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Complex Quantum Groups and Their Quantum Enveloping Algebras

Bernhard Drabant[1], **Michael Schlieker**[2], **Wolfgang Weich**[2], **and Bruno Zumino**[3]

[1] Max-Planck-Institut für Physik, Werner-Heisenberg-Institut, Föhringer Ring 6, W-8000 München 40, Federal Republic of Germany
[2] Sektion Physik der Universität München, Lehrstuhl Professor Wess, Theresienstrasse 37, W-8000 München 2, Federal Republic of Germany
[3] Department of Physics, University of California, Berkeley, California, CA 94720, USA

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Abstract. We construct complexified versions of the quantum groups associated with the Lie algebras of type A_{n-1}, B_n, C_n, and D_n. Following the ideas of Faddeev, Reshetikhin and Takhtajan we obtain the Hopf algebras of regular functionals $U_{\mathcal{R}}$ on these complexified quantum groups. In the special example A_1 we derive the q-deformed enveloping algebra $U_q(sl(2,\mathbb{C}))$. In the limit $q \to 1$ the undeformed $U(sl(2,\mathbb{C}))$ is recovered.

1. Introduction

For quantum groups associated with the Lie algebras g of type A_{n-1}, B_n, C_n, and D_n there exist well defined correlations between the quantum group itself and the corresponding q-deformed universal enveloping algebra $U_q(g)$ [Dri, FRT]. Coming from the quantum group, one can construct the algebra of regular functionals which is shown to be the algebra $U_q(g)$ for a certain completion. Though the q-deformed Lorentz group already exists in at least two versions [CSSW, PW], there is not yet such a straightforward procedure like in the case of compact Lie groups to derive the corresponding quantized universal enveloping algebra. However this q-deformed algebra is the very object of interest since it should be fundamental for the construction of a q-deformed relativistic field theory.

In this paper we present the quantized universal enveloping algebra $U_q(sl(2,\mathbb{C}))$ of the q-deformed Lorentz group $Sl_q(2,\mathbb{C})$. In Sect. (2) we construct complex quantum groups for the Lie algebras A_{n-1}, B_n, C_n, and D_n. These are complexifications of the original quantum groups. The algebraic relations can be written in a generalized RTT-formulation and the usual determinant or metric relations. Following the ideas of [FRT] this fact is used in Sect. (3) to build up the algebra of regular functionals on the complex quantum groups[1]. The approach in this paper is purely algebraic without

[1] The same universal enveloping algebra corresponding to the complex quantum group is constructed by analyzing the algebra of the fundamental bicovariant bicomodule [CW]

considering the C^*-structure which has been investigated in [Pod, PW]. In Sect. (4) we derive as a special example the algebra $U_q(sl(2,\mathbb{C}))$[2]. We investigate the limit $q \to 1$ in Sect. (5) and recover $U(sl(2,\mathbb{C}))$.

2. Complexified Quantum Groups

In the approach of [FRT] the quantum group is a Hopf algebra with comultiplication Φ, counit e and antipode κ [Abe], generated by the matrix elements $t^i{}_j$ $(i,j=1,\ldots,N$; $N=n$ for A_{n-1} and $N=2n+1$ for B_n, $N=2n$ for C_n, $D_n)$ with the relations

$$I_{t,t}{}^{ij}_{st} := \hat{R}_q^{ij}{}_{kl} t^k{}_s t^l{}_t - t^i{}_v t^j{}_w \hat{R}_q^{vw}{}_{st} = 0 \tag{2.1}$$

and

$$\left\{\begin{array}{ll} \det(t^i{}_j) = \dfrac{(-1)^{n-1}}{[n]_q!} q^{-\binom{n}{2}} \varepsilon^{k_1\ldots k_n} t^{l_1}{}_{k_1} \cdot \ldots \cdot t^{l_n}{}_{k_n} \varepsilon_{l_1\ldots l_n} = \mathbf{1} & \text{for } A_{n-1} \\ t^i{}_s (C^{-1})^{sk} t^l{}_k C_{lj} = (C^{-1})^{ik} t^l{}_k C_{ls} t^s{}_j = \delta^i_j \mathbf{1} & \text{for } B_n, C_n, D_n\,. \end{array}\right. \tag{2.2}$$

where $\varepsilon_{i_1\ldots i_n} = (-1)^{n-1} \cdot \varepsilon^{i_1\ldots i_n} = (-q)^{l(\sigma)}$, $[n]_q!$ is the usual q-factorial [CSWW] and C_{ij} is the usual metric [FRT]. The $\hat{R}$-matrices for the respective quantum groups are taken from [FRT] with $q > 0$ real.

