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$(T^*G)_t$: A Toy Model for Conformal Field Theory*

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Abstract. We study a chiral operator algebra of conformal field theory and quantum deformation of the finite-dimensional Lie group to obtain the definition of $(T^*G)_t$ and its representation.

The closeness of the Kač-Moody algebras, constituting the chiral operator algebra of a typical (and generic) conformal field theory model, namely the WZNW model, and quantum deformation of corresponding finite-dimensional Lie group G has become more and more evident in recent years [1–5]. This in particular prompts further investigation of the differential geometry of such deformations. The notion of tangent and cotangent bundles is basic in classical differential geometry. It is only natural that the quantum deformations of TG and T^*G are to be introduced alongside those for G itself. Physical ideas could be useful for this goal.

Indeed, the T^*G can be interpreted as a phase space for a kind of a top, generalizing the usual top associated with $G=SO(3)$. The classical mechanics is a natural language to describe differential geometry, whereas the usual quantization is nothing but the representation theory.

In this paper we put corresponding formulas in such a fashion that their deformation becomes almost evident, given the experience in this domain. As a result we get the definition of $(T^*G)_t$ and its representation (t is the deformation parameter).

To make the exposition most simple and formulas transparent we shall work on an example of $G=sl(2)$ and present results in such a way that the generalizations become evident. We shall stick to generic complex versions, real and especially compact forms requiring some additional consideration, not all of which are self-evident.

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I. Classical Differential Geometry

Let G be $sl(2, \mathbf{C})$ and g be a generic point in G. A point in the phase space T^*G is given by $m=(g, \omega)$, where ω belongs to the space $\mathcal{G}^*$ dual to the Lie algebra $\mathcal{G}$. The one-form

$$\alpha = \langle \omega | dg\, g^{-1} \rangle ,$$

where $\langle \cdot | \cdot \rangle$ is a pairing of $\mathcal{G}^*$ and $\mathcal{G}$, defines the symplectic structure on T^*G. To write down the Poisson structure one can use coordinates ω^a, $a=1,2,3$ in $\mathcal{G}^*$, corresponding to a chosen basis t^a in $\mathcal{G}$ with the structure constants f_c^{ab},

$$[t^a, t^b] = f_c^{ab} t^c .$$

We have non-vanishing brackets

$$\{\omega^a, g\} = t^a g , \qquad \{\omega^a, \omega^b\} = -f_c^{ab} \omega^c ,$$

where $t^a g$ is interpreted as a product of infinitesimal group element t^a and g.

In the following it will be convenient to use the two-dimensional representation for g and t^a. We can choose t^a as being proportional to the Pauli matrices

$$t^a = \tfrac{1}{2} \sigma^a ,$$

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and interpret $\langle \,|\, \rangle$ as a trace. Then the dual basis in $\mathcal{G}^*$ is given by

$$t_a = \sigma^a .$$

Introducing the matrix

$$\omega = \omega^a t_a ,$$

we can rewrite the Poisson brackets in the convenient form

$$\begin{aligned} &\{g^1, g^2\} = 0 , \\ &\{\omega^1, g^2\} = C g^2 , \\ &\{\omega^1, \omega^2\} = -\tfrac{1}{2} [C, \omega^1 - \omega^2] , \end{aligned} \tag{1}$$

where g^i, ω^i, $i=1,2$ are considered as matrices in $\mathbf{C}^2 \otimes \mathbf{C}^2$ defined as

$$\begin{aligned} g^1 &= g \otimes I ; \qquad g^2 = I \otimes g ; \\ \omega^1 &= \omega \otimes I ; \qquad \omega^2 = I \otimes \omega ; \end{aligned}$$

and C is a matrix Casimir operator

$$C = t^a \otimes t_a .$$

Alternatively we could use the "right" choice of variables. We denote for the time being

$$\omega = \omega_L$$

and introduce

$$\omega_R = g^{-1} \omega_L g .$$

Then we have

$$\{\omega_R^1, g^2\} = g^2 C,$$
$$\{\omega_R^1, \omega_R^2\} = \tfrac{1}{2}[C, \omega_R^1 - \omega_R^2].$$

Moreover, ω_L and ω_R commute

$$\{\omega_L^1, \omega_R^2\} = 0.$$

Now we introduce a change of variables (g, ω) which will be used to describe and justify the deformation in the next section. We begin with diagonalization of ω_L and ω_R. Let

$$\omega_L = uPu^{-1}, \quad \omega_R = v^{-1}Pv,$$

where P is a diagonal matrix and u, v are elements from G/H, where H is a Cartan subgroup associated with P. It is easy to see that

$$g = uQv,$$

where Q belongs to H. It is defined uniquely as soon as a concrete choice is made for the representatives u_0 and v_0 for u and v.

