Commun. Math. Phys. 139, 559–588 (1991)

Communications in
Mathematical
Physics
© Springer-Verlag 1991

Universality in Orbit Spaces of Compact Linear Groups*

G. Sartori and V. Talamini

Dipartimento di Fisica, Università di Padova, and INFN, Sezione di Padova, I-35100 Padova, Italy

Received July 2, 1990; in revised form February 28, 1991

Abstract. If $\{p^1(x), \ldots, p^q(x)\}$ is a minimal integrity basis of the ideal of polynomial invariants of a compact coregular linear group G, the orbit map

$$p=(p^1(x), \ldots, p^q(x)) : \mathbb{R}^n \to \mathbb{R}^q ,$$

yields a diffeomorphic image $\bar{\mathcal{S}} = p(\mathbb{R}^n) \subseteq \mathbb{R}^q$ of the orbit space $\mathbb{R}^n/G$. Starting from this fact, we point out some properties which are common to the orbit spaces of all the compact coregular linear groups of transformations of $\mathbb{R}^n$. In particular we show that a contravariant metric matrix $\hat{P}(p)$ can be defined in the interior of $\bar{\mathcal{S}}$, as a polynomial function of $(p^1, \ldots, p^q)$. We prove that the matrix $\hat{P}(p)$, which characterizes the set $\bar{\mathcal{S}}$, as it is positive semi-definite only for $p \in \bar{\mathcal{S}}$, can be determined as a solution of a canonical differential equation, which, for every compact coregular linear group, depends only on the number q and on the degrees of the elements of the minimal integrity bases. This allows to determine all the isomorphism classes of the orbit spaces of the compact coregular linear groups through a determination of the equivalence classes of the corresponding matrices $\hat{P}(p)$. For $q \leqq 3$ (orbit spaces with dimensions $\leqq 3$), the solutions $\hat{P}(p)$ of the canonical equation are explicitly determined and the number of their equivalence classes is shown to be finite. It is also shown that, with a convenient choice of the minimal integrity basis, the polynomial matrix elements of $\hat{P}(p)$ have only integer coefficients. Arguments are given in favour of the conjecture that our conclusions hold true for all values of q. Our results are relevant and lead to universality properties in the physics of spontaneous symmetry breaking.

1. Introduction

In theories in which the ground state of the system is determined by a stationary point of a potential which is invariant under the transformations of a compact

---

* Partially supported by INFN and Ministero della Pubblica Istruzione

linear group (henceforth abbreviated in CLG) G, the characterization of the schemes of spontaneous symmetry breaking rests on the determination of the stationary points of the potential. Owing to the high number of variables involved and the degeneracy of the stationary points along the orbits of G, this is generally a difficult problem to solve, even if a polynomial approximation is used for the potential.

A complete elimination of the degeneracies associated with the invariance properties of the potential is obtained if one uses as basic variables of the problem a set of polynomial invariant functions forming a minimal integrity basis for the ring of polynomial invariant functions of the group [G, J 1, 2, AS 1, 2]. In fact, it has been proved that any invariant polynomial or C^∞-function V can be written as a polynomial [H, N] or respectively C^∞-function [Sc 1] $\hat{V}$ of the elements of a minimal integrity basis. The function $\hat{V}$ has the same range of V, but is not plagued by the same degeneracies. Thus, $\hat{V}$ can be substituted for V in the determination of the stationary points [G, J 1], provided one knows the range $\bar{\mathscr{S}}$ of the values taken on by the elements of the minimal integrity basis [AS 1, 2]. The set $\bar{\mathscr{S}}$ has been shown to be semi-algebraic and to yield a concrete model of the orbit space of G [AS 1, 2, PS 1, 2], the geometric primary stratification of $\bar{\mathscr{S}}$ being strictly related to the isotropy type stratification of the orbit space. The elements of a minimal integrity basis can therefore be thought of, with some caution (there may be local or even global algebraic relations among the elements of a minimal integrity basis), as coordinates in orbit space.

As long as the potential is not specified, each point of $\bar{\mathscr{S}}$ can be seen as the representative of a possible ground state of the system, points lying on the same stratum representing ground states whose invariance groups are conjugated subgroups of G. The semi-algebraic set $\bar{\mathscr{S}}$ yields therefore a geometric picture of the possible configurations (phases) of the system after spontaneous symmetry breaking.

Often, invariance properties are the only bounds which are imposed on the potential beyond regularity and stability properties and/or bounds on the degree when the potential is a polynomial function. If the symmetry groups of the potentials of different theories share isomorphic orbit spaces, the potentials have the same formal expression and the same domain when written as functions in orbit space, despite the completely different physical meaning of the variables and parameters involved in the definition of the potentials. Thus, the problems of determining the geometric features of the phase space, the location and stability properties of the minima of the potential, the number of phases and the allowed phase transitions are identical in all these theories [AS 2, JMS, Sa 2].

The main purpose of this paper is to prove in detail a universality property of the orbit spaces of coregular CLG's (hereafter abbreviated in CCLG's), pointed out in [Sa 2]. In particular, we shall show that a polynomial contravariant metric matrix can be defined in the image $\bar{\mathscr{S}}$ of the orbit space of every CCLG. The polynomial equations and inequalities defining the semi-algebraic set $\bar{\mathscr{S}}$ can be obtained from semi-positivity conditions for the metric matrix [AS 1, 2, PS 1]. If the linear group is coregular, the metric matrix can be determined as a solution of a canonical differential equation [Sa 2]. We shall analyse the structure of the canonical equation and determine the initial conditions which must be imposed in order to select, among all its solutions, those corresponding to matrices which are positive semi-definite only in a semi-algebraic set with the same geometric structure of the orbit spaces of the CCLG's.

We have solved the canonical equation in all cases corresponding to 2- and 3-dimensional orbit spaces of compact coregular linear transformation groups with no fixed points. The method of solution is straightforward; therefore, in spite of the length of the calculations, which increases quickly with the dimensions q of the orbit spaces, it is conceivable that there should be no difficulty, at least in principle, in the solution of the canonical equation for higher values of q too.

The solutions we have determined share the following features:

1. For each choice of the number q of elements of the minimal integrity basis and of their homogeneity degrees $d_1 \geqq d_2 \geqq \ldots \geqq d_q = 2$, there is only a *finite* number of non-equivalent (with respect to minimal integrity basis transformations) solutions. This implies that the number of non-isomorphic orbit spaces for all the CCLG's whose integrity bases are characterized by the same numbers $(d_1, \ldots, d_q)$ is also finite.

2. With a convenient choice of the minimal integrity basis, the polynomial contravariant metric matrix has only integer coefficients.

3. Given the number q of the elements of a minimal integrity basis, only some sets of homogeneity degrees $(d_1, \ldots, d_q)$ are allowed: there are selection rules.

We have solved the canonical equation also for $q=4$, $d_1 \leqq 5$, running on a Vax 8600 an ad hoc Fortran program, which requires as input only the numbers $d_1, \ldots, d_q$. The results we have obtained cannot be written in a compact form and will not be reported in this paper; we have checked, however, that they satisfy the three conditions listed above.

The properties listed under items 1. and 2. have been conjectured in [Sa 2] to hold true for all q. For $q \leqq 3$ and for $q=4$, $d_1 \leqq 5$ the validity of the conjecture has therefore been checked.

The paper will be organized in the following way. In Sect. 2 we shall fix our notations and recall some more or less known results concerning compact transformation groups and the geometric approach to invariant theory, which will be relevant for our subsequent analysis. In Sect. 3 we shall prove that a polynomial contravariant metric matrix can be defined in the interior of the image $\bar{\mathcal{S}}$ of the orbit space of a CCLG and derive the canonical equation for CCLG's. The main properties of the canonical equation and of its solutions will be derived and discussed in Sect. 4. Sections 5 and 6 will be devoted to a reformulation of the problem for the action of the group G on the unit sphere of $\mathbb{R}^n$. The initial conditions to be imposed on the solutions of the canonical equation to make them acceptable will be discussed in Sect. 7 and in Sect. 8 we shall give all the acceptable solutions of the canonical equation for $q \leqq 3$. In Sect. 9 we shall draw our conclusions.

2. Mathematical Background

In this section, we shall first define most of our notations and recall some results concerning invariant theory and the geometry of orbit spaces of CLG's (see for instance [Br, Sc 3] and references therein), then we shall introduce the first definitions and the basic tools for our subsequent analysis.

Let G be a compact group of $n \times n$ matrices acting linearly in the Euclidean space $\mathbb{R}^n$. It will not be restrictive to assume that $G \subseteq SO_n(\mathbb{R})$. We shall denote by

$x=(x^1,\dots,x^n)$ a point of $\mathbb{R}^n$, by $g\cdot x$ the action of $g\in G$ on x and by $G_x=\{g\in G|g\cdot x=x\}$ the isotropy subgroup of G at x (the little group of x). As explained in [AS 2], the assumptions of reality and orthogonality are not restrictive for the physical applications described in the introduction.

As is well known, the isotropy subgroups $G_{g\cdot x}$, at all the points laying on the orbit $\Omega(x)=\{g\cdot x|g\in G\}$ through x, are conjugated to $G_x: G_{g\cdot x}=gG_xg^{-1}$, $\forall g\in G$. Therefore a whole class $[H]$ of conjugated subgroups of G (called an *orbit type*), can be associated to each orbit of G. All the points $x\in\mathbb{R}^n$ laying on orbits with the same orbit type form an *isotropy type stratum of the action of* G *in* $\mathbb{R}^n$, hereafter called simply a *stratum of* $\mathbb{R}^n$. Strata are in a one-to-one correspondence with orbit types and all their connected components are smooth manifolds with the same dimensions.

The orbit space of the action of G in $\mathbb{R}^n$ is the quotient space $\mathbb{R}^n/G$ defined by the equivalence relation between points belonging to the same orbit, endowed with the quotient topology and differentiable structure. The images in orbit space of the strata of $\mathbb{R}^n$ will be called *(isotropy type) strata of* $\mathbb{R}^n/G$; all their connected components are smooth submanifolds of $\mathbb{R}^q$ with the same dimensions.

Almost all the orbits of G, considered as points of $\mathbb{R}^n/G$, belong to a unique stratum Σ_p of $\mathbb{R}^n/G$, the *principal stratum*, which is a connected open dense subset of $\mathbb{R}^n/G$. The boundary $\bar{\Sigma}_p\backslash\Sigma_p$ of the principal stratum of $\mathbb{R}^n/G$ is the union of disjoint *singular* strata. All the strata laying on the boundary of a stratum Σ are open in $\bar{\Sigma}\backslash\Sigma$, if $\bar{\Sigma}\backslash\Sigma\neq\phi$.