To find the complexified versions of these quantum groups one has to introduce the complex conjugates $t^{*i}{}_j$ of the generators $t^i{}_j$ as additional generators with the complex conjugate versions of the relations (2.1) and (2.2) above [CSWW]. With the definition:

$$\hat{t}^i{}_j := (\kappa(t^j{}_i))^* \tag{2.3}$$

we get

$$I_{\hat{t},\hat{t}}{}^{ij}_{st} := (\hat{R}_q^{-1})^{ij}{}_{kl} \hat{t}^k{}_s \hat{t}^l{}_t - \hat{t}^i{}_v \hat{t}^j{}_w (\hat{R}_q^{-1})^{vw}{}_{st} = 0\,, \tag{2.4}$$

$$\left\{\begin{array}{ll} \det(\hat{t}^i{}_j) = \dfrac{(-1)^{n-1}}{[n]_q!} q^{-\binom{n}{2}} \varepsilon^{k_1\ldots k_n} \hat{t}^{l_1}{}_{k_1} \cdot \ldots \cdot \hat{t}^{l_n}{}_{k_n} \varepsilon_{l_1\ldots l_n} = \mathbf{1} & \text{for } A_{n-1} \\ \hat{t}^i{}_s (C^{-1})^{sk} \hat{t}^l{}_k C_{lj} = (C^{-1})^{ik} \hat{t}^l{}_k C_{ls} \hat{t}^s{}_j = \delta^i_j \mathbf{1} & \text{for } B_n, C_n, D_n\,. \end{array}\right. \tag{2.5}$$

One still has to define commutation relations between the generators $t^i{}_j$ and their complex conjugates:

$$I_{\hat{t},t}{}^{ij}_{st} := \hat{R}_q^{ij}{}_{kl} \hat{t}^k{}_s t^l{}_l - t^i{}_v \hat{t}^j{}_w \hat{R}_q^{vw}{}_{st} = 0\,. \tag{2.6}$$

With this choice of commutation relations one can identify the function algebra over the unitary group as the quotient $\hat{t}^i{}_j = t^i{}_j$. There is a second possibility interchanging the role of $t^i{}_j$ and $\hat{t}^i{}_j$ in (2.6) which is equivalent to the first.

Summarizing we are considering the following quantum group:

$$\mathscr{A} := \mathbb{C}\langle t^i{}_j, \hat{t}^i{}_j\rangle / (I_{t,t}{}^{ij}_{st}, I_{\hat{t},t}{}^{ij}_{st}, I_{\hat{t},\hat{t}}{}^{ij}_{st}, (2.2), (2.5))\,. \tag{2.7}$$

[2] This algebra also has been investigated in [SWZ, OSWZ] by an alternative approach

Proposition 1. *The algebra $\mathscr{A}$ becomes a $*$-Hopf algebra with comultiplication Φ, counit e and antipode κ which are defined on the generators through*

$$\begin{aligned}
&\Phi(t^i{}_j) = t^i{}_k \otimes t^k{}_j\,, \\
&e(t^i{}_j) = \delta^i_j\,, \\
&\kappa(t^i{}_j) = \begin{cases} \dfrac{q^{-\binom{n}{2}}}{[n-1]_q!}\,\varepsilon^{ik_1\ldots k_{n-1}} t^{l_1}{}_{k_1}\cdot\ldots\cdot t^{l_{n-1}}{}_{k_{n-1}}\varepsilon_{l_1\ldots l_{n-1}j} & \textit{for } A_{n-1} \\ (C^{-1})^{ik} t^l{}_k C_{lj} & \textit{for } B_n, C_n, D_n \end{cases}
\end{aligned} \tag{2.8}$$

and

$$\begin{aligned}
\phi(t^{*i}{}_j) &= t^{*i}{}_k \otimes t^{*k}{}_j\,, \\
e(t^{*i}{}_j) &= \delta^i_j\,, \\
\kappa(t^{*i}{}_j) &= (\kappa^{-1}(t^i{}_j))^*\,.
\end{aligned} \tag{2.9}$$

It is convenient to introduce an RTT-formulation for this complexified quantum group. Set $(I) := (i, \bar{\imath})$, $\bar{I} := (\bar{\imath}, \bar{\bar{\imath}}) = (\bar{\imath}, i)$, $(i, \bar{\imath} = 1, \ldots, N)$ and define the $2N \times 2N$-matrix,

$$T^I{}_J := \begin{pmatrix} t & 0 \\ 0 & \hat{t} \end{pmatrix}^I{}_J\,. \tag{2.10}$$

Correspondingly one defines the $\mathscr{R}$-matrix,

$$\hat{\mathscr{R}}_q{}^{IJ}{}_{KL} := \begin{pmatrix} \alpha_0 \hat{R}_q & 0 & 0 & 0 \\ 0 & 0 & \alpha_1 \hat{R}_q & 0 \\ 0 & \alpha_2 \hat{R}_q^{-1} & 0 & 0 \\ 0 & 0 & 0 & \alpha_3 \hat{R}_q^{-1} \end{pmatrix} \tag{2.11}$$

with $\alpha_i \in \mathbb{C}$.