In this way we get the full number (namely six) of new variables: one in P, one in Q and two pairs in u_0 and v_0.

Making a change of variables

$$(g, \omega) \rightarrow (u_0, v_0, P, Q)$$

in one-form α leads to the Poisson brackets for the new variables. They can be easily calculated using some particular parametrization, e.g. Euler angles for u_0 and v_0,

$$u_0 = \begin{pmatrix} e^{i\alpha}\cos\beta & e^{i\alpha}\sin\beta \\ -e^{-i\alpha}\sin\beta & e^{-i\alpha}\cos\beta \end{pmatrix}$$
$$v_0 = \begin{pmatrix} \cos\gamma e^{i\delta} & \sin\gamma e^{-i\delta} \\ -\sin\gamma e^{i\delta} & \cos\gamma e^{-i\delta} \end{pmatrix}$$

and

$$P = -\frac{i}{2}\begin{pmatrix} p & 0 \\ 0 & -p \end{pmatrix}, \quad Q = \begin{pmatrix} e^{iq} & 0 \\ 0 & e^{-iq} \end{pmatrix}$$

for P and Q, in terms of which α acquires the form

$$\alpha = pdq + p(\cos 2\beta d\alpha + \cos 2\gamma d\delta)$$

so that (p, q), $(m = p\cos 2\beta, \alpha)$ and $(n = p\cos 2\gamma, \delta)$ are canonical pairs.

The brackets look most interesting in terms of the combinations

$$u = u_0 Q, \quad v = Q v_0$$

through which g is given by

$$g = uQ^{-1}v.$$

We have

$$\{u^1, u^2\} = u^1 u^2 r_0(p), \quad \{v^1, v^2\} = -r_0(p) v^1 v^2 \tag{2}$$

and

$$\{u_0^1, v_0^2\} = 0, \quad \{p, q\} = 1, \quad \{u_0, p\} = \{v_0, p\} = 0. \tag{3}$$

The brackets of u_0, v_0 with q are less transparent and we shall not need them in the following.

Here $r_0(p)$ is 4×4 matrix given by

$$r_0(p) = -\frac{i}{p}(\sigma_+ \otimes \sigma_- - \sigma_- \otimes \sigma_+),$$

$\sigma_\pm = (\sigma_1 \pm i\sigma_2)/2$ or more explicitly

$$r_0(p) = -\frac{i}{p}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The main feature of these formulas, namely the appearance of the quadratic Poisson algebra for u and v, is unexpected but quite welcome. Indeed it can be used as a point of departure for quantization and deformation.

II. t-Deformation

The existing experience of work with the quadratic Poisson algebras gives a natural proposal for t-deformation of the last formulas in Sect. I. We are compelled to use the letter t instead of the usual q which is already used for the coordinate conjugate to p. It is enough to write down consistent quadratic relations and check that they turn into (2) in the limit $t \to 1$. We shall follow this tactic and propose formulas in the already-quantized form.

Consider the quadratic relations

$$\begin{aligned} R_t u^1 u^2 &= u^2 u^1 R_t(p), \\ v^1 v^2 R_t &= R_t(p) v^2 v^1 \end{aligned} \tag{4}$$

for the operator-valued matrices u and v. Here R_t is the usual $sl(2)$ R-matrix

$$R_t = \begin{pmatrix} t^{1/2} & 0 & 0 & 0 \\ 0 & t^{-1/2} & 0 & 0 \\ 0 & t^{1/2} - t^{-3/2} & t^{-1/2} & 0 \\ 0 & 0 & 0 & t^{1/2} \end{pmatrix},$$

and $R_t(p)$ is the corresponding $6j$ symbol

$$R_t(p) = \begin{pmatrix} t^{1/2} & 0 & 0 & 0 \\ 0 & t^{-1/2}\sqrt{1 - \left(\dfrac{t - t^{-1}}{e^{ip} - e^{-ip}}\right)^2} & \dfrac{t^{1/2} - t^{-3/2}}{1 - e^{2ip}} & 0 \\ 0 & \dfrac{t^{1/2} - t^{-3/2}}{1 - e^{-2ip}} & t^{-1/2}\sqrt{1 - \left(\dfrac{t - t^{-1}}{e^{ip} - e^{-ip}}\right)^2} & 0 \\ 0 & 0 & 0 & t^{1/2} \end{pmatrix}$$

(see e.g. [5]).

