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Lines in Space-Times

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Abstract. We construct a complete timelike maximal geodesic ("line") in a timelike geodesically complete spacetime M containing a compact acausal spacelike hypersurface S which lies in the past of some S-ray. An S-ray is a future complete geodesic starting on S which maximizes Lorentzian distance from S to any of its points. If the timelike convergence condition (strong energy condition) holds, a line exists only if M is static, i.e. it splits geometrically as space $\times$ time. So timelike completeness must fail for a nonstatic spacetime with strong energy condition which contains a "closed universe" S with the above properties.

1. Introduction

Let M be a timelike geodesically complete time-oriented Lorentzian manifold containing a compact spacelike acausal hypersurface S. A conjecture stated by R. Bartnik [B] says: If M satisfies the timelike convergence condition (strong energy condition), then M splits isometrically as space $\times$ time. (In fact, Bartnik assumes S to be a Cauchy hypersurface.) By the Lorentzian splitting theorem [N], this statement is true if we can construct a timelike line, i.e. an inextendible maximal timelike geodesic. However, without the timelike convergence condition, such a line need not exist (cf. [EG]). It is the aim of the present paper to construct a timelike line if S lies in the past of some S-ray, i.e. a future inextendible causal curve γ starting on S such that $\gamma \mid [0, t]$ is a curve of maximal length between S and $\gamma(t)$ for all $t > 0$.

The main results are stated and proved in Sect. 5; the ingredients are given in Sects. 2–4. For standard facts in Lorentzian geometry and for standard notation (such as I^+, J^+, D^+, H^+) we refer to [HE, BE].

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2. Limit Curves

Let (M, g) be a space-time, i.e. a time-oriented Lorentzian manifold. Additionally, we choose a complete Riemannian metric h on M. All nonspacelike curves are rectifiable and (with the possible exception of certain limit curves which inherit a limit parameter) we will always parameterize them by arc length with respect to h. Clearly, a causal curve γ is (future and past) inextendible if and only if it is parametrized on $(-\infty, \infty)$.

Limit Curve Lemma for Inextendible Nonspacelike Curves. *Let $\gamma_n : (-\infty, \infty) \to M$ be a sequence of inextendible nonspacelike curves (parametrized by arc length in h). Suppose that $p \in M$ is an accumulation point of the sequence $(\gamma_n(0))$. Then there exists an inextendible nonspacelike curve $\gamma : (-\infty, \infty) \to M$ such that $\gamma(0) = p$ and a subsequence (γ_m) which converges uniformly (with respect to h) to γ on compact subsets of $\mathbb{R}$. γ is called a limit curve of (γ_n).*

Comment. The proof of this lemma is an application of Arzela's theorem and is essentially contained in the proof of Proposition 2.18 in [BE]. One advantage of the parametrization with respect to the background metric h is that one can establish the upper semicontinuity of the Lorentzian length functional *without* invoking the assumption of strong causality:

Proposition. *The Lorentzian arc length functional is upper semicontinuous with respect to the topology of uniform convergence on compact subsets, i.e. if a sequence $\gamma_n : [a, b] \to M$ of nonspacelike curves converges uniformly to the nonspacelike curve $\gamma : [a, b] \to M$, then*

$$L(\gamma) \geq \limsup_{n \to \infty} L(\gamma_n).$$

Comment. The idea behind circumventing the strong causality assumption is this: One can partition $[a, b]$ as $a = t_0 < t_1 < \cdots < t_n = b$ so that each subsegment $\gamma \mid [t_{i-1}, t_i]$ is contained in a normal neighborhood N_i of M. (N_i, g), viewed as a space-time in its own right, is strongly causal. By the uniform convergence, $\gamma_n \mid [t_{i-1}, t_i] \subset N_i$ for all sufficiently large n. Now apply the known upper semicontinuity of the Lorentzian arc length functional on the strongly causal space-time (N_i, g) to conclude,

$$L(\gamma \mid [t_{i-1}, t_i]) \geq \limsup_{n \to \infty} L(\gamma_n \mid [t_{i-1}, t_i]).$$

Now sum over i to get the desired result.