Then the relations (2.1), (2.4), and (2.6) can be written in compact form as

$$\hat{\mathscr{R}}_q{}^{IJ}{}_{KL} T^K{}_R T^L{}_S = T^I{}_V T^J{}_W \hat{\mathscr{R}}_q{}^{VW}{}_{RS}\,, \tag{2.12}$$

and $\hat{\mathscr{R}}_q$ fulfills the Yang-Baxter-Equation:

$$(\mathbf{E} \otimes \hat{\mathscr{R}}_q)(\hat{\mathscr{R}}_q \otimes \mathbf{E})(\mathbf{E} \otimes \hat{\mathscr{R}}_q) = (\hat{\mathscr{R}}_q \otimes \mathbf{E})(\mathbf{E} \otimes \hat{\mathscr{R}}_q)(\hat{\mathscr{R}}_q \otimes \mathbf{E}) \tag{2.13}$$

with $\mathbf{E}^I_J = \delta^I_J$.

There are three further possibilities for the choice of the $\hat{\mathscr{R}}_q$-matrix which we disregard here, since one of them yields equivalent results and the others do not admit a simple involution on the algebra of regular functionals.

3. The Algebra of Regular Functionals

The dual space $\mathscr{A}^*$ of the Hopf algebra $\mathscr{A}$ is an algebra with the convolution product. One can introduce an antimultiplication involution "$\dagger$" on $\mathscr{A}^*$: For $f \in \mathscr{A}^*$ one sets

$$\forall a \in \mathscr{A} : f^\dagger(a) := \overline{f(\kappa^{-1}(a^*))}\,. \tag{3.1 }$$

In the following we are working mostly with the multiplicative involution "$^-$":

$$\bar{f} := f^\dagger \circ \kappa^{-1}\,. \tag{3.2}$$

It is also possible to consider an involution where κ^{-1} is replaced by κ in (3.1) and (3.2). Since $\kappa((\kappa(a^*))^*) = \kappa^{-1}((\kappa^{-1}(a^*))^*) = a \; \forall a \in \mathcal{A}$ the multiplicative involutions coincide for both cases. This is also true for the antimultiplicative ones for $q \to 1$. We now construct the algebra of regular functionals on $\mathcal{A}$. We define functionals $L^{\pm I}{}_J \in \mathcal{A}^*$ through their action on the generators of $\mathcal{A}$:

$$\begin{aligned} L^{\pm I}{}_J(1) &:= \delta^I_J\,, \\ L^{\pm I}{}_J(T^k{}_L) &:= \mathcal{R}_q^{\pm 1 IK}{}_{LJ} \end{aligned} \tag{3.3}$$

and their comultiplication

$$\forall a, b \in \mathcal{A} : L^{\pm I}{}_J(ab) = L^{\pm I}{}_K(a) L^{\pm K}{}_J(b)\,. \tag{3.4}$$

This definition is compatible with the algebra relations in $\mathcal{A}$ and it holds

Proposition 2.

$$\begin{aligned} L^{\pm i}{}_{\bar{j}} = L^{\pm \bar{j}}{}_i &= 0 \quad \forall i, \bar{j}\,, \\ \mathcal{R}_q^{JI}{}_{LK} L^{\pm K}{}_V L^{\pm L}{}_W &= L^{\pm I}{}_A L^{\pm J}{}_B \mathcal{R}_q^{BA}{}_{WV}\,, \\ \mathcal{R}_q^{JI}{}_{LK} L^{+K}{}_V L^{-L}{}_W &= L^{-I}{}_A L^{+J}{}_B \mathcal{R}_q^{BA}{}_{WV}\,. \end{aligned} \tag{3.5}$$

The equations (2.2) and (2.5) partly determine the coefficients α_i in Eq. (2.11):

Proposition 3. *For A_{n-1} one has*

$$(\alpha_0)^{-n} = (\alpha_1)^{-n} = (\alpha_2)^n = (\alpha_3)^n = q\,. \tag{3.6}$$

In the cases of B_n, C_n, D_n, one gets

$$(\alpha_0)^2 = (\alpha_1)^2 = (\alpha_2)^2 = (\alpha_3)^2 = 1\,. \tag{3.7}$$

Definition. The algebra $U_{\mathcal{R}}$ of regular functionals on $\mathcal{A}$ is the unital algebra generated by $\{L^{\pm I}{}_J\}$.