The self-consistency of these relations follows from the Yang-Baxter relation for R_t,

$$R_t^{12}R_t^{13}R_t^{23}=R_t^{23}R_t^{13}R_t^{12}$$

and its generalization for $R_t(p)$,

$$(Q^1)^{-1}R^{23}(p)Q^1R^{13}(p)(Q^3)^{-1}R^{12}(p)Q^3=R^{12}(p)(Q^2)^{-1}R^{13}(p)Q^2R^{23}(p).$$

Here the matrix Q,

$$Q=\begin{pmatrix} e^{iq} & 0 \\ 0 & e^{-iq} \end{pmatrix}, \tag{5}$$

consists of the shift operators for variable p, i.e.

$$e^{-iq}f(p)e^{iq}=f\left(p+\frac{1}{i}\ln t\right).$$

The relation for $R_t(p)$ was first introduced in [6].

The classical relations of Sect. I follow from (4) in the contraction limit. If we write

$$t=e^{i\gamma\hbar},$$

where $\hbar$ is the ordinary Planck constant and γ measures the t-deformation, and take into account that

$$\{\cdot\,,\cdot\}=\lim_{\hbar\to 0} i\frac{[\cdot\,,\cdot]}{\hbar},$$

we shall get (2) in the limit $\gamma=0$ and $\hbar=0$, if we renormalize p as

$$p\to\gamma p.$$

There are also two intermediate cases:

1. $\hbar=0$, $\gamma\neq 0$, namely a classical t-deformed top; and
2. $\gamma=0$, $\hbar\neq 0$, namely the quantum version of a top, or the regular representation of the group G.

We shall comment on them in the Discussion.

Other relations of Sect. I, namely (3) are left unchanged, in particular

$$u_0^1v_0^2=v_0^2u_0^1,$$

where

$$u_0=uQ^{-1}, \quad v_0=Q^{-1}v,$$

and Q is given now in (5).

Now we can introduce the analogues of the original variables g and ω. As always after deformation, the Lie-algebraic variable ω is to be substituted by the Lie-group like one. So we introduce

$$g=uQ^{-1}v$$

and

$$\Omega=uDu^{-1}, \tag{6}$$

where

$$D=\begin{pmatrix} e^{ip} & 0 \\ 0 & e^{-ip} \end{pmatrix}.$$

The classical variable ω appears in the limit

$$\Omega=1-2\gamma\omega+\dots.$$

Let us find the relations for g and Ω.

Beginning with the relations for g we have a chain of commutations:

$$\begin{aligned} g^1g^2&=u^1(Q^1)^{-1}v^1u^2(Q^2)^{-1}v^2 \\ &=u^1v_0^1u_0^2v^2=u^1u_0^2v_0^1v^2 \\ &=u^1u^2(Q^1Q^2)^{-1}v^1v^2 \\ &=R_t^{-1}u^2u^1R_t(p)(Q^1Q^2)^{-1}R_t(p)^{-1}v^2v^1R_t. \end{aligned}$$

Now from the explicit form of Q^1Q^2

$$Q^2Q^2=\begin{pmatrix} e^{2iq} & & & \\ & 1 & & \\ & & 1 & \\ & & & e^{-2iq} \end{pmatrix}=Q^2Q^1,$$

it follows that $R_t(p)$ commutes with Q^1Q^2, so that $R_t(p)$ cancels in the last line. We continue

$$\begin{aligned} R_tg^1g^2&=u^2u^2(Q^1)^{-1}(Q^2)^{-1}v^2v^1R_t \\ &=u^2u_0^1v_0^2v^1R_t=u^2v_0^2u_0^1v^1R_t \\ &=g^2g^1R_t. \end{aligned}$$

Thus we get the defining relation of the quantum group

$$R_tg^1g^2=g^2g^1R_t. \tag{7}$$

Let us come now to the relation between g and Ω. We have

$$\begin{aligned} g^1\Omega^2&=u^1v_0^1u^2D^2(u^2)^{-1}=u^1u^2D^2(u^2)^{-1}v_0^2 \\ &=R_t^{-1}u^2u^1R_t(p)D^2(u^2)^{-1}v_0^1. \end{aligned}$$

It follows from the relation between p and D that

$$D^2u^1=u^1D^2\sigma,$$

where

$$\sigma=t^{\sigma_3\otimes\sigma_3}=\begin{pmatrix} t & & & \\ & t^{-1} & & \\ & & t^{-1} & \\ & & & t \end{pmatrix}.$$

Now the matrix

$$\tilde{R}_t(p)=(D^2)^{-1}\sigma^{-1}R_t(p)D^2$$

has essentially the same properties as $R_t(p)$. More explicitly

$$\tilde{R}_t(p) = PR_t(p)^{-1}P$$

which can be checked directly. The relation (4) can be rewritten as

$$\tilde{R}_t u^1 u^2 = u^2 u^1 \tilde{R}_t(p)$$

or

$$(u^2)^{-1}\tilde{R}_t u^1 = u^1 \tilde{R}_t(p)(u^2)^{-1},$$

where

$$\tilde{R}_t = PR_t^{-1}P$$

so that we can continue

$$\begin{aligned} R_t g^1 \Omega^2 &= u^2 D^2 u^1 \tilde{R}_t(p)(u^2)^{-1} v_0^1 \\ &= u^2 D^2 (u^2)^{-1} \tilde{R}_t u^1 v_0^1 \\ &= \Omega^2 \tilde{R}_t g^1 . \end{aligned}$$

Once more the $6j$ symbols disappear in the final formula.

With more symmetric notations

$$R_t^+ = R_t; \qquad R_t^- = \tilde{R}_t$$

we have the relation

$$g^1 \Omega^2 = (R_t^+)^{-1} \Omega^2 R_t^- g^1 . \tag{8}$$

The relation between Ω's can be calculated in a similar way and we present the final result

$$R_t^+ \Omega^1 (R_t^+)^{-1} \Omega^2 = \Omega^2 R_t^- \Omega_t^1 (R_t^-)^{-1} . \tag{9}$$

The formulas (7), (8), and (9) are defining formulas of the relations for the generators of the algebra $\mathrm{Fun}(T^*G)_t$ of functions over the deformed phase space $(T^*G)_t$. It is instructive to check how the classical relations (1) follow in the contraction limit. To this end one is to observe that the Casimir C can be written as

$$C = r^+ - r^- ,$$

where

$$R_t^{\pm} = 1 + i\hbar\gamma r^{\pm} + \dots .$$

The final formulas are quite general and can be used to define $(T^*G)_t$ for any group G, for which its t-deformed analogue G_t is known.

III. Discussion

1. Relation to Conformal Field Theory

One can immediately see the analogy of the formulas (6) and the corresponding chiral decomposition of local field and monodromy matrix in the WZNW model (see e.g. [7]). This is why we call (T^*G) a toy model for conformal field theory. More seriously, let us mention that as was shown recently [8] the value of the local field $g(x)$ at some point, say $x=0$ and monodromy Ω of its left chiral component $u(x)$

through the whole period $(0, 2\pi)$ comprise a $(T^*G)_t$ pair for

$$t = \exp\left\{\frac{i\pi}{l+\hbar}\right\},$$

where l is level and h the Coxeter number of the group in question. In particular, the spectrum of monodromy, defining the eigenvalues of the zero-mode momentum (or spin), can be read off from the finite-dimensional quantum-mechanical problem, associated with this top.

2. *Quantum Non-Deformed Case*

The limit $\gamma = 0$ with $\hbar \neq 0$ corresponds to the theory of representations of the group G and matrix u satisfies the contracted relation

$$u^1 u^2 = u^2 u^1 R_0(p),$$

where $R_0(p)$ now is given by

$$R_0(p) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \dfrac{\sqrt{(p+\hbar)(p-\hbar)}}{p} & -\dfrac{\hbar}{p} & 0 \\ 0 & \dfrac{\hbar}{p} & \dfrac{\sqrt{(p+\hbar)(p-\hbar)}}{p} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Naturally $R_0(p)$ and the matrix elements u_{ik} of u are to have some interpretation in terms of the theory of representations. We shall describe this interpretation in the case of unitary reduction, which is given by the conditions

$$p = -p^*, \quad u_{11} = u_{22}^*; \quad u_{12} = -u_{21}^*.$$

The full Hilbert space is given by

$$\mathscr{H} = \sum_{\mathscr{J}} \oplus (H_{\mathscr{J}} \otimes H_{\mathscr{J}}),$$

but u and ω_L act non-trivially only in the model space

$$\mathscr{H}_M = \sum_{\mathscr{J}} \oplus H_{\mathscr{J}},$$

ω giving an irreducible representation in each $H_{\mathscr{J}}$.

It is easy to see that the relation between u and ω

$$[\omega_L, u] = i\hbar C\omega$$

allows one to interpret the matrix element $\langle \mathscr{J}', m | u_{ij} | \mathscr{J}, m\rangle$ as a $3j$ symbol

$$\langle \mathscr{J}', m' | u_{ij} | \mathscr{J}, m\rangle = \begin{Bmatrix} \mathscr{J} & 1/2 & \mathscr{J}' \\ m & i & m' \end{Bmatrix},$$

$$\mathscr{J}' - \mathscr{J} = j, \quad j = \pm 1/2,$$
$$m' - m = i, \quad i = \pm 1/2,$$

and then $R_0(p)$ is nothing but the corresponding $6j$ symbol.

3. *Deformed Classical Case*

The contraction limit $\hbar=0$, $\gamma \neq 0$ of (7–9) gives the non-degenerate Poisson structure on the manifold of pairs (g, Ω)

$$\{g^1, g^2\} = \gamma[r^+, g^1 g^2],$$
$$\{g^1, \Omega^2\} = \gamma(r^+ g^1 \Omega^2 - \Omega^2 r^- g^1),$$
$$\{\Omega^1, \Omega^2\} = \gamma(\Omega^1 \Omega^2 r^+ + r^- \Omega^1 \Omega^2 - \Omega^2 r^+ \Omega^1 - \Omega^2 r^- \Omega^1).$$

It will be interesting to invert these relations to obtain a symplectic form. This would enable us to introduce a Lagrangian for the deformed top.

4. *Triangular Decomposition*

It was convenient to use triangular matrices in the description of the deformed Lie-algebra [9]. A similar trick can be used for T^*G. Namely, instead of Ω one can consider a pair of triangular matrices Ω_+, Ω_- with the relations

$$\begin{aligned} R\Omega_+^1 \Omega_+^2 &= \Omega_+^2 \Omega_+^1 R, \\ R\Omega_+^1 \Omega_-^2 &= \Omega_-^2 \Omega_+^1 R, \\ R\Omega_-^1 \Omega_-^2 &= \Omega_-^2 \Omega_-^1 R, \end{aligned} \tag{10}$$

and

$$R^{\pm} g^1 \Omega_{\pm}^2 = \Omega_{\pm}^2 g^1 .$$

It is easy to show that

$$\Omega = \Omega_+ (\Omega_-)^{-1}$$

satisfy the relations (9). The reduction of (9) to (10) was already done [10].

One can also decompose g in a similar fashion (see [11]). The pair g_+, Ω_- can then be considered as a deformed cotangent bundle for the Borel subgroup $(T^*B)_t$. This object is very tightly connected with Kač-Moody algebra, as was shown recently [8].

5.

One cannot help feeling that the language and technique developed in this article will be useful in the description of Tannaka-Krein duality. We plan to return to this purely mathematical question in the future.

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References

1. Knizhnik, V.G., Zamolodchikov, A.B.: Nucl. Phys. B **247** (1984)
2. Kohno, T.: Ann. Inst. Fourier (Grenoble) **37**, 4 (1987)
3. Tsuchiya, A., Kanie, Y.: Vertex operators in conformal field theory on P^1 and monodromy representations of the braid group. In: Conformal field theory and solvable lattice models. Adv. Stud. Pure Math. **16** (1987)
4. Alekseev, A., Shatashvili, S.: Commun. Math. Phys. **133** (1990)
5. Faddeev, L.D.: Commun. Math. Phys. **132** (1990)
6. Gervais, J.-L., Neveu, A.: Nucl. Phys. B **238** (1984)
7. Faddeev, L.D.: In: Fields and particles. Proceedings of the XXIX Winter School in Nuclear Physics. Mitter, H., Schweiger, W. (eds.). Schladming, Austria, March 1990. Berlin, Heidelberg, New York: Springer 1990
8. Alekseev, A., Faddeev, L.D., Semenov-Tian-Shansky, M., Volkov, A.: The Unravelling of the Quantum Group Structure in the WZNW Theory. CERN preprint TH 5981/91 (1991)
9. Faddeev, L.D., Reshetikhin, N.Yu., Takhtajan, L.A.: Algebra and Analysis **1** (1989) (in Russian)
10. Reshetikhin, N.Yu., Semenov-Tian-Shansky, M.A.: Lett. Math. Phys. **19** (1990)
11. Faddeev, L.D., Reshetikhin, N.Yu., Takhtajan, L.A.: In: Braid group, knot theory and statistical mechanics. Yang, C.N., Ge, M.L. (eds.). Singapore: World Scientific 1989

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