The orbit type $[H]$ of a stratum Σ is contained in the orbit types $[H_b]$ of the strata Σ_b laying in its boundary, in the sense that every element of $[H]$ is conjugated to a proper subgroup of an element of $[H_b]$. The number of distinct orbit types of G is finite and there is a unique minimum orbit type, the *principal orbit type*, corresponding to the principal stratum.

A basic result of the geometric approach to invariant theory allows to build a faithful image of $\mathbb{R}^n/G$ in the following way.

According to a well known theorem due to Hilbert [H] (see also [N] for finite groups) and extended by Schwarz [Sc 1], any invariant polynomial [C^∞-] function $F:\mathbb{R}^n\to\mathbb{R}$, can be expressed as a polynomial [C^∞-] function $\hat{F}:\mathbb{R}^q\to\mathbb{R}$, of a minimal *finite* set of polynomial invariant functions $p(x)=(p^1(x),\dots,p^q(x))$, yielding an integrity basis for the ring $\mathbb{R}[\mathbb{R}^n]^G$ of the G-invariant polynomial functions of $x\in\mathbb{R}^n$:

$$\hat{F}(p(x))=F(x), \qquad x\in\mathbb{R}^n. \tag{2.1}$$

Only in order to simplify the presentation of our results, in the following we shall assume that G has *no fixed points* in $\mathbb{R}^n$. In this assumption the degrees of all the polynomials invariants are necessarily $\geqq 2$ and one of the elements of the minimal integrity basis, for instance p^q, can always be identified with the quadratic invariant $\|x\|^2$:

$$p^q(x)=\sum_1^n{}_i\,(x^i)^2. \tag{2.2}$$

The elements $p^a(x)$ of a minimal integrity basis can, and will always, be chosen to be homogeneous polynomial functions of x. There is obviously a certain freedom in the choice of the $p^a(x)$'s, however their number q and their degrees $d_1,\dots,d_q$ are only determined by the linear group G. In the following the elements of a minimal integrity basis will always be ordered so that

$$d_1\geqq d_2\geqq\dots\geqq d_q=2. \tag{2.3}$$

Hereafter, by a MIB we shall always mean a *minimal homogeneous integrity basis* for which the conventions of Eqs. (2.2) and (2.3) are satisfied.

In order to translate the possible homogeneity properties of the invariant polynomial $q(x)$ into corresponding restrictions on the polynomial $\hat{q}(p)$, related to $q(x)$ by Hilbert's theorem, we shall define the *weight* $w(m)$ *of a monomial* $m(p)$:

$$m(p)=c(p^1)^{k_1}\dots(p^q)^{k_q}, \qquad 0 \neq c \in \mathbb{R}, \tag{2.4a}$$

as the number $w(m)$:

$$w(m)=\sum_{a=1}^{q} d_a k_a . \tag{2.4b}$$

The *weight of a polynomial* $q(p)$ will be defined as the maximum weight of its monomials and the polynomial will be said to be *w-homogeneous* if all its monomials have the same weight.

For $q \geqq 2$, the conventions of Eqs. (2.2) and (2.3) are not sufficient to fix a MIB for a CLG. By Hilbert's theorem two MIB's, $\{p(x)\}$ and $\{p'(x)\}$, are connected by a relation of the following kind:

$$\begin{aligned} p'^{\alpha}(x)&=\hat{p}'^{\alpha}(p(x)), \qquad \alpha=1,\dots,q-1; \\ p'^{q}(x)&=p^{q}(x); \end{aligned} \tag{2.5}$$

where the $\hat{p}'^{\alpha}(p)$ are w-homogeneous polynomials in $p^1,\dots,p^q$.

If $\{p(x)\}$ is a MIB, the *orbit map*, $p=(p^1,\dots,p^q):\mathbb{R}^n\to\mathbb{R}^q$, maps all the points of $\mathbb{R}^n$ laying on the same orbit onto a unique point of $\mathbb{R}^q$ and induces a diffeomorphism of $\mathbb{R}^n/G$ onto a *semi-algebraic connected closed* subset $\bar{\mathcal{S}}=p(\mathbb{R}^n)$ of $\mathbb{R}^q$. As any semialgebraic set, the set $\bar{\mathcal{S}}$ is the disjoint union of finitely many connected semialgebraic differentiable varieties $\{E_i\}$, called the primary strata, such that the boundary of each E_i is empty or the union of lower dimensional primary strata, and each primary stratum is open in its closure. The orbit map maps in a one-to-one manner the connected components of the isotropy type strata of $\mathbb{R}^n$ onto the primary strata of $\bar{\mathcal{S}}$; the interior $\mathcal{S}$ of $\bar{\mathcal{S}}$ represents the principal stratum, the boundary $\bar{\mathcal{S}}\backslash\mathcal{S}$ hosts the singular strata. The set $\bar{\mathcal{S}}$ depends obviously on the choice of the MIB.

When there are no algebraic relations among the elements of a minimal integrity basis of G, the linear group G is said to be *coregular*.

Hereafter, unless differently stated, we shall limit our statements to CCLG's. For such groups, each choice of a MIB corresponds to the choice of a global coordinate system in the interior of $\mathbb{R}^n/G$ and these are the only coordinate systems we shall consider in the following.

A change of MIB (see Eq. (2.5)) induces a coordinate transformation in $\mathbb{R}^n/G$ and in $\mathbb{R}^q$, for which we shall use the following simplified notations:

$$\begin{aligned} p'^{a}&=p'^{a}(p), \qquad a=1,\dots,q-1; \\ p'^{q}&=p^{q}. \end{aligned} \tag{2.6}$$

where $p'^{a}(p)$ is a w-homogeneous polynomial function which only depends on the p^b's with weights $d_b \leqq d_a$. The transformations on $(p^1,\dots,p^q)$, defined in Eqs. (2.6), will be called MIB *transformations* (hereafter abbreviated in MIBT's).

The images $\bar{\mathcal{S}}$ of all the one dimensional $(q=1)$ orbit spaces of the CLG's with no fixed points reduce to the semi-axis $p_1 \geqq 0$. Therefore, in the following we shall only consider the less trivial cases $q \geqq 2$.

The Jacobian matrix $J(p)$ of a MIB transformation,

$$J^a{}_b(p) = \frac{\partial p'^a(p)}{\partial p^b} = \partial_b p'^a(p), \qquad a, b = 1, \dots, q, \tag{2.7}$$

is an upper block-triangular matrix. The diagonal blocks and the determinant do not depend on p. Owing to Eq. (2.6), the last row of $J(p)$ is fixed to be:

$$J^q{}_a(p) = \delta^q_a, \qquad a = 1, \dots, q. \tag{2.8}$$

The set of all MIBT's forms a group. In the following we shall be interested in polynomial tensor fields on $\mathbb{R}^q$ with respect to the transformations of this group. These tensor fields can be considered as natural extensions to $\mathbb{R}^q$ of tensor fields defined in the interior of $\mathbb{R}^n/G$.

Definition. *Let τ be a* rank n *contravariant tensor field with respect to* MIBT's. *It will be said w-homogeneous if, in any coordinate system, all its non-null components are w-homogeneous polynomials which satisfy the following conditions:*

$$w(\tau^{a_1 \dots a_n}) - w(\tau^{b_1 \dots b_n}) = \sum_1^n{}_k (d_{a_k} - d_{b_k}). \tag{2.9}$$

The weight of each component of a w-homogeneous contravariant tensor field is the same in all the coordinate systems in which the component does not vanish identically and will be defined as the weight of that component.

Since the differential of an orbit map maps the tangent space to the stratum of $\mathbb{R}^n$ at x onto the tangent space to the stratum of $\bar{\mathscr{S}}$ at $p = p(x)$ [Sa 1], the set $\bar{\mathscr{S}}$ can be characterized by means of a matrix $\hat{P}(p)$, defined in the following way [AS 1, 2]:

$$P^{ab}(x) = \langle \partial p^a(x), \partial p^b(x) \rangle = \hat{P}^{ab}(p(x)), \qquad a, b = 1, \dots, q, \tag{2.10}$$

where $\langle\, , \rangle$ denotes the Euclidean scalar product in $\mathbb{R}^n$ and, in the right-hand side the matrix elements $P^{ab}(x)$, which are G-invariant polynomials, have been expressed as polynomial functions of the elements of the MIB.

Equation (2.10) defines a polynomial matrix $\hat{P}(p)$ on $\mathbb{R}^q$. As shown in [AS 2] and [PS 1], for a given MIB, the image $\bar{\mathscr{S}}$ of the orbit space of a CCLG is the unique connected semi-algebraic subset of $\mathbb{R}^q$, where the matrix $\hat{P}(p)$ is positive semi-definite. The k-dimensional primary strata of $\bar{\mathscr{S}}$ are the connected components of the set

$$\hat{W}^{(k)} = \{p \in \mathbb{R}^q \mid \hat{P}(p) \geqq 0, \operatorname{rank} \hat{P}(p) = k\};$$

they coincide with the images of the connected components of the k-dimensional isotropy type strata of $\mathbb{R}^n/G$. A system of polynomial equations and inequalities defining $[\bar{\mathscr{S}}]\mathscr{S}$ are given by any set of conditions assuring the [semi-]positivity of $\hat{P}(p)$.

The correspondence between orbit spaces and matrices $\hat{P}$ is *not* one-to-one, owing to the freedom in the choice of the MIB. In fact, if $\{p(x)\}$ and $\{p'(x)\}$ are MIB's related to the same CCLG, then from Eqs. (2.10), (2.6), and (2.7) one obtains:

$$\hat{P}'(p'(p)) = J(p)\hat{P}(p)J^T(p), \tag{2.11}$$

where T denotes transposition.

Definition. *Two matrices $\hat{P}(p)$ and $\hat{P}'(p')$ satisfying Eq.* (2.11), *where $p' = p'(p)$ is a* MIBT, *will be said to be equivalent.*

We would like to stress that, in the definition just given, the two matrices are not necessarily associated to the same CCLG.

Definition. *The images* $\bar{\mathcal{S}}$ *and* $\bar{\mathcal{S}}'$ *in* $\mathbb{R}^q$ *of the orbit spaces of the* CCLG's G *and* G' *will be said to be isomorphic if* $\bar{\mathcal{S}}' = p'(\bar{\mathcal{S}})$ *and Eq.* (2.11) *is satisfied,* $p' = p'(p)$ *being a* MIBT. *The orbit spaces of the* CCLG's G *and* G' *will be said to be isomorphic if their images in* $\mathbb{R}^q$ *are isomorphic.*

There is a one-to-one correspondence between isomorphism classes of orbit spaces of CCLG's and the equivalence classes of matrices $\hat{P}$ just defined; thus, in order to classify the isomorphism classes of the orbit spaces of all the CCLG's, it is sufficient to classify the equivalence classes of matrices $\hat{P}(p)$.