The limit curve lemma was discussed for inextendible causal curves. There is an obvious version for future (respectively past) inextendible causal curves $\gamma_n : [0, \infty) \to M$.

Let d denote the Lorentzian distance function, i.e.

$$d(p, q) = \sup\{L(\mu);\ \mu \in C(p, q)\} \leq \infty,$$

where $C(p, q)$ denotes the set of future directed causal curves from p to q. The Lorentzian distance function is known to be lower semicontinuous. A sequence $\gamma_n : [a_n, b_n] \to M$ of causal curves is called *limit maximizing* if

$$L(\gamma_n) \geq d(\gamma_n(a_n), \gamma_n(b_n)) - \varepsilon_n$$

for some sequence $\varepsilon_n \to 0$. Suppose that γ_n converges uniformly to $\gamma : [a, b] \to M$ on some subinterval $[a, b] \subset \bigcap_n [a_n, b_n]$. Since L is upper and d lower semicontinuous, there is a sequence $\delta_n \to 0$ such that

$$\begin{aligned} L(\gamma_n) - \delta_n \le L(\gamma) \le d(\gamma(a), \gamma(b)) \\ \le d(\gamma_n(a), \gamma_n(b)) + \delta_n \le L(\gamma_n) + \varepsilon_n + \delta_n, \end{aligned}$$

thus

$$\lim L(\gamma_n) = L(\gamma) = d(\gamma(a), \gamma(b)) = \lim d(\gamma_n(a), \gamma_n(b))$$

and in particular, γ is maximal. (Beem and Ehrlich introduced the notion of limit maximizing curves in the strongly causal setting; cf. [BE, Chap. 7].)

3. Rays, Co-Rays and Busemann Function

A *ray* in M is a maximal future inextendible causal geodesic $\gamma : [0, \infty) \to M$. Rays often arise from limit constructions:

Lemma 1. *Let z_n be a sequence in M with $z_n \to z$. Let $p_n \in I^+(z_n)$ with finite $d(z_n, p_n)$. Let $\gamma_n : [0, a_n] \to M$ be a limit maximizing sequence of causal curves with $\gamma_n(0) = z_n$ and $\gamma_n(a_n) = p_n$. Let $\bar{\gamma}_n : [0, \infty) \to M$ be any future inextendible extension of γ_n. Suppose either*

(a) $p_n \to \infty$, *i.e. no subsequence is convergent,*
or
(b) $d(z_n, p_n) \to \infty$.

Then any limit curve $\gamma : [0, \infty) \to M$ of the sequence $\bar{\gamma}_n$ is a ray starting at z.

Proof. All we have to show is that $a_n \to \infty$. Suppose not. By passing to a subsequence, we may assume $a_n \to a < \infty$. Since γ_n are parametrized by arc length for h, all γ_n are contained in a compact subset $K \subset M$, e.g. the closed h-ball of radius $2a$ around z. This is clearly impossible in Case (a). In Case (b), let T be a timelike unit vector field [i.e. $g(T, T) = -1$] on M and $\tau = g(., T)$. Consider the Riemannian metric

$$h_0 = g + 2\tau \otimes \tau = g^\perp + \tau \otimes \tau.$$

Note that for any causal curve segment σ,

$$L(\sigma) = L_g(\sigma) \le L_{h_0}(\sigma),$$

where L_{h_0} denotes the length with respect to h_0. By assumption, $L(\gamma_n) \ge d(z_n, p_n) - \varepsilon_n \to \infty$, hence $L_{h_0}(\gamma_n) \to \infty$. Since K is compact, there exists $\lambda > 0$ such that $h \ge \lambda \cdot h_0$ on $TM \mid K$. Therefore $a_n = L_h(\gamma_n) \to \infty$ which is a contradiction.

S-Rays. Let $\gamma : [0, \infty) \to M$ be a ray. Let $S \subset M$ be a subset containing $\gamma(0)$ such that γ maximizes distance to S, i.e. for any $t \in [0, \infty)$,

$$L(\gamma \mid [0, t]) = d(S, \gamma(t)),$$

where $d(S,x) = \sup\{d(q,x);\ \ q \in S\}$. Then γ is called an *S-ray:* E.g., any ray γ is a $\{\gamma(0)\}$-ray. Observe that for any $x \in I^-(\gamma) \cap J^+(S)$ and all sufficiently large t,

$$d(S,x) + d(x,\gamma(t)) \leq d(\gamma(0),\gamma(t)) < \infty. \tag{$*$}$$

Co-Rays. Let $\gamma : [0,\infty) \to M$ be a future inextendible S-ray and let $z \in I^-(\gamma) \cap J^+(S)$. Let $z_n \to z$ in $J^+(S)$ and put $p_n = \gamma(r_n)$ for some sequence $r_n \to \infty$. Then $z_n \in I^-(p_n)$ for sufficiently large n, and $d(z_n, p_n) < \infty$ by $(*)$. Assume either

$$\text{(a)}\quad p_n \to \infty \qquad \text{or} \qquad \text{(b)}\quad d(z_n, p_n) \to \infty.$$

[Note that (b) holds if γ has infinite length.] Consider a limit maximizing sequence μ_n of causal curves from z_n to p_n. By Lemma 1, any limit curve $\mu : [0,\infty) \to M$ of the μ_n is a ray starting at z. Such a ray is called a *co-ray* of γ. Note that μ is contained in the closure of $I^-(\gamma)$. (In fact, if $\mu(t) \in \partial I^-(\gamma)$, then $\mu \mid [t,\infty)$ is a future inextendible null geodesic generator of $\partial I^-(\gamma)$.)

Busemann Functions. Let $\gamma : [0,\infty) \to M$ be a timelike S-ray and $b : I^-(\gamma) \to [-\infty,\infty)$ the associated Busemann function, namely $b(x) = \lim_{t\to\infty} b_t(x)$, where

$$b_t(x) = d(\gamma(0),\gamma(t)) - d(x,\gamma(t)).$$

Recall that $b_t(x)$ decreases monotonely with t, since for $s > t$ we have

$$\begin{aligned} d(x,\gamma(s)) &\geq d(x,\gamma(t)) + d(\gamma(t),\gamma(s)),\\ d(\gamma(0),\gamma(s)) &= d(\gamma(0),\gamma(t)) + d(\gamma(t),\gamma(s)). \end{aligned}$$

Further, for $x \in I^-(\gamma) \cap J^+(S)$, we have

$$b(x) \geq d(S,x) \geq 0,$$

since $(*)$ shows $b_t(x) \geq d(S,x)$ for any t. Recall that d is lower semicontinuous, hence b_t is upper semicontinuous, and since b is the decreasing limit of the b_t, it is also upper semicontinuous.

Lemma 2. *Let $\gamma : [0,\infty) \to M$ be a timelike S-ray and $\mu : [0,\infty) \to M$ a co-ray with $\mu(0) = z \in I^-(\gamma) \cap J^+(S)$. Then we have for any $s > 0$ and any $x \in I^-(\mu(s))$*

$$b(x) \leq b(z) + d(z,\mu(s)).$$

In particular, if μ is a null ray, then $b(x) \leq b(z)$ for any $x \in I^-(\mu)$.

Proof. Let $\mu = \lim \mu_n$ where μ_n is a limit maximizing sequence from z_n to $\gamma(r_n)$. Let $b_n := b_{r_n}$. Then

$$b_n(x) = d(\gamma(0),\gamma(r_n)) - d(x,\gamma(r_n)).$$

Since $\mu_n(s) \to \mu(s)$, we have $x \in I^-(\mu_n(s))$ and

$$d(x, \gamma(r_n)) \geq d(x, \mu_n(s)) + d(\mu_n(s), \gamma(r_n))$$

which shows

$$b_n(x) \leq -d(x, \mu_n(s)) + b_n(\mu_n(s)) \leq b_n(\mu_n(s)).$$

For two real sequences (a_n), (b_n) we will write $a_n \approx b_n$ if $a_n - b_n$ is converging to zero. Since μ_n is maximal up to an error ε_n, we have

$$\begin{aligned} b_n(\mu_n(s)) - b_n(z_n) &= d(z_n, \gamma(r_n)) - d(\mu_n(s), \gamma(r_n)) \\ &\approx d(z_n, \mu_n(s)). \end{aligned}$$

Thus

$$b_n(x) \leq b_n(\mu_n(s)) \approx b_n(z_n) + d(z_n, \mu_n(s)).$$

Now for any $y \in I^+(z) \cap I^-(\gamma)$ we have $y \in I^+(z_n)$ for large n and therefore

$$d(z_n, y) + d(y, \gamma(r_n)) \leq d(z_n, \gamma(r_n)),$$

which shows $d(y, \gamma(r_n)) \leq d(z_n, \gamma(r_n))$, hence $b_n(y) \geq b_n(z_n)$. So we obtain

$$b_n(x) \leq b_n(y) + d(z_n, \mu_n(s)) + \varepsilon_n.$$

Taking the limit as $n \to \infty$, we get the result; note that $d(z_n, \mu_n(s)) \to d(z, \mu(s))$ since $\mu_n \mid [0, s]$ is limit maximizing, and use the upper semicontinuity of b.

Comment. Lemma 2 replaces the well known fact in Riemannian geometry that the Busemann function grows with unit speed (with respect to arc length) along co-rays. This still holds in Lorentzian geometry provided that d is continuous and μ timelike (cf. [E, p. 480]).

4. Spacelike Hypersurfaces

Definition. A subset $S \subset M$ is called a *spacelike hypersurface* if for each $p \in S$ there is a neighborhood U of p in M such that $S \cap U$ is acausal and edgeless in U.

Comment. A spacelike hypersurface is necessarily an embedded topological submanifold of M with codimension one. A smooth hypersurface with timelike normal vector is a spacelike hypersurface in the sense of our definition.

Lemma 3. *Let $S \subset M$ be an acausal spacelike hypersurface. Then*

$$I^+(S) = J^+(S) \setminus S.$$

Consequently, any S-ray is timelike.

Proof. Clearly, $I^+(S) \subset J^+(S) \setminus S$. So let $p \in J^+(S) \setminus S$ and let μ be any causal past directed curve from p to S. Let $q \in S$ be the past end point of μ. There exists a neighborhood U of q and a coordinate chart $x = (x_0, \ldots, x_d) : U \to I^{d+1}$ such that

$\partial/\partial x_0$ is timelike, and $x^{-1}(S \cap U)$ is a graph over I^d. Let $q' \in \mu \cap U$, $q' \neq q$, and replace the segment of μ between q' and q by the x_0-parameter line through q' which also meets S. Thus $q' \in I^+(S)$, hence $p \in I^+(S)$. This shows that $I^+(S) = J^+(S) \setminus S$. If γ is an S-ray, it cannot stay in S since S is locally acausal. So $\gamma(t) \in I^+(S)$ for some $t > 0$ which implies that $d(\gamma(0), \gamma(t)) > 0$. Hence γ is timelike.

Lemma 4. *Let $S \subset M$ be a compact acausal spacelike hypersurface. Then there exists a timelike S-ray in $D^+(S)$. If $H^+(S) \neq 0$, we find such a ray in $I^-(p) \cap D^+(S)$ for any $p \in H^+(S)$.*

Proof. If $H^+(S) \neq \emptyset$, this is true by the "Main Lemma" in [G2]. So it remains to consider the (easier) case where $H^+(S) = \emptyset$. Let $p \in S$ and $\mu : [0, \infty) \to M$ be a future inextendible timelike geodesic with $\mu(0) = p$. Since $H^+(S) = \emptyset$, we have $\overline{\mu((0, \infty))} \subset D^+(S)$. Let $r_n \to \infty$ and $p_n = \mu(r_n)$. Then $p_n \to \infty$ since $p_m \to p \in D^+(S)$ (for some subsequence (p_m) of (p_n)) would be a violation of strong causality. By compactness of S, there are maximal curves γ_n from S to $p_n \in D^+(S)$. Let $z_n = \gamma_n(0) \in S$. We may assume that $z_n \to z \in S$. By Lemma 1, the γ_n accumulate to an S-ray γ. By Lemma 3, γ is timelike.

Lemma 5. *Assume M is future timelike geodesically complete. Let S be a compact acausal spacelike hypersurface in M. Then each S-ray γ is contained in $D^+(S)$ and any co-ray β of γ is timelike.*

Proof. If γ is not contained in $D^+(S)$, it will leave $D^+(S)$ at some point $o = \gamma(t) \in H^+(S)$. By Lemma 4, there exists a timelike S-ray of infinite length (by completeness) in $I^-(o) \cap D^+(S)$. Therefore, $d(S, o) = \infty$ which contradicts the fact that γ is an S-ray.

Now let β be a co-ray of γ with $\beta(0) = q \in J^+(S)$. Since S is acausal, we have $\beta(t) \in J^+(S) \setminus S = I^+(S)$ (cf. Lemma 3) for any $t > 0$. Choose a sequence $t_n \to \infty$ and put $p_n = \beta(t_n)$. We will show that

$$d(S, p_n) \to \infty\,. \tag{$*$}$$

By perturbing the sequence (p_n) slightly to the past and using the lower semi-continuity of d, one can easily construct a sequence $(q_n) \subset I^-(\gamma) \cap J^+(S)$ with $q_n \in I^-(p_n)$ for all n, such that $d(S, q_n) \to \infty$. This implies that β cannot be null: Otherwise, for the Busemann function β of γ we would get $b(q_n) \leq b(q) < \infty$ (cf. Lemma 2), but on the other hand, $b(q_n) \geq d(S, q_n) \to \infty$ (cf. Sect. 3), a contradiction.

In order to show $(*)$, we may assume $d(S, p_n) < \infty$ for all n. Let $\sigma_n : [0, a_n] \to M$ be a limit maximizing sequence of curves from S to p_n, i.e. $L(\sigma_n) \geq d(S, p_n) - \varepsilon_n$ with $\varepsilon_n \to 0$. Let $\sigma_n(0) = z_n \in S$. By compactness, we may assume $z_n \to z \in S$.

Case 1. $p_n \to \infty$. Then by Lemma 1, $a_n \to \infty$, and σ_n accumulate to an S-ray $\sigma : [0, \infty) \to M$. By Lemma 3, σ is timelike and has infinite length (by completeness). So we have for any $a > 0$ and for large enough n,

$$d(S, p_n) \geq L(\sigma_n \mid [0, a_n]) \geq L(\sigma_n \mid [0, a]) \to L(\sigma \mid [0, a])$$

(cf. Sect. 2). Since $L(\sigma \mid [0, a]) \to \infty$ as $a \to \infty$, we get $(*)$.

Case 2. $p_n \to p \in M$. The coray β is contained in $\overline{D^+(S)}$, thus $p \in \overline{D^+(S)}$. Since strong causality is violated at p, it cannot lie in $D^+(S)$, hence $p \in H^+(S)$. Applying Lemma 4 again gives an S-ray $\mu \subset I^-(p) \cap D^+(S)$ of infinite length. In particular, we have $p \in I^+(\mu(t))$ for any $t > 0$ and therefore $p_n \in I^+(\mu(t))$ for large n. Hence $d(S, p_n) \to \infty$.

5. The Main Theorem

Recall that a *line* is a (future and past) inextendible geodesic γ such that any compact segment $\gamma \mid [a,b]$ is maximal, i.e. $L(\gamma \mid [a,b]) = d(\gamma(a),\gamma(b))$.

Theorem A. *Let M be a spacetime which is timelike geodesically complete and contains a compact acausal spacelike hypersurface S. Suppose that there exists an S-ray γ such that $S \subset I^-(\gamma)$. Then M contains a timelike line.*

Proof. Let $\beta : [0,\infty) \to M$ be a past directed S-ray in $D^-(S)$ which exists by the time dual of Lemma 4. Since $\beta(0) \in S \subset I^-(\gamma)$, we have $\beta(s) \in I^-(\gamma(t))$ for all s and sufficiently large t. Pick monotone sequences $t_n, s_n \to \infty$ and set $q_n = \gamma(t_n)$ and $p_n = \beta(s_n)$. Let $\mu_n : [a_n, b_n] \to M$ be a limit maximizing causal curves from p_n to q_n. Since $p_n \in D^-(S)$ and $q_n \in J^+(S)$, the curve μ_n must intersect S, say at z_n, and we choose the parameter so that $z_n = \mu_n(0)$. By compactness, we may assume that $z_n \to z \in S$. Let μ be a limit curve of complete extensions of the μ_n's (cf. Sect. 2). We have to show that $b_n \to \infty$, $a_n \to -\infty$ (then μ is a line) and that μ is timelike.

Note that $\mu^+ = \mu \mid [0,\infty)$ is a co-ray of γ, and in particular, $b_n \to \infty$ (cf. proof of Lemma 1). Thus μ^+ is a timelike ray (cf. Lemma 5), and moreover, there exists $0 < \delta < \liminf |a_n|$ such that $\mu \mid [-\delta,\infty)$ is maximizing, hence also a timelike ray.

In order to see that $\mu^- : [0,\infty) \to M$, $\mu^-(t) = \mu(-t)$ is a (past directed) co-ray of β we have to show that $z \in I^+(\beta)$. But since $\mu_n \mid [-\delta,0] \to \mu \mid [-\delta,0]$ which is a timelike geodesic, we have $\mu_n(s) \in I^-(z)$ for sufficiently large n and suitable $s \in [-\delta,0]$, hence $z \in I^+(\beta(s_n)) \subset I^+(\beta)$. Hence μ^- is a co-ray of β, and in particular, $a_n \to -\infty$. Thus μ is a line, and since μ^+ is timelike, μ must be timelike.

Remark. The proof shows that the assumption of timelike geodesic completeness can be replaced by the assumptions that $J^+(S)$ is future timelike geodesically complete and $J^-(S)$ is strongly causal.

As a consequence of Theorem A and the Lorentzian splitting theorem [N], we get immediately the following rigidity result:

Theorem B. *Let M be a spacetime which contains a compact acausal spacelike hypersurface S, and which satisfies the timelike convergence condition, i.e.* $\mathrm{Ric}(v,v) \geq 0$ *for all timelike vectors $v \in TM$. If M is timelike geodesically complete and there exists an S-ray γ such that $S \subset I^-(\gamma)$ then M splits, i.e. M is isometric to $(\mathbb{R} \times V,\ -dt^2 \oplus h)$, where (V,h) is a compact Riemannian manifold.*

Remark. There are numerous corollaries one can point out. The S-ray condition is implied by any of the following assumptions:

(a) *For every future inextendible timelike geodesic γ in $J^+(S)$, S is contained in $I^-(\gamma)$.*
(b) *For every future inextendible timelike geodesic γ in $J^+(S)$, $I^-(\gamma) = M$.*
(c) *There exists $t > 0$ such that $S \subset I^-(x)$ for any $x \in I^+(S)$ with $d(S,x) \geq t$.*

Conditions (a) and (b) both weaken the "no observer horizon" condition of Theorem 1.1 in [G1] (which, in addition, requires S to be Cauchy). Conditions (b) and (c) actually imply that S is a future Cauchy surface, i.e. $J^+(S) = D^+(S)$ or equivalently $H^+(S) = \emptyset$.

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