Commun. Math. Phys. 140, 399–414 (1991)

Communications in Mathematical Physics

Quantum Group A_∞

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Received January 15, 1991

Abstract. The quantum groups gl_∞ and A_∞ are construted. The representation theory of these algebras is developed and the universal R-matrix is presented.

0.1. The Lie algebra gl_∞ and its extension A_∞ play an important role in the theory of nonlinear equations [DJKM]. They are of interest as an example of Kac-Moody-Lie algebras of infinite type [K, FF]. Therefore it is natural to ask: what are the quantum analogues of these algebras in the sense of the quantum groups theory of Drinfeld [D1]? The answer is trivial for $gl_\infty = \varinjlim_n gl_n$, but this is not the case for A_∞. Some non-triviality is due tot the fact that there is no Lie algebra gl_∞ in the quantum group case [we have the quantized universal enveloping algebra $U_h(gl_\infty)$ only]. Hence one must analyse the completion of gl_∞ and the central extension of the corresponding algebra $\overline{gl}_\infty$ in terms of $U_h(gl_\infty)$ only. Moreover we need the Hopf Algebra structure in $U_h(A_\infty)$. This is essential in the case $h = 0$ already, because, for example, the well-known KP hierarchy is related to the equations for the orbit of highest vector in $L(\Lambda_0) \otimes L(\Lambda_0)$ where $L(\Lambda_0)$ is the basic representation of A_∞ [K, Chap. 14]. For the same reason we want to obtain $U_h(A_\infty)$ as the quasitriangular topological Hopf algebra [D1].

The purpose of the paper is to construct $U_h(gl_\infty)$ and $U_h(A_\infty)$ as quasitriangular topological Hopf algebras and investigate the representation theory of these algebras. Some results along this lines have been obtained by Hayashi in [H]. Note that there are no constructions of $U_h(gl_\infty)$ and $U_h(A_\infty)$ as quantum groups in his paper.

0.2. Let us describe the contents. In Sect. 1 we construct the Hopf algebra $U_h(gl_\infty)$. This is the quantum analogue of gl_∞. The representations of $U_h(gl_\infty)$ in the spaces of sequences and (quantum) semi-infinite forms are given in Sect. 2. The Hopf algebra $U_h(A_\infty)$ (and some related algebras) is constructed in Sect. 3. This construction is more complicated than in the non-quantum case [K]. The representation theory of $U_h(A_\infty)$ is presented in Sect. 4. Our class of representations is the same as in [FF, Chap. 3]. For example, we construct the representations in the space of quantum semifinite forms and in the space of the usual semifinite forms. The vertex operators for $U_h(A_\infty)$ is constructed also. In the last section, Sect. 5 we construct the universal quantum R-matrix for $U_h(A_\infty)$ and the related quantum analogue of Casimir operator [D2].

0.3. Concluding remarks. We deal with algebras and modules over formal power series $\mathbb{C}[[h]]$. It is easy to see that all the results of Sect. 1–4 remain true for fixed $h \notin \pi i \mathbb{Q}$.

We can't construct the embedding of Hopf algebras $U_h(A_n^{(1)}) \to U_h(A_\infty)$. This differs strikingly from the case $h = 0$. Still, this embedding exists in certain representation space (cf. [H, Sect. 6]).

0.4. We wish to express our thanks to V. Drinfeld for useful discussions.

1. The P.B.W. Basic for $U_h(gl_\infty)$

1.1. Definition. Let $\mathbb{C}[[h]]$ be the ring of formal power series in h. $U_h(gl_\infty)$ denotes the Hopf algebra, which is a topologically free module over $\mathbb{C}[[h]]$ (complete in h-adic topology), with generators $\{X_{i,i+1}, X_{i+1,i}, E_{ii}\}_{i\in\mathbb{Z}}$ and fundamental relations

$$[E_{ii}, E_{jj}] = 0, \tag{1.1}$$

$$\begin{aligned}[E_{ii}, X_{j,j+1}] &= (\delta_{ij} - \delta_{i,j+1})\, X_{j,j+1},\\ [E_{ii}, X_{j+1,j}] &= (-\delta_{ij} + \delta_{i,j+1})\, X_{j+1,j},\end{aligned} \tag{1.2}$$

$$[X_{i,i+1}, X_{j+1,j}] = \delta_{ij} \frac{q^{H_{i,i+1}} - q^{-H_{i,i+1}}}{q - q^{-1}}, \tag{1.3}$$

where $H_{ij} = E_{ii} - E_{jj}$, $q = \exp(h/2)$

$$\begin{gathered}[X_{i,i+1}, X_{j,j+1}] = 0, \quad |i-j| > 1,\\ X_{i,i+1}^2 X_{j,j+1} - (q^2 + 1 + q^{-2})\, X_{i,i+1} X_{j,j+1} X_{i,i+1} + X_{j,j+1} X_{i,i+1}^2 = 0,\\ |i-j| = 1,\end{gathered} \tag{1.4}$$

the formulae (1.4) with pairs of indices $(i+1, i)$, $(j+1, j)$ substituted for pairs $(i, i+1)$, $(j, j+1)$. (1.5)

The coproduct map is defined on generators by

$$\begin{aligned}\Delta E_{ii} &= E_{ii} \otimes 1 + 1 \otimes E_{ii},\\ \Delta X_{i,i+1} &= X_{i,i+1} \otimes q^{+H_{i,i+1}/2} + q^{-H_{i,i+1}/2} \otimes X_{i,i+1},\\ \Delta X_{i+1,i} &= X_{i+1,i} \otimes q^{+H_{i,i+1}/2} + q^{-H_{i,i+1}/2} \otimes X_{i+1,i},\end{aligned} \tag{1.6}$$

and the counit ε and the antipode S are defined by

$$\begin{gathered}\varepsilon(E_{ii}) = \varepsilon(X_{i,i+1}) = \varepsilon(X_{i+1,i}) = 0,\\ S(E_{ii}) = -E_{ii}, \quad S(X_{i,i+1}) = -q X_{i,i+1}, \quad S(X_{i+1,i}) = -q^{-1} X_{i+1,i}.\end{gathered} \tag{1.7}$$

1.2. The adjoint representation ad: $U_h(gl_\infty) \to \mathrm{End}\, U_h(gl_\infty)$ is given by $\mathrm{ad}_a(x) = \Delta(a) \circ (x)$, where $(a \otimes b) \circ x = a x S(b)$. Starting with the opposite coproduct Δ' and the related antipode S', we obtain another adjoint action ad$'$. We introduce the new generators $E_{i,i+1} = X_{i,i+1} \cdot q^{-H_{i,i+1}/2}$, $F_{i,i+1} = X_{i+1,i} \cdot q^{H_{i,i+1}/2}$ and define the quantum analogues of root vectors by induction: for $i < j-1$,

$$E_{ij} = \mathrm{ad}_{E_{i,i+1}}(E_{i+1,i}), \quad F_{ij} = \mathrm{ad}'_{F_{i,i+1}}(F_{i+1,j}). \tag{1.8}$$

From (1.8), (1.7), (1.2) it follows that

$$E_{ij} = [E_{i,i+1}, E_{i+1,j}]_q, \quad F_{ij} = [F_{i,i+1}, F_{i+1,j}]_q, \tag{1.9}$$

where $[A, B]_q = AB - qBA$, and

$$[E_{kk}, E_{ij}] = (\delta_{ki} - \delta_{kj})\, E_{ij}, \quad [E_{kk}, F_{ij}] = (-\delta_{ki} + \delta_{kj})\, F_{ij}. \tag{1.10}$$

In the next subsections we state and prove the communication relations for root vectors.