3. Properties of the Matrix $\hat{P}(p)$

In this section we shall exhibit a set of necessary conditions which must be satisfied by the matrices $\hat{P}(p)$ of all CCLG's. They will be listed below under items P1–P5. Conditions P1–P4 will be derived as more or less immediate consequences of the definition of $\hat{P}(p)$ and of the homogeneity of the MIB. Condition P5 will be expressed as a differential equation, whose form, *for every* CCLG, only depends on the numbers q and $d_1, \ldots, d_q$. The *universality properties* of the orbit spaces of CLG's, proclaimed in the title of this paper, originate from this fact.

P1. *Symmetry*. The matrix $\hat{P}$ is a $q \times q$ symmetric matrix.

P2. *Homogeneity*. The matrix elements $\hat{P}^{ab}(p)$ are real w-homogeneous polynomial functions of p and

$$w(\hat{P}^{ab}) = d_a + d_b - 2\,; \tag{3.1a}$$

moreover, the form of the last row and column of $\hat{P}(p)$ is determined by the convention of Eq. (2.2):

$$\hat{P}^{qa}(p) = \hat{P}^{aq}(p) = 2d_a p^a\,, \qquad a = 1, \ldots, q\,. \tag{3.1b}$$

P3. *Tensor properties.* The matrix elements of $\hat{P}(p)$ transform as the components of a rank 2 contravariant tensor under MIBT's.

As recalled at the end of the preceding section, the image, $\bar{\mathcal{S}}$, of the orbit space, obtained through the orbit map associated to the MIB $\{p(x)\}$, can be characterized in the following way:

P4. *Positivity.* The matrix $\hat{P}(p)$, is positive semi-definite only on the closed connected semi-algebraic subset $\bar{\mathcal{S}} \subset \mathbb{R}^q$; in the interior $\mathcal{S}$ of $\bar{\mathcal{S}}$, the rank of $\hat{P}(p)$ is q; on the boundary it is lower and almost everywhere equal to $(q-1)$ [AS 1, 2, PS 1].

From P1, P4 and Eqs. (2.11) and (2.7) the following facts emerge:

1. The matrix $\hat{P}'(p')|_{p'=p'(p)}$ is positive [semi-]definite if and only if $\hat{P}(p)$ is.

2. The matrix $\hat{P}(p)$ yields the contravariant components of a positive definite metric tensor in $\mathcal{S}$ [Sa 2].

3. The determinant of the matrix $\hat{P}(p)$ is multiplied by a positive constant factor when the MIB is changed. It is therefore a relative invariant of the group of MIBT's.

In the following we shall adopt the standard convention of tensor calculus: sums from 1 to q on repeated indices $a, b, \ldots$, will be understood when this does not cause ambiguities. We shall also use the notation ∂^a for the contravariant derivative:

$$\partial^a = \hat{P}^{ab}(p)\frac{\partial}{\partial p^b}, \qquad a=1,\ldots,q. \tag{3.2}$$

The last necessary condition on $\hat{P}(p)$ we shall need in order to derive our results is the following:

P5. [Sa 2] *Boundary conditions.* If an irreducible polynomial factor $\imath(p)$ of $\det\hat{P}(p)$ vanishes on a $(q-1)$-dimensional component of the boundary of $\bar{\mathscr{S}}$, then it satisfies the following equation:

$$\partial^a \imath(p) = \lambda^a(p)\imath(p), \qquad a=1,\ldots,q, \tag{3.3}$$

where $\lambda(p)$ is a w-homogeneous contravariant vector field, depending on $\imath(p)$. We shall call Eq. (3.3) *canonical equation.*

Proof of P5. The determinant of $\hat{P}(p)$ is a w-homogeneous polynomial function of p and can be written as a product of (real) irreducible w-homogeneous polynomial factors. According to P4, at least one of these factors, say $\imath(p)$, has to vanish in a $(q-1)$-dimensional component of the boundary $\bar{\mathscr{S}}\backslash\mathscr{S}$ of $\mathscr{S}$. The algebraic set $W_\imath = \{x\in\mathbb{R}^n \mid \imath(p(x))=0\}$ is a component of the inverse image $W=p^{-1}(\bar{\mathscr{S}}\backslash\mathscr{S})\subset\mathbb{R}^n$ of the boundary of $\bar{\mathscr{S}}$. The gradient of $\imath(p(x))$ at a regular point $x\in W_\imath$ is *orthogonal* to $W_\imath$ at x. But the gradient at $x\in W_\imath$ of an invariant function is *tangent* to $W_\imath$ at x (see for instance [Br]). Therefore the gradient of $\imath(p(x))$ at $x\in W_\imath$ must vanish and, with the help of Eqs. (2.10), we obtain, for every $x\in W_\imath$:

$$0=\langle\partial p^a(x), \partial\imath(p(x))\rangle = \hat{P}^{ab}(p)\left.\frac{\partial\imath(p)}{\partial p^b}\right|_{p=p(x)}, \qquad a=1,\ldots,q, \tag{3.4}$$

whence:

$$\partial^a\imath(p)|_{\imath(p)=0}=0, \qquad a=1,\ldots,q, \tag{3.5}$$

which is equivalent to Eqs. (3.3), since $\imath(p)$ is irreducible.

Now, both $\det\hat{P}(p)$ and its irreducible factors are relative invariants under MIBT's. Therefore, if Eq. (3.3) is satisfied, $\lambda(p)$ is a w-homogeneous contravariant vector field. □

4. A Preliminary Analysis of the Canonical Equation

In the preceding section we started from the point of view that the group G and one of its MIB's were given, so that the matrix $\hat{P}(p)$ could be explicitly constructed and conditions P1–P5 where derived as *necessary* conditions.

In this section we shall analyse the properties of the polynomial solutions $[a(p), \lambda(p)]$ of the canonical equation:

$$\hat{P}^{ab}(p)\partial_b a(p) = \lambda^a(p)a(p), \qquad a=1,\ldots,q, \tag{4.1}$$

under the following assumption:

I. The numbers $d_1,\ldots,d_q$, and the matrix $\hat{P}(p)$ are given and satisfy conditions P1–P4 of Sect. 3.

For our subsequent analysis it will be advantageous to enlarge our analysis also to complex solutions $[a(p), \lambda(p)]: a(p), \lambda^a(p) \in \mathbb{C}[\mathbb{C}^q]$. As usual, when referring to a complex [real] polynomial, irreducibility will be meant on $\mathbb{C}$ [$\mathbb{R}$]. When not explicitly specified, our statements will be understood to hold both for complex and real solutions.

We shall need the following definitions:

Definition. *A solution $[a(p), \lambda(p)]$ of the canonical equation will be said to be irreducible if the polynomial $a(p)$ is irreducible.*

Definition. *The solutions $[a(p), \lambda(p)]$ and $[a'(p), \lambda'(p)]$ of the canonical equation will be said to be prime if $a(p)$ and $a'(p)$ are prime.*

The following theorem summarizes some relevant properties of the solutions of the canonical equation.

Theorem 4.1. *Let $[a(p), \lambda(p)]$ and $[a_1(p), \lambda_1(p)]$ be polynomial solutions of the canonical equation. Then the following statements* i)–v) *hold true:*

i) *$a(p)$ is a w-homogeneous relative invariant and $\lambda(p)$ a w-homogeneous contravariant vector field, with respect to* MIBT's, *and $\lambda^q(p) = 2w(a) = \text{const}$.*

ii) *The product $[ca(p)a_1(p), \lambda(p) + \lambda_1(p)]$, $0 \neq c \in \mathbb{C}$, and the complex conjugate $[\bar{a}(p), \bar{\lambda}(p)]$ are also solutions.*

iii) *If*

$$a(p) = \prod_1^k{}_r\, \iota_r(p)^{m_r}, \qquad m_r \in \mathbb{N}, \tag{4.2a}$$

is a decomposition of $a(p)$ into prime irreducible factors $\iota_r(p)$, then, for each $r = 1, \ldots, k$, there exists a w-homogeneous contravariant vector field $\lambda_r(p)$ such that $[\iota_r(p), \lambda_r(p)]$ is an irreducible solution of the canonical equation and

$$\lambda(p) = \sum_1^k m_r \lambda_r(p). \tag{4.2b}$$

iv) *If $a(p)$ is irreducible, it is a factor of* $\det \hat{P}(p)$.

v) *If $a(p) \in \mathbb{R}[\mathbb{R}^q]$ is irreducible and vanishes on a $(q-1)$-dimensional component $\mathscr{B}_a$ of the boundary of $\mathscr{S}$, then it has odd multiplicity as a factor of* $\det \hat{P}(p)$.

Proof of i). From Eq. (3.1b) and (4.1) we obtain

$$\partial^q a(p) = \sum_1^q{}_a\, 2d_a p^a \partial_a a(p) = \lambda^q(p) a(p), \tag{4.3a}$$

which assures that $a(p)$ is w-homogeneous and

$$\lambda^q(p) = 2w(a). \tag{4.3b}$$

Then, from Eq. (4.1) and the w-homogeneity of $a(p)$,

$$w(\lambda^a) = d_a - 2, \qquad a = 1, \ldots, q. \tag{4.3c}$$

Moreover, using Eq. (2.11) it is immediate to check that $[a'(p'), \lambda'(p')]$ is a solution of the canonical equation in the MIB $p' = \hat{p}'(p)$, provided that $a'(p') = a(p)$ and $\lambda'(p') = J(p)\lambda(p)$, where $J(p)$ is the Jacobian matrix of the MIBT.

Proof of ii). The proof reduces to a trivial check, which makes use, of the fact that $\hat{P}(p)$ is real.

Proof of iii). For each r let us set

$$a(p)=\sigma_r(p)\iota_r(p)^{m_r}, \tag{4.4}$$

where the polynomial $\sigma_r(p)$ is prime to $\iota_r(p)$. Then, from Eqs. (4.4) and (4.1),

$$m_r\sigma_r\partial^a\iota_r=\iota_r(-\partial^a\sigma_r+\lambda^a\sigma_r);$$

since $\iota_r(p)$ is prime to $\sigma_r(p)$ this implies:

$$\partial^a\iota_r(p)=\lambda_r^a(p)\iota_r(p), \qquad r=1,\dots,k;\ a=1,\dots,q, \tag{4.5}$$

where $\lambda_r(p)$ is a w-homogeneous contravariant vector field.

Equation (4.2b) is an immediate consequence of Eqs. (4.2a), (4.5) and of item ii) of this theorem.

