alpha}\rho^\alpha(x),\end{aligned}$$

where Lemma 3.3 has been used.

In particular, if $\alpha > 1$, then φ_i must be constant in each connected component of N in M.

Proposition 3.1. (*Smallness of the normalized energy*). *Let $(\varphi_1, \ldots, \varphi_m)$ be the solution of* (3.1) *with $|\varphi_m|$ uniformly bounded. Then for any $\varepsilon > 0$, x in M, there is a σ_x between $e^{-3C/\varepsilon}$ and $\frac{1}{4}$ such that*

$$\sigma_x^{-n+2} \int_{B_{\sigma_x}(x)} \left(|\nabla\varphi_m|^2 + \rho^{-2\alpha}\sum_{i=1}^{m-1} |\nabla\varphi_i|^2\right)dV \leqq \varepsilon, \tag{3.16}$$

where C is the uniform constant in (3.6).

Proof. Define

$$f(\sigma) = \frac{1}{\sigma^{n-2}} \int_{B_\sigma(x)} \left(|\nabla\varphi_m|^2 + \rho^{-2\alpha}\sum_{i=1}^{m-1} |\nabla\varphi_i|^2\right)dV.$$

Then

$$\int_0^{\frac14}\frac{f(\sigma)}{\sigma}d\sigma=\int_0^{\frac14}\sigma^{-n+1}\left(\int_{B_\sigma(x)}\left\{|\nabla\varphi_m|^2+\rho^{-2\alpha}\sum_{i=1}^{m-1}|\nabla\varphi_i|^2\right\}dV\right)d\sigma$$

$$=-(n-2)\int_0^{\frac14}\frac{d}{d\sigma}(\sigma^{-n+2})\int_{B_\sigma(x)}\left(|\nabla\varphi_m|^2+\rho^{-2\alpha}\sum_{i=1}^{m-1}|\nabla\varphi_i|^2\right)dV\cdot d\sigma$$

$$=-(n-2)\sigma^{-n+2}\int_{B_\sigma(x)}\left(|\nabla\varphi_m|^2+\rho^{-2\alpha}\sum_{i=1}^{m-1}|\nabla\varphi_i|^2\right)dV\Bigg|_{0+}^{\frac14}$$

$$+(n-2)\int_0^{\frac14}\sigma^{-n+2}\left(\int_{\partial B_\sigma(x)}\left(|\nabla\varphi_m|^2+\rho^{-2\alpha}\sum_{i=1}^{m-1}|\nabla\varphi_i|^2\right)dS\right)d\sigma$$

$$=-(n-2)\sigma^{-n+2}\int_{B_\sigma(x)}\left(|\nabla\varphi_m|^2+\rho^{-2\alpha}\sum_{i=1}^{m-1}|\nabla\varphi_i|^2\right)dV\Bigg|_{0+}^{\frac14}$$

$$+(n-2)\int_{B_{\frac14}(x)}r_x(z)^{-n+2}\left(|\nabla\varphi_m|^2+\rho^{-2\alpha}\sum_{i=1}^{m-1}|\nabla\varphi_i|^2\right)(z)dV(z)\,.$$

By Lemma 3.2 and Lemma 3.3, all last three integrals are uniformly bounded, so it follows

$$\int_0^{\frac14}\frac{f(\sigma)}{\sigma}d\sigma\leqq C\,. \tag{3.17}$$

Then the lemma just follows from a simple estimate on the lower bound of the integral in (3.17).

4. Hölder Estimates

All the notations in Sect. 3 will be adopted in this section. The aim of this section is to show the Hölder continuity of the solution $(\varphi_1,\ldots,\varphi_m)$ of (3.1) at the points of N where h_m is of the form (3.2). Such a Hölder estimate will follow from a strengthened version of energy estimate in Proposition 3.1. Usually, this can be accomplished by means of a monotonicity formula, for instance, in the case of harmonic maps. But this required monotonicity formula is not at hand in our case, so we first need to derive it.

Let $(\varphi_1,\ldots,\varphi_m)$ be a fixed solution of (3.1) with $|\varphi_m|\leqq C$ as in Sect. 3, and $B_4(x_0)$ is a geodesically convex ball at x_0 in N, in which h_m can be written as in (3.2). Also, since Theorem 1.1 is local, we may assume that $M=R^n$ and $N=R^{n-2}\subset M$.

Lemma 4.1. *For any β, $0\leqq\sigma\leqq\frac14$, $\varepsilon>0$ and $x\in B_{\frac12}(x_0)\cap N$, there is the following inequality.*

$$E_{\frac{\sigma}{2}}(x)\leqq\varepsilon E_\sigma(x)+\frac{C}{\varepsilon\sigma^n}\int_{B_\sigma(x)\backslash B_{\frac{\sigma}{2}}(x)}\left\{|\varphi_m-\beta|^2+h_m^{-2}e^{-2\varphi_m}\sum_{i=1}^{m-1}|\varphi_i-\beta_i|^2\right\}dV$$
$$+C_\alpha\sigma^{1-\alpha} \tag{4.1}$$

where C_α is a constant which is zero if $\alpha \geqq 1$ and we define

$$E_\sigma(x) = \sigma^{-n+2} \int_{B_\sigma(x)} \left\{ |\nabla\varphi_m|^2 + h_m^{-2} e^{-2\varphi_m} \sum_{i=1}^{m-1} |\nabla\varphi_i|^2 \right\} dV. \tag{4.2}$$

Proof. Let η_μ: $R^1 \to R^1$ be cut-off functions ($\mu > 1$) satisfying:

$$\eta_\mu(t) \equiv 1 \text{ for } t \leqq 1; \eta_\mu(t) \equiv 0 \text{ for } t \geqq \mu; \quad \eta_\mu(t) \geqq 0;$$

$$|\eta'_\mu(t)| \leqq \frac{1}{\mu-1}; \quad |\eta''_\mu(t)| \leqq \left(\frac{1}{\mu-1}\right)^2. \tag{4.3}$$

There are two cases: (i) $B_{\frac{3}{4}\sigma}(x) \cap N = \emptyset$; (ii) $B_{\frac{3}{4}\sigma}(x) \cap N \neq \emptyset$. Presumably, the first case is easier. Let us consider (ii) and then indicate why (4.1) is also true in case (i). In case $B_{\frac{3}{4}\sigma}(x) \cap N \neq \emptyset$, all β_i $(i = 1, 2, \ldots, m-1)$ are zeroes according to our assumption.

We observe that there is σ_0 between $\frac{3}{4}\sigma$ and $\frac{7}{8}\sigma$ such that

$$\int_{\partial B_{\sigma_0}(x)} h_m^{-2} e^{-2\varphi_m} \sum_{i=1}^{m-1} |\nabla\varphi_i|^2 dV \leqq \frac{8}{\sigma} \int_{B_{\frac{7}{8}\sigma}(x)\setminus B_{\frac{3}{4}\sigma}(x)} h_m^{-2} e^{-2\varphi_m} \sum_{i=1}^{m-1} |\nabla\varphi_i|^2 dV \tag{4.4}$$

and

$$\int_{\partial B_{\sigma_0}(x)} h_m^{-2} e^{-2\varphi_m} \sum_{i=1}^{m-1} |\varphi_i|^2 \leqq \frac{8}{\sigma} \int_{B_{\frac{7}{8}\sigma}(x)\setminus B_{\frac{3}{4}\sigma}(x)} h_m^{-2} e^{-2\varphi_m} \sum_{i=1}^{m-1} |\varphi_i|^2 dV. \tag{4.5}$$

Let λ be a positive number $\leqq \frac{1}{16}$, specified later. We call $(\varphi_1^*, \ldots, \varphi_m^*)$ an admissible map if it is in the domain of the functional F (cf. Sect. 1, 2), in particular, for such a map, we have

$$F(\varphi_1, \ldots, \varphi_m) \leqq F(\varphi_1^*, \ldots, \varphi_m^*).$$

Define

$$\varphi_m^*(z) = \left(1 - \eta_{\frac{3}{2}}\left(\frac{3}{2\sigma_0} r_x(z)\right)\right)\varphi_m(z) + \eta_{\frac{3}{2}}\left(\frac{3}{2\sigma_0} r_x(z)\right)\beta. \tag{4.6}$$

In order to define φ_i^* for $1 \leqq i \leqq m-1$, we need to first introduce $\tilde{\varphi}_i$ as follows: define $\tilde{\varphi}_i(z)$ to be $\varphi_i(z) - h_i(z)$ for $z \notin B_{\sigma_0}(x)$ and $(\varphi_i - h_i)\left(x + \dfrac{\sigma_0(z-x)}{z-x}\right)$ for z in $B_{\sigma_0}(x)$. In case $\alpha \geqq 1$, we may assume h_i are constants in $B_1(x_0)$.

Now we define

$$\varphi_i^*(z) = \left(1 - \eta_{\frac{1}{1-\lambda}}\left(\frac{r_x(z)}{(1-\lambda)r_0}\right)\right)\tilde{\varphi}_i(z) + h_i, \; i = 1, 2, \ldots, m-1. \tag{4.7}$$

Using the fact that $\rho\left(x + \dfrac{\sigma_0(z-x)}{|z-x|}\right)$ is uniformly equivalent to $\rho(z)$ for z in $B_{\sigma_0}(x)\setminus B_{\sigma_0/2}(x)$, one can easily check that $(\varphi_1^*, \ldots, \varphi_m^*)$ is an admissible map, moreover, φ_i^* and the derivatives of φ_i^* $(i = 1, 2, \ldots, m-1)$ coincide with φ_m and those of φ_i outside $B_{\sigma_0}(x)$, respectively. Therefore, we have

$$\begin{aligned}
&\int_{B_{\sigma_0}(x)} \left\{ |\nabla\varphi_m|^2 + h_m^{-2} e^{-2\varphi_m} \sum_{i=1}^{m-1} |\nabla\varphi_i|^2 \right\} dV \\
&\leqq \int_{B_{\sigma_0}(x)} \left\{ |\nabla\varphi_m^*|^2 + h_m^{-2} e^{-2\varphi_m^*} \sum_{i=1}^{m-1} |\nabla\varphi_i^*|^2 \right\} dV \\
&= \int_{B_{\sigma_0}(x)} \left\{ \left(1 - \eta_{\frac{3}{2}}\left(\frac{3}{2\sigma_0} r_x\right)\right) |\nabla\varphi_m|^2 + \frac{9}{4}\left(\eta_{\frac{3}{2}}' \frac{1}{\sigma_0^2} |\varphi_m - \beta|^2 \right.\right. \\
&\quad - \frac{3}{\sigma_0} \eta_{\frac{3}{2}}' \left(\frac{3}{2\sigma_0} r_x\right) \nabla r_x \cdot \nabla\varphi_m(\varphi_m - \beta) + h_m^{-2} e^{-2\varphi_m^*} \\
&\quad \left. \cdot \sum_{i=1}^{m-1} \left| \nabla h_i + \left(1 - \eta_{\frac{1}{1-\lambda}}\left(\frac{r_x}{(1-\lambda)\sigma_0}\right)\right) \nabla\tilde{\varphi}_i - \frac{\tilde{\varphi}_i}{(1-\lambda)\sigma_0} \eta_{\frac{1}{1-\lambda}}' \cdot \nabla r_x \right|^2 \right\} dV \\
&\leqq \int_{B_{\sigma_0}(x)} \left\{ \left(1 - \eta_{\frac{3}{2}}\left(\frac{3r_x}{2\sigma_0}\right)\right) |\nabla\varphi_m|^2 + \frac{20}{\sigma_0^2} \left((\eta_{\frac{3}{2}}')^2 + |\eta_{\frac{3}{2}}''|\right) |\varphi_m - \beta|^2 \right\} dV \\
&\quad + 4 \int_{B_{\sigma_0}(x) \setminus B_{(1-\lambda)\sigma_0}(x)} h_m^{-2} e^{-2\varphi_m^*} \sum_{i=1}^{m-1} |\nabla\tilde{\varphi}_i|^2 \, dV \\
&\quad + \frac{1}{\lambda^2\sigma_0^2} \int_{B_{\sigma_0}(z) \setminus B_{(1-\lambda)\sigma_0}(x)} h_m^{-2} e^{-2\varphi_m^*} \sum_{i=1}^{m-1} \tilde{\varphi}_i^2 \, dV \\
&\quad + 2 \int_{B_{\sigma_0}(x)} h_m^{-2} e^{-2\varphi_m^*} \sum_{i=1}^{m-1} |\nabla h_i|^2 \, dV.
\end{aligned} \tag{4.8}$$

In particular, it implies

$$\begin{aligned}
&\int_{B_{\frac{3\sigma_0}{4}}(x)} |\nabla\varphi_m|^2 dV + \int_{B_{\sigma_0}(x)} h_m^{-2} e^{-2\varphi_m} \sum_{i=1}^{m-1} |\nabla\varphi_i|^2 dV \leqq \frac{200}{\sigma_0^2} \int_{B_{\sigma_0}(x) \setminus B_{\frac{3\sigma_0}{4}}(x)} |\varphi_m - \beta|^2 dV \\
&\quad + 2\tilde{C} \int_{B_{\sigma_0}(x) \setminus B_{(1-\lambda)\sigma_0}(x)} h_m^{-2} \sum_{i=1}^{m-1} |\nabla\tilde{\varphi}_i|^2 dV + \frac{2\tilde{C}}{\lambda^2\sigma_0^2} \int_{B_{\sigma_0}(x) \setminus B_{(1-\lambda)\sigma_0}(x)} h_m^{-2} \sum_{i=1}^{m-1} \tilde{\varphi}_i^2 dV,
\end{aligned} \tag{4.9}$$

where $\tilde{C}$ is a constant depending only on the supremum of $|\varphi_m|$. Since $x \in N$ and $\lambda \leqq \frac{1}{16}$, two quantities $h_m(z)$ and $h_m\left(x + \frac{\sigma_0(z-x)}{|z-x|}\right)$ are uniformly equivalent for z in $B_{\sigma_0}(x) \setminus B_{(1-\lambda)\sigma_0}(x)$, say

$$C_1^{-1} h_m(z) \leqq h_m\left(x + \frac{\sigma_0(z-x)}{|z-x|}\right) \leqq C_1 h_m(z), \tag{4.10}$$

where C_1 is a universal constant. Thus by using

$$\begin{aligned}
&\int_{B_{\sigma_0}(x) \setminus B_{(1-\lambda)\sigma_0}(x)} h_m^{-2} \sum_{i=1}^{m-1} |\nabla\tilde{\varphi}_i|^2 dV \leqq C_1 \int_{\partial B_{\sigma_0}(x)} h_m^{-2} \sum_{i=1}^{m-1} |\nabla\varphi_i|^2 dS \cdot \int_{1-\lambda}^{1} t^{n-1} dt \\
&\leqq \lambda C_1 \tilde{C} \int_{\partial B_{\sigma_0}(x)} h_m^{-2} e^{-2\varphi_m} \sum_{i=1}^{m-1} |\nabla\varphi_i|^2 dV \\
&\leqq 8\lambda C_1 \tilde{C} \int_{B_{\sigma_0}(x) \setminus B_{\frac{3}{4}\sigma_0}(x)} h_m^{-2} e^{-2\varphi_m} \sum_{i=1}^{m-1} |\nabla\varphi_i|^2 dV.
\end{aligned} \tag{4.11}$$

Similarly, we also have

$$\int_{B_{\sigma_0}(x)\setminus B_{(1-\lambda)\sigma_0}(x)} h_m^{-2}\sum_{i=1}^{m-1}\tilde{\varphi}_i dV \leqq 8\lambda C_1\tilde{C}\int_{B_{\frac{7}{8}\sigma}(x)\setminus B_{\frac{3}{4}\sigma}(x)} h_m^{-2}e^{-2\varphi_m}\sum_{i=1}^{m-1}\varphi_i^2 dV. \quad (14.12)$$

Combining (4.9), (4.11) and (4.12), we obtain

$$\begin{aligned}\int_{B_{\frac{3\sigma_0}{4}}(x)} |\nabla\varphi_m|^2 dV &+ \int_{B_{\sigma_0}(x)} h_m^{-2}e^{-2\varphi_m}\sum_{i=1}^{m-1}|\nabla\varphi_i|^2 dV \leqq \frac{200}{\sigma_0^2}\int_{B_{\sigma_0}(x)\setminus B_{\frac{3\sigma_0}{4}}(x)} |\varphi_m-\beta|^2 dV\\ &+16\lambda\tilde{C}^2C_1\left\{\int_{B_{\frac{7}{8}\sigma}(x)\setminus B_{\frac{3}{4}\sigma}(x)} h_m^{-2}e^{-2\varphi_m}\sum_{i=1}^{m-1}|\nabla\varphi_i|^2 dV\right.\\ &\left.+\frac{1}{\lambda^2\sigma_0^2}\int_{B_{\frac{7}{8}\sigma}(x)\setminus B_{\frac{3}{4}\sigma}(x)} h_m^{-2}e^{-2\varphi_m}\sum_{i=1}^{m-1}\varphi_i^2 dV\right\}. \end{aligned}\quad (4.13)$$

Now we choose λ such that $16\lambda\tilde{C}^2C_1 \leqq \varepsilon$. Then there is a uniform constant C_0 such that

$$\begin{aligned}&\int_{B_{\frac{3}{4}\sigma_0}(x)}\left\{|\nabla\varphi_m|^2 + h_m^{-2}e^{-2\varphi_m}\sum_{i=1}^{m-1}|\nabla\varphi_i|^2\right\}dV\\ &\leqq \varepsilon E_\sigma(x) + \frac{C_0}{\varepsilon\sigma^2}\left\{\int_{B_\sigma(x)\setminus B_{\frac{3}{4}\sigma}(x)}\left(|\varphi_m-\beta|^2 + h_m^{-2}e^{-2\varphi_m}\sum_{i=1}^{m-1}|\nabla\varphi_i|^2\right)dV\right\}.\end{aligned}$$

Note that $\frac{3}{4}\sigma \leqq \sigma_0 \leqq \frac{7\sigma}{8}$, so $\frac{3\sigma_0}{4} \geqq \frac{1}{2}\sigma$, $\sigma_0 \leqq \sigma$, so the lemma follows from the above inequality.

Lemma 4.2. *For any β, $0<\sigma\leqq\frac{1}{4}$, $\varepsilon>0$ and $x\in B_{\frac{1}{2}}(x_0)$, there is the following inequality*:

$$\begin{aligned}E_{\frac{\sigma}{2}}(x) \leqq \varepsilon E_\sigma(x) &+ \frac{C}{\varepsilon\sigma^n}\int_{B_\sigma(x)\setminus B_{\frac{\sigma}{8}}(x)}\left\{|\varphi_m-\beta|^2 + h_m^{-2}e^{-2\varphi_m}\sum_{i=1}^{m-1}|\varphi_i-\beta_i|^2\right\}dV\\ &+ C_\alpha\sigma^{1-\alpha},\end{aligned}$$

where $\beta_1,\ldots,\beta_{m-1}$ are any constants if $B_{\sigma/5}(x)\cap N = \emptyset$; zeroes otherwise, C_α is a constant which is zero if $\alpha\geqq 1$ and we define

Proof. There are two cases: (i) $B_{\sigma/5}(x)\cap N \neq \emptyset$; (ii) $B_{\sigma/5}(x)\cap N = \emptyset$. In the first case, let y be a point in $B_{\sigma/5}(x)\cap N$, then

$$B_{\frac{\sigma}{8}}(x)\subset B_{\frac{3}{8}\sigma}(y), \qquad B_{\frac{3}{4}\sigma}(y)\subset B_\sigma(x).$$

Applying Lemma 4.1 with ε replaced by $\varepsilon' = 4^{-n+2}\varepsilon$, we have

$$\begin{aligned}E_{\frac{\sigma}{8}}(x) &\leqq 3^{n-2}\left\{\varepsilon' E_{\frac{3\sigma}{4}}(y) + \frac{C}{\varepsilon'\sigma^n}\int_{B_{\frac{3}{4}\sigma}(y)\setminus B_{\frac{3}{8}\sigma}(y)}\left\{|\varphi_m-\beta|^2 + h_m^{-2}e^{-2\varphi_m}\sum_{i=1}^{m-1}|\varphi_i-\beta_i|^2\right\}dV\right.\\ &\quad\left.+ C_\alpha\sigma^{1-\alpha}\right\}\\ &\leqq \varepsilon E_\sigma(x) + \frac{C}{\varepsilon\sigma^n}\int_{B_\sigma(x)\setminus B_{\frac{\sigma}{8}}(x)}\left\{|\varphi_m-\beta|^2 + h_m^{-2}e^{-2\varphi_m}\sum_{i=1}^{m-1}|\varphi_i-\beta_i|^2\right\}dV\\ &\quad + C_\alpha\sigma^{1-\alpha}.\end{aligned}$$

In the second case, we may simply take

$$\varphi_j^*(z) = \left(1 - \eta_{\frac{4}{3}}\left(\frac{8r_x}{\sigma}\right)\right)\varphi_j + \eta_{\frac{4}{3}}\left(\frac{8r_x}{\sigma}\right)\beta_j, \quad 1 \leqq j \leqq m$$

as comparison functions in the derivation of (4.8) and proceed as in the proof of Lemma 4.1; we can still obtain (4.12).

Next, we give a weighted Poincaré inequality. It should be well-known.

Lemma 4.3. *Fix $0 < \sigma < \frac{1}{4}$, x in $B_1(x_0)$ with $B_{\frac{\sigma}{4}}(x) \cap N \neq \emptyset$. Then*

$$\int_{B_\sigma(x)\backslash B_{\frac{1}{2}\sigma}(x)} \frac{\psi^2}{\rho^{2\alpha+2}}\, dV \leqq \gamma \int_{B_\sigma(x)\backslash B_{\frac{\sigma}{2}}(x)} \frac{|\nabla\psi|^2}{\rho^{2\alpha}}\, dV \tag{4.14}$$

for any C^1 function ψ in $B_\sigma(x)$ vanishing on $B_\sigma(x) \cap N$, where γ is a universal constant possibly depending only on n.

Proof. First let us make a few observations: (i) ds^2 is very close to the Euclidean one in $B_1(x_0)$ by the assumption at the beginning of Sect. 3. Therefore, it suffices to show (4.14) in case of Euclidean space with N being a subspace; (ii) (4.14) is invariant under scaling, so we may take σ to be 2.

Choose Euclidean coordinates $(x_1, \ldots, x_n)$ such that $x = (0, \ldots, 0)$ and N is defined by $x_{n-1} = 0$, $x_n = \mu > 0$. Since $B_{\frac{1}{2}}(x) \cap N \neq \emptyset$, $\mu < \frac{1}{2}$. Now $\rho = \sqrt{x_{n-1}^2 + (x_n - \mu)^2}$. Let $\tilde{r} = \sqrt{x_1^2, + \ldots + x_{n-2}^2}$, $\tilde{\rho} = \sqrt{\tilde{r}^2 + x_{n-1}^2 + x_n^2}$.

Let η be a function on $B_2(x)\backslash B_1(x)$ satisfying: $\eta \equiv 1$ if $\tilde{r} \leqq \frac{1}{4}$; $\eta \equiv 0$ if $\tilde{r} \geqq \frac{1}{2}$; $|\nabla\eta| \leqq 10$. Then $\eta\psi$ vanishes outside $\tilde{r} \leqq \frac{1}{2}$, so the standard Poincaré inequality implies: for a uniform constant C,

$$\begin{aligned} \int_{B_2(x)\backslash B_1(x)} (\eta\psi)^2\, dV &\leqq C \int_{B_2(x)\backslash B_1(x)} |\nabla(\eta\psi)|^2\, dV \\ &\leqq 20C\left(\int_{B_2(x)\backslash B_1(x)} |\nabla\psi|^2\, dV + \int_{\substack{B_2(x)\backslash B_1(x) \\ \frac{1}{2} \geqq \tilde{r} \geqq \frac{1}{4}}} \psi^2\, dV\right). \end{aligned} \tag{4.15}$$

However, in case $\tilde{r} \leqq \frac{1}{4}$, $3 \geqq \rho \geqq \frac{1}{4}$. It follows that for some $\tilde{C}$,

$$\int_{\substack{B_2(x)\backslash B_1(x) \\ \tilde{r} \leqq \frac{1}{4}}} \frac{\psi^2}{\rho^{2\alpha+2}}\, dV \leqq \tilde{C}\left(\int_{B_2(x)\backslash B_1(x)} \frac{|\nabla\psi|^2}{\rho^{2\alpha}}\, dV + \int_{\substack{B_2(x)\backslash B_1(x) \\ \tilde{r} \geqq \frac{1}{4}}} \frac{\psi^2}{\rho^{2\alpha+2}}\, dV\right). \tag{4.16}$$

Therefore, to prove (4.14), it suffices to show

$$\int_{\substack{B_2(x)\backslash B_1(x) \\ \tilde{r} \geqq \frac{1}{4}}} \frac{\psi^2}{\rho^{2\alpha+2}}\, dV \leqq C' \int_{B_2(x)\backslash B_1(x)} \frac{|\nabla\psi|^2}{\rho^{2\alpha}}\, dV. \tag{4.17}$$

Using the spherical coordinates for $(x_1, \ldots, x_{n-2})$ and polar coordinates (ρ, θ) for $(x_{n-1}, x_n - \mu)$, one can easily reduce (4.17) to the following inequality:

$$\begin{aligned} &\iint_{\substack{1 \leqq \tilde{r}^2 + (\rho + \mu\sin\theta)^2 + \mu^2\cos^2\theta \leqq 4 \\ \tilde{r} \geqq \frac{1}{4}}} \tilde{r}^{n-3}\rho^{-2\alpha-1}|\psi|^2\, d\tilde{r}\, d\rho \\ &\leqq C' \iint_{1 \leqq \tilde{r}^2 + (\rho + \mu\sin\theta)^2 + \mu^2\cos^2\theta \leqq 4} \tilde{r}^{n-3}\rho^{-2\alpha-1}|\nabla\psi|^2\, d\tilde{r}\, d\rho, \end{aligned} \tag{4.18}$$

where $0 \lneq \theta \leqq 2\pi$. Note that using $\mu \leqq \frac{1}{2}$ so we may simply write μ for $\mu \sin\theta$ and put $b = 4 - \mu^2\cos^2\theta \leqq 4$, $a = 1 - \mu^2\cos^2\theta \geqq \frac{3}{4}$, then $b - a = 3$. Write $t = \rho + \mu$, then $\rho = t - \mu$; we obtain the equivalent form of (4.18),

$$\iint_{\substack{a \leqq \tilde r^2 + t^2 \leqq b \\ \tilde r \geqq \frac{1}{4}}} \tilde r^{n-3}|t-\mu|^{-2\alpha-1}|\psi|^2\, d\tilde r dt \leqq \gamma \iint_{a \leqq \tilde r^2 + t^2 \leqq b} \tilde r^{n-3}|t-\mu|^{-2\alpha+1}|\nabla\psi|^2\, d\tilde r dt\,. \tag{4.19}$$

Using the polar coordinates $(\tilde\rho, \tilde\theta)$ of $(\tilde r, t)$, we can easily see that (4.19) follows from the inequalities:

$$\begin{aligned}&\int_{\tilde\rho\cos\tilde\theta \geqq \frac{1}{4}} (\cos\tilde\theta)^{n-3}\left|\sin\tilde\theta - \frac{\mu}{\tilde\rho}\right|^{-2\alpha-1}|\psi|^2\, d\tilde\theta \\ &\leqq \gamma \int_{\tilde\rho\cos\tilde\theta \geqq \frac{1}{4}} (\cos\tilde\theta)^{n-3}\left|\sin\tilde\theta - \frac{\mu}{\tilde\rho}\right|^{-2\alpha+1}\left|\frac{d\psi}{d\tilde\theta}\right|^2 d\tilde\theta\,,\end{aligned} \tag{4.20}$$

where $\frac{3}{4} \leqq \tilde\rho \leqq 4$, $|\mu| \leqq \frac{1}{2}$. Note that ψ vanishes at those points with $\sin\tilde\theta = \mu/\tilde\rho$, which are indeed in the path $\{\tilde\rho\cos\tilde\theta \geqq \frac{1}{4}\}$.

The inequality (4.20) can be proved by using integration by parts, the Schwartz inequality and the fact that $\cos\tilde\theta$ is bounded from below.

Corollary 4.1. *For $1 < \sigma < \frac{1}{4}$, x in $B_1(x_0)$ with $B_{\frac{\sigma}{4}}(x_0) \cap N \neq \emptyset$. Then*

$$\int_{B_\sigma(x)} \frac{\psi^2}{\rho^{2\alpha+2}}\, dV \leqq \gamma \int_{B_\sigma(x)} \frac{|\nabla\psi|^2}{\rho^{2\alpha}}\, dV \tag{4.21}$$

for any C^1 function ψ in $B_\sigma(x)$ vanishing on $B_\sigma(x) \cap N$, where $\tilde\gamma$ is a universal constant depending only on n.

Proof. Choose a point $\tilde x$ in $B_{\sigma/4}(x) \cap N$, then

$$B_{\frac{\sigma}{2}}(x) \subset B_{\frac{3}{4}\sigma}(\tilde x) \subset B_\sigma(x)\,.$$

According to Lemma 4.3, we obtain

$$\int_{B_\sigma(x)\setminus B_{\frac{\sigma}{2}}(x)} \frac{\psi^2}{\rho^{2\alpha+2}}\, dV \leqq \gamma \int_{B_\sigma(x)\setminus B_{\frac{\sigma}{2}}(x)} \frac{|\nabla\psi|^2}{\rho^{2\alpha}}\, dV$$

$$\begin{aligned}\int_{B_{\frac{3}{4}\sigma}(\tilde x)} \frac{\psi^2}{\rho^{2\alpha+2}}\, dV &= \sum_{i=0}^{\infty} \int_{B_{\frac{1}{2^i}\frac{3}{4}\sigma}(\tilde x)\setminus B_{\frac{1}{2^{i+1}}\frac{3}{4}\sigma}(\tilde x)} \frac{\psi^2}{\rho^{2\alpha+2}}\, dV \\ &\leqq \gamma \sum_{i=0}^{\infty} \int_{B_{\frac{1}{2^i}\frac{3}{4}\sigma}(\tilde x)\setminus B_{\frac{1}{2^{i+1}}\frac{3}{4}\sigma}(\tilde x)} \frac{|\nabla\psi|^2}{\rho^{2\alpha}}\, dV \\ &= \int_{B_{\frac{3}{4}\sigma}(\tilde x)} \frac{\psi^2}{\rho^{2\alpha}}\, dV\,,\end{aligned}$$

therefore (4.21) follows with $\tilde\gamma = 2\gamma$.

Proposition 4.1. *There are ε_0 and λ_0, independent of x in $B_{1/2}(x_0)$, such that whenever*

$$E_\sigma(x) \leqq \varepsilon_0^2, \quad 0 < \sigma \leqq \frac{1}{4}, \quad x \in B_{\frac{1}{2}}(x_0)\,, \tag{4.22}$$

we have

$$E_{\lambda_0\sigma}(x) \leqq \frac{1}{2} E_\sigma(x) . \tag{4.23}$$

Proof. We will prove this proposition in case $\alpha \geqq 1$. The proof of the remaining case $\alpha < 1$ is completely identical except that we use $\varphi_i - h_i$ in place of φ_i $(i = 1, 2, \ldots, m-1)$. Since h_i are assumed to be smooth across N for $i = 1, 2, \ldots, m-1$, this modification won't affect any argument below.

In case $\alpha \geqq 1$, we may assume that φ_i are zero along N for $i = 1, 2, \ldots, m-1$ (cf. Sect. 2).

We prove (4.23) by contradiction. Suppose that this proposition is not true. Then there are sequences of $\{x_i\}_{i \geqq 1}$ in $B_{1/2}(x_0)$, $\{\varepsilon_i\}_{i \geqq 1}$, and $\{\sigma_i\}_{i \geqq 1}$ such that $0 < \sigma_i \leqq \frac{1}{4}$, $\lim_{i\to\infty} \varepsilon_i = 0$, and

$$\varepsilon_i^2 = E_{\sigma_i}(x_i) , \tag{4.24}$$

$$E_{\lambda_0\sigma_i}(x_i) > \frac{1}{2} E_{\sigma_i}(x_i) = \frac{1}{2}\varepsilon_i^2 , \tag{4.25}$$

where λ_0 is a small number, determined later.

For simplicity of notations, we assume that $B_1(x_0)$ is the Euclidean ball in R^n with x_0 being the origin, ds^2 is just the Euclidean metric and $N = R^{n-2} \subset R^n$. The proof in general case is completely analogous.

By taking a subsequence of $\{x_i\}$, we may assume that x_i converge to a point $\tilde{x}$ in $B_{1/2}(x_0)$. Since the solution $(\varphi_1, \ldots, \varphi_m)$ has been known to be regular outside N, such a point $\tilde{x}$ must be in $B_{1/2}(x_0) \cap N$.

First let us consider the case: there is a subsequence of $\{x_i\}$ such that for any x_i in this sequence, $B_{\sigma_i/l_0}(x_i) \cap N = \emptyset$, where l_0 will be determined later prior to the determination of λ_0. We remark that this l_0 will be independent of x in $B_{1/2}(x_0)$. For simplicity, assume $\{x_i\}$ is just this sequence. Then $\rho(x) \geqq \sigma_i/2l_0$ for y in $B_{\sigma_i/2l_0}(x_i)$. Define functions on $B_1(0) \subset R^n$ as follows: $i \geqq 1$,

$$\psi_j^i(x) = \frac{1}{\rho(x_i)^\alpha \varepsilon_i}(\varphi_j(x_i + \sigma_i x) - \bar{\varphi}_j^i), \quad x \in B_1(0), \quad j = 1, 2, \ldots, m ,$$

$$\psi_m^i(x) = \frac{1}{\varepsilon_i}(\varphi_m(x_i + \sigma_i x) - \bar{\varphi}_m^i), \quad x \in B_1(0) , \tag{4.26}$$

where $\bar{\varphi}_j^i$, $\bar{\varphi}_m^i$ are the averages of φ_j, φ_m over $B_{\sigma_i/2l_0}(x_i)$, respectively.

Now $E_{\sigma_i}(x_i) = \varepsilon_i^2$ implies that

$$\int_{B_{\frac{1}{2l_0}}(0)} |\nabla \psi_j^i|^2 \, dV \leqq 4 \quad \text{for } j = 1, 2, \ldots, m; \quad i \geqq 1 . \tag{4.27}$$

By taking the subsequence, we may assume that ψ_j^i converge weakly to ψ_j in H_2^1-norms. Note that we need to use the fact that the averages of ψ_j^i are zeroes in $B_{1/2l_0}(0)$. Moreover, the standard argument shows that ψ_j^i converge strongly to ψ_j in L^p-norms for any $p < \dfrac{2n}{n-2}$ in case $n > 2$ or $p < \infty$ in case $n = 2$. Using (3.1), one can easily show that in the sense of distribution,

$$\begin{cases} \Delta \psi_m = 0 \\ \Delta \psi_j = 2 \nabla \psi_j \cdot \nabla \theta, \end{cases} \quad j = 1, 2, \ldots, m-1 , \tag{4.28}$$

where θ is a smooth, uniformly bounded function. Then we conclude from the standard elliptic theory (cf. [GT]) that

$$|\psi_j(x) - \psi_j(0)| \leqq C_1|x| \quad \text{for } x \text{ in } B_{\frac{1}{2l_0}}(0) \quad j = 1, 2, \ldots, m\,. \tag{4.29}$$

Let λ_0 be smaller than $\frac{1}{32}$. Applying (4.1) with $\varepsilon = \frac{1}{8}$ and $\varepsilon_i\rho(x_i)^\alpha\psi_j(0)$ as β_j for $j \leqq m-1$, and $\varepsilon_i\psi_m(0)$ as β_m, we obtain

$$\begin{aligned}
\varepsilon_i^2 = E_{\lambda_0\sigma_i}(x_i) &\leqq \frac{1}{8}E_{8\lambda_0\sigma_i}(x_i) + \frac{C}{(\lambda_0\sigma_i)^n}\int_{B_{8\lambda_0\sigma_i}(x_i)\backslash B_{\lambda_0\sigma_i}(x_i)} \{|\varphi_m - \varepsilon_i\psi_m(0)|^2 \\
&\quad + \rho^{-2\alpha}\sum_{j=1}^{m-1}|\varphi_j - \varepsilon_i\rho(x_i)^\alpha\psi_j(0)|^2\}\,dV \\
&\leqq \frac{1}{8l}E_{8^l\lambda_0\sigma_i}(x_i) + 8C\sum_{s=1}^{l}\frac{1}{8^s(8^s\lambda_0\sigma_i)^n}\int_{B_{8^s\lambda_0\sigma_i}(x_i)\backslash B_{8^{s-1}\lambda_0\sigma_i}(x_i)} \{|\varphi_m - \varepsilon_i\psi_m(0)|^2 \\
&\quad + \rho^{-2\alpha}\sum_{j=1}^{m-1}|\varphi_j - \varepsilon_i\rho(x_i)^\alpha\psi_j(0)|^2\}\,dV
\end{aligned}$$

(choose l such that $8^l\lambda_0 \leqq \frac{1}{8}$)

$$\begin{aligned}
&\leqq \frac{1}{8^l}E_{8^l\lambda_0\sigma_i}(x_i) \\
&\quad + 2^{6\alpha+3}C\varepsilon_i^2\sum_{s=1}^{l}\frac{1}{8^s(8^s\lambda_0)^n}\int_{B_{8^s\lambda_0}(0)\backslash B_{8^{s-1}\lambda_0}(0)}\sum_{j=1}^{m}|\psi_j^i - \psi_j(0)|^2)\,dV \\
&\leqq \frac{1}{8^l}E_{8^l\lambda_0\sigma_i}(x_i) \\
&\quad + 2^{6\alpha+4}C\varepsilon_i^2\sum_{s=1}^{l}\frac{1}{8^s(8^s\lambda_0)^n}\int_{B_{8^s\lambda_0}(0)\backslash B_{8^{s-1}\lambda_0}(0)}\left(\sum_{j=1}^{m}|\psi_j^i - \psi_j|^2\right. \\
&\quad \left. + |\psi_j - \psi_j(0)|^2\right)dV \\
&\leqq \frac{1}{8^l}E_{8^l\lambda_0\sigma_i}(x_i) \\
&\quad + C_2\varepsilon_i^2\lambda_0^2\sum_{s=1}^{l}\frac{1}{8^s} + 2^{6\alpha+4}C\varepsilon_i^2\frac{1}{(\lambda_0)^n}\int_{B_{8^l\lambda_0}(0)\backslash B_{\lambda_0}(0)}\sum_{j=1}^{m}|\psi_j^i - \psi_j|^2\,dV\,,
\end{aligned} \tag{4.30}$$

where C_2 depends only on C and C_1. Note that C always denotes a uniform constant, although its value may vary in different places.

Now we choose λ_0, l such that $C_2\lambda_0^2 \leqq 1$, $\dfrac{1}{4l_0} \leqq 8^l\lambda_0 \leqq \dfrac{1}{2l_0}$ and $\dfrac{1}{8l}(4l_0)^{n-2} \leqq \dfrac{1}{8}$, then take i sufficiently large such that the last term in (4.30) is less than $\frac{1}{4}$. Here we use the convergence of ψ_j^i to $\psi_j (j = 1, 2, \ldots, m)$. Therefore, we have

$$E_{\lambda_0\sigma_i}(x_i) \leqq \frac{1}{2}\varepsilon_i^2\,.$$

It contradicts to the inequality (4.25).

Next, we may assume that $B_{\frac{1}{l_0}\sigma_i}(x_i) \cap N \neq \emptyset$ for all i. This time we define

$$\psi_j^i(x) = \frac{1}{\sigma_i^\alpha \varepsilon_i} \varphi_j(x_i + \sigma_i x), \quad j = 1, 2, \ldots, m-1,$$

$$\psi_m^i(x) = \frac{1}{\varepsilon_i}(\varphi_m(x_i + \sigma_i x) - \bar{\varphi}_m), \tag{4.31}$$

where x is in $B_1(0)$ and $\bar{\varphi}_m$ denotes the average of φ_m over $B_{\sigma_i}(x_i)$.

Under the transformations: $x \to x_i + \sigma_i x$ in R^n, the preimages of N are N_i parallel to the subspace $R^{n-2} \subset R^n$ and in the distance $\pi(x_i)/\sigma_i$, where π is the orthogonal projection from R^n onto R^{n-2}. Since $B_{\sigma_i/l_0}(x_i) \cap N \neq \varphi$, we have $|\pi(x_i)/\sigma_i| \leqq 1/l_0 \leqq \frac{1}{4}$. Thus we may assume that N_i converge to an affine subspace N_∞ in R^n within the distance $1/l_0$ from R^{n-2}. Let ρ_i, ρ_∞ be the distance from N_i, N_∞, respectively. Then

$$\int_{B_1(0)} \left\{ |\nabla \psi_m^i|^2 + 2^{2\varphi_m(x_i + \sigma_i x)} \left(\sum_{j=1}^{m-1} |\nabla \psi_j^i|^2 \right) \rho_i^{-2\alpha} \right\} dV = 1. \tag{4.32}$$

Note that φ_m is uniformly bounded. Apply the Corollary 4.1, we have

$$\int_{B_1(0)} \left\{ |\psi_m^i|^2 + \rho_i^{-2\alpha-2} \sum_{j=1}^{m-1} |\psi_j^i|^2 \right\} dV \leqq C. \tag{4.33}$$

Recall our convention that C always denotes a uniform constant. As in the previous case, these ψ_m^i, ψ_j^i $(1 \leqq j \leqq m-1)$ converge to ψ_m, ψ_j in H_2^1-norm and H_{2,ρ^α}^1 norms respectively, where

$$\|\psi\|_{H_2^1} = \int_{B_1(0)} \left(\frac{|\nabla \psi|^2}{\rho^{2\alpha}} + \frac{\psi^2}{\rho^{2\alpha+2}} \right) dV. \tag{4.34}$$

Moreover, the functions $\tilde{\psi}_m = \psi_m$ and $\tilde{\psi}_j = \dfrac{1}{\rho_\infty^\alpha} \psi_j$ satisfy the following equations in the weak sense:

$$\Delta \tilde{\psi}_m = 0, \tag{4.35}$$

$$\Delta \tilde{\psi}_j = 2 \nabla \tilde{\psi}_j \nabla \theta_j + \frac{\alpha \tilde{\psi}_j}{\rho_\infty} \left(\frac{\alpha + 1}{\rho_\infty} - \nabla \theta_j \nabla \rho_\infty \right), \tag{4.36}$$

where θ_j are smooth outside N_∞ in $B_1(0)$ and $|\nabla \theta_j| \leqq C'_\delta \rho_\infty^{-\delta}$ for any $\delta < 1$ and some constant C'_δ. So we still have the estimate (4.29) for this $\tilde{\psi}_m$. Also, these $\tilde{\psi}_j$ are uniformly bounded outside the set

$$U = \left\{ \rho_\infty^{-1} > 2 \max \max_{1 \leqq j \leqq m-1} \{ \nabla \theta_j \nabla \rho_\infty(x) | x \in B_1(0) \} \right\},$$

since ρ_∞ is smooth here. On the other hand, for any x in U,

$$\left(\frac{\alpha + 1}{\rho_\infty} - \nabla \theta_j \nabla \rho_\infty \right)(x) > 0.$$

Applying Moser's iteration (cf. [GT], Chap. 8), one can show

$$\sup_{B_{\frac{1}{2}}(0)} |\tilde{\psi}_j| \leqq C\left(1 + \int_{B_1(0)} |\tilde{\psi}_j|^2 dV\right)^{\frac{1}{2}}, \quad j = 1, 2, \ldots, m-1 . \tag{4.37}$$

In particular, these $\tilde{\psi}_j$ are uniformly bounded, or equivalently, $\rho_\infty^{-\alpha}\psi_j$ are uniformly bounded. It also implies that ψ_j vanishes along $N_\infty \cap B_1(0)$.

Next, we need the following lemma, whose proof will be given later.

Lemma 4.4. *Let v be a smooth function outside $N_\infty \cap B_1(0)$ and $|\nabla v| \leqq C'\rho_\infty^{-\beta}$ with $\beta < 1$, f be a smooth function in $B_1(0)$ with $\max_{B_1(0)}(\rho_\infty^{2-2\alpha}|f|) \leqq C'$. Then for any $\varepsilon > 0$, there is a uniform constant C_ε, depending only on C', β, ε, such that if u is a solution of the equation*

$$\Delta u = 2\nabla u\left(\frac{\alpha \Delta \rho_\infty}{\rho_\infty} + \nabla v\right) + f \quad \text{in } B_1(0) \tag{4.38}$$

and $|u|_{C^0(\partial B_{2/3}(0))} \leqq C'$, $u|_{N_\infty} \equiv 0$, then

$$|u(x)| \leqq C_\varepsilon \rho_\infty^{2\alpha - 2\varepsilon}(x) \quad \text{for } x \text{ in } B_{\frac{1}{2}}(0) . \tag{4.39}$$

Applying the lemma to ψ_j ($j = 1, 2, \ldots, m-1$) with $\varepsilon = \alpha/4$, we obtain

$$|\psi_j(x)| \leqq C\rho_\infty(x)^{\frac{3\alpha}{2}}, \quad x \in B_{\frac{1}{2}}(0)$$
$$j = 1, 2, \ldots, m-1 . \tag{4.40}$$

Here, as usual, C denotes a constant independent of ψ_j and x.

Now we use (4.1) with $\beta_j = 0$ for $j \leqq m-1$ and $\varepsilon_i \psi_m(0)$ for β_m, and deduce as we did for (4.30),

$$\begin{aligned}
E_{\lambda_0\sigma_i}(x_i) &\leqq \frac{1}{8^l} E_{8^l\lambda_0\sigma_i}(x_i) \\
&\quad + C_3\varepsilon_i^2 \sum_{s=1}^{l} \frac{1}{8^s(8^s\lambda_0)^n} \int_{B_{8^l\lambda_0}(0)\setminus B_{8^{l-1}\lambda_0}(0)} \left\{\rho_\infty^{-2\alpha}\sum_{j=1}^{m-1}|\psi_j^i - \psi_j|^2 + |\psi_j|^2\right. \\
&\quad \left. + |\psi_m - \psi_m(0)|^2 + |\psi_m^i - \psi_m|^2\right\} dV \\
&\leqq \frac{1}{8^l} E_{8^l\lambda_0\lambda_i}(x_i) \\
&\quad + \frac{C_3\varepsilon_i^2}{\lambda_0^n} \int_{B_{8^l\lambda_0}(0)\setminus B_{\lambda_0}(0)} \left(\rho_\infty^{-2\alpha}\sum_{j=1}^{m-1}(|\psi_j^i - \psi_j|^2 + |\psi_m^i - \psi_m|^2)\right) dV \\
&\quad + C_4\varepsilon_i^2\left(8^{2l}\lambda_0^2 + (m-1)\sum_{s=1}^{l-1}\frac{1}{8^s}\sup_{B_{8^s\lambda_0}(0)\setminus B_{8^{s-1}\lambda_0}(0)}\rho_\infty^\alpha\right) \\
&\quad \text{(use (4.29) and (4.40))} \\
&\leqq \frac{1}{8^l} E_{8^l\lambda_0\sigma_i}(x_i) + \frac{C_{\frac{1}{8}}\varepsilon_i^2}{\lambda_0^n} \int_{B_{8^l\lambda_0}(0)\setminus B_{\lambda_0}(0)} \left(|\psi_m^i - \psi_m|^2 + \rho_\infty^{-2\alpha}\sum_{j=1}^{m-1}|\psi_j^i - \psi_j|^2\right) \\
&\quad + C_4\varepsilon_i^2\left(\lambda_0^2 + (m-1)\sup_{B_{8^l\lambda_0}(0)}\rho_\infty^\alpha\right) .
\end{aligned} \tag{4.41}$$

Note that both C_3 and C_4 here are uniform constants depending only on n, $C_{1/8}$ in (4.1) and C in (4.40). Now we take l_0 to be $2(8C_4(m-1))^{1/\alpha}$, and then choose λ_0, l as before such that $1/4l_0 \leqq 2^l\lambda_0 \leqq 1/l_0$ and $(4l_0)^{n-2} \leqq 8^{l-1}$. Since $B_{2^l\lambda_0}(0) \subset B_{1/2l_0}(0)$ and $B_{1/l_0(0)} \cap N_\infty \neq \varphi$, we have

$$(m-1)C_4 \sup_{B_{2^l\lambda_0}(0)} \rho_\infty^\alpha \leqq \frac{1}{8}\,.$$

We further assume $C_0\lambda_0 \leqq \frac{1}{8}$. Then by the strong convergence of ψ_j^i to ψ_j in L^2-norms, for i sufficiently large, we have

$$E_{\lambda_0\sigma_i}(x_i) \leqq \frac{1}{2}\varepsilon_i^2\,.$$

A contradiction! The proposition is proved. Note that λ_0, ε_0 are obviously independent of x in $B_{\frac{1}{2}}(x_0)$.

Proof of Lemma 4.3. We will construct the barrier to obtain the decay estimate (4.39). First, we remark that it suffices to show (4.39) in a neighborhood of $N_\infty \cap B_{\frac{1}{2}}(0)$ in $B_{\frac{1}{2}}(0)$, since the coefficients of the linear equation (4.38) are smooth in any compact subset outside N_∞.

Easy computations yield

$$\begin{aligned}\left(\Delta - 2\left(\frac{\alpha\nabla\rho_\infty}{\rho_\infty} + \nabla v\right)\cdot\nabla\right)\rho_\infty^\gamma &= \gamma(\gamma-2\alpha)\rho_\infty^{\gamma-2} - 2\alpha\gamma\nabla v\nabla\rho_\infty\cdot\rho_\infty^{\gamma-1}\\ &= \gamma\rho_\infty^{\gamma-2}\{(\gamma-2\alpha) - 2\alpha\rho_\infty\nabla v\cdot\nabla\rho_\infty\}\,, \qquad (4.42)\end{aligned}$$

$$\begin{aligned}\left(\Delta - 2\left(\frac{\alpha\nabla\rho_\infty}{\rho_\infty} + \nabla v\right)\cdot\nabla\right)\rho_\infty^{\gamma-\delta}\tilde{r}_{\bar{x}_0}(x)^2 &= \tilde{\gamma}_{\bar{x}_0}(x)^2\{(\gamma-\delta)(\gamma-2\alpha-\delta)\rho_\infty^{\gamma-\delta-2}\\ &\quad - 2\alpha(\gamma-\delta)\rho_\infty^{\gamma-\delta-1}\nabla v\nabla\rho_\infty\}\\ &\quad + 2(n-2)\rho_\infty^{\gamma-\delta} - 2\alpha\rho_\infty^{\gamma-\delta}\nabla v\nabla\tilde{r}_{\bar{x}_0}(x)^2\\ &\quad + 2(\gamma-\delta)\rho_\infty^{\gamma-\delta-1}\nabla\rho_\infty\nabla\tilde{r}_{\bar{x}_0}(x)^2\,,\end{aligned} \qquad (4.43)$$

where $\tilde{r}_{\bar{x}_0}(x)$ is the projection onto N_∞ of the distance between $\bar{x}_0$ and x for any fixed $\bar{x}_0$ in $B_{\frac{1}{2}}(0)$.

Therefore, for any given $\delta \leqq \gamma < 2\alpha$ and $0 < \delta < 1$, there is a neighborhood U_γ of $N \cap B_{\frac{1}{2}}(0)$, depending only on ε, γ, β, such that

$$\left(\Delta - 2\left(\frac{\alpha\nabla\rho_\infty}{\rho_\infty} + \nabla v\right)\cdot\nabla\right)\left(\rho_\infty^\gamma + \rho_\infty^{\gamma-\delta}\tilde{r}_{\bar{x}_0}(x)^2\right) \leqq -C'\rho_\infty^{\gamma-2} \quad \text{in } U_\gamma\,. \qquad (4.44)$$

Note that the assumption $|\nabla v| \leqq C\rho_\infty^{-\beta}$ $(\beta < 1)$ is used here. The function $\rho_\infty^\gamma + \rho_\infty^{\gamma-\delta}\tilde{r}_{x_0}(x)^2 \geqq (\frac{2}{3})^2\rho_\infty^{\gamma-\delta}$ for those x with $\tilde{r}_{\bar{x}_0}(x) \geqq \frac{2}{3}$. So these $\rho_\infty^\gamma + \rho_\infty^{\gamma-\varepsilon}\tilde{r}_{\bar{x}_0}(x)^2$ can be served as upper barriers for u. Now fix $\tilde{\varepsilon}$ such that $l\tilde{\varepsilon} = 2\alpha - 2\varepsilon$ for some integer $l > 0$. First we take $\delta = \gamma = \tilde{\varepsilon}$. Using maximum principle and comparing u with $\rho_\infty^\gamma + \tilde{r}_{\bar{x}_0}(x)^2$ for any $\bar{x}_0$ in $N_\infty \cap B_{\frac{1}{2}}(0)$, we obtain

$$|u(x)| \leqq C_\alpha\rho_\infty(x)^{\tilde{\varepsilon}} \quad \text{in } U_\alpha\,.$$

Then we repeat this argument successively with $\gamma = j\tilde{\varepsilon}$, $\delta = \tilde{\varepsilon}$ for $2 \leqq j \leqq l$ to conclude the proof of this lemma.

Remark. Proposition 4.1 can easily be modified in the axisymmetric case, where we need to construct an axisymmetric comparison map.

Now we are ready to give the main theorem of this section.

Theorem 4.1. *There are two uniform constants C and $\delta > 0$ such that for any x in $B_{\frac{1}{4}}(x_0)$, $0 < \sigma \leqq \frac{1}{8}$,*

$$E_\sigma(x) \leqq C\sigma^{2\delta} . \tag{4.45}$$

In particular, it implies that φ_m is δ-Hölder continuous (cf. [GT], *Chap.* 8).

Proof. Fix any x in $B_{\frac{1}{4}}(x_0)$. Let ε_0, λ_0 be given in the last proposition. Then by Proposition 3.1, there is a σ_x between e^{-3C/ε_0^2} and $\frac{1}{4}$, where C is independent of x in $B_{\frac{1}{2}}(\alpha_0)$ and $(\varphi_1, \ldots, \varphi_m)$, such that

$$E_{\sigma_x}(x) \leqq \varepsilon_0^2 .$$

Combining this with the last proposition, we have

$$E_{\lambda_0^k\sigma_x}(x) \leqq \left(\frac{1}{2}\right)^k E_{\sigma_x}(x) = \left(\frac{1}{2}\right)^k \varepsilon_0^2 \quad \text{for } k \geqq 1 .$$

Now for any $\sigma \leqq \frac{1}{8}$ and $\sigma \leqq \sigma_x$, choose k such that $\lambda_0^{k+1}\sigma_x < \sigma \leqq \lambda_0^k\sigma_x$, then

$$\begin{aligned} E_\sigma(x) &\leqq \lambda_0^{-(n-2)} E_{\lambda_0^k\sigma_x}(x) \leqq \lambda_0^{-(n-2)}\varepsilon_0^2\left(\frac{1}{2}\right)^k \\ &\leqq \lambda_0^{-(n-2)}\varepsilon_0^2\left(\frac{1}{2}\right)^{-\frac{\log\sigma_x}{\log\lambda_0}} \sigma^{\frac{-\log 2}{\log\lambda_0}} . \end{aligned}$$

Put $2\delta = -\dfrac{\log 2}{\log\lambda_0} > 0$, $C' = \lambda_0^{-(n-2)}\varepsilon_0^2\left(\dfrac{1}{2}\right)^{-\frac{\log\sigma_x}{\log\lambda_0}}$, then

$$E_\sigma(x) \leqq C'\sigma^{2\delta} \quad \text{for } \sigma \leqq \min\left\{\frac{1}{8}, \sigma_z\right\} . \tag{4.46}$$

On the other hand, σ_x is uniformly bounded from below, therefore, the estimate (4.45) for $0 \leqq \sigma \leqq \frac{1}{8}$ follows from (4.46).

5. Higher-order Estimates

We will fix the solution $(\varphi_1, \ldots, \varphi_m)$ of (3.1) with $|\varphi_m| \leqq C$ in a geodesic ball $B_1(x_0)$ as in Sects. 3 and 4. Note that this center x_0 is in $N \subset M$ and h_m is of the form (3.2) in the ball. In this section, we complete the proof of our main theorem stated in the introduction. We will always use C to denote a uniform constant depending only on $(h_1, \ldots, h_m)$, N, etc.

Lemma 5.1. *There is an $\varepsilon_0 > 0$, independent of $(\varphi_1, \ldots, \varphi_m)$ and the point x in $B_{\frac{1}{2}}(x_0)$, such that*

$$|\nabla\varphi_m| \leqq C\rho(x)^{-1+\varepsilon_0}, \quad x \text{ in } B_{\frac{1}{2}}(x_0)\backslash N . \tag{5.1}$$

Proof. Given any point x in $B_{\frac{1}{2}}(x)\backslash N$, put $2\sigma = \rho(x)$. Then $\rho \geqq \sigma$ in $B_\sigma(x)$. Let $G_z(y)$ be the Green function in $B_\sigma(x)$ with Dirichlet boundary condition and singularity at z. Then one can easily check

$$\int_{B_\sigma(x)} |\nabla G_x(y)|^\beta \, dV_y \leqq C_\beta \quad \text{for } \beta < \frac{n}{n-1}, \tag{5.2}$$

$$\int_{\partial B_\sigma(x)} \left| \nabla \frac{\partial G_x(y)}{\partial n_y} \right| dV_y \leqq C, \tag{5.3}$$

where C_β depends only on β and n_y denotes the outward normal vector of $\partial B_\sigma(x)$.

Using the first equation in (3.1), we have

$$\begin{aligned} \varphi_m(z) - \varphi_m(x) = & \int_{\partial B_\sigma(x)} (\varphi_m(y) - \varphi_m(x)) \frac{\partial G_z(y)}{\partial n_y} dV_y \\ & - \int_{B_\sigma(x)} h_m^{-2} e^{-2\varphi_m} \sum_{i=1}^{m-1} |\nabla \varphi_i|^2(y) G_z(y) \, dV_y . \end{aligned} \tag{5.4}$$

It follows

$$\begin{aligned} |\nabla \varphi_m(x)| \leqq & \int_{\partial B_\sigma(x)} |\varphi_m(y) - \varphi_m(x)| \left| \nabla \left(\frac{\partial G_x}{\partial n_y} \right) \right| (y) \, dV_y \\ & + C_1 \int_{B_\sigma(x)} \rho^{-2\alpha} \sum_{i=1}^{m-1} |\nabla \varphi_i|^2(y) |\nabla G_x(y)| \, dV_y , \\ \text{(Theorem 4.1 and (5.3))} \leqq & \; C_2 \sigma^\delta + C_1 \left(\int_{B_\sigma(x)} \rho^{-2\alpha} \sum_{i=1}^{m-1} |\nabla \varphi_i|^2 |\nabla G_x(y)| \, dV_y \right), \end{aligned} \tag{5.5}$$

where $\delta > 0$ is given in Theorem 4.1.

By Hölder inequality, we have

$$\begin{aligned} \int_{B_\sigma(x)} \left(\rho^{-2\alpha} \sum_{i=1}^{m-1} |\nabla \varphi_i|^2 \right) |\nabla G_x(y)| \, dV_y \leqq & \left(\int_{B_{-\sigma}(x)} \left(\rho^{-2\alpha} \sum_{i=1}^{m-1} |\nabla \varphi_i|^2 \right)^{\frac{\beta}{\beta-1}} dV_y \right)^{\frac{\beta-1}{\beta}} \\ & \times \left(\int_{B_\sigma(x)} |\nabla G_x(y)|^\beta \, dV_y \right)^{\frac{1}{\beta}} \quad \left(\beta < \frac{n}{n-1} \right), \\ \text{(Lemma 3.3)} \qquad \leqq & \; C_\beta^{\frac{1}{\beta}} \cdot C_\beta^{\frac{\beta-1}{\beta}} \sigma^{-\frac{2}{\beta}} \left(\int_{B_\sigma(x)} \rho^{-2\alpha} \sum_{i=1}^{m-1} |\nabla \varphi_i|^2 \, dV_y \right)^{\frac{\beta-1}{\beta}}, \\ \text{(Theorem 4.1)} \qquad \leqq & \; C_\beta^{\frac{1}{\beta}} \cdot C_4^{\frac{\beta-1}{\beta}} \sigma^{-\frac{2}{\beta} + (n-2+2\delta)\frac{\beta-1}{\beta}} . \end{aligned} \tag{5.6}$$

Now choose $1 < \beta < \frac{n}{n-1}^{\frac{n}{n-1}}$ such that $(n-1)\beta - n + 2\delta(\beta - 1) = \beta\varepsilon_0 > 0$ for some $\varepsilon_0 > 0$. Such a β can be taken only dependent of δ, so is ε_0. Then $-\frac{2}{\beta} + (n-2+\delta)\frac{\beta-1}{\beta} \geqq -1 + \varepsilon_0$, and by (5.5), (5.6), we obtain

$$|\nabla \varphi_m(x)| \leqq C_1 \sigma^\delta + C_5 \sigma^{-1+\varepsilon_0} \leqq C \rho(x)^{-1+\varepsilon_0} . \tag{5.7}$$

The lemma is proved.

Corollary 5.1. *For any $\varepsilon > 0$, $\varepsilon < \alpha$, there is a uniform constant $C_\varepsilon > 0$ such that*

$$|\varphi_j(x) - h_j(\pi(x))| \leqq C_\varepsilon \rho(x)^{2\alpha - 2\varepsilon}, \quad x \text{ in } B_{\frac{1}{2}}(x_0), \tag{5.8}$$

where $j = 1, 2, \ldots, m-1$ and π is the projection from $B_1(x_0)$ into N as given in Corollary 3.1.

Proof. In case $\alpha \geqq 1$, $h_j \equiv$ const. in $B_1(x_0) \cap N$. Then (5.8) follows from the previous lemma and Lemma 4.3. In case $\alpha < 1$, we may extend h_j such that $\nabla h_j \nabla \rho \leqq C\rho$ near N. Then we apply Lemma 4.3 to $\varphi_j - h_j$ and obtain the estimate (5.8). Note that $\rho_\infty = \rho$ and $N_\infty = N$ in the application of Lemma 4.3.

In particular, the solution $(\varphi_1, \ldots, \varphi_m)$ is δ-Hölder continuous and its δ-Hölder norm is uniformly bounded, where $\delta = \delta(\alpha)$ depends only on α. However, in case $\alpha > 1$, we can have more estimates on the second derivatives of this solution.

Corollary 5.2. *Suppose that $\alpha > 1$ and h_m is of $C^{1,\beta}$, where $l \in Z_+$ and $0 < \beta < 1$. Then for any $\varepsilon > 0$, $\varepsilon < 2\alpha$, there is a uniform constant $C_\varepsilon > 0$ such that*

$$\max_{1 \leqq j \leqq m} \{ \|\varphi_j\|_{C^{k_\alpha, \lambda_\alpha}(B_{\frac{1}{2}}(x_0))} \} \leqq C_\varepsilon, \tag{5.9}$$

where $k_\alpha = \min\{[2\alpha - 2\varepsilon], l+1\}$, and $\lambda_\alpha = \min\{2\alpha - 2\varepsilon - k_\alpha, \beta\}$.

Proof. Note that $h_j \equiv$ const. for $j = 1, 2, \ldots, m-1$ in case $\alpha > 1$. By Corollary 5.1, it suffices to bound on the $(k_\alpha, \lambda_\alpha)$-Hölder norm of φ_m in $B_{\frac{1}{2}}(x_0)$. In the first equation of (3.1), φ_m is Hölder continuous and $h_m^{-2} \sum_{j=1}^{m-1} |\nabla \varphi_j|^2$ is $([2\alpha - 2 - 2\varepsilon], 2\alpha - 2 - 2\varepsilon - [2\alpha - 2\varepsilon - 2])$-Hölder continuous by Corollary 5.1. Therefore the bound on the $(k_\alpha, \lambda_\alpha)$-Hölder norm of φ_m follows from the standard Schauder estimate (cf. [GT]).

Theorem 1.1 follows from Corollary 5.1 and 5.2.

In the following, we will prove that the solution $(\varphi_1, \ldots, \varphi_m)$ has more regularity along the tangential directions of N in $B_{\frac{1}{2}}(x_0)$, precisely, by a local diffeomorphism, we may assume that $B_1(x_0)$ is an open neighborhood of the origin in R^n with $N \cap B_1(x_0)$ being in the subspace $R^{n-2} \subset R^n$, let $(x_1, \ldots, x_n)$ be the Euclidean coordinates, then $\dfrac{\partial^l \varphi_j}{\partial x_1^{l_1} \ldots \partial x_{n-2}^{l_{n-2}}}$ $(l_1 + \cdots + l_{n-2} = l)$ have the same estimate as (5.9) in the half-ball $B_{\frac{1}{2}}(x_0)$ as long as h_m, N and the metric ds^2 are of C^{l+2} in $B_1(x_0)$. To avoid the complexity from the presence of curvature of N and ds^2, we simply consider the case where $M = R^n$. $N = R^{n-2} \subset R^n$ and ds^2 is the flat metric. The proof for general cases is identical. Furthermore, we assume that $\log h_m$ is harmonic outside N. By adding a C^∞-function to φ_m, we may take h_m to be ρ^α, where $\alpha > 1$ and $\rho(x_1, \ldots, x_n) = \sqrt{x_{n-1}^2 + x_n^2}$. The equations in (3.1) become

$$\Delta \varphi_m = \rho^{-2\alpha} e^{-2\varphi_m} \sum_{i=1}^{m-1} |\nabla \varphi_j|^2, \tag{5.10}$$

$$\Delta \varphi_j = 2 \nabla \varphi_j \left(\frac{\alpha \nabla \rho}{\rho} + \nabla \varphi_m \right), \quad j = 1, 2, \ldots, m-1. \tag{5.11}$$

Lemma 5.2. *For any integer $l \geqq 1$, and nonnegative integers $l_1, \ldots, l_{n-2}$ with $l = l_1 + \cdots + l_{n-2}$, we have*

$$\Delta\left(\frac{\partial^l \varphi_m}{\partial x_1^{l_1} \ldots \partial x_{n-2}^{l_{n-2}}}\right) = -2\rho^{-2\alpha} e^{-2\varphi_m} \sum_{j=1}^{m-1} |\nabla \varphi_j|^2 \cdot \frac{\partial^l \varphi_m}{\partial x_1^{l_1} \ldots \partial x_{n-2}^{l_{n-2}}}$$
$$+ 2\rho^{-2\alpha} e^{-2\varphi_m} \sum_{j=1}^{m-1} \nabla \varphi_j \nabla\left(\frac{\partial^l \varphi_j}{\partial x_1^{l_1} \ldots \partial x_{n-2}^{l_{n-2}}}\right)$$
$$+ P^m_{l_1, \ldots, l_{n-2}} \cdot \rho^{-2\alpha} e^{-2\varphi_m}, \tag{5.12}$$

$$\Delta\left(\frac{\partial^l \varphi_j}{\partial x_1^{l_1} \ldots \partial x_{n-2}^{l_{n-2}}}\right) = 2\nabla\left(\frac{\partial^l \varphi_j}{\partial x_1^{l_1} \ldots \partial x_{n-2}^{l_{n-2}}}\right) \cdot \left(\frac{\alpha \nabla}{\rho} + \nabla \varphi_m\right)$$
$$+ 2\nabla \varphi_j \nabla\left(\frac{\partial^l \varphi_m}{\partial x_1^{l_1} \ldots \partial x_{n-2}^{l_{n-2}}}\right)$$
$$+ P^j_{l_1, \ldots, l_{n-2}}, \quad j = 1, 2, \ldots, m-1, \tag{5.13}$$

where $P^j_{l_1, \ldots, l_{n-2}}$, $P^m_{l_1, \ldots, l_{n-2}}$ are polynomials of degree at most $l + 2$ in $\dfrac{\partial^k \varphi_m}{\partial x_1^{k_1} \ldots \partial x_{n-2}^{k_{n-2}}}$ and $\nabla\left(\dfrac{\partial^k \varphi_m}{\partial x_1^{k_1} \ldots \partial x_{n-2}^{k_{n-2}}}\right)$, where $0 \leqq k \leqq l-1$, $k_1 + \cdots + k_{n-2} = k$. Furthermore, $P^m_{l_1, \ldots, l_{n-2}}$ is at least quadratic on the derivatives of φ_j $(j = 1, 2, \ldots, m-1)$.

Proof. These equations follow from the direct computations, (5.10) and (5.11).

Theorem 5.1. *Let $N = R^{n-2} \subset R^n = M$ be the subspace, $h_1, \ldots, h_m$ be C^∞-functions with $h_m = \rho^\alpha$ in $B_1(0)$. Then for any $(n-2)$-tuple of nonnegative integers $(l_1, \ldots, l_{n-2})$ and $\varepsilon > 0$, there is a constant $C_{\varepsilon, l}$, depending only on ε and $l = l_1 + \cdots + l_{n-2}$, such that*

$$\max_{1 \leqq j \leqq m} \left\{ \left\| \frac{\partial^l \varphi_j}{\partial z_1^{l_1} \ldots \partial z_{n-2}^{l_{n-2}}} \right\|_{C^{k_\alpha, \lambda_\alpha}(B_{\frac{1}{2}}(0))} \right\} \leqq C_{\varepsilon, l}, \tag{5.14}$$

where $k_\alpha = [2\alpha - 2\varepsilon]$, $\lambda_\alpha = 2\alpha - 2\varepsilon - k_\alpha$ as in Corollary 5.2.

Proof. We will prove (5.14) by induction on $l = \sum_{i=1}^{n-2} l_i$ for fixed small $\varepsilon > 0$. In case $l = 0$, (5.14) is just (5.9). Now we suppose that (5.14) is true for all $(l'_1, \ldots, l'_{n-2})$ with $l'_1, \ldots, l'_{n-2} < l$. By induction, we may further assume that on $B_{\frac{3}{4}}(0)$,

$$\left| \frac{\partial^k \varphi_j}{\partial x_1^{k_1} \ldots \partial x_{n-2}^{k_2}} \right| (x) \leqq C_{k, \varepsilon} \rho(x)^{2\alpha - 2\varepsilon} \tag{5.15}$$

and

$$\left| \nabla\left(\frac{\partial^k \varphi_j}{\partial x_1^{k_1} \ldots \partial x_{n-2}^{k^2}}\right) \right| (x) \leqq C_{k, \varepsilon} \rho(x)^{2\alpha - 2\varepsilon - 1}, \tag{5.16}$$

where $0 \leqq k \leqq l-1$, $1 \leqq j \leqq m-1$ and $C_{k, \varepsilon}$ are constants independent of the solution.

Then by Lemma 5.2, in Eqs. (5.12) and (5.13), for some uniform constant C,

$$|P^m_{l_1,\dots,l_{n-2}}(x)| \leqq C\rho^{4\alpha-4\varepsilon-2}(x) \quad x \text{ in } B_{\frac{1}{2}}(0)\,, \tag{5.17}$$

$$|P^j_{l_1,\dots,l_{n-2}}(x)| \leqq C\rho^{2\alpha-2\varepsilon-1}(x), \quad j=1,2,\dots,m-1\,. \tag{5.18}$$

On the other hand, by induction

$$\left|\nabla\left(\frac{\partial^l\varphi_m}{\partial x_1^{l_1}\dots\partial x_{n-2}^{l_{n-2}}}\right)\right|(x) \leqq \max\left\|\frac{\partial^{l-1}\varphi_m}{\partial x_1^{k_1}\dots\partial x_{n-2}^{k_{n-2}}}\right\|_{C^2(B_{\frac{1}{2}}(0))|k_1+\dots+k_{n-2}=l-1}$$
$$\leqq C_{\varepsilon,l-1}\,.$$

Therefore

$$\left|2\nabla\varphi_j\nabla\left(\frac{\partial^l\varphi_m}{\partial x_1^{l_1}\dots\partial x_{n-2}^{l_{n-2}}}\right)+P^j_{l_1,\dots,l_{n-2}}\right|(x) \leqq C\rho^{2\alpha-2\varepsilon-1} \leqq C\rho^{2\alpha-2}\,.$$

Now we can apply Lemma 4.3 to (5.13) with $u=\dfrac{\partial^l\varphi_j}{\partial x_1^{l_1}\dots\partial x_{n-2}^{l_{n-2}}}$ for $j=1,2,\dots,m-1$ to conclude

$$\left|\frac{\partial^l\varphi_j}{\partial x_1^{l_1}\dots\partial x_{n-2}^{l_{n-2}(x)}}\right| \leqq C\rho(x)^{2\alpha-2\varepsilon}, \quad x \text{ in } B_{\frac{1}{2}}(0)\,. \tag{5.19}$$

Note that the estimate for $\dfrac{\partial^l\varphi_j}{\partial x_1^{l_1}\dots\partial x_{n-2}^{l_{n-2}(x)}}$ corresponding to (5.5) and (5.16) can be easily derived from (5.19). Therefore, (5.14) follows from (5.19), (5.12), (5.13) and the standard Schauder estimates (cf. [GT], Chap. 6).

Theorem 5.1′ (*The version of Theorem* 5.1 *in general case*). *Assume that M is a C^∞-manifold, N is a C^∞-submanifold and $(h_1,\dots,h_m)$ are C^∞-smooth. Let $B_1(x_0)$ be the geodesic ball with x_0 in N in which h_m is of form* (3.2). *Then for any $\varepsilon>0$, any vector fields $T_1,\dots,T_l$ in $B_1(x_0)$ which are tangential to N along $B_1(x_0)\cap N$, there is a constant $C_\varepsilon=C_\varepsilon(T_1,\dots,T_l)$, independent of the solution $(\varphi_1,\dots,\varphi_m)$, such that*

$$\sup_{1\leqq j\leqq m}\|T_1\dots T_l\varphi_j\|_{C^{k_\alpha,\lambda_\alpha}(B_{\frac{1}{2}}(x_0))} \leqslant C_{\varepsilon,l}\,, \tag{5.20}$$

where k_α, λ_α are given as in Theorem 5.1.

The proof of it is identical to that of Theorem 5.1 except for some possible complexity due to the bending of M and N.

6. An Application

In this section, we apply the regularity theorem in Sect. 5 for harmonic maps into H^2 to the uniqueness problem of axially symmetric, asymptotically flat, stationary spacetimes (cf. [Ca, We] for the definition) in general relativity.

The Einstein vacuum field equations of general relativity are

$$\mathrm{Ric}(g)=0\,, \tag{6.1}$$

where (M, g) is a 4-dimensional Lorentzian manifold and $\mathrm{Ric}(g)$ denotes its Ricci curvature. In view of the great difficulties involved in the study of these equations, one is led to consider special cases with symmetry. The solutions of Schwarzschild are found in 1916, the first explicit ones parameterized by the mass. They are all static and spherically symmetric. The Kerr family of solutions to (6.1), discovered in 1963, has two parameters, the mass and the aungular momentum, and is both stationary and axially symmetric. In the 70's Robinson shows that the Kerr solutions are unique among all asymptotically flat axially symmetric stationary spacetimes that have a connected event horizon. His proof is based on some results of Ernst [Er] and Carter [Ca]. They reduce the Einstein vacuum equation in the asymptotically flat, axially symmetric stationary case to an axially symmetric harmonic map from 3-dimensional euclidean space into a hyperbolic plane H^2. Then the uniqueness of Kerr solutions is the same as that of the harmonic map into H^2. Robinson affirmed the later by using the convexity of the distance function on H^2.

After this striking result of Robinson, there is a "little" problem left, i.e., is it possible to drop the extra assumption on the connectedness of the event horizon. Following a suggestion of D. Christodoulou, G. Weinstein considered this problem in his thesis [We1]. In order to describe his basic result, we need to first introduce some notations.

Let (ρ, φ, z) be the cylindrical coordinates for R^3 and A be the z-axis. Choose parameters $\{a_i, b_i, c_i\}_{1 \leqq i \leqq L}$ for any fixed positive integer L satisfying:

$$-\infty < a_1 < b_1 < a_2 < \cdots < b_{L-1} < a_L < b_L = 0\,. \tag{6.2}$$

Define $\Gamma = A \backslash \{(0, \varphi, z) | a_i < z < b_i \text{ for some } i\}$. Then the distance function $d((\rho, \varphi, z), \Gamma)$ is Lipschitz in R^3 and the Laplacian of its logarithm is bounded except at those boundary points. By Fourier transformation, one can find a smooth function u outside $A \backslash \Gamma$ such that $h = 2\log d(\cdot, \Gamma) + u$ is harmonic outside Γ and h is asymptotic to $2\log d(\cdot, \Gamma)$ at the interior points of Γ. In fact, such a h can be explicitly obtained by superposition of Schwarzschild metrics. Namely, we can write

$$h = 2\log\rho + \sum_{i=1}^{L} u_i\,,$$

$$u_i = -\log\left(1 - \frac{b_i - a_i}{r_i}\right)\,, \tag{6.3}$$

where (r_i, θ_i) are given by transformations

$$\rho = r_i\left(1 - \frac{b_i - a_i}{r_i}\right)^{\frac{1}{2}} \sin\theta_i\,, \tag{6.4}$$

$$z - \frac{b_i + a_i}{2} = \left(r_i - \frac{b_i - a_i}{2}\right)\cos\theta_i \tag{6.5}$$

(cf. [We1]).

Let (X, Y) be a harmonic map from $R^3 \backslash A$ into the upper half plane with standard hyperbolic metric, satisfying.

X is smooth across Γ away from its boundary points and ,

$$X(0, \varphi, z) = c_i \quad \text{for } b_i < z < a_{i+1}, \quad i = 1, 2, \ldots, L-1 ,$$

$$X(0, \varphi, z) = c_L \quad \text{for } z > b_L ,$$

$$X(0, \varphi, z) = -c_L \quad \text{for } z < a_1 , \tag{6.6}$$

both X and Y are axially symmetric, i.e., they are independent of φ . (6.7)

Y is of the form e^{h+y} and y is smooth across Γ away from its boundary points. (6.8)

Both X and y are of order $O\left(\frac{1}{\sqrt{\rho^2+z^2}}\right)$ as (ρ, φ, z) near infinity . (6.9)

By the harmonicity of (X, Y) and (6.3)–(6.5), one can easily check that the differential 1-form Ψ is closed and smooth in $R^3\backslash(\overline{A\backslash\Gamma})$, where $\overline{A\backslash\Gamma}$ denotes the closure of $A\backslash\Gamma$ and

$$\Psi = \frac{1}{4}\rho Y^{-2}\left\{\left(\frac{\partial Y}{\partial\rho}\right)^2 + \left(\frac{\partial X}{\partial\rho}\right)^2 - \left(\frac{\partial Y}{\partial z}\right)^2 - \left(\frac{\partial X}{\partial z}\right)^2\right\} d\rho$$

$$+ \frac{1}{2}\rho X^{-2}\left\{\left(\frac{\partial Y}{\partial\rho}\right)\left(\frac{\partial Y}{\partial z}\right) + \left(\frac{\partial X}{\partial\rho}\right)\left(\frac{\partial X}{\partial z}\right)\right\} dz - d\log\rho . \tag{6.10}$$

Therefore, there is a smooth function ω in $R^3\backslash(\overline{A\backslash\Gamma})$ satisfying $d\omega = \Psi$. Such an ω is unique up to constants, so we may normalize $(\omega - u - y)(0, \varphi, 1) = 0$. The following proposition is essentially due to Ernst [Er] and Carter [Ca], but the form of it presented here is formulated in [We1] with some changes on the notations.

Proposition 6.1. *The asymptotically flat axially symmetric stationary solutions of EVE (6.1) with n connected components of even horizon are equivalent to those harmonic maps described as above with one extra condition*

$$\beta \equiv 0 \quad \text{on } \Gamma\backslash\partial\Gamma , \tag{6.11}$$

where β is defined to be $\omega - u - y$ in $R^3\backslash(\overline{A\backslash\Gamma})$.

Note that our *EVE* solutions always have a nondegenerate event horizon in the sense in [Ca]. Also, the condition (6.11) is nothing else but an interpretation of the regularity of the *EVE* solutions along the axis away from event horizon.

The parameters $\{a_i, b_i, c_i\}_{1 \leqq i \leqq L}$ have the following physical interpretations. The differences $a_{i+1} - b_i$ $(1 \leqq i \leqq L-1)$ are the distances between two adjacent components of the event horizon, the length $b_i - a_i$ of the interval $[a_i, b_i]$ can be regarded as the mass of the i^{th}-component of an event horizon. Finally, the angular momentum J_i of the i^{th}-component is $\frac{1}{8}(c_i - c_{i-1})$ for $i \geqq 2$ and $\frac{1}{8}(c_1 + c_L)$ for $i = 1$. In particular, the total angular momentum is $\frac{1}{4}c_L$. We refer the readers to Sect. 6 in [We1] for details. The solutions of *EVE* (6.1) without rotation, i.e., all c_i vanishes, were discussed a long time ago by Bach and Weyl [BW]. The corresponding harmonic maps are of form $\left(0, \rho^2 e^{\sum_{i=1}^{L} u_i}\right)$, where u_i are given by (6.3)–(6.5). Let us first compute β on the bounded components of Γ for these particular nonrotational solutions.

Lemma 6.1. *Let* $\left(0, \rho^2 e^{\sum_{i=1}^{L} u_i}\right)$ *be the harmonic map corresponding to one solution of Bach and Weyl,* i.e. *all* c_i *vanish. Then for* $z \in (b_i, a_{i+1})$, $i = 1, 2, \ldots, L-1$,

$$\beta(0, \varphi, z) = -\sum_{k=i+1}^{L} \sum_{j=1}^{i} \log \frac{(b_k - b_j)(a_k - a_j)}{(b_k - a_j)(a_k - b_j)}$$
$$< 0$$
$$\beta(0, \varphi, z) = 0 \quad \text{for either } z > b_L \quad \text{or } z < a_1 . \tag{6.12}$$

Proof. Let C_ε^i be a path in the (ρ, z)-plane defined by the equation $r_i = 2m_i + \varepsilon$, where r_i is defined in (6.4) and (6.5), m_i is equal $\frac{b_i - a_i}{2}$. Put $d_i = \frac{a_i + b_i}{2}$. Then by computations, we find

$$\frac{\partial u_i}{\partial \rho} = \frac{-2m_i}{r_i^{\frac{1}{2}}(r_i - 2m_i)^{\frac{1}{2}}}(r_i - m_i)\sin\theta_i \{(r_i - m_i)^2 \sin^2\theta_i$$
$$+ r_i(r_i - 2m_i)\cos^2\theta_i\}^{-1} , \tag{6.13}$$

$$\frac{\partial u_i}{\partial z} = -2\cos\theta_i \{(r_i - m_i)^2 \sin^2\theta_i$$
$$+ r_i(r_i - 2m_i)\cos^2\theta_i\}^{-1} , \tag{6.14}$$

where $i = 1, 2, \ldots, L$.

In particular, it implies that both $\frac{\partial u_i}{\partial z}$ and $\frac{\partial u_i}{\partial \rho}$ are smooth in the region $\{r_i > 2m_i\}$. Since $(0, \rho^2, e^{u_i})$ corresponds to the Schwarzschild solution, we have

$$\frac{1}{4}\int_{C_\varepsilon^i} \left\{\rho\left(\left(\frac{\partial u_i}{\partial \rho}\right)^i - \left(\frac{\partial u_i}{\partial z}\right)^i\right) d\rho + 2\rho \frac{\partial u_i}{\partial \rho}\frac{\partial u_i}{\partial z} dz\right\} = 0 . \tag{6.15}$$

Therefore, using (6.13), (6.14) and (6.15), for z_2 in (b_i, a_{i+1}) and z_1 in (b_{i-1}, a_i), we have

$$\beta(0, \varphi, z_2) - \beta(0, \varphi, z_1) = \lim_{\varepsilon \to 0+} \int_{C_\varepsilon^i} d\beta$$
$$= \frac{1}{4}\lim_{\varepsilon \to 0+} \int_{C_\varepsilon^i} \left\{\rho\left(\left(\frac{\partial u}{\partial \rho}\right)^2 - \left(\frac{\partial u}{\partial z}\right)^2\right) d\rho\right.$$
$$\left. + 2\rho \frac{\partial u}{\partial \rho}\frac{\partial u}{\partial z} dz\right\} . \tag{6.15}$$
$$= \frac{1}{2}\lim_{\varepsilon \to 0+} \int_{C_\varepsilon^i} \left(\sum_{j \neq i} du_j\right) \cdot \frac{\partial u_i}{\partial \rho}$$
$$= -\sum_{j \neq i} u_j(b_i) - u_j(a_i)$$
$$= \sum_{j > i} \log \frac{(a_j - b_i)(b_j - a_i)}{(b_j - b_i)(a_j - a_i)}$$
$$+ \sum_{j < i} \log \frac{(b_i - b_j)(a_i - a_j)}{(b_i - a_j)(a_i - b_j)} . \tag{6.16}$$

Now our normalization says $\beta(0, \varphi, 1) = 0$, so (6.12) follows from (6.16).

In particular, any nonrotational asymptotically flat axially symmetric stationary of (6.1) is not regular on the bounded components of the axis away from event horizon.

In [Wel], G. Weinstein considered the construction of harmonic maps from $R^3\backslash A$ into H^2 of form (X, e^{h+y}), where h is equal to $2\log\rho + \sum_{i=1}^{L} u_i$ with u_i given in (6.3)–(6.5). His idea is to minimize the following functional

$$H(X, y) = \int_{R^3\backslash A} \left(|\nabla y|^2 + e^{-2h-2y}|\nabla X|^2 \right) dV \tag{6.17}$$

in the space $H_{1,h} \times H_1$ (cf. Sect. 2 or Sect. 6 in [Wel]).

Proposition 6.2. *For any set of numbers $\{a_i, b_i, c_i\}_{1\leqq i\leqq L}$ satisfying* (6.2), *there is a unique axially symmetric harmonic map of form (X, e^{h+y}) from $R^3\backslash A$ into H^2 satisfying:*

$$\int_{R^3} \left(|\nabla y|^2 + e^{-2h-2y}|\nabla X|^2 \right) dV \leqq C \tag{6.18}$$

and

$$\sup_{R^3\backslash A} |y| \leqq C\,, \tag{6.19}$$

where C is a constant depending only on $b_L - a_1$ and $\max_{1\leqq i\leqq L}|c_i|$.

Proof. This is due to Weinstein (cf. in [Wel]).

The following is just a special case of Theorem 1.1.

Theorem 6.1. *For any set of numbers $\{a_i, b_i, c_i\}_{1\leqq i\leqq L}$ satisfying* (6.2), *there is a unique harmonic map from $R^3\backslash A$ into H^2 such that* (6.8)–(6.9) *hold. Moreover, the $C^{2,\frac{1}{2}}$-norms of (X, y) in any compact subset $K \subset\subset R^3\backslash A$ are uniformly bounded by a constant depending only on K and C in* (6.18) *and* (6.19).

Theorem 6.2. *Given any two numbers $\lambda_1, \lambda_2 > 0$, there is an $\varepsilon = \varepsilon(\lambda_1, \lambda_2) > 0$ such that for any set of $\{a_i, b_i, c_i\}_{1\leqq i\leqq L}$ satisfying* $\max_{1\leqq i\leqq L}|c_i| \leqq \varepsilon$, $b_L - a_1 \leqq \lambda_2$ *and*

$$\min\left\{ \inf_{1\leqq i\leqq L} (b_i - a_i),\quad \inf_{1\leqq i\leqq L-1} (a_{i+1} - b_i) \right\} \geqq \lambda_1\,,$$

the function β defined in Proposition 6.1 *is negative in each bounded component of Γ. Equivalently, there is no regular asymptotically flat axially symmetric stationary solution of EVE* (6.1) *such that its event horizon has L connected components disjoint from each other in the distance at least λ_1 and at most λ_2, and each component has mass $\geqq \lambda_1$ and angular momentum less than ε.*

Proof. We observe that the $C^{2,\frac{1}{2}}$-estimates for (X, y) in Sect. 5 are uniform if (6.18) and (6.19) hold (cf. Theorem 6.1). On the other hand, the harmonic map (X, Y) does satisfy (6.18)–(6.19) for a constant C depending only on λ_1, λ_2 under our assumption on a_i, b_i, c_i $(1 \leqq i \leqq L)$ (cf. Sect. 2). Therefore, this theorem follows from a continuity argument and Lemma 6.1.

Theorem 6.3. *Let (X_α, Y_α) be a sequence of harmonic maps from $R^3\backslash A$ ino H^2 satisfying* (6.6)–(6.9) *for $\{a_{\alpha i}, b_{\alpha i}, c_{\alpha i}\}_{1\leqq i\leqq L}$. Suppose that*

(1) $\sup_{\alpha, i}\{|b_{\alpha i} - a_{\alpha i}|, |c_{\alpha i}|\} \leqq C\,,$

(2) $\exists i_0$, s.t. $\lim_{\alpha\to\infty} (a_{\alpha i_0+1} - b_{\alpha i_0}) = \infty$ *and* $\sup_{\alpha, i \neq i_0} (|a_{\alpha i+1} - b_{\alpha i}|) \leqq C\,,$

where C is a uniform constant. Then (X_α, Y_α) converges to the union of two harmonic maps from $R^3\backslash A$ into H^2 satisfying (6.6)–(6.9) *for two sets of numbers*

$\{a_{\infty i}, b_{\infty i}, c_{\infty i}\}_{i \leqq i \leqq i_0}$ *and* $\{a_{\infty i}, b_{\infty i}, c_{\infty i}\}_{i_0 \leqq i \leqq L}$, *respectively, where*

$$a_{\infty i} = \lim_{\alpha \to \infty} (a_{\alpha i} - b_{\alpha i_0}), \qquad b_{\infty i} = \lim_{\alpha \to \infty} (b_{\alpha i} - b_{\alpha i_0}) \quad \text{for } i \leqq i_0,$$

$$a_{\infty i} = \lim_{\alpha \to \infty} (a_{\alpha i} - a_{\alpha i_0 + 1}), \qquad b_{\infty i} = \lim_{\alpha \to \infty} (b_{\alpha i} - b_{\alpha i_0 + 1}) \quad \text{for } i > i_0,$$

$$c_{\infty i} = \lim_{\alpha \to \infty} c_{\alpha i}.$$

We omit the proof. It is simply a corollary of the results in Sect. 2, the regularity theorem in Sect. 5 and some standard arguments.

In particular, this last theorem implies that the solution of EVE (6.1) with two black holes constructed in Sect. 8 of [We] converges to the union of two Kerr's solutions with opposite total angular momentum as the distance of two black holes approaches to infinity.

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