1.3. Theorem. Let $i < j < k < m$. *Then*

$$[E_{ij}, E_{kk}]_q = E_{ik}, \tag{1.11}$$

$$[E_{ik}, E_{jk}]_{q^{-1}} = 0, \quad [E_{ij}, E_{ik}]_{q^{-1}} = 0, \tag{1.12}$$

$$[E_{ik}, E_{jm}] = (q^{-1} - q)\, E_{im} E_{jk}, \tag{1.13}$$

$$[E_{ij}, E_{km}] = 0, \quad [E_{im}, E_{jk}] = 0, \tag{1.14}$$

formulae (1.11)–(1.14) *with the letter F substituted for the letter E.* (1.15)

Proof. Formulae (1.11)–(1.14) were proved in [R] and (1.15) is their consequence since linear Cartan involution ω_0 defined on generators by

$$\omega_0(h) = h, \quad \omega_0(E_{ii}) = -E_{ii}, \quad \omega_0(X_{i,i+1}) = -X_{i+1,i}, \quad \omega_0(X_{i+1,i}) = -X_{i,i+1}$$

extends to Hopf algebra isomorphism, $\omega_0\colon (U_h(gl_\infty), \delta) \to (U_h(gl_\infty), \Delta')$ and

$$\omega_0(E_{ij}) = (-1)^{j-i} F_{ij}, \quad \omega_0(F_{ij}) = (-1)^{j-i} E_{ij}.$$

Set $K_{ij} = q^{H_{ij}/2}$.

1.4. Theorem. a) *For* $i < j < k < m$,

$$[E_{ij}, F_{km}] = 0, \quad [E_{km}, F_{ij}] = 0. \tag{1.16}$$

b) *For* $i < j$

$$[E_{ij}, F_{ij}] = \frac{(-q^2)^{j-i}}{1-q^2}(K_{ij}^2 - K_{ij}^{-2}). \tag{1.17}$$

c) *For* $i < j < k < m$,

$$[E_{ik}, F_{jk}] = -(-q^2)^{k-j} E_{ij} K_{jk}^{-2}, \tag{1.18}$$

$$[E_{im}, F_{ij}] = (-q^2)^{j-i} K_{ij}^2 E_{jm}, \tag{1.19}$$

$$[E_{jm}, F_{im}] = (-q^2)^{m-j} F_{ij} K_{jm}^2, \tag{1.20}$$

$$[E_{ij}, F_{im}] = -(-q^2)^{j-i} K_{ij}^{-2} F_{jm}, \tag{1.21}$$

$$[E_{im}, F_{jk}] = [E_{jk}, F_{im}] = 0. \tag{1.22}$$

Proof. a) (1.16) is an easy consequence of (1.3), (1.9).
b) For $j-i=1$, (1.17) is just (1.3) and the general case can be proven by induction, use being made of the formulae (1.11), (1.15), (1.10).
c) Formulae (1.18)–(1.22) follows from Theorem 1 and the formulae (1.17), (1.11).

Below now consider the action of the coproduct on root vectors.

1.5. Theorem. *For $i<j$,*

$$\Delta(E_{ij}) = E_{ij}\otimes 1 + (1-q^2)\sum_{i<m<j} E_{im}K_{mj}^{-2}\otimes E_{mj} + K_{ij}^{-2}\otimes E_{ij}, \quad (1.23)$$

$$\Delta(F_{ij}) = 1\otimes F_{ij} + (1-q^2)\sum_{i<m<j} F_{mj}\otimes F_{im}K_{mj}^2 + F_{ij}\otimes K_{ij}^2. \quad (1.24)$$

Proof. Formula (1.23) was proved in [R], and (1.24) follows from (1.23) since $\omega_0\colon (U_h(gl_\infty),\Delta)\to(U_h(gl_\infty),\Delta')$ is Hopf algebra isomorphism and $\omega_0(K_{ij}^{-2}) = K_{ij}^2$, $\omega_0(E_{ij}) = (-1)^{j-i}F_{ij}$.

1.6. Set

$$\tilde{E}_{ij} = \begin{cases}1, & i\geqq j\\ (1-q^2)E_{ij}, & i<j\end{cases}, \quad \tilde{F}_{ij} = \begin{cases}1, & i\geqq j\\ (1-q^2)F_{ij}, & i<j\end{cases},$$

and rewrite (1.23), (1.24) in the more convenient fashion:

$$\Delta(\tilde{E}_{ij}) = \sum_{i\leqq m\leqq j}\tilde{E}_{im}K_{mj}^{-2}\otimes\tilde{E}_{mj}, \quad \Delta(\tilde{F}_{ij}) = \sum_{i\leqq m\leqq j}\tilde{F}_{mj}\otimes\tilde{F}_{im}K_{mj}^2. \quad (1.29)$$

Define the homomorphisms

$$\Delta^{(j)}\colon U_h(gl_\infty)\to U_h(gl_\infty)^{\otimes(j+1)}$$

by induction:

$$\Delta^{(1)} = \Delta, \quad \Delta^{(j+1)} = (\Delta\otimes\mathrm{id}^{\otimes j})\Delta^{(j)} = (\mathrm{id}^{\otimes j}\otimes\Delta)\Delta^{(j)}, \quad j\geqq 1.$$

Due to (1.29)

$$\Delta^{(l)}(\tilde{E}_{ij}) = \sum_{i\leqq r_1\leqq r_2\leqq\cdots\leqq r_l\leqq j}\tilde{E}_{ir_1}K_{r_1j}^{-2}\otimes\tilde{E}_{r_1r_2}K_{r_2j}^{-2}\otimes\cdots\otimes\tilde{E}_{r_lj}, \quad (1.30)$$

$$\Delta^{(l)}(\tilde{F}_{ij}) = \sum_{i\leqq r_1\leqq r_2\leqq\cdots\leqq r_l\leqq j}\tilde{F}_{r_lj}\otimes\tilde{F}_{r_{l-1}r_l}K_{r_lj}^2\otimes\cdots\otimes\tilde{F}_{ir_1}K_{r_1j}^2, \quad (1.31)$$

and due to (1.6)

$$\Delta^{(l)}(E_{ii}) = E_{ii}\otimes 1^{\otimes l} + 1\otimes E_{ii}\otimes 1^{\otimes(l-1)} + \cdots + 1^{\otimes l}\otimes E_{ii}, \quad (1.32)$$

$$\Delta^{(l)}(K_{ij}^p) = K_{ij}^p\otimes\cdots\otimes K_{ij}^p. \quad (1.33)$$

1.7. Set for $i<j$ $E_{ji} = F_{ij}$ and introduce in $\mathbb{Z}^2$, the ordering as follows:
1) if $i<j$, $l<k$, $r<s$, then

$$(j,i)<(l,l)<(k,k)<(r,s),$$

2) let $r'<s'$, $r<s$; then

$$(r',s')<(r,s) \quad \text{iff} \quad r'>r \quad \text{or} \quad r'=r \quad \text{and} \quad s'>s$$

and

$$(s',r')>(s,r) \quad \text{iff} \quad r'>r \quad \text{or} \quad r'=r \quad \text{and} \quad s'>s.$$

1.8. Theorem. *The set of ordered monomials*

$$E^n = \prod_{(i,j)\in\mathbb{Z}^2} E_{ij}^{n_{ij}}$$

with finitely many non-zero exponents $n_{ij}\in\mathbb{Z}_+$ *form a basis in* $\mathbb{C}[[h]]$*-module* $U_h(gl_\infty)$.

Proof is essentially the same as that for $U_h(sl(n))$ in [R, Theorems 1.3–1.5] being used.

2. The Representations of $U_h(gl_\infty)$ in $(\bar{\mathbb{C}}^\infty_-)_h$ and in $\Lambda^\infty_{(s),h}$

2.1. Definition. Let A be an algebra and $\mathbb{C}[[h]]$-module. Let V be topologically free $\mathbb{C}[[h]]$-module. Then a $\mathbb{C}[[h]]$-module homomorphism $\varrho: A\to \operatorname{End} V$ is called a representation of A in V provided ϱ is continuous in the h-adic topology.

2.2. Definition. $\bar{\mathbb{C}}^\infty_-$ denotes the vector space of sequences $(u_i)_{i\in\mathbb{Z}}$ with finitely many non-zero u_i for $i>0$. We consider $\bar{\mathbb{C}}^\infty_-$ as a topological vector space, the fundamental system of neighbourhoods of zero being $\{V^r \mid r\in\mathbb{Z}\}$, where

$$V^r = \{u \mid u_i = 0 \text{ for } i > -r\}.$$

$\mathbb{C}^\infty$ denotes the subspace consisting of $\{u_i\}$ with finitely many non-zero u_i. It's evident that $\mathbb{C}^\infty$ is dense in $\bar{\mathbb{C}}^\infty_-$.

2.3. Let l_{ij} denote the matrix which is 1 in (i,j) entry and zero everywhere else. Such matrices act in $\bar{\mathbb{C}}^\infty_-$ and we can define the representation of $U_h(gl_\infty)$ in $\bar{\mathbb{C}}^\infty_-\otimes\mathbb{C}[[h]] = (\bar{\mathbb{C}}^\infty_-)_h$ by

$$\pi(X_{i,i+1}) = l_{i,i+1},\quad \pi(X_{i+1,i}) = l_{i+1,i},\quad \pi(E_{ii}) = l_{ii}.$$

By (1.9) for $i<j$,

$$\pi(E_{ij}) = q^{(j-i)/2}\, l_{ij},\quad \pi(E_{ji}) = (-1)^{j-i-1}\, q^{3(j-i)/2-1}\cdot l_{ji},$$

and by (1.30)–(1.32) the representation in $(\bar{\mathbb{C}}^\infty_-)^{\otimes(l+1)}\otimes\mathbb{C}[[h]]$ is given by

$$\pi^{(l+1)}(E_{ii}) = l_{ii}\otimes 1^{\otimes l} + 1\otimes l_{ii}\otimes 1^{\otimes(l-1)} + \cdots + 1^{\otimes l}\otimes l_{ii}, \tag{2.1}$$

$$\pi^{(l+1)}(E_{ij}) = q^{(j-i)/2}\sum_r (q^{-1}-q)^{\mu(r)-1}\,\hat{l}_{ir_1}\otimes\hat{l}_{r_1r_2}\otimes\cdots\otimes\hat{l}_{r_lj}, \tag{2.2}$$

$$\pi^{(l+1)}(E_{ji}) = (-1)^{j-i-1}\, q^{3(j-i)/2-1}\sum_r (q-q^{-1})^{\mu(r)-1}\,\hat{l}_{jr_l}\otimes\hat{l}_{r_lr_{l-1}}\otimes\cdots\otimes\hat{l}_{r_1i}, \tag{2.3}$$

where $\hat{l}_{ps} = I$ for $p=s$, $\hat{l}_{ps} = l_{ps}$ otherwise, and $\mu(r)$ is the number of $\hat{l}_{ps}\neq I$ in summand of (2.2), (2.3), l_{ij} are the matrix units.

2.4. Let $\{f_i\}$ be the standard basis in $\mathbb{C}^\infty$. Denote by $\Lambda^\infty_{(s),h}$ the $\mathbb{C}[[h]]$-module generated by all expressions of the form $u_0\wedge u_{-1}\wedge u_{-2}\wedge\cdots$, where $u_i\in\mathbb{C}^\infty$ and $u_{-i} = f_{-i+s}$ for sufficiently large i, the following identification being assumed: if $i<j$ then

$$\cdots\wedge f_i\wedge f_j\wedge\cdots = -q^{-1}\cdots\wedge f_j\wedge f_i\wedge\cdots. \tag{2.4}$$

If we start with expressions $u = u_0 \wedge u_{-1} \wedge \cdots \wedge u_{-l}$, where $u_j \in \mathbb{C}^\infty$, then we get the definition of the $\mathbb{C}[[h]]$-module $\Lambda_h^{l+1}(\mathbb{C}^\infty)$.

2.5. Define the action $\hat{\pi}_{(s)}$: $U_h(gl_\infty) \to \operatorname{End} \Lambda^\infty_{(s),h}(\mathbb{C}^\infty)$ on generators E_{ii}, E_{ij}, E_{ji} $(i < j)$ by

$$\hat{\pi}_{(s)}(E_{ii})(u_0 \wedge u_{-1} \wedge \cdots) = l_{ii} u_0 \wedge u_{-1} \wedge \cdots + u_0 \wedge l_{ii} u_{-1} \wedge \cdots + \cdots, \tag{2.5}$$

$$\hat{\pi}_{(s)}(E_{ij})(u_0 \wedge u_{-1} \wedge \cdots) = q^{(j-i)/2} \sum_{l \geqq 0} \sum_{i \leqq k_1 \leqq \cdots \leqq k_l \leqq j} (q^{-1} - q)^{\mu(k)-1} \cdot \hat{l}_{ik_1} u_0 \wedge \hat{l}_{k_1 k_2} u_{-1} \wedge \cdots \wedge \hat{l}_{k_l j} u_{-l} \wedge u_{-l-1} \wedge \cdots, \tag{2.6}$$

$$\hat{\pi}_{(s)}(E_{ij})(u_0 \wedge u_{-1} \wedge \cdots) = (-1)^{j-i-1} q^{3(j-i)/2-1} \sum_{l \geqq 0} \sum_{i \leqq k_1 \leqq \cdots \leqq k_l \leqq j} \cdot (q - q^{-1})^{\mu(k)-1} \hat{l}_{jk_l} u_0 \wedge \hat{l}_{k_l k_{l-1}} u_{-1} \wedge \cdots \wedge \hat{l}_{k_2 k_1} u_{-l+1} \wedge \hat{l}_{k_1 i} u_{-l} \wedge \cdots. \tag{2.7}$$

2.6. Theorem. *Formulae* (2.5)–(2.7) *define the representation of* $U_h(gl_\infty)$.

Proof. For a fixed u, in (2.5)–(2.7) there are finitely many non-zero summands. Hence, it suffices to prove that the formulae (2.1)–(2.3) define the representation of $U_h(gl_\infty)$ in $\Lambda_h^{l+1}(\mathbb{C}^\infty)$. Since the latter formulae define the representation in $\mathbb{C}[[h]] \otimes (\mathbb{C}^\infty)^{\otimes(l+1)}$, it suffices to show that the subspace in $\mathbb{C}[[h]] \otimes (\mathbb{C}^\infty)^{\otimes(l+1)}$ generated by the expressions

$$\cdots \otimes f_i \otimes f_j \otimes \cdots + q^{-1} \cdots \otimes f_j \otimes f_i \otimes \cdots, \qquad i < j,$$

is stable under all of the E_{kk}, $E_{i,i+1}$, $E_{i+1,i}$. But this is easily verified by straightforward calculations.

2.7. In this subsection we'll simplify the formulae (2.6), (2.7) for $u = f_{i_1} \wedge f_{i_2} \wedge \cdots \wedge f_{i_t} \wedge \cdots$ with $i_1 > i_2 > \cdots$. Denote by $\varkappa(i,j) = \varkappa(i,j,u)$ the number of indices $i_k \in (i,j)$ and note that if $j \neq i_r$ for all r or $i = i_t$ for some t, then all the terms in (2.6) vanish, otherwise all but one of them are zero. Hence, we obtain

$$\hat{\pi}_{(s)}(E_{ij}) u = q^{(j-i)/2} (-q^{-1})^{\varkappa(i,j)} \cdots \wedge f_{i_{r-1}} \wedge f_{i_{r+1}} \wedge \cdots \wedge f_i \wedge \cdots, \tag{2.8}$$

the indices on the right-hand side being ordered.

Further, $\hat{\pi}_{(s)}(E_{ij}) u = 0$ unless $i = i_r$ for some r and $j \neq i_t$ for all t; of these two conditions hold, then in (2.7) the number of non-zero summands with fixed $\mu = \mu(k)$ is $C_{\mu-1}^{\varkappa(j,i)}$, and each non-zero term is of the form

$$(-1)^{j-i-1} q^{3(j-i)/2-1} (q - q^{-1})^{\mu-1} \cdots \wedge l_{jv_1} f_{v_1} \wedge l_{v_1 v_2} f_{v_2} \wedge \cdots \wedge f_{i_{r-1}} \wedge l_{v_{\mu-1}, i} f_i \wedge f_{i_{r+1}} \wedge \cdots,$$

where $j > v_1 > \cdots > v_{\mu-1} > i$ (and $v_1 = i$ if $\mu = 1$).

By using (2.4) we get

$$\hat{\pi}_{(s)}(E_{ji}) u = (-1)^{j-i-1} q^{3(j-i)/2-1} (-q)^{-\varkappa(i,j)} \sum_{1 \leqq \mu \leqq \varkappa(i,j)+1} \cdot C_{\mu-1}^{\varkappa(i,j)} (1 - q^2)^{\mu-1} \cdots \wedge f_j \wedge \cdots \wedge f_{i_{r-1}} \wedge f_{i_{r+1}} \wedge \cdots = (-1)^{j-i-1} \cdot q^{3(j-i)/2-1} ((2 - q^2)(-q^{-1}))^{\varkappa(i,j)} \cdots \wedge f_j \wedge \cdots \wedge f_{i_{r-1}} \wedge f_{i_{r+1}} \wedge \cdots. \tag{2.9}$$

the indices on the right-hand side being ordered.

2.8. Define $\Lambda^\infty_{(s)}(\mathbb{C}^\infty)$ as the $\mathbb{C}$-span of all expressions of the form $u_0 \wedge u_{-1} \wedge u_{-2} \wedge \cdots$ with the identification

$$\cdots \wedge f_i \wedge f_j \wedge \cdots = - \cdots \wedge f_j \wedge f_i \wedge \cdots$$

for $i \leqq j$. Next, define the $\mathbb{C}[[h]]$-module isomorphism $j\colon \Lambda^\infty_{(s),h}(\mathbb{C}^\infty) \to \Lambda^\infty_{(s)}(\mathbb{C}^\infty) \otimes \mathbb{C}[[h]]$ by $f_{i_1} \wedge f_{i_2} \wedge \cdots \wedge \cdots \to f_{i_1} \wedge f_{i_2} \wedge \cdots \wedge \cdots$, and denote by $\varrho_{(s)}$ the usual representation of gl_∞ in $\Lambda^\infty_{(s)}(\mathbb{C}^\infty)$:

$$\varrho_{(s)}(l_{ij})\, u = l_{ij} u_0 \wedge u_{-1} \wedge \cdots + u_0 \wedge l_{ij} u_{-1} \wedge u_{-2} \wedge \cdots .$$

Now, if we define

$$K(i,j) = \exp\left(\frac{h}{2} \sum_{i+1 \leqq r \leqq j-1} l_{rr}\right) \in U(gl_\infty) \otimes \mathbb{C}[[h]],$$

then the formulae

$$\pi_s(E_{ij}) = \varrho_{(s)}(l_{ii}), \tag{2.10}$$

$$\pi_{(s)}(E_{ij}) = q^{(j-i)/2-1}(-\varrho_{(s)}(K(i,j)^{-1})\, \varrho_{(s)}(l_{ij}), \tag{2.11}$$

$$\pi_{(s)}(E_{ji}) = (-1)^{j-i-1}\, q^{3(j-i)/2-1}(2\varrho_{(s)}(K(i,j)^{-1}) - \varrho_{(s)}(K(i,j)))\, \varrho_{(s)}(l_{ji}) \tag{2.12}$$

define the representation $\pi_{(s)}\colon U_h(gl_\infty) \to \mathrm{End}(\Lambda^\infty_{(s)} \otimes \mathbb{C}[[h]])$ [see (2.5), (2.8), (2.9)].

3. The Algebras $U_h(g'(A_\infty))$, $U_h(g(A_\infty))$

3.1. Definition. $U_h(g'(A_\infty))_f$ is the topologically free $\mathbb{C}[[h]]$-module, complete in h-adic topology, and the unital algebra with generators $\{c, E_{ii}, E_{ij}, E_{ji} = F_{ij}\}_{i<j,\,(i,j)\in\mathbb{Z}^2}$ and relations

1) $[c, \text{everything}] = 0;\quad [E_{ii}, E_{jj}] = 0,\quad \text{all}\quad i,j,$ (3.1)

2) formulae (1.10)–(1.15); (3.2)

3) formulae (1.16)–(1.22) with

$$\mathring{E}_{ii} = \begin{cases} E_{ii}, & i>0 \\ E_{ii}+c, & i\leqq 0 \end{cases}, \quad \mathring{H}_{ij} = \mathring{E}_{ii} - \mathring{E}_{jj}, \quad \mathring{K}_{ij} = q^{\mathring{H}_{ij}},$$

substituted for E_{ii}, H_{ij}, K_{ij}. (3.3)

3.2. $U_h(g'(A_\infty))_f$ can be equipped with a Hopf algebra structure, the coproduct being defined on generators by formulae

$$\Delta c = c \otimes 1 + 1 \otimes c, \quad \Delta E_{ii} = E_{ii} \otimes 1 + 1 \otimes E_{ii} \tag{3.4}$$

and by

formulae (1.23), (1.24) with $\mathring{K}_{ij}$ substituted for K_{ij}. (3.5)

One easily gets the following analogue of Theorem 1.8.

3.3. Theorem. *The set of ordered monomials*

$$c^l E^n = c^l \prod_{i,j} E_{ij}^{n_{ij}}$$

with finitely many non-zero exponents $n_{ij} \in \mathbb{Z}_+$, $l \in \mathbb{Z}_+$ *form a basis in the* $\mathbb{C}[[h]]$*-module* $U_h(g'(A_\infty))_f$.

3.4. Set $f' = \bigoplus_i \mathbb{C} E_{ii}$, define linear functionals $\varepsilon_i\colon h' \to \mathbb{C}$ by $\varepsilon_i(E_{jj}) = \delta_{ij}$ and set $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$, $\mathbb{Q}'_+ = \bigoplus_i \mathbb{Z}_+ \alpha_i$. Denote by $U_h(n_+)$ (respectively $U_h(n_-)$) the unital subalgebra in $U_h(g'(A_\infty))_f$ generated by $\{E_{ij}\}_{i<j}$ (respectively $\{E_{ji}\}_{i<j}$). Evidently,

$$U_h(n_\pm) = \bigoplus_{\alpha \in \mathbb{Q}'_+} U_h(n_\pm)_{\pm\alpha},$$

where

$$U_h(n_\pm)_{\pm\alpha} = \{x \in U_h(n_\pm) \mid [h, x] = \pm\,\alpha(h)\,x \ \forall h \in h'\}$$

for $\alpha \neq 0$, and $U_h(n_\pm)_0 = \mathbb{C}$. By Theorem 3.3, any element $u \in U_h(g'(A_\infty))_f$ can be represented as follows:

$$u = \sum_{k \in \mathbb{Z}_+} h^k \sum_{0 \leqq l \leqq l(k)} c^l \sum_{\alpha, \beta \in \mathbb{Q}'_+} \sum_{\gamma(k,l) \in \mathbb{Z}_+^\infty} \sum_{1 \leqq t \leqq t(\beta)} \mathscr{F}_{\alpha,k,l} \prod_i E_{ii}^{\gamma(k,l)_i} \mathscr{E}_{\beta,k,l,t}. \tag{3.6}$$

Here $\mathscr{F}_{\alpha,k,l} \in U_h(n_-)_{-\alpha}$, $\mathscr{E}_{\beta,k,l,t} \in U_h(n_+)_\beta$ and for fixed k, l there are finitely many non-zero summands in (3.6).

To obtain the completed algebra $U_h(g'(A_\infty))$ we replace the sums over (α, β, γ) for the series but impose certain conditions on pairs (α, γ) corresponding to non-zero summands (note that the set of such pairs is uniquely defined by the series u).

3.5. Set for $\alpha = \sum_i m_i \alpha_i \in \mathbb{Q}'_+$, $\gamma = (\gamma_i) \in \mathbb{Z}_+^\infty$,

$$S(\alpha) = \{i \mid m_i \neq 0\}, \quad S(\gamma) = \{i \mid \gamma_i \neq 0\}, \quad S(\alpha, \gamma) = S(\alpha) \cup S(\gamma).$$

By connecting i, j for $|i - j| = 1$, we can view $S(\alpha, \gamma)$ as a graph. Denote by $\mathscr{I}(\alpha, \gamma)$ the set of its connected components and set for $p \in \mathbb{Z}_+$,

$$\mathscr{I}(u, p) = \cup\, \mathscr{I}(\alpha, \gamma),$$

where the union is taken over non-zero summands with $k \leqq p$ in (3.6).

For $i \in \mathbb{Z}$ and $p \in \mathbb{Z}_+$ set $\mathrm{Int}(u, p, i) = \{I \in \mathscr{I}(u, p) \mid i \in I\}$.

Recall that $\mathscr{E}_{\beta,k,l,t}$ (respectively $\mathscr{F}_{\alpha,k,l}$) are expressed via E_{ii} and $E_{i,i+1}$ (respectively $E_{i+1,i}$), $i \in \mathbb{Z}$, and, for $r \in \mathbb{N}$, define the series $u(r)$ by substituting 0 for all E_{ii} ($i \leqq -r$ or $i \geqq r+1$) and for all $E_{i,i+1}$, $E_{i+1,i}$, ($|i| \geqq r$).

3.6. Definition. The series u of the form (3.6) is said to belong to $U_h(g'(A_\infty))$ provided the following conditions hold

a) for all $p \in \mathbb{Z}_+$, $i \in \mathbb{Z}$ the sets $\mathrm{Int}(u, p, i)$ are finite,

b) $u(r) \in U_h(g'(A_\infty))_f$ for all $r \in \mathbb{N}$.

3.7. Definition. Let $\mathscr{I}_{p,i}$ be finite sets of finite integer intervals containing i ($i \in \mathbb{Z}$, $p \in \mathbb{Z}_+$), and let $r \in \mathbb{N}$.

We say that $u \in U_h(g'(A_\infty))$ belongs to the neighbourhood of zero $V(\{\mathscr{I}_{p,i}\}_{i \in \mathbb{Z}, p \in \mathbb{Z}_+}, r)$ provided

a) $\mathrm{Int}(u, p, i) \subset \mathscr{I}_{p,i}$, $\forall p \in \mathbb{Z}_+$, $\forall i \in \mathbb{Z}$,

b) $u(r) = 0$.

3.8. We introduce in $U_h(g'(A_\infty))$ the topology by declaring $\{V(\{\mathscr{I}_{p,i}\}_{i\in\mathbb{Z},\, p\in\mathbb{Z}_+}, r)\}$ to be the fundamental system of neighbourhoods of zero.

3.9. Proposition. *$U_h(g'(A_\infty))_f \subset U_h(g'(A_\infty))$ densely.*

Proof. $u(r) \to u$ as $r \to \infty$.

3.10. Theorem. *Let $u_i \in U_h(g'(A_\infty))$, $i = 1, 2$, and let $\{u_i^j\}_{j\geqq 0} \subset U_h(g'(A_\infty))_f$ be a sequence having the limit u_i. Then the sequence $\{u_1^j u_2^j\}$ has the limit, denote it u, in $U_h(g'(A_\infty))$ and u is independent of the choice of sequences $\{u_1^j\}$, $\{u_2^j\}$.*

Proof. Write the expression (3.6) for u_i^j and $u^j \stackrel{\text{def}}{=} u_1^j u_2^j$ in the form

$$u_i^j = \sum_k \sum_l \sum_{\alpha,\beta} \sum_\gamma \sum_t u_i^j(k, l, \alpha, \beta, \gamma, t), \tag{3.7}$$

$$u^j = \sum_k \sum_l \sum_{\alpha,\beta} \sum_\gamma \sum_t u^j(k, l, \alpha, \beta, \gamma, t), \tag{3.8}$$

and fix a tuple $(k, l, \alpha, \beta, \gamma, t)$. From Theorems 1.3, 1.4 and from Definition 3.6 it follows that $u^j(k, l, \alpha, \beta, \gamma, t)$ depends on finitely many summands in (3.7), $i = 1, 2$. Moreover, the number of these summands is bounded uniformly in j. From Definitions 3.7, 3.8, it follows that $u^j(k, l, \alpha, \beta, \gamma, t)$ is independent of j, provided j is sufficiently large: $u^j(k, l, \alpha, \beta, \gamma, t) = u(k, l, \alpha, \beta, \gamma, t)$ for $j \geqq j_0$, where j_0 depends on $(k, l, \alpha, \beta, \gamma, t)$. Hence, $u(k, l, \alpha, \beta, \gamma, t)$ is independent of a choice of sequences $\{u_1^j\}$, $\{u_2^j\}$. Now we see that the omission of upper indices in (3.8) gives the formula u; clearly, it's independent of a choice of sequences.

The close inspection of the above arguments shows that u obeys the conditions of Definition 3.6.

3.11. Let $s \in \mathbb{N}$. Consider a formal series

$$\begin{aligned} u = \sum_{k\in\mathbb{Z}_+} h^k \sum_{\substack{0\leqq l_j \leqq l_j(k)\\(1\leqq j\leqq s)}} c^{l_1} \otimes \cdots \otimes c^{l_s} \sum_{\alpha,\beta\in(\mathbb{Q}'_+)^s} \sum_{\gamma(k,l)\in(\mathbb{Z}_+^\infty)^t} \\ \cdot \sum_{\substack{1\leqq t_j\leqq t_j(\beta^1)\\(1\leqq j\leqq s)}} \mathscr{F}_{\alpha^1,k,l} \prod_i E_{ii}^{\gamma(k,l)_i^1} \mathscr{E}_{\beta^1,k,l,t_1} \otimes \\ \cdots \otimes \mathscr{F}_{\alpha^s,k,l} \prod_i E_{ii}^{\gamma(k,l)_i^s} \mathscr{E}_{\beta^s,k,l,t_s}\, . \end{aligned} \tag{3.9}$$

Non-zero summands of this series determine the set of tuples of pairs $(\alpha, \gamma) = ((\alpha^1, \gamma^1), (\alpha^2, \gamma^2), \ldots, (\alpha^s, \gamma^s))$. Set for $1 \leqq j \leqq s$, $p \in \mathbb{Z}_+$,

$$\mathscr{I}_j(u, p) = \cup\, \mathscr{I}(\alpha^j, \gamma^j),$$

where the union is taken over non-zero summands with $k \leqq p$ in (3.9).

For $i \in \mathbb{Z}^s$ and $p \in \mathbb{Z}_+$ set

$$\operatorname{Int}(u, p, i) = \{(I^1, I^2, \ldots, I^s) \in \mathscr{I}_1(u, p) \times \cdots \times \mathscr{I}_s(u, p) \mid i_1 \in I^1, \ldots, i_s \in I^s\}.$$

For $r \in \mathbb{N}$ define the series $u(r)$ by substituting 0 for all E_{ii} ($i \leqq -r$ or $i \geqq r + 1$) and all $E_{i,i+1}$, $E_{i+1,i}$ ($|i| \geqq r$).

3.12. Definition. The series u of the form (3.9) is said to belong to $U_h(g'(A_\infty))^{\otimes s}$ provided the following conditions hold

a) for every $p \in \mathbb{Z}_+$, $i \in \mathbb{Z}^s$ the set $\operatorname{Int}(u, p, i)$ is finite;
b) $u(r) \in U_h(g'(A_\infty))_f^{\otimes s}$ for all $r \in \mathbb{N}$.

3.13. Definition. Let $\mathscr{I}_{p,i_j}$ be finite sets of finite integer intervals, containing i_j ($p \in \mathbb{Z}_+$, $1 \leqq j \leqq s$, $i_j \in \mathbb{Z}$) and let $r \in \mathbb{N}$.

We say that $u \in U_h(g'(A_\infty))^{\otimes s}$ belongs to the neighbourhood of zero $V(\{\mathscr{I}_{p,i_1} \times \cdots \times \mathscr{I}_{p,i_s}\}, r)$, provided

a) $\mathrm{Int}(u,p,i) \subset \mathscr{I}_{p,i_1} \times \cdots \times \mathscr{I}_{p,i_s}$ $\forall i \in \mathbb{Z}^s$, $\forall p \in \mathbb{Z}_+$,

b) $u(r) = 0$.

3.14. We introduce in $U_h(g'(A_\infty))^{\otimes s}$ the topology by declaring $\{V(\{\mathscr{I}_{p,i_1} \times \cdots \times \mathscr{I}_{p,i_s}\}, r\}$ to be the fundamental system of neighbourhoods of zero.

3.15. The analogues of Proposition 3.9 and Theorem 3.10 for $U_h(g'(A_\infty))^{\otimes s}$ are obvious.

3.16. Theorem. *Let $u \in U_h(g'(A_\infty))$ and let $\{u^j\} \subset U_h(g'(A_\infty))_f$ be a sequence having the limit u.*

Then the sequence $\{\Delta(u^j)\} \subset U_h(g'(A_\infty))_f^{\otimes 2}$ has the limit, denote it $\Delta(u)$, in $U_h(g'(A_\infty))^{\otimes 2}$, and it is independent of a choice of a sequence.

Proof is similar to that of Theorem 3.10, use being made of Theorem 1.5.

One can easily state the analogues of Theorem 3.16 for the maps $\mathrm{id} \otimes \Delta$, $\Delta \otimes \mathrm{id}$: $U_h(g'(A_\infty))^{\otimes 2} \to U_h(g'(A_\infty))^{\otimes 3}$.

Since $U_h(g'(A_\infty))_f$ is a Hopf algebra, from Theorems 3.10, 3.16 and their analogues the next theorem immediately follows.

3.17. Theorem. *$U_h(g'(A_\infty))$ is a topological Hopf algebra with the product map*

$$U_h(g'(A_\infty))^{\otimes 2} \ni u_1 \otimes u_2 \mapsto u \in U_h(g'(A_\infty)),$$

and the coproduct map

$$U_h(g'(A_\infty)) \ni u \mapsto \Delta(u) \in U_h(g'(A_\infty))^{\otimes 2}.$$

3.18. If we set in all constructions of this section $c = 0$ then we get another Hopf algebra which can be naturally denoted by $U_h(\overline{gl_\infty})$. Note that $U_h(g'(A_\infty))$ can be naturally viewed as the central extension of $U_h(\overline{gl_\infty})$.

Now we extend $U_h(g'(A_\infty))$ be derivation d.

3.19. Definition. $U_h(g(A_\infty))_f$ is a topologically free $\mathbb{C}[[h]]$-module, complete in h-adic topology, and an unital algebra with generators $\{c, d\} \cup \{E_{ij}\}_{i,j \in \mathbb{Z}}$ and relations

1. formulae (3.1)–(3.3);
2. $[d, E_{i,i+1}] = \delta_{i_0} E_{i,i+1}$, $[d, E_{i+1,i}] = -\delta_{i_0} E_{i+1,i}$, $[d, c] = 0$, $[d, E_{ii}] = 0$ all i.

$U_h(g(A_\infty))_f$ can be equipped with a Hopf algebra structure, the coproduct being defined by (3.4), (3.5) and by $\Delta(d) = d \otimes 1 + 1 \otimes d$.

3.20. Now, in the complete analogy with the definition of $U_h(g'(A_\infty))$ we can define the Hopf algebra $U_h(g(A_\infty))$, in the definition of polynomials in c being replaced for polynomials in two variables c, d [see (3.6)].

3.21. Below we shall need the subspaces $\tilde{h}' = h' \oplus \mathbb{C}c \subset U_h(g'(A_\infty))_f \subset U_h(g'(A_\infty))$, $\tilde{h} = \tilde{h}' \oplus \mathbb{C}d \subset U_h(g(A_\infty))_f \subset U_h(g(A_\infty))$ and the subalgebras

$$U_h(b'_\pm)_f \subset U_h(g'(A_\infty))_f, \quad U_h(b'_\pm) \subset U_h(g'(A_\infty)),$$
$$U_h(b_\pm)_f \subset U_h(g(A_\infty))_f, \quad U_h(b_\pm) \subset U_h(g(A_\infty))$$

defined in an obvious way.

4. Representations of the Algebras $U_h(g'(A_\infty))$, $U_h(g(A_\infty))$

4.1. Definition. A representation of the algebra $U_h(g(A_\infty))$ in a topologically free $\mathbb{C}[[h]]$-module V is said to be restricted if for a given vector $v = \sum_{j\geqq 0} h^j v_j \in V$ there exist $r_j \in \mathbb{N}$, $j = 0, 1, \ldots$, such that for every j vector v_j is killed by the following subspaces:
1. $U_h(n_+)_\alpha$ provided $S(\alpha) \not\subset [-r_j, r_j]$ or $ht\,\alpha > r_j$,
2. $U_h(n_-)_{-\alpha}$ provided $S(\alpha) \subset (-\infty, -r_j)$ or $ht\,\alpha > r_j$ or $S(\alpha) \subset (r_{j+1}, +\infty)$,
3. $\mathbb{C} \cdot E_{ii}$ provided $i < -r_j$ or $i > r_j + 1$ [for definitions of $U_h(n_\pm)_{\pm\alpha}$ and $S(\alpha)$, see 3.3, 3.4].

Restricted representations of the algebras $U_h(g'(A_\infty))_f$, $U_h(g'(A_\infty))$, $U_h(g(A_\infty))_f$ are defined by the same conditions.

4.2. Theorem. a) *A restricted representation σ_f of the algebra $U_h(g'(A_\infty))_f$ extends uniquely to a restricted representation σ of the algebra $U_h(g'(A_\infty))$ and to restricted representation $\tilde{\sigma}$ of the algebra $U_h(g(A_\infty))$, the action of d being defined by*

$$\tilde{\sigma}(d) = -\sum_{j>0} \sigma_f(E_{jj}). \tag{4.1}$$

b) *A restricted representation $\tilde{\sigma}_f$ of the algebra $U_h(g(A_\infty))_f$ extends uniquely to a restricted representation $\tilde{\sigma}$ of the algebra $U_h(g(A_\infty))$.*

Proof. Evident.

It is clear that every submodule or quotient of a restricted module is restricted, and that the direct sum or tensor product of a finite number of restricted modules is also restricted.

4.3. Example. The formulae

$$\sigma_{(s)}(c) = 1, \quad \sigma_{(s)}(E_{ii}) = \begin{cases} \varrho_{(s)}(l_{ii}), & i > s \\ \varrho_{(s)}(l_{ii}) - I, & i \leqq s \end{cases}, \tag{4.2}$$

$$\sigma_{(s)}(E_{ij}) = q^{(j-i)/2}\, q^{\sum_{i+1\leqq r\leqq j-1} \varrho_{(s)}(l_{rr})}\, \varrho_{(s)}(l_{ij}), \tag{4.3}$$

$$\sigma_{(s)}(E_{ji}) = (-1)^{j-i-1} q^{3(j-i)/2-1} \left(2q^{-\sum_{i+1\leqq r\leqq j-1} \varrho_{(s)}(l_{rr})} - q^{\sum_{i+1\leqq r\leqq j-1} \varrho_{(s)}(l_{rr})}\right) \varrho_{(s)}(l_{ji}), \tag{4.4}$$

where $i < j$, define restricted representations of the algebras $U_h(g'(A_\infty))_f$, $U_h(g'(A_\infty))$ in $\Lambda^\infty_{(s)}(\mathbb{C}^\infty) \otimes \mathbb{C}[[h]]$ [cf. (2.10)–(2.12)].

4.4. Example. The formulae (4.2)–(4.4) together with formula

$$\sigma_{(s)}(d) = -\sum_{j>0} \sigma_{(s)}(E_{jj})$$

define restricted representations of the algebras $U_h(g(A_\infty))_f$, $U_h(g(A_\infty))$.

4.5. In what follows the linear functional Λ on $\tilde{h}' = h' \oplus \mathbb{C}c$ is supposed to satisfy the conditions $\Lambda(H_j) \in \mathbb{Z}_+$ and $\Lambda(H_j) > 0$ for finitely many j.

The functional Λ_s is defined by conditions

$$\Lambda_s(H_j) = \delta_{sj}, \quad \Lambda_s(c) = 1.$$

4.6. Definition. A $U_h(g'(A_\infty))$-module V is called a highest weight module with highest weight Λ if there exists a non-zero vector $v \in V$ such that

$$U_h(n_+)v = 0, \quad h(v) = \Lambda(h)v \quad \text{for } h \in \tilde{h}'.$$

and $U_h(g'(A_\infty))(v) = V$.

The vector V is called a highest weight vector.

A highest weight module over $U_h(g(A_\infty))$ is defined in the similar fashion.

4.7. Example. The representation $\sigma_{(s)}$ of Example 4.3 is a highest weight representation with the highest weight Λ_s, the highest weight vector being $f_s \wedge f_{s-1} \wedge f_{s-2} \wedge \cdots$.

Denote by $L(\Lambda_s)_h$ the corresponding $U_h(g'(A_\infty))$-module and recall that the representation $\tilde{\varrho}_{(s)}$: $U(g'(A_\infty)) \to \operatorname{End} \Lambda^\infty_{(s)}(\mathbb{C}^\infty)$ defined by $\tilde{\varrho}_{(s)}(c) = 1$,

$$\varrho_{(s)}(l_{ij}) = \begin{cases} \varrho_{(s)}(l_{ij}) - I, & i = j < 0 \\ \varrho_{(s)}(l_{ij}) & \text{otherwise,} \end{cases}$$

is the classical highest weight representation $L(\Lambda_s)$ with the highest weight Λ_s, the highest weight vector being $f_s \wedge f_{s-1} \wedge f_{s-2} \wedge \cdots$.

Recall also the following classical result [K].

4.7. Theorem. *The space of the basic representation $L(\Lambda_0)$ can be identified with the space of polynomials $\mathbb{C}[x_1, x_2, \ldots]$ so that $c \mapsto 1$ and*

$$\sum_{i,j} u^i v^{-j} E_{ij} \mapsto \frac{u}{u-v}(\Gamma(u,v) - 1),$$

where $\Gamma(u,v)$ is the following vertex operator:

$$\Gamma(u,v) = \exp\left(\sum_{j \geqq 1} (u^j - v^j)x_j\right) \exp\left(-\sum_{j \geqq 1} \frac{1}{j}(u^{-j} - v^{-j})\frac{\partial}{\partial x_j}\right).$$

Hence, from formulae (4.2)–(4.4) and the definition of the representation $\sigma_{(0)}$ we obtain the following.

4.8. Theorem. *The space of the representation $L(\Lambda_0)_h$ over $U_h(g'(A_\infty))$ can be identified with the space $\mathbb{C}[[h]] \otimes \mathbb{C}[x_1, x_2, \ldots]$ so that $c \mapsto 1$ and*

$$\sum_{i,j} u^i v^{-j} \hat{E}_{ij} \mapsto \frac{u}{u-v}(\Gamma(u,v) - 1),$$

where for $i < j$

$$\hat{E}_{ii} = E_{ii},$$

$$\hat{E}_{ij} = q^{(i-j)/2}\left(-q^{\sum_{i+1 \leqq r \leqq j-1} \mathring{E}_{rr}}\right) E_{ij},$$

$$\hat{E}_{ij} = (-1)^{j-i-1} q^{1-3(j-i)/2}\left(2 - q^{2\sum_{i+1 \leqq r \leqq j-1} \mathring{E}_{rr}}\right)^{-1} q^{-\sum_{i+1 \leqq r \leqq j-1} \mathring{E}_{rr}} \cdot E_{ij}.$$

Here $\mathring{E}_{rr} = E_{rr}$ if $r > 0$, and $\mathring{E}_{rr} = E_{rr} + c$ if $r \leqq 0$.

In particular, for $k \in \mathbb{N}$,

$$q^{-k/2} \sum_{i \in \mathbb{Z}} \left(-q^{\sum_{i+1 \leqq r \leqq i+k-1} \mathring{E}_{rr}}\right) E_{i,i+k} \mapsto \frac{\partial}{\partial x_k}$$

$$(-1)^{k-1} q^{1-3k/2} \sum_{i \in \mathbb{Z}} \left(2 - q^{2 \sum_{i+1 \leqq r \leqq i+k-1} \mathring{E}_{rr}}\right)^{-1} q^{-\sum_{i+1 \leqq r \leqq i+k-1} \mathring{E}_{rr}} \cdot E_{i+k,i} \mapsto x_k .$$

5. Quantum R-Matrices and Quantum Casimir Operators for the Algebras $U_h(g'(A_\infty))$, $U_h(g(A_\infty))$

5.1. Set for finite set $\{L_{ij}\}_{\substack{1 \leqq i \leqq s \\ 1 \leqq j \leqq p}}$ of restricted $U_h(g'(A_\infty))_f$-modules and for $v_{ij} \in L_{ij}$ $(1 \leqq i \leqq s,\ 1 \leqq j \leqq p)$,

$$V(\{L_{ij}, v_{ij}\}) = \{u \in U_h(g'(A_\infty))^{\otimes s} \mid u(v_{1j} \otimes \cdots \otimes v_{sj}) = 0,\ 1 \leqq j \leqq p\},$$

and introduce in $U_h(g'(A_\infty))_f^{\otimes s}$ the topology by declaring $\{V(\{L_{ij}, v_{ij}\})\}$ to be the fundamental system of neighbourhoods of zero. The completion of $U_h(g'(A_\infty))_f^{\otimes s}$ with respect to this topology will be denoted by $U_h(g'(A_\infty))^{\hat{\otimes} s}$. Clearly, $U_h(g'(A_\infty))^{\otimes s} \hookrightarrow U_h(g'(A_\infty))^{\hat{\otimes} s}$ continuously and the product in $U_h(g'(A_\infty))_f^{\otimes s}$ has the unique continuous extension to the product in $U_h(g'(A_\infty))^{\hat{\otimes} s}$. Also, the maps

$$\Delta, \quad \mathrm{id} \otimes \Delta, \quad \Delta \otimes \mathrm{id}$$

have unique continuous extensions to the maps

$$\hat{\Delta}, \quad \mathrm{id} \otimes \hat{\Delta}, \quad \hat{\Delta} \otimes \mathrm{id}.$$

In complete analogy with this definition we define $U_h(g(A_\infty))^{\hat{\otimes} s}$.

5.2. Theorem. a) *$U_h(g'(A_\infty))$ is a quasitriangular Hopf algebra, i.e. there exists invertible $R \in U_h(g'(A_\infty))^{\hat{\otimes} 2}$ such that*

$$(\hat{\Delta} \otimes \mathrm{id})(R) = R_{12} R_{23}, \quad (\mathrm{id} \otimes \hat{\Delta})(R) = R_{13} R_{12}, \tag{5.1}$$

$$\hat{\Delta}'(u) = R \hat{\Delta}(u) R^{-1}, \quad u \in U_h(g'(A_\infty)). \tag{5.2}$$

b) *The statement* a) *holds for $U_h(g(A_\infty))$.*

5.3. Remark. Writing $R = \sum_k R_k^{(1)} \otimes R_k^{(2)}$, the notation used is $R_{ij} = \sum_k 1 \otimes \cdots \otimes R_k^{(1)} \otimes \cdots \otimes R_k^{(2)} \otimes 1 \otimes \cdots$ with the non-unit factors at i and j entries.

5.4. Proof of Theorem 5.2. We'll construct an R-matrix for $U_h(g(A_\infty))$; the R-matrix for $U_h(g'(A_\infty))$ can be obtained from the R-matrix for $U_h(g(A_\infty))$ by substituting $-\sum_{j>0} E_{jj}$ for d (see Theorem 4.2).

Since $U_h(g(A_\infty))_f$ is dense in $U_h(g(A_\infty))$, it suffices to construct the R-matrix for $U_h(g(A_\infty))_f$.

5.5. We'll use the quantum double construction [D1]. Recall that the R-matrix is the image of the canonical element from $\mathscr{D}(U_h(b_+))_f \otimes \mathscr{D}(U_h(b_+)_f)^*$ under projection to $U_h(g(A_\infty))_f^{\otimes s}$. Here the subalgebra $U_h(b_+)_f \subset U_h(g(A_\infty))_f$ is a subalgebra generated by c, d, $\{E_{ij}\}_{i \leqq j}$ and the double $\mathscr{D}(A)$ of the Hopf algebra A is defined in [D1]. We omit the details. The realization of Drinfeld's approach to construction of the R-matrix in a finite-dimensional situation can be found in [R] or [Le S], [KR].

5.6. The basis in the $\mathbb{C}[[h]]$-module $U_h(b_+)_f$ consists of ordered monomials

$$\{\prod_{i,j} E_{ij}^{n_{ij}} c^k d^l\}$$

with finitely many non-zero exponents. Define linear functionals on $U_h(b_+)_f$ by the following conditions:

$$\begin{aligned} \langle \eta_{ij}, E_{ij}\rangle &= 1, \quad \text{and} \quad = 0 \quad \text{on other monomials;}\\ \langle \xi_c, c\rangle &= 1, \quad \text{and} \quad = 0 \quad \text{on other monomials;}\\ \langle \xi_d, d\rangle &= 1, \quad \text{and} \quad = 0 \quad \text{on other monomials;} \end{aligned}$$

and set $\eta_i = \eta_{i,i+1}$, $\xi_i = \eta_{ii}$. The same arguments as those in [R] give the following formula for the canonical element of $\mathcal{D}(U_h(b_+)_f) \otimes \mathcal{D}(U_h(b_+)_f)^*$

$$R = \prod_{i<j} \exp_{q^{-2}}(E_{ij} \otimes \eta_{ij}) \exp(\sum_i E_{ii} \otimes \xi_i + c \otimes \xi_c + d \otimes \xi_d). \tag{5.3}$$

5.7. Now, to derive from (5.3) the formula for the R-matrix, we have to establish the isomorphism φ: $U_h(b_+)_f^0 \to U_h(b_-)_f$. For this purpose we derive commutation relations between η_i, ξ_j, ξ_c, ξ_d and compute $\Delta\eta_i$, $\Delta\xi_j$, $\Delta\xi_c$, $\Delta\xi_d$.

5.8. Lemma. a) ξ_i, ξ_j, ξ_c, ξ_d *commute for all* i, j;

b) $[\xi_i, \eta_j] = -\frac{h}{2}(\delta_{ij} - \delta_{i,j+1})\eta_j$,

c) $[\xi_c, \eta_j] = -\frac{h}{2}\delta_{j0} h_j$,

d) $[\eta_i, \eta_j] = 0$ *if* $|i-j| > 1$ *and* $\eta_i^2 \eta_{i\pm 1} - (q+q^{-1})\eta_i \eta_{i\pm 1}\eta_i + \eta_{i\pm 1}\eta_i^2 = 0$,

e) $[\eta_i, \eta_{i+1,j}]_q = (1-q^2)\eta_{ij}$.

The proof is essentially the same as those of Lemma 2 and the corollary following it in [R].

5.9. Lemma. a) $\Delta\xi_i = \xi_i \otimes 1 + 1 \otimes \xi_i$,

$$\Delta\xi_c = \xi_c \otimes 1 + 1 \otimes \xi_c, \quad \Delta\xi_d = \xi_d \otimes 1 + 1 \otimes \xi_d.$$

b) $\Delta\eta_j = \eta_j \otimes 1 + \exp(\xi_j - \xi_{j+1} + \delta_{j0}\xi_d) \otimes \eta_j$.

Proof. a) is immediate.
b) $\delta\eta_j$ takes a non-zero value on $E_{j,j+1} \otimes 1$: $\langle \Delta\eta_j, E_{j,j+1} \otimes 1\rangle = 1$ and, possibly, on $\prod_i E_{ii}^{n_i} c^l d^k \otimes E_{j,j+1}$:

$$\begin{aligned} \langle \Delta\eta_j, \prod_i E_{ii}^{n_i} c^l d^k \otimes E_{j,j+1}\rangle &= \langle \eta_j, \prod_i E_{ii}^{n_i} c^l d^k \otimes E_{j,j+1}\rangle \\ &= \langle \eta_j, E_{j,j+1} \prod_i (E_{ii} + \delta_{ij} - \delta_{i,j+1})^{n_i} (E_{ii} + \delta_{j0})^k c^l\rangle \\ &= \delta_{l0} \prod_i (\delta_{ij} - \delta_{ij+1})^{n_i} \delta_{j0}^k. \end{aligned}$$

Hence,

$$\Delta\eta_j = \eta_j \otimes 1 + \sum_{l,k,n_i} \frac{(\xi_j - \xi_{j+1})^{n_i}}{n_i!} \frac{\xi_d^k}{k!} \otimes \eta_j$$

and b) is proved.

5.10. Lemma 5.8, d) shows that we can set $\varphi(\eta_j) = \lambda_j F_{j,j+1}$, where $\lambda_j \in \mathbb{C}[[h]]$ are invertible. By Lemma 5.8, c) we must set $\varphi(\xi_c) = \frac{h}{2} d$, and since ξ_d commutes with everything, we must have $\varphi(\xi_d) = \lambda_c$ with $\lambda \in \mathbb{C}[[h]]$ invertible. Further, we see that the conditions in Lemma 5.8, b) are satisfied with $\varphi(\xi_i) = \frac{h}{2} E_{ii}$; hence, the equality in Lemma 5.9, b) is satisfied with $\varphi(\xi_d) = \frac{h}{2} c$.

So, it remains to calculate λ_j, but this can be done as in [R]. The result is $\lambda_j = (1 - q^{-2})$, and, from Lemma 5.8, e) we derive easily $\varphi(\eta_{ij}) = (1 - q^{-2}) F_{ij}$.

Now we derive from (5.3) the formula for R-matrix for $U_h(g(A_\infty))_f$ (and, hence, for $U_h(g(A_\infty)))$:

$$R = \prod_{i<j} \exp_{q^{-2}}((1 - q^{-2}) E_{ij} \otimes E_{ji}) \cdot q^{\sum_i E_{ii} \otimes E_{ii} + c \otimes d + d \otimes c}. \tag{5.4}$$

Finally, note that (5.4) with $d = -\sum_{j>0} E_{jj}$ gives the formula for the R-matrix for $U_h(g'(A_\infty))$.

5.11. Set $\varrho = \sum_i j E_{jj}$. Then the square of the antipode equals to $\mathrm{Ad}(e^{h\varrho})$ and the general formula (valid in any quasitriangular Hopf algebra) give quantum Casimir element [D2]:

$$e^{-hc/2} = e^{-h\varrho} u, \quad u = \sum_k S(R_k^{(2)}) R_k^{(1)},$$

and the formula for action of the coproduct on it:

$$\Delta(e^{-hc/2}) = (e^{-hc/2} \otimes e^{-hc/2})(R_{21} R)^{-1}.$$

Using this result one can try to obtain the quantum analogue of the KP hierarchy (see [K, Chap. 14]).

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Communicated by N. Yu. Reshetikhin