Proof of iv). Let us first assume that $a(p)\in\mathbb{C}[\mathbb{C}^q]$ is irreducible. Then, its gradient $\partial a(p)$ is almost everywhere $\neq 0$ on $\mathscr{V}_a=\{p\in\mathbb{C}^q|a(p)=0\}$ and from Eq. (4.1) one easily realizes that the null space of the matrix $\hat{P}(p)$ is non-trivial for almost all $p\in\mathscr{V}_a$. This assures that $\det\hat{P}(p)$ vanishes everywhere in $\mathscr{V}_a$ and $a(p)$, which is irreducible, is necessarily a factor of $\det\hat{P}(p)$.

If $a(p)$ is real irreducible on $\mathbb{R}$, but reducible on $\mathbb{C}$, then it is proportional to the squared modulus of an irreducible complex polynomial. Owing to item ii), this polynomial and its complex conjugate are components of two irreducible complex solutions of the canonical equation. From what we have just proved, they must be both factors of $\det\hat{P}(p)$.

Proof of v). Let γ be an oriented path, crossing $\mathscr{B}_a$ at a regular point, p_0, of $\mathscr{B}_a$. If we shift p along γ, one and only one of the eigenvalues of $\hat{P}(p)$ changes its sign when p crosses $\mathscr{B}_a$ at p_0, since $\operatorname{rank}\hat{P}(p_0)=q-1$ and $\hat{P}(p)$ looses its semi-positivity outside $\mathscr{S}$. Therefore, $\det\hat{P}(p)$ changes its sign at p_0. Since, in our assumptions, the only irreducible factor of $\det\hat{P}(p)$ which vanishes at p_0 is $a(p)$, its multiplicity must be odd.

An obvious corollary of item iv) of Theorem 4.1 is the following:

Corollary 4.1. *The number of prime irreducible solutions* $[\iota(p),\lambda(p)]$ *of the canonical equation is finite and* $w(\iota)\leqq w(\det\hat{P})$.

It will be worthwhile to note the following facts:

Remark 4.1. From item iii) of Theorem 4.1 we learn that all the solutions of the canonical equation can be obtained from products of irreducible ones.

Remark 4.2. If $p_0\in\mathscr{S}$, then $\det\hat{P}(p_0)\neq 0$ and, from items iii) and iv) of Theorem 4.1, for all solutions $[a(p),\lambda(p)]$ of Eq. (4.1) $a(p_0)\neq 0$ too. So we can require that *all* the solutions of Eq. (4.1) be normalized to 1 at the same point p_0. A convenient choice for p_0 will be indicated later on.

Remark 4.3. As a consequence of the last argument in the proof of item iv) of Theorem 4.1, if $[\iota(p),\lambda(p)]$ is a real irreducible solution which is reducible on $\mathbb{C}$, then the algebraic set $\mathscr{V}_\iota=\{p\in\mathbb{R}^q|\iota(p)=0\}$ has dimension $\leqq q-2$.

We shall conclude this section by proving a proposition which will make much easier the derivation of our subsequent results and the calculation of explicit solutions of the canonical equation.

Proposition 4.1. *Let λ be a real w-homogeneous vector field on $\mathbb{R}^q$ such that $\lambda^q = 2d \neq 0$ is a constant. Then there exists a class Λ of coordinate systems $\{p\}$ in which $\lambda^\alpha(p) = 0$, $\alpha = 1, \ldots, q-1$. Any couple $\{p\}$ and $\{p'\}$ of elements of Λ is related by a* MIBT $p' = p'(p)$, *in which the functions $p'^\alpha(p)$, $\alpha = 1, \ldots, q-1$, do not depend on p^q.*

Proof. In our assumptions, $\lambda^a(p)$ can only depend on the p^b's whose weight is $\leqq d_a - 2$ and, in our conventions, this implies $b > a$. Let us first consider $\lambda^{q-1}(p)$. If d_{q-1} is odd, then $\lambda^{q-1}(p) = 0$. If d_{q-1} is even, then

$$\lambda^{q-1}(p) = c(p^q)^{(d_{q-1}-2)/2}, \qquad c \in \mathbb{R};$$

thus the following MIBT:

$$\begin{aligned} p'^\alpha &= p^\alpha, \quad \text{for} \quad \alpha = 1, \ldots, q-2; \\ p'^{q-1} &= p^{q-1} - \frac{c}{dd_{q-1}} (p^q)^{d_{q-1/2}}; \end{aligned} \tag{4.6}$$

leads to $\lambda'^{q-1}(p') = 0$.

Let us now assume inductively that $\lambda^a(p) = 0$, for $a = k+1, k+2, \ldots, q-1$, while $\lambda^k(p) \neq 0$.

The following MIBT:

$$\begin{aligned} p'^\alpha &= p^\alpha, \quad \text{for} \quad 1 \leqq \alpha \leqq q, \alpha \neq k, \\ p'^k &= p^k + h^k(p); \end{aligned} \tag{4.7}$$

where $h^k(p)$ is a w-homogeneous polynomial of weight d_k, leads to

$$\begin{aligned} \lambda'^a(p') &= \lambda^a(p), \quad \text{for} \quad a \neq k; \\ \lambda'^k(p') &= \lambda^k(p) + 2d\partial_q h^k(p). \end{aligned} \tag{4.8}$$

With the choice

$$h^k(p) = -\frac{1}{2d} \int_0^{p^q} dz \lambda^k(p^1, \ldots, p^{q-1}, z), \tag{4.9}$$

we obtain $\lambda'^a(p') = 0$, for $a = k, k+1, \ldots, q-1$, as we liked. An iteration of this procedure leads to a coordinate system $\{p''\}$ in which $\lambda''^\alpha(p'') = 0$, $\alpha = 1, \ldots, q-1$.

To prove the last claim in Proposition 4.1, let us assume $\lambda^\alpha(p) = 0$, for $\alpha = 1, \ldots, q-1$. A MIBT $p' = p'(p)$ leads to:

$$\lambda'^\alpha(p') = 2d\partial_q p'^\alpha(p), \qquad \alpha = 1, \ldots, q-1.$$

Therefore, $\lambda'^\alpha(p') = 0$, for $\alpha = 1, \ldots, q-1$, if and only if the $p'^\alpha(p)$ do not depend on p^q. □

According to Proposition 4.1, for each solution $[a(p), \lambda(p)]$ of Eq. (4.1), we can find a class of coordinate systems in which Eq. (4.1) reduces to

$$\partial^\alpha a(p') = 0, \qquad \alpha = 1, \ldots, q-1; \tag{4.10a}$$

$$\partial^q a(p') = 2w(a)a(p'). \tag{4.10b}$$

Definition. *Let $[a(p), \lambda(p)]$ be a solution of the canonical equation* (4.1) *in a given coordinate system $\{p\}$; if $\lambda^\alpha(p) = 0$ for all $\alpha = 1, \ldots, q-1$, then $\{p\}$ will be called an a-basis.*

We shall not need the following result, we think however it may have some interest. The symbol $(,)$ will denote the Euclidean scalar product in $\mathbb{R}^q$.

Proposition 4.2. *Let $[a(p), \lambda(p)]$ and $[a'(p), \lambda'(p)]$ be real solutions of the canonical equation. Then the following conditions* a) *and* b) *are equivalent:*

a) *The polynomials $a(p)$ and $a'(p)$ are prime.*

b) *The covector fields $\partial a(p)$ and $\partial a'(p)$ satisfy the following orthogonality relation:*

$$(\partial a(p), \hat{P}(p)\partial a'(p)) = 0\,. \tag{4.11}$$

Proof.
(a)→(b): Let us set

$$\Gamma(p) = (\partial a(p), \hat{P}(p)\partial a'(p))\,, \tag{4.12}$$

where $a(p)$ and $a'(p)$ are prime polynomials. Making use of the canonical equation and of the symmetry of $\hat{P}(p)$, we obtain for the w-homogeneous polynomial $\Gamma(p)$ the following two expressions:

$$\Gamma(p) = a(p)\,(\lambda(p), \partial a'(p)) = (\partial a(p), \lambda'(p))a'(p)\,.$$

Suppose now $\Gamma(p) \neq 0$; then the polynomials $a(p)$ and $a'(p)$, which are prime, must both be factors of $\Gamma(p)$ and this implies

$$w(\Gamma) \geqq w(a) + w(a')\,. \tag{4.13}$$

But the weight of each term in the sum on the second member of Eq. (4.12) is

$$\begin{aligned} w(\Gamma) &= w(a) + w(a') + (d_a + d_b - 2) - d_a - d_b \\ &= w(a) + w(a') - 2\,, \end{aligned}$$

which is inconsistent with Eq. (4.13), unless $\Gamma = 0$.

(b)→(a): Let

$$\begin{aligned} a(p) &= c \prod_r \iota_r^{m_r}(p)\,; \\ a'(p) &= c' \prod_r \iota_r^{m'_r}(p)\,; \end{aligned} \tag{4.14}$$

be a decomposition of $a(p)$ and $a'(p)$ into prime irreducible factors. In Eq. (4.14) the m_r's and the m'_r's are non-negative integers and the ι_r run on a complete set $\mathscr{I}$ of prime irreducible factors of $\det \hat{P}(p)$, normalized to 1 at a fixed point of $\mathscr{S}$. Then, from Eqs. (4.11) and (4.14) we obtain:

$$a(p)a'(p) \sum_{rr'} m_r m'_{r'} \iota_r(p)^{-1} \iota_{r'}(p)^{-1} (\partial \iota_r(p), \hat{P}(p)\partial \iota_{r'}(p)) = 0\,. \tag{4.15}$$

But, for $r \neq r'$, the polynomials $\iota_r(p)$ and $\iota_{r'}(p)$ are prime and, as just proved, Eq. (4.11) holds for their gradients. Thus, Eq. (4.15) reduces to

$$a(p)a'(p) \sum_r m_r m'_r \iota_r(p)^{-2} (\partial \iota_r(p), \hat{P}(p)\partial \iota_r(p)) = 0\,. \tag{4.16}$$

Now, for all $p \in \mathscr{S}$ and all r, the following inequalities hold (see Remark 4.2): $a(p)a'(p) \neq 0 \neq \iota_r(p)$; therefore also $\partial \iota_r \neq 0$ in $\mathscr{S}$, since $\iota_r(p)$ is a w-homogeneous polynomial. Since in $\mathscr{S}$, $\hat{P}(p) > 0$, we can conclude that, $(\partial \iota_r(p), \hat{P}(p)\partial \iota_r(p)) > 0$. As a consequence, the sum in the right-hand side of Eq. (4.16) can vanish only if $m_r m'_r = 0$ for all r. This means that $a(p)$ and $a'(p)$ are prime. □

5. Reduction to the Unit Sphere

Owing to the linearity of the action of G, the isotropy subgroups of G at points lying on the same straight line through the origin of $\mathbb{R}^n$ coincide; thus an essentially complete specification of the structure of the orbit space of the action of G in $\mathbb{R}^n$ is obtained from the orbit space of the action of G on the unit sphere $S^{n-1} \subset \mathbb{R}^n$. In fact, there is a bijection Φ mapping the set $\{\sigma\}$ of strata of S^{n-1}/G onto the set $\{\Sigma\}$ of strata of $(\mathbb{R}^n \backslash \{0\})/G$, such that $\Sigma = \Phi(\sigma)$ is homeomorphic to $\sigma \times \mathbb{R}_+$.