Proposition 4. *The algebra $U_{\mathcal{R}}$ becomes a bialgebra with comultiplication $\Delta : U_{\mathcal{R}} \to U_{\mathcal{R}} \otimes U_{\mathcal{R}}$ and counit $\varepsilon : U_{\mathcal{R}} \to \mathbb{C}$ through the definitions*

$$\begin{aligned} \Delta(L^{\pm I}{}_J) &:= L^{\pm I}{}_K \otimes L^{\pm K}{}_J\,, \\ \varepsilon(L^{\pm I}{}_J) &:= \delta^I_J\,, \end{aligned} \tag{3.8}$$

on the generators of $U_{\mathcal{R}}$.

Consider now the map $\tilde{S} : \mathcal{A}^* \to \mathcal{A}^*$ defined by

$$\tilde{S} := \cdot \circ \kappa\,. \tag{3.9}$$

With this definition we get the following

Proposition 5.

$$\tilde{S}(U_{\mathcal{R}}) = U_{\mathcal{R}} \tag{3.10}$$

and

$$\tilde{S}(L^{\pm i}{}_j) = \begin{cases} \dfrac{q^{-\binom{n}{2}}}{[n-1]_q!} \varepsilon^{k_{n-1}\dots k_1 i} L^{\pm l_1}{}_{k_1} \dots L^{\pm l_{n-1}}{}_{k_{n-1}} \varepsilon_{j l_{n-1}\dots l_1} & \text{for } A_{n-1} \\ (C^{-1})^{ki} L^{\pm l}{}_k C_{jl} & \text{for } B_n, C_n, D_n\,, \end{cases} \tag{3.11}$$

$$\tilde{S}(L^{\pm \bar{i}}{}_{\bar{j}}) = \begin{cases} \dfrac{q^{-\binom{n}{2}}}{[n-1]_q!} \varepsilon^{\bar{k}_{n-1}\dots \bar{k}_1 \bar{i}} L^{\pm \bar{l}_1}{}_{\bar{k}_1} \dots L^{\pm \bar{l}_{n-1}}{}_{\bar{k}_{n-1}} \varepsilon_{\bar{j} \bar{l}_{n-1}\dots \bar{l}_1} & \text{for } A_{n-1} \\ (C^{-1})^{\bar{k}\bar{i}} L^{\pm \bar{l}}{}_{\bar{k}} C_{\bar{j}\bar{l}} & \text{for } B_n, C_n, D_n\,. \end{cases}$$

it Consequently the algebra $U_{\mathscr{R}}$ becomes a Hopf algebra with antipode $S := \tilde{S}_{|U_{\mathscr{R}}}$. And it holds

$$L^{\pm I}{}_J S(L^{\pm I}{}_K) = \delta^I_K e\,. \tag{3.12}$$

Proposition 6. *The involution on the generators of* $U_{\mathscr{R}}$ *is*

$$\overline{L^{\pm J}{}_I} = L^{\pm \bar{I}}{}_{\bar{J}} \tag{3.13}$$

if

$$\overline{\alpha_0} \cdot \alpha_3 = 1 \tag{3.14}$$

and

$$\overline{\alpha_2} \cdot \alpha_1 = 1\,. \tag{3.15}$$

With the involution "†" $U_{\mathscr{R}}$ becomes a $*$-Hopf algebra. Nevertheless the coefficients α_i are not yet completely fixed. For further calculations we introduce the so-called root-of-unity-homomorphisms $e_{r,s}$ which are elements of $\mathscr{A}^*$ and are defined multiplicatively on the generators of $\mathscr{A}$ as follows:

$$\begin{aligned} e_{r,s}(1) &:= 1\,, \\ e_{r,s}(t^a{}_b) &:= e^{2\pi i \cdot r/\theta} \cdot \delta^a_b\,, \\ e_{r,s}(\hat{t}^{\bar{a}}{}_{\bar{b}}) &:= e^{2\pi i \cdot s/\theta} \cdot \delta^{\bar{a}}_{\bar{b}}\,, \end{aligned} \tag{3.16}$$

where $r, s \in \mathbb{Z}$, $\Theta := \begin{cases} n & \text{for } A_{n-1} \\ 2 & \text{for } B_n, C_n, D_n \end{cases}$.

One can easily check the following

Proposition 7. 1. $e_{r,s}$ *is a well defined algebra homomorphism,*

2. $e_{0,0} = (e_{r,s})^{\Theta} = e$,
3. $e_{l,k} \cdot e_{m,n} = e_{l+m,k+n}$,
4. $[e_{r,s}, f] = 0 \ \forall f \in \mathscr{A}^*$,
5. $\overline{e_{r,s}} = e_{s,r}$.

Using the special form of the $\mathscr{R}_q$-matrix and the form of the matrices $\hat{R}_q$ for A_{n-1}, B_n, C_n, or D_n, we get

Proposition 8. 1. $L^{+i}{}_j$ *is upper-triangular,* $L^{+\bar{i}}{}_j$ *is lower-triangular.*

2. $L^{+i}{}_i \cdot L^{+\bar{i}}{}_{\bar{i}} = L^{+\bar{i}}{}_{\bar{i}} \cdot L^{+i}{}_i = e_{l,l}$, *where* $\alpha_0 \cdot \alpha_2 = \alpha_1 \cdot \alpha_3 = e^{2\pi i \cdot l/\Theta}$.
3. $L^{-\bar{i}}{}_{\bar{j}} = L^{-i}{}_j \cdot e_{r,-r}$, *where* $\alpha_0 \cdot {\alpha_1}^{-1} = (\alpha_2 \cdot {\alpha_3}^{-1})^* = e^{2\pi i \cdot r/\Theta}$.
4. $[L^{+i}{}_i, L^{+j}_j] = [L^{+\bar{i}}{}_{\bar{i}}, L^{+\bar{j}}_{\bar{j}}] = 0$.
5. $L^{+1}{}_1 \cdot \ldots \cdot L^{+N}{}_N = L^{+\bar{1}}{}_{\bar{1}} \cdot \ldots \cdot L^{+\bar{N}}{}_{\bar{N}} = e$ *for* A_{n-1}, C_n, D_n *and* $(L^{+1}{}_1 \cdot \ldots \cdot L^{+N}{}_N)^2 = (L^{+\bar{1}}{}_{\bar{1}} \cdot \ldots \cdot L^{+\bar{N}}{}_{\bar{N}})^2 = e$ *for* B_n.

4. The Hopf Algebra $U_q(sl(2,\mathbb{C}))$

To illustrate the above developed formalism we now investigate the easiest example, that is the Hopf algebra $U_q(sl(2,\mathbb{C}))$ with the additional choice $\alpha_0 = \alpha_1$. The other possibility, $\alpha_0 = -\alpha_1$, would provide the additional algebra homomorphism $e_{1,1}$ in $U_{\mathscr{R}}$. We do not consider this case in this paper. As a consequence of these restrictions we get $\alpha_0 \cdot \alpha_2 = \alpha_0 \cdot (\alpha_1)^{-1} = 1$ and thus the equations in Proposition 8 only contain $e_{0,0} = e$. Therefore in the case A_1 we only have to consider the unit e and the generators

$$L^{+1}{}_1, L^{+1}{}_2, L^{+\bar{2}}{}_{\bar{1}}, L^{-1}{}_1, L^{-1}{}_2, L^{-2}{}_1, L^{-2}{}_2, (L^{+1}{}_1)^{-1}\,. \tag{4.1}$$

For further considerations we define the element

$$\Delta := L^{-1}{}_2 \cdot L^{-2}{}_1 \in U_{\mathscr{R}}\,. \tag{4.2}$$

Proposition 9. 1. $\{\Delta^n \,|\, n \in \mathbb{N}^0\}$ *is a linearly independent set in* $\mathscr{A}^*$.
2. $\Delta^n = 0$ *for monomials* $t^{g_1}\hat{t}^{g_2}$ *with* $\min(g_1, g_2) < n$.

Property 2 of Proposition 9 allows us to handle power series in $\mathscr{A}^*$ of the form

$$\begin{aligned} \Lambda^1{}_1 &= L^{-1}{}_1\left(e + \sum_{n=1}^{\infty} \alpha_n \Delta^n\right), \\ \Lambda^2{}_2 &= L^{-2}{}_2\left(e + \sum_{n=1}^{\infty} \beta_n \Delta^n\right), \end{aligned} \tag{4.3}$$

where α_n, β_n are arbitrary complex numbers. Because of this fact, property 1 of Proposition 9 and (3.12) we obtain

Proposition 10. $L^{-1}{}_1$ *is invertible and* $(L^{-1}{}_1)^{-1} = L^{-2}{}_2\left(e + \sum_{n=1}^{\infty} (-q)^{-n}\Delta^n\right)$ *is an element of a certain minimal extension of* $U_{\mathscr{R}}$.

Consequently there remain six essential generators since $L^{-2}{}_2$ can now be expressed through Δ and $(L^{-1}{}_1)^{-1}$. Using (3.5) and (3.12) we get the following algebra relations:

$$\begin{gathered} [L^{-1}{}_2, L^{-2}{}_1] = 0 \qquad [L^{-1}{}_1, L^{+1}{}_1] = 0\,, \qquad [L^{+1}{}_2, L^{-1}{}_2] = [L^{-2}{}_1, L^{+\bar{2}}{}_{\bar{1}}] = 0\,, \\ [L^{-2}{}_1, L^{+1}{}_2] = (q - q^{-1})\{(L^{+1}{}_1)^{-1}L^{-1}{}_1 - (L^{-1}{}_1)^{-1}(e + q^{-1}\Delta)L^{+1}{}_1\}\,, \\ [L^{-1}{}_2, L^{+\bar{2}}{}_{\bar{1}}] = (q - q^{-1})\{L^{-1}{}_1 L^{+1}{}_1 - (L^{+1}{}_1)(L^{-1}{}_1)^{-1}(e + q^{-1}\Delta)\}\,, \\ [L^{+1}{}_2, L^{+\bar{2}}{}_{\bar{1}}] = (q - q^{-1})\{(L^{+1}{}_1)^2 - (L^{+1}{}_1)^{-2}\}\,, \\ L^{\pm 1}{}_1 L^{\pm 1}{}_2 = q^{-1}L^{\pm 1}{}_2 L^{\pm 1}{}_1\,, \;\; L^{-1}{}_1 L^{-2}{}_1 = q^{-1}L^{-2}{}_1 L^{-1}{}_1\,, \;\; L^{-1}{}_2 L^{+1}{}_1 = qL^{+1}{}_1 L^{-1}{}_2\,, \\ L^{-2}{}_1 L^{+1}{}_1 = q^{-1}L^{+1}{}_1 L^{-2}{}_1\,, \qquad L^{+\bar{2}}{}_{\bar{1}} L^{+1}{}_1 = q^{-1}L^{+1}{}_1 L^{+\bar{2}}{}_{\bar{1}}\,, \\ L^{-1}{}_1 L^{+\bar{2}}{}_{\bar{1}} - qL^{+\bar{2}}{}_{\bar{1}} L^{-1}{}_1 = (q^{-1} - q)L^{-2}{}_1(L^{+1}{}_1)^{-1}\,, \\ L^{-1}{}_1 L^{+1}{}_2 - qL^{+1}{}_2 L^{-1}{}_1 = (q^{-1} - q)L^{-1}{}_2 L^{+1}{}_1\,. \end{gathered} \tag{4.4}$$

In the next step we make an ansatz similar to [FRT] with H_i, $X_i^{\pm}$; $i = 1, 2$. We set

$$\begin{aligned} &L^{+1}{}_1 = \exp(h/2H_1)\,, \quad L^{-1}{}_1 = \exp(h/2H_2)\,, \quad L^{+1}{}_2 = -(q - q^{-1})X_1^-\,, \\ &L^{+\bar{2}}{}_{\bar{1}} = (q - q^{-1})X_1^+\,, \quad L^{-2}{}_1 = (q - q^{-1})X_2^-\,, \quad L^{-1}{}_2 = -(q - q^{-1})X_2^+\,, \end{aligned} \tag{4.5}$$

where $q = e^h$.

The equations (4.4) and (4.5) yield the following algebra relations:

$$
\begin{aligned}
&[H_1, H_2] = [X_1^{\pm}, X_2^{\mp}] = [X_2^+, X_2^-] = 0\,, \qquad [H_1, X_1^{\pm}] = \pm 2X_1^{\pm}\,,\\
&[H_1, X_2^{\pm}] = \mp 2X_2^{\pm}\,, \qquad [H_2, X_2^{\pm}] = -\,2X_2^{\pm}\,,\\
&[H_2, X_1^{\pm}] = 2X_1^{\pm} - 4X_2^{\mp}\exp(\mp h/2(H_1 \pm H_2))\,,\\
&[X_1^+, X_1^-] = \frac{\exp(hH_1) - \exp(-hH_1)}{(q-q^{-1})}\,,\\
&[X_1^{\pm}, X_2^{\pm}] = \frac{\exp(\pm h/2(H_1 \pm H_2)) - \exp(\mp h/2(H_1 \pm H_2))}{(q-q^{-1})}\\
&\qquad + (1-q^2)\exp(\mp h/2(H_1 \pm H_2))X_2^+X_2^-\,.
\end{aligned}
\tag{4.6}
$$

Coming from H_i, $X_i^{\pm}$ one can argue that this algebra is a certain completion of $U_{\mathscr{R}}$ and a $*$-Hopf algebra with coproduct Δ,

$$
\begin{aligned}
&\Delta(X_1^{\pm}) = X_1^{\pm} \otimes \exp(-h/2H_1) + \exp(h/2H_1) \otimes X_1^{\pm}\,,\\
&\Delta(X_2^{+}) = X_2^{+} \otimes \exp(-h/2H_2)\,(e - q^{-2}\mathscr{D}) + \exp(h/2H_2) \otimes X_2^{+}\,,\\
&\Delta(X_2^{-}) = X_2^{-} \otimes \exp(h/2H_2) + \exp(-h/2H_2)\,(e - q^{-2}\mathscr{D}) \otimes X_2^{-}\,,\\
&\Delta(H_1) = H_1 \otimes e + e \otimes H_1\,,\\
&\Delta(H_2) = \frac{2}{h}\sum_{k=1}^{\infty}\frac{1}{k}(-1)^{k-1}(\exp(h/2(H_2 \otimes e + e \otimes H_2))\\
&\qquad - (q-q^{-1})^2 X_2^+ \otimes X_2^- + -e \otimes e)^k\,,
\end{aligned}
\tag{4.7}
$$

antipode S

$$
\begin{aligned}
&S(X_1^{\pm}) = -\exp(-h/2H_1)X_1^{\pm}\exp(h/2H_1)\,,\\
&S(X_2^{\pm}) = -\exp(\mp h/2H_2)X_2^{\pm}\exp(\pm h/2H_2)\,,\\
&S(H_1) = -\,H_1\,,\\
&S(H_2) = \frac{2}{h}\sum_{k=1}^{\infty}\frac{1}{k}(-1)^{k-1}(\exp(-h/2H_2)\,(e - q^{-2}\mathscr{D}) - e)^k\,,
\end{aligned}
\tag{4.8}
$$

and counit ε

$$
\varepsilon(H_1) = \varepsilon(H_2) = 0\,, \qquad \varepsilon(X_1^{\pm}) = \varepsilon(X_2^{\pm}) = 0\,, \tag{4.9}
$$

where $\mathscr{D} := q(q-q^{-1})^2X_2^+X_2^-$.

As a formal power series in h the generators H_1 and H_2 are well defined and unique [Ogi].

5. The Limit $q \to 1$

We investigate the limit $q \to 1$ for the algebra relations (4.6). A short computation yields

$$
\begin{aligned}
&[H_1, H_2] = [X_1^{\pm}, X_2^{\mp}] = [X_2^+, X_2^-] = 0\,, \qquad [H_1, X_1^{\pm}] = \pm 2X_1^{\pm}\,,\\
&[H_1, X_2^{\pm}] = \mp 2X_2^{\pm}\,, \qquad [H_2, X_2^{\pm}] = -\,2X_2^{\pm}\,,\\
&[H_2, X_1^{\pm}] = 2X_1^{\pm} - 4X_2^{\mp}\,, \qquad [X_1^+, X_1^-] = H_1\,,\\
&[X_2^-, X_1^-] = 1/2(H_1 - H_2)\,, \qquad [X_1^+, X_2^+] = 1/2(H_1 + H_2)\,.
\end{aligned}
\tag{5.1}
$$

To recover the usual $U(sl(2,\mathbb{C}))$-structure, we transform the Lie algebra (5.1) linearly

$$\begin{aligned} \hat{H}_1 &:= 1/2(H_1 - H_2), \quad & \hat{H}_2 &:= 1/2(H_1 + H_2), \quad & \hat{X}_1^+ &:= X_2^-, \\ \hat{X}_1^- &:= (X_1^- - X_2^+), \quad & \hat{X}_2^+ &:= (X_1^+ - X_2^-), \quad & \hat{X}_2^- &:= X_2^+. \end{aligned} \tag{5.2}$$

With (5.1) and (5.2) we get the relations

$$\begin{aligned} &[\hat{H}_i, \hat{X}_i^\pm] = \pm 2\hat{X}_i^\pm, \quad [\hat{X}_i^+, \hat{X}_i^-] = \hat{H}_i, \quad [\hat{H}_1, \hat{H}_2] = 0, \\ &[\hat{H}_1, \hat{X}_2^\pm] = [\hat{H}_2, \hat{X}_1^\pm] = 0, \quad [\hat{X}_1^\pm, \hat{X}_2^\pm] = [\hat{X}_1^\pm, \hat{X}_2^\mp] = 0 \end{aligned} \tag{5.3}$$

and the involution

$$\hat{H}_1^\dagger = \hat{H}_2, \quad (\hat{X}_1^\pm)^\dagger = \hat{X}_2^\mp. \tag{5.4}$$

Considering comultiplication and antipode in this limit one recovers the universal enveloping algebra of $sl(2,\mathbb{C})$.

6. Concluding Remarks

Apart from our work there are three further papers which deal with the same object [SWZ, OSWZ, CW] and, closely related, with the q-deformed Poincaré algebra [LNRT]. In [SWZ, OSWZ] a q-deformed version of the Lorentz algebra is derived via linear representations of the algebra on the complex spinor quantum plane. This yields a 7-generator algebra with additional parameter [SWZ]. This algebra can be found in the enveloping algebra of a 6-generator formulation of $U_q(sl(2,\mathbb{C}))$ [OSWZ].

Using the algebra $U_{\mathscr{R}}$ a differential calculus is developed in [CW] (see footnote in Sect. (1)). This algebra of differential operators is another formulation of the algebra $U_q(sl(2,\mathbb{C}))$.

Another approach uses the q-generalization of the adjoint representation of Lie groups to derive the analogous of the linear functionals in [Wor, CSWW, Jur] which correspond to the left invariant vector fields on the Lie group in the limit $q \longrightarrow 1$. This is now under investigation [CDSWZ].

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References

[Abe] Abe, E.: Hopf algebras. Cambridge tracts in Mathematics, Vol. 74. Cambridge: Cambridge University Press (1980)

[CDSWZ] Chryssomalakos, C., Drabant, B., Schlieker, M., Weich, W., Zumino, B.: Vector fields on complex quantum groups. Commun. Math. Phys. **147**, 635–646 (1992)

[CSSW] Carow-Watamura, U. Schlieker, M., Scholl, M., Watamura, S.: Tensor representation of the quantum group $SL_q(2,\mathbb{C})$ and quantum Minkowski space. Z. Phys. C **48**, 159–165 (1990);
Carow-Watamura, U. Schlieker, M., Scholl, M., Watamura, S.: A quantum Lorentz group. Int. J. Mod. Phys. A **6**, No. 17, 3081–3108 (1991)

[CSWW] Carow-Watamura, U. Schlieker, M., Watamura, S., Weich, W.: Bicovariant differential calculus on quamtum groups $SU_q(N)$ and $SO_q(N)$. Commun. Math. Phys. **142**, 605–641 (1991)

[CW] Carow-Watamura, U. Watamura, S.: Tokhu preprint (1991)

[Dri] Drinfel'd V.G.: Quantum groups. In: Proceedings of the International Congress of Mathematicians, Berkeley 1986, pp. 798–820. Berkeley 1986

[FRT] Faddeev, L.D., Reshetikhin, N.Yu., Takhtajan, L.A.: Quantization of Lie groups and Lie algebras. In: Algebra Analysis **1**, 178 (1987)
[Jur] Jurčo, B.: Differential calculus on quantized simple Lie groups. Lett. math. Phys. **22**, 177 (1991)
[LNRT] Lukierski, J., Nowicki, A., Ruegg, H., Tolstoy, V.N.: Geneva preprints VGVA-DPT 1991/02-710, VGVA-DPT 1991/08-740 to appear in Phys. Lett. B (1991)
[Ogi] Ogievetsky, O.: Private communication
[OSWZ] Ogievetsky, O., Schmidke, W.B., Wess, J., Zumino, B.: Six generator q-deformed Lorentz algebra. MPI/Ph/91-51 (1991)
[Pod] Podlés, P.: Complex quantum groups. Preprint RIMS 754 (1991)
[PW] Podlés, P., Woronowicz, S.L.: Quantum deformation of Lorentz group. Commun. Math. Phys. **130**, 381–431 (1990)
[SWZ] Schmidke, W.B., Wess, J., Zumino, B.: A q-deformed Lorentz algebra. Preprint MPI-Ph/91-15 (1991)
[Wor] Woronowicz, S.L.: Differential calculus on compact matrix pseudogroups (quantum groups). Commun. Math. Phys. **122**, 125–170 (1989)

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