In our assumptions, in correspondence with each coordinate system in $\mathbb{R}^n/G$, an image of the orbit space S^{n-1}/G is yielded by the compact connected set $\bar{\mathscr{S}}_\pi$:

$$\bar{\mathscr{S}}_\pi = p(S^{n-1}) = \bar{\mathscr{S}} \cap \Pi ; \qquad \Pi = \{ p \in \mathbb{R}^q \,|\, p^q = 1 \} . \tag{5.1}$$

The projection I:

$$I : (p^1, \ldots, p^{q-1}, p^q) \to (p^1, \ldots, p^{q-1}) ,$$

induces a bijection of $\bar{\mathscr{S}}_\pi$ onto a *compact connected semi-algebraic* subset $\bar{\mathscr{S}}_1$ of $\mathbb{R}^{q-1}$:

$$\bar{\mathscr{S}}_1 = I(\bar{\mathscr{S}}_\pi) . \tag{5.2}$$

The set $\bar{\mathscr{S}}_1$, is a diffeomorphic image of the orbit space S^{n-1}/G.

In this section the results obtained until now will be adapted to the action of G on the unit sphere of $\mathbb{R}^n$.

We shall denote by π a generic point of $\mathbb{R}^{q-1}$. The restriction of a polynomial function $F(p)$ to Π will be identified to a polynomial function of $\pi = I(p)$ and denoted with the same symbol F:

$$F(\pi) \stackrel{\text{def}}{=} F(p)|_{p \in \Pi} , \qquad \pi = I(p) . \tag{5.3}$$

In the following, by a *polynomial function of* $\pi = I(p)$ we shall always understand the *restriction to* Π *of a w-homogeneous polynomial function of* p.

Analogously to what we have done for polynomial functions of p, we shall define the *weight* $w_1(m)$ *of the monomial* $m(\pi)$, $\pi = (\pi^1, \ldots, \pi^{q-1})$:

$$m(\pi) = c(\pi^1)^{k_1} \ldots (\pi^{q-1})^{k_{q-1}} , \qquad 0 \neq c \in \mathbb{R} , \tag{5.4a}$$

as the number $w_1(m)$:

$$w_1(m) = \sum_{1}^{q-1}{}_{\alpha} d_\alpha k_\alpha . \tag{5.4b}$$

The *weight* $w_1(a)$ *of a polynomial* $a(\pi)$ will be defined to be the maximum weight of its monomials and the polynomial will be said to be w_1-*homogeneous* if all its monomials have the same weight.

Remark 5.1. The restriction of a w-homogeneous polynomial function $a(p)$ to Π does not define, in general, a w_1-homogeneous polynomial function $a(\pi)$ and $w_1(a) = w(a) - 2k_q$, where k_q is the exponent of the power of p^q that factorizes in $a(p)$.

In analogy with what we have done with $\bar{\mathscr{S}}$, the semi-algebraic set $\bar{\mathscr{S}}_1$ and its stratification can be characterized through a matrix $\hat{Q}(\pi)$, defined in terms of the projections $\{\xi^\alpha(x)\}_{\alpha=1,\ldots,q-1}$ of the gradients of the elements $\{p^\alpha(x)\}_{\alpha=1,\ldots,q-1}$ of a MIB on the tangent space at $x\in\mathbb{R}^n$ to a sphere centered at 0:

$$\xi^\alpha(x)=\partial p^\alpha(x)-\frac{\langle x,\partial p^\alpha(x)\rangle}{\langle x,x\rangle}x\,,\qquad \alpha=1,\ldots,q-1\,. \tag{5.5}$$

Let us set

$$\hat{Q}^{\alpha\beta}(p(x))=\langle\xi^\alpha(x),\xi^\beta(x)\rangle\,\|x\|^2\,,\qquad \alpha,\beta=1,\ldots,q-1\,. \tag{5.6}$$

Since $\hat{Q}(p(x))$ is a polynomial function of $p(x)$ it admits a natural extension, $\hat{Q}(p)$, to $\mathbb{R}^q$. Using Eqs. (5.3), (5.5), and (5.6), for $\pi\in\mathbb{R}^{q-1}$ we obtain

$$\hat{Q}^{\alpha\beta}(\pi)=\hat{P}^{\alpha\beta}(\pi)-d_\alpha d_\beta\pi^\alpha\pi^\beta\,,\qquad \alpha,\beta=1,\ldots,q-1\,. \tag{5.7}$$

It will be useful to note that $\hat{Q}(p)$ can be obtained from $\hat{P}(p)$ through the following transformation:

$$\begin{pmatrix}\hat{Q}(p) & 0^T\\ 0 & 4(p^q)^2\end{pmatrix}=p^qK(p)\hat{P}(p)K(p)^T\,, \tag{5.8a}$$

where the left-hand side is expressed as a block matrix (0 denotes the $1\times(q-1)$ zero matrix) and

$$K(p)=\begin{pmatrix}1 & 0 & 0 & \ldots & -d_1p^1/2p^q\\ 0 & 1 & 0 & \ldots & -d_2p^2/2p^q\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & 0 & 1\end{pmatrix}. \tag{5.8b}$$

Equations (5.8) assure that $\hat{P}(p)$ and $\hat{Q}(p)$ have the same positivity properties and

$$(p^q)^{q-2}\det\hat{P}(p)=4\det\hat{Q}(p)\,. \tag{5.9}$$

The matrix $\hat{Q}(\pi)$, $\pi\in\mathbb{R}^{q-1}$, is symmetric and it is positive semi-definite *only* for $\pi\in\mathscr{S}_1$; the rank of $\hat{Q}(\pi)$ is $q-1$ in the interior of $\mathscr{S}_1$ and, on the boundary, it is $\leqq(q-2)$ and almost everywhere equal to $q-2$.

The MIBT's on $\{p\}$ induce on $\{\pi\}$ the following MIB *transformations*, which are simply obtained from Eq. (2.6) by setting $p^q=1=p'^q$:

$$\pi'^\alpha=\pi'^\alpha(\pi)\,,\qquad \alpha=1,\ldots,q-1\,, \tag{5.10}$$

where $\pi'^\alpha(\pi)$ is a polynomial function of $\pi\in\mathbb{R}^{q-1}$. The polynomial $\pi'^\alpha(\pi)$ depends only on the π^β's with weights $d_\beta\leqq d_\alpha$. As a consequence the Jacobian matrix $J(\pi)$ of the transformation,

$$J^\alpha{}_\beta(\pi)=\partial_\beta\pi'^\alpha(\pi)\,,\qquad \alpha,\beta=1,\ldots,q-1\,, \tag{5.11}$$

is a $(q-1)\times(q-1)$ upper block-triangular matrix and the diagonal blocks and the determinant do not depend on π.

The matrix elements of $\hat{Q}(\pi)$ yield the components of a rank two contravariant symmetric tensor field with respect to MIBT's on π,

$$\hat{Q}'(\pi'(\pi))=J(\pi)\hat{Q}(\pi)J^T(\pi)\,. \tag{5.12}$$

The matrix $\hat{Q}(\pi)$ can therefore be used as a contravariant metric matrix in $\mathscr{S}_1$.

The restriction of the canonical equation to Π can also be expressed in terms of the matrix $\hat{Q}(\pi)$. To this end, let us note that $a(p)$ cannot factorize p^q, since p^q is an irreducible polynomial that does not solve the canonical equation. Therefore (recall Remark 5.1), in Eq. (4.3b) the weight $w_1(a)$ of $a(\pi)$ can be substituted for the weight $w(a)$ of $a(p)$ and using also Eq. (5.7), the canonical equation can be rewritten, for $p^q=1$, in the following form:

$$\sum_{\beta=1}^{q-1} \hat{Q}^{\alpha\beta}(\pi)\partial_\beta a(\pi)=[\lambda^\alpha(\pi)-w_1(a)d_\alpha\pi^\alpha]a(\pi)=\mu^\alpha(\pi)a(\pi), \qquad \alpha=1,\dots,q-1. \tag{5.13}$$

In Eq. (5.13) *no sum over repeated greek indices is understood.* The same convention will be followed in the rest of the paper. Equation (5.13) will be called the *canonical equation* for the action of G in S^{n-1}.

Remark 5.2. In the matrix elements of $\hat{Q}(\pi)$ the terms of dominant weight are the terms $d_\alpha d_\beta \pi^\alpha\pi^\beta$; they cannot be cancelled by terms coming from matrix elements of $\hat{P}(\pi)$, which have lower weight.

The characterization of the solutions of the canonical equation obtained in the preceding sections can be easily translated into analogous results concerning the solutions of Eq. (5.13).

In particular, from Theorem 4.1 we obtain the results collected in the following theorem:

Theorem 5.1. *Let* $[a(\pi),\mu(\pi)]$ *and* $[a'(\pi),\mu'(\pi)]$ *be polynomial solutions of the canonical equation* (5.13). *Then the following statements* i)–v) *hold true:*

i) *$a(\pi)$ is the restriction to $\mathbb{R}^{q-1}$ of a w-homogeneous polynomial function $a(p)$, $p\in\mathbb{R}^q$, and $w_1(a)=w(a)$; $\mu^\alpha(\pi)$ is a contravariant vector field.*

ii) *The product* $[ca(\pi)a'(\pi),\mu(\pi)+\mu'(\pi)]$, $0\neq c\in\mathbb{C}$, *and the complex conjugate* $[\bar{a}(\pi),\bar{\mu}(\pi)]$ *are also solutions.*

iii) *If* $a(\pi)=\prod_{r=1}^{k}\iota_r(\pi)^{m_r}$, $m_r\in\mathbb{N}$, *is a decomposition of $a(\pi)$ into prime irreducible factors $\iota_r(\pi)$, then, for each $r=1,\dots,k$, there exists a contravariant polynomial vector field $\mu_r(\pi)$ such that $[\iota_r(\pi),\mu_r(\pi)]$ is an irreducible solution of the canonical equation and* $\mu(\pi)=\sum_{r=1}^{k} m_r\mu_r(\pi)$.

iv) *If $a(\pi)$ is irreducible, it is a factor of* $\det\hat{Q}(\pi)$.

v) *If $a(\pi)\in\mathbb{R}[\mathbb{R}^{q-1}]$ is irreducible and vanishes on a $(q-2)$-dimensional component $\mathscr{B}_a$ of the boundary of $\mathscr{S}_1$, then it has odd multiplicity as a factor of* $\det\hat{Q}(\pi)$.

Definition. *If $\{p\}$ is an a-basis then $\{\pi\}=(p^1,\dots,p^{q-1})$ will also be called an a-basis.*

According to Proposition 4.1, two a-bases $\{\pi\}$ and $\{\pi'\}$ are related by w_1-homogeneous MIBT's. In an a-basis $\{\pi\}$, Eq. (5.13) reduces to the following simpler form:

$$\sum_{\beta=1}^{q-1} \hat{Q}^{\alpha\beta}(\pi)\partial_\beta a(\pi)=-w_1(a)d_\alpha\pi^\alpha a(\pi), \qquad \alpha=1,\dots,q-1. \tag{5.14}$$

6. Complete Active Factors

In this section we shall continue our analysis of the canonical equation for the action of G in S^{n-1}. In particular we shall define the notion and point out the properties of complete active factors of $\det\hat{Q}(\pi)$. As in the preceding sections the matrix $\hat{P}(p)$ will be assumed to be given and to satisfy conditions P1–P4 of Sect. 3.

Definition. *The normalized (see Remark 4.2) factor $a(\pi)$ of $\det\hat{Q}(\pi)$ will be said to be active if, for some $\mu(\pi)$, the couple $[a(\pi), \mu(\pi)]$ is a solution of the canonical equation. An active factor $A(\pi)$ will be said to be a complete factor if it vanishes on the whole boundary of $\bar{\mathscr{S}}_1$.*

If $A(\pi)$ is a complete factor, the set of its divisors includes all the irreducible active factors.

Lemma 6.1. *An active factor $a(\pi)$ has at most one stationary point, π_0, outside the set of its zeroes. If π_0 is a stationary point of $a(\pi)$ and $a(\pi_0)\neq 0$, then, in all a-bases, $\pi_0=0$.*

Proof. Let $a(\pi)$ be an active factor. If, in an a-basis $\{\pi\}$, $a(\pi)$ has a stationary point at π_0 then, from Eq. (5.14),

$$\pi_0^\alpha a(\pi_0)=0\,, \qquad \alpha=1,\dots,q-1\,. \tag{6.1}$$

Owing to the covariance of $\partial_\alpha a(\pi)$, in all coordinate systems the stationary point will still be unique. □

A fundamental result for our subsequent analysis is stated in the following proposition:

Proposition 6.1. *Every complete factor $A(\pi)$ of $\det\hat{Q}(\pi)$ has a unique absolute maximum at a point π_0, which lays in $\mathscr{S}_1$. In all A-bases, $\pi_0=0$.*

Proof. Since the polynomial $A(\pi)$ is a complete factor of $\det\hat{Q}(\pi)$, it is positive in $\mathscr{S}_1$ and vanishes on the whole boundary of $\bar{\mathscr{S}}_1$. Being $\bar{\mathscr{S}}_1$ a compact connected subset of $\mathbb{R}^{q-1}$, $A(\pi)|_{\pi\in\hat{\mathscr{S}}_1}$ has a maximum at an interior point of $\bar{\mathscr{S}}_1$ and degenerate minima on the boundary. But, according to Lemma 6.1, $A(\pi)$ has at most one stationary point outside the set of its zeroes, at the point $\pi=0$ in all A-bases. □

The results stated in Proposition 6.1 offer a natural choice for the point where to normalize all the active factors $a(\pi)$. Hereafter, for every active factor we shall require

$$a(0)=1\,, \quad \text{in all } a\text{-bases}\,. \tag{6.2}$$

After differentiating Eq. (5.14) at $\pi=0$, and defining the matrices D and $H(\pi)$:

$$D^\alpha_\beta=d^\beta\delta^\alpha_\beta\,; \qquad H_{\alpha\beta}(\pi)=\partial_\alpha\partial_\beta A(\pi)\,; \qquad \alpha,\beta=1,\dots,q-1\,, \tag{6.3a}$$

with the help of Lemma 6.1 and Eq. (6.2) we obtain, for every complete factor $A(\pi)$ in an A-basis $\{\pi\}$, the following expression for the Hessian matrix at $\pi=0$:

$$H(0)=-w_1(A)\hat{Q}^{-1}(0)D\,, \tag{6.3b}$$

so that $H(0)<0$, in agreement with Proposition 6.1.

Proposition 6.2. *Let $A(\pi)$ be a complete factor of* $\det\hat{Q}(\pi)$. *Then, in all A-bases the matrix $\hat{Q}(\pi)$ is block diagonal at $\pi=0$ and in a subclass Λ_0 of A-bases $\hat{Q}(0)=\mathbb{1}$. Two elements of Λ_0 are related by a w_1-homogeneous* MIBT *whose Jacobian matrix $J(\pi)$ is orthogonal at $\pi=0$.*

Proof. Let us define the cofactor matrix $\hat{Q}^*(\pi)$:

$$\hat{Q}^*(\pi)=\hat{Q}^{-1}(\pi)\det\hat{Q}(\pi), \tag{6.4a}$$

and

$$\det\hat{Q}(\pi)=\sigma(\pi)A(\pi). \tag{6.4b}$$

Then, in a generic A-basis $\{\pi\}$ we obtain from Eq. (5.14)

$$\sigma(\pi)\partial_\alpha A(\pi)=-w_1(A)\sum_{1}^{q-1}{}_\beta\,\hat{Q}^*_{\alpha\beta}(\pi)d_\beta\pi^\beta. \tag{6.4c}$$

After differentiating Eq. (6.4c) with respect to π^γ and taking the antisymmetric part in the couple of indices $\alpha\gamma$, we obtain

$$\begin{aligned}0=&\,\partial_\alpha\sigma(\pi)\partial_\gamma A(\pi)-\partial_\gamma\sigma(\pi)\partial_\alpha A(\pi)+w_1(A)\hat{Q}^*_{\alpha\gamma}(\pi)(d_\alpha-d_\gamma)\\&+w_1(A)\sum_{1}^{q-1}{}_\beta(\partial_\alpha\hat{Q}^*_{\gamma\beta}(\pi)-\partial_\gamma\hat{Q}^*_{\alpha\beta}(\pi))d_\beta\pi^\beta.\end{aligned} \tag{6.5}$$

At $\pi=0$ Eq. (6.5) reduces to

$$(d_\alpha-d_\gamma)\hat{Q}^*_{\alpha\gamma}(0)=0, \tag{6.6}$$

which implies

$$\hat{Q}_{\alpha\gamma}(0)=0, \quad \text{for} \quad d_\alpha\neq d_\gamma. \tag{6.7}$$

In the coordinate system we have chosen, the matrix $\hat{Q}(0)$ is block diagonal.

In order to prove the existence of the subclass Λ_0 of A-bases in which $\hat{Q}(0)=\mathbb{1}$, let us denote by $\mathscr{D}=\{d^{(1)},\dots,d^{(r)}\}$, $d^{(1)}>\dots>d^{(r)}$, the set of all distinct weights $d_1,\dots,d_{q-1}$, and let $\pi_{(i)}=(\pi^{1+n_{i-1}},\dots,\pi^{n_i})$ denote a vector whose components are all the π^α with the same weight $d^{(i)}$, $i=1,\dots,r$, where $q-1=n_r>n_{r-1}>\dots>n_0=0$. Then the general form of a linear MIBT can be written, in matrix notation,

$$\pi'_{(i)}=J_{(i)}\pi_{(i)}, \qquad i=1,\dots,r, \tag{6.8}$$

where $J_{(i)}$ is an arbitrary regular $(n_i-n_{i-1})\times(n_i-n_{i-1})$ real constant matrix.

From Eqs. (5.12) and (6.8),

$$\hat{Q}'(0)=J(0)\hat{Q}(0)J^T(0)=\mathrm{diag}(J_{(i)}\hat{Q}_{(i)}(0)J^T_{(i)}). \tag{6.9}$$

Now, $J_{(i)}$ can always be factorized in the following way:

$$J_{(i)}=O_{(i)}K_{(i)}O'_{(i)}, \qquad i=1,\dots,r, \tag{6.10}$$

where $O_{(i)}$ and $O'_{(i)}$ are real orthogonal matrices and $K_{(i)}$ is a regular real diagonal matrix. Moreover, since $\hat{Q}_{(i)}(0)$ is real symmetric and positive, $O'_{(i)}$ and $K_{(i)}$ can be chosen in such a way that $K_{(i)}O'_{(i)}\hat{Q}_{(i)}(0)O'^T_{(i)}K_{(i)}=\mathbb{1}$, and, consequently, $\hat{Q}'(0)=\mathbb{1}$, whichever is the orthogonal matrix $O_{(i)}$. □

7. Allowable Matrices $\hat{P}(p)$

In the preceding sections we started from the point of view that the matrix $\hat{P}(p)$ was known in a given MIB and analysed the properties of the solutions of the canonical equation in $\mathbb{R}^q$ and in $\mathbb{R}^{q-1}=I(\Pi)$.

Now, we have got all the elements we need to enlarge the analysis of the real solutions of the canonical equation to the following situation. Using as input only the numbers $d_1 \geqq d_2 \geqq \ldots \geqq d_q = 2$, $(q \geqq 2)$, which fix the number and weights of the elements of an arbitrary MIB of a generic CCLG, and conditions P1–P3 of Sect. 3, which partially fix the form of the contravariant metric matrix $\hat{P}(p)$, we shall try to determine the initial conditions to be imposed on the *real* solutions of the canonical equation, in order to single out the contravariant metric matrices $\hat{P}(p)$ which satisfy the following conditions A1–A2:

A1) The matrix $\hat{P}(p)$ is positive definite *only* in the interior points of a connected q-dimensional closed semi-algebraic subset $\bar{\mathscr{R}}$ of $\mathbb{R}^q$.

A2) The set $\bar{\mathscr{R}}_1 = I(\bar{\mathscr{R}} \cap \Pi)$ is compact.

Definition. *A solution* $[\hat{P}(p), A(p), \lambda(p)]$ *of the canonical equation and the associated matrix* $\hat{P}(p)$, *will be said to be allowable, if* $\hat{P}(p)$ *satisfies conditions* A1–A2.

All the allowable solutions of the canonical equation, for $q \leqq 3$, will be determined in the following section.

From the results obtained until now, it is not difficult to realize that all the allowable solutions of the canonical equation can be obtained, through MIBT's, from a subset of the solutions of the canonical equation written in the following form (see Eq. (5.14)):

$$\sum_{1}^{q-1}{}_{\beta} \hat{Q}^{\alpha\beta}(\pi)\partial_\beta A(\pi) = -w_1(A) d_\alpha \pi^\alpha A(\pi)\,, \tag{7.1}$$

where we attach the following meaning to the symbols:

i) $\hat{Q}(\pi)$ is a $(q-1)\times(q-1)$ matrix defined in terms of a $q \times q$ matrix $\hat{P}(p)|_{p^q=1}$ as in Eq. (5.7). The matrix $\hat{P}(p)$ is assumed to satisfy conditions P1–P3 of Sect. 3.

ii) The matrix $\hat{Q}(\pi)$ is given and positive definite for $\pi=0$; we shall assume (in agreement with Proposition 6.2 and the fact that it is always possible to rescale the p^α, $\alpha=1,\ldots,q-1$)

$$\hat{Q}(0) = D^2\,. \tag{7.2}$$

iii) $A(\pi)$ is a real polynomial function of $\pi \in \mathbb{R}^{q-1}$.

A couple $[\hat{Q}(\pi), A(\pi)]$ will be considered to be a solution of Eq. (7.1) *only if* $\hat{Q}(\pi)$ and $A(\pi)$ satisfy also conditions i)–iii) above. A solution will be said to be *complete*, if $A(\pi)$ is a complete factor of $\det \hat{Q}(\pi)$.

It will be worthwhile to recall that a complete active factor $A(\pi)$ satisfies, in particular, the following conditions:

$$A(\pi)|_{\pi=0} = 1\,; \tag{7.3a}$$

$$w_1(A) \leqq w(\det \hat{P}) = 2\sum_{1}^{q}{}_{\alpha} d_\alpha - 2q\,. \tag{7.3b}$$

From the results proved in the preceding sections, we already know that, in correspondence with every given allowable matrix $\hat{P}(\pi)$, there is at least one complete solution of Eq. (7.1) in an A-basis. In the following section we shall show that all the complete solutions of Eq. (7.1) are indeed allowable solutions for $q \leqq 3$. We have checked that the same result holds true for $q=4$, $d_1 \leqq 5$ too; we do not know if its validity extends to all natural values of q, below (see Theorems 7.1 and 7.2), however, we shall show that the restrictions coming from conditions i)–iii) following Eq. (7.1), are indeed quite restrictive, if used in the aim of selecting, among all the solutions of the canonical equation, the allowable ones.

Theorem 7.1. *If $\hat{Q}(\pi)$ is defined by the conditions* i)–ii) *following Eq.* (7.1), *the set $\bar{\mathscr{R}}_1 = \{\pi \in \mathbb{R}^{q-1} \mid Q(\pi) \geqq 0\}$ is a compact semi-algebraic $(q-1)$-dimensional subset of $\mathbb{R}^{q-1}$.*

Since the proof of Theorem 7.1 is quite long, it will be postponed to the statement of Theorem 7.2; this includes in its assumptions the validity of the following condition iv), which is clearly a property of all the allowable solutions of the canonical equation:

iv) If $\varrho(\pi)$ is an irreducible *non-active* factor of $\det \hat{Q}(\pi)$ and

$$\mathscr{V}_\varrho = \{\pi \in \mathbb{R}^{q-1} \mid \varrho(\pi) = 0\},$$

then the restriction of $\hat{Q}(\pi)$ to $\mathscr{V}_\varrho$ can be positive semi-definite at most on an algebraic subset of $\mathscr{V}_\varrho$ of dimension $\leqq q-3$, where also $A(\pi)=0$.

For $q \leqq 3$ and $q=4$, $d_1 \leqq 5$, we have checked that condition iv) is a consequence of the structure of Eq. (7.1) and of the conditions i)–iii) following it. This is likely to be true for all values of q but, by now, we have not been able to find a general proof of this conjecture. In the following Theorem 7.2, therefore, condition iv) will be accepted as an additional assumption.

Theorem 7.2. *If $[\hat{Q}(\pi), A(\pi)]$ is a complete solution of Eq.* (7.1) *satisfying also condition* iv), *there is a unique $(q-1)$-dimensional compact connected semi-algebraic subset $\bar{\mathscr{R}}_1 \subset \mathbb{R}^{q-1}$, where $\hat{Q}(\pi) \geqq 0$.*

Proof of Theorem 7.1. The continuity of $\hat{Q}(\pi)$ and condition ii) assure that $\bar{\mathscr{R}}_1$ is non-vacuum and has dimension $(q-1)$; moreover, the condition $\hat{Q}(\pi) \geqq 0$ can only define a closed subset of $\mathbb{R}^{q-1}$. Therefore, in order to prove the compactness of $\bar{\mathscr{R}}_1$ we need only prove its boundedness.

A necessary condition for the semi-positivity of $\hat{Q}(\pi)$ is clearly the following:

$$\hat{Q}^{\alpha\alpha}(\pi) \geqq 0, \qquad \alpha = 1, \ldots, q-1. \tag{7.4}$$

We shall prove that these conditions can only be satisfied in a bounded subset of $\mathbb{R}^{q-1}$.

The general form of $\hat{Q}^{\alpha\alpha}(\pi)$ allowed by our present assumptions is the following:

$$d_\alpha^{-2} Q^{\alpha\alpha}(\pi) = -(\pi^\alpha)^2 + \sum_k a_k^{(\alpha)} \prod_{1}^{q-1}{}_\beta (\pi^\beta)^{k_\beta}, \qquad \alpha = 1, \ldots, q-1, \tag{7.5a}$$

where

$$k = (k_1, \ldots, k_{q-1}) \in \mathbb{N}_0^{q-1} \tag{7.5b}$$

and the real coefficient $a_k^{(\alpha)}$ may be $\neq 0$ only if the weight of the associated monomial is an even number that does not exceed the weight, $(2d_\alpha-2)$, of $\hat{P}^{\alpha\alpha}(p)$:

$$\sum_1^{q-1}{}_\beta\, d_\beta k_\beta \leqq 2d_\alpha - 2\,, \qquad \alpha=1,\dots,q-1\,. \tag{7.5c}$$

Let us denote by $\mathbb{N}^{(\alpha)}$, the set of all the solutions $k\in\mathbb{N}_0^{q-1}$ of Eqs. (7.5c), by $|a|$ the maximum absolute value of the numerical coefficients in all the diagonal matrix elements $\hat{P}^{\alpha\alpha}(\pi)$, and by v the maximum number of elements in every $\mathbb{N}^{(\alpha)}$, i.e. the maximum allowed number of non-zero monomials in every $\hat{P}^{\alpha\alpha}(\pi)$. Then, defining also

$$|\pi^0| = v|a|\,, \tag{7.6}$$

from Eqs. (7.5) and (7.6), for every arbitrarily fixed point $\pi\in\mathbb{R}^{q-1}$, we obtain the following trivial majorizations:

$$\begin{aligned} 0 \leqq d_\alpha^{-2}\hat{Q}^{\alpha\alpha}(\pi) &\leqq -(\pi^\alpha)^2 + \sum_{k\in\mathbb{N}^{(\alpha)}} \left| a_k^{(\alpha)} \prod_1^{q-1}{}_\beta (\pi^\beta)^{k_\beta} \right| \\ &\leqq -(\pi^\alpha)^2 + |\pi^0| \max_{k\in\mathbb{N}^{(\alpha)}} \left\{ \prod_1^{q-1}{}_\beta |\pi^\beta|^{k_\beta} \right\} \\ &= -(\pi^\alpha)^2 + \prod_0^{q-1}{}_B |\pi^B|^{k_B^{(1;\alpha)}}\,, \qquad \alpha=1,\dots,q-1\,, \end{aligned} \tag{7.7a}$$

where

$$k_0^{(1;\alpha)} = 1\,, \qquad \alpha=1,\dots,q-1\,, \tag{7.7b}$$

and $k^{(1;\alpha)}=(k_1^{(1;\alpha)},\dots,k_{q-1}^{(1;\alpha)})$ denotes the choice of $k\in\mathbb{N}^{(\alpha)}$ for which the product $\prod_1^{q-1}{}_\beta |\pi^\beta|^{k_\beta}$ assumes the maximum value (for the given π). The apex 1 on the exponents $k_B^{(1;\alpha)}$ will be used to enumerate the successive steps of an iterative procedure, whose starting points are Eqs. (7.7) and the following Eq. (7.9).

Since Eq. (7.5c) implies

$$\sum_1^{q-1}{}_\beta\, d_\beta k_\beta^{(1;\alpha)} \leqq 2d_\alpha - 2\,, \qquad \alpha=1,\dots,q-1\,, \tag{7.8a}$$

and, therefore,

$$k_\beta^{(1;\alpha)} < 2\,, \quad \text{for} \quad \beta \leqq \alpha = 1,\dots,q-1\,, \tag{7.8b}$$

for $\alpha=q-1$, Eqs. (7.7) yield the following necessary condition for the semi-positivity of $\hat{Q}^{\alpha\alpha}(\pi)$:

$$|\pi^{q-1}| \leqq \left(\prod_0^{q-2}{}_B |\pi^B|^{k_B^{(1;q-1)}} \right)^{1/(2-k_{q-1}^{(1;q-1)})}\,. \tag{7.9}$$

For $q=2$, the right-hand side of Eq. (7.9) is a constant and the theorem is therefore proved. For $q>2$ the exponent $(2-k_{q-1}^{(1;q-1)})^{-1}$, in Eq. (7.9), is positive and $|\pi^{q-1}|$ is majorized by a product of rational powers of $|\pi^B|$, $B=1,\dots,q-2$, whose total w_1-weight is strictly lower than the weight of $|\pi^{q-1}|$. Here, and in the rest of the proof of Theorem 7.1, we make use of a natural extension of the notion of w_1-weight to the case of rational powers of the π^α's.

If now, in the last member of Eq. (7.7a) the expression forming the right-hand side of Eq. (7.9) is substituted for $|\pi^{q-1}|$, the product will be majorized by a product of rational powers of $|\pi^B|$, $B=1,\dots,q-2$, of non-larger weight:

$$0 \leqq d_\alpha^{-2}\hat{Q}^{\alpha\alpha}(\pi) \leqq -(\pi^\alpha)^2 + \prod_{0}^{q-2}{}_B |\pi^B|^{k_B^{(2;\alpha)}}, \qquad \alpha=1,\dots,q-2, \tag{7.10a}$$

where we have defined

$$k_B^{(2;\alpha)} = k_B^{(1;\alpha)} + \frac{k_B^{(1;q-1)} k_{q-1}^{(1;\alpha)}}{2-k_{q-1}^{(1;q-1)}}, \qquad B=0,\dots,q-2. \tag{7.10b}$$

Now, for $\alpha=q-1$ Eqs. (7.8) imply

$$\sum_{1}^{q-2}{}_\beta \frac{d_\beta k_\beta^{(1;q-1)}}{2-k_{q-1}^{(1;q-1)}} \leqq \frac{2d_{q-1}-2-d_{q-1}k_{q-1}^{(1;q-1)}}{2-k_{q-1}^{(1;q-1)}} \leqq d_{q-1}-1 \tag{7.11a}$$

and, consequently, the non-negative rational numbers $k_\beta^{(2;\alpha)}$ satisfy the following inequalities, analogous to Eqs. (7.8):

$$\sum_{1}^{q-2}{}_\beta d_\beta k_\beta^{(2;\alpha)} \leqq 2d_\alpha - 2, \qquad \alpha=1,\dots,q-2; \tag{7.11b}$$

$$k_\beta^{(2;\alpha)} < 2, \quad \text{for} \quad \beta \leqq \alpha = 1,\dots,q-2. \tag{7.11c}$$

Following the same procedure as above, from Eqs. (7.7a), for $\alpha=q-2, q-3, \dots$, we can get majorizations successively for $|\pi^{q-2}|$, in terms of $\{|\pi^B|\}_{0\leqq B\leqq q-3}$, for $|\pi^{q-3}|$, in terms of $\{|\pi^B|\}_{0\leqq B\leqq q-4}$, and so on. At the m^{th} $(1\leqq m\leqq q-1)$ step

$$0 \leqq d_\alpha^{-2} Q^{\alpha\alpha}(\pi) \leqq -(\pi^\alpha)^2 + \prod_{0}^{q-m}{}_B |\pi^B|^{k_B^{(m;\alpha)}}, \qquad \alpha=1,\dots,q-m, \tag{7.12a}$$

where we have set

$$k_B^{(m;\alpha)} = k_B^{(m-1;\alpha)} + \frac{k_B^{(m-1;q-m+1)} k_{q-m+1}^{(m-1;\alpha)}}{2-k_{q-m+1}^{(m-1;q-m+1)}}, \qquad B=0,1,\dots,q-m. \tag{7.12b}$$

For $\alpha=q-m$, we obtain from Eqs. (7.12):

$$|\pi^{q-m}| \leqq \prod_{0}^{q-m-1}{}_B |\pi^B|^{k_B^{(m;q-m)}/(2-k_{q-m}^{(m;q-m)})}, \qquad m=1,2,\dots,q-1, \tag{7.13a}$$

where the non-negative rational numbers $k_\beta^{(m;\alpha)}$ satisfy the following inequalities:

$$\sum_{1}^{q-m}{}_\beta d_\beta k_\beta^{(m;\alpha)} \leqq 2d_\alpha - 2, \qquad \alpha=1,\dots,q-m; \tag{7.13b}$$

$$k_\beta^{(m;\alpha)} < 2, \quad \text{for} \quad \beta \leqq \alpha = 1,\dots,q-m. \tag{7.13c}$$

Consequently, in Eq. (7.13a), the weight of the left-hand side is strictly lower than the weight of the right-hand side.

The iterative procedure ends up with the following relation:

$$|\pi^1| \leqq |\pi^0|^{k_0^{(q-1;1)}/(2-k_1^{(q-1;1)})} \leqq |\pi^0|^{d_1^q/2}, \tag{7.14a}$$

where the majorization

$$k_0^{(q-1;1)}/(2-k_1^{(q-1;1)}) \leqq \frac{d_1^q}{2}, \tag{7.14b}$$

can be justified in the following way.

One first proves, inductively that

$$k_0^{(m;\alpha)} \leqq d_\alpha^{m-1}, \qquad \alpha = 1, \ldots, q-1. \tag{7.15}$$

In fact, Eq. (7.15) is true for $m=1$, owing to the definition in Eq. (7.7b). Let us assume it holds true for a given m. Then, making use in Eq. (7.12b), for $B=0$, of the following immediate consequence of Eq. (7.13b):

$$k_{q-m}^{(m;\alpha)} \leqq 2\frac{d_\alpha - 1}{d_{q-m}}, \qquad \alpha = 1, \ldots, q-m, \tag{7.16}$$

one obtains

$$\begin{aligned} k_0^{(m+1;\alpha)} &= k_0^{(m;\alpha)} + \frac{k_0^{(m;q-m)} k_{q-m}^{(m;\alpha)}}{2 - k_{q-m}^{(m;q-m)}} \\ &\leqq d_\alpha^{m-1} + (d_\alpha - 1) d_{q-m}^{m-1} \\ &\leqq d_\alpha^m; \qquad \alpha = 1, \ldots, q-m. \end{aligned}$$

Once Eq. (7.15) has been checked, using also Eq. (7.16), for $\alpha = 1$ one gets Eq. (7.14b).

At this point, it is clear that, owing to Eqs. (7.14a) and (7.13), the matrix $\hat{Q}(\pi)$ can be *positive semi-definite only if all the* $|\pi^\alpha|$, $\alpha = 1, \ldots, q-1$, *are bounded by* constants, depending only on $d_1, \ldots, d_{q-1}$. This achieves the proof of the boundedness of $\mathscr{R}$. □

Proof of Theorem 7.2. Let us consider the open set

$$\mathscr{R}_1^+ = \{\pi \in \mathbb{R}^{q-1} \,|\, \hat{Q}(\pi) > 0\}. \tag{7.17}$$

Its closure $\bar{\mathscr{R}}_1^+$ is clearly contained in the set $\bar{\mathscr{R}}_1$, defined in Theorem 7.1 and is consequently compact.

The set $\bar{\mathscr{R}}_1^+$ is also connected. This fact results from the following arguments. In the interior points of each connected component of $\bar{\mathscr{R}}_1^+$, the matrix $\hat{Q}(\pi)$ is positive definite and the determinant of $\hat{Q}(\pi)$ and $A(\pi)$ (see item iii) of Theorem 5.1) have constant sign. On the boundary $\mathscr{B} = \bar{\mathscr{R}}_1^+ \setminus \mathscr{R}_1^+$ of $\bar{\mathscr{R}}_1^+$ the matrix $\hat{Q}(\pi)$ is $\geqq 0$ and has almost everywhere rank $q-2$. Therefore, owing to condition iv), $A(\pi)|_{\mathscr{B}}$ vanishes identically, like $\det \hat{Q}(\pi)|_{\mathscr{B}}$, since $A(\pi)$ is a complete active factor of $\det \hat{Q}(\pi)$.

In the interior of each connected component of $\bar{\mathscr{R}}_1^+$, therefore, $A(\pi)$ has a stationary point. According to Lemma 6.1 this can only occur at $\pi = 0$. This allows us to conclude that $\bar{\mathscr{R}}_1^+$ has only one connected component. □

We would like to stress that we have not proved that $\bar{\mathscr{R}}_1^+ = \bar{\mathscr{R}}_1$ since we have no reliable arguments to exclude pathological situations in which there exist algebraic subsets of dimension $\leqq (q-3)$ and $\pitchfork \mathscr{B}$, where $\hat{Q}(\pi) \geqq 0$ and det $\hat{Q}(\pi) = 0$. This depends on the fact that we could not prove that the algebraic set of the zeroes of an active factor necessarily intersects the boundary of $\bar{\mathscr{R}}_1^+$. As already pointed out, however, we have checked that all the solutions of Eq. (7.1), for $q \leqq 3$ and for $q=4$, $d_1 \leqq 5$, are not affected by these pathologies. These would suggest the conjecture that this is a general property of the solutions of the canonical equation satisfying only the initial conditions i)–iii) following Eq. (7.1).

8. Allowable Matrices $\hat{P}(p)$ for $q \leqq 3$

In this section we shall provide an explicit form of a representative for each equivalence class of allowable matrices $\hat{Q}(\pi)$ for $q \leqq 3$. All the allowable matrices $\hat{Q}(\pi)$ can be obtained from the matrices we shall list by means of general MIB transformations.

The procedure to solve the canonical equation is very lengthy for $q \geqq 3$, but requires only standard analytic manipulations, therefore we shall limit ourselves to resume under the following items 1.–3., the main steps of our calculations for $q \leqq 3$; in these cases we have obtained complete results.

1. We wrote down $\hat{Q}^{\alpha\beta}(\pi)$, $\alpha, \beta = 1, \ldots, q-1$, and $A(\pi)$ as the most general polynomials satisfying Eqs. (7.3) and the conditions i)–iii) following Eq. (7.1). In so doing, the dependence on a variable π^α, was written explicitly only when, for dimensional reasons, its maximal exponent could be determined independently of the particular values assumed by $d_1, \ldots, d_q$. This was possible, for instance, for all the variables of maximum weight.

2. When possible, we exploited the freedom in the choice of an A-basis in order to pick up a coordinate system in which the form of $\hat{Q}(\pi)$ is particularly simple. This was achieved by choosing the arbitrary parameters of a general w_1-homogeneous MIBT so that the highest possible number of unknown coefficients in the polynomials $Q^{\alpha\beta}(\pi)$ were fixed to conveniently chosen values.

3. We used the following properties a)–d) of the solutions of the canonical equation in an A-basis to determine all the terms of degree $\leqq 2$ in the general form of $A(\pi)$: a) $A(0)$ was determined from Eq. (7.3a); b) dimensionally allowed linear terms were ignored in $A(\pi)$, since it must be maximum at $\pi = 0$; c) the quadratic terms of $A(\pi)$ were determined from Eqs. (6.3), in terms only of $\hat{Q}(0) = D^2$ and $w_1(A)$; d) since $A(\pi)$ is the restriction to $\mathbb{R}^{q-1}$ of a w-homogeneous polynomial function of $p \in \mathbb{R}^q$ and $A(0) = 1$, $w_1(A)$ must be even.

The simplified forms of $\hat{Q}(\pi)$ and $A(\pi)$, obtained after the operations described under items 1., 2., and 3., were substituted into Eq. (7.1). The resulting system of coupled algebraic and/or differential equations could be solved only in terms of the integer parameters d_α, $\alpha = 1, \ldots, q$.

In particular, for $q = 2, 3$, we obtained the following results:

Case $q = 2$. The following theorem has been proved in [Sa 2]:

Theorem 8.1. *The orbit spaces of all the* CLG's *whose* MIB's *are formed by only two elements with given weights,* d_1, d_2 $(d_1 \geqq d_2 = 2)$, *are isomorphic.*

For completeness let us rederive this result in the S^{n-1}/G setting of the problem. This will allow us to illustrate the procedure described in items 1.–3. above.

Proof. For $q = 2$, the matrix $\hat{Q}(\pi)$ reduces to a scalar function of the unique component x of π. The most general form it can assume, in agreement with conditions i) and ii) following Eq. (7.1), is the following:

$$\hat{Q}(x) = d_1^2(1 + \xi x - x^2), \qquad \xi \in \mathbb{R}, \tag{8.1}$$

where the unknown parameter ξ can be $\neq 0$ only if d_1 is even.

Since the irreducible factors of $A(x)$ are factors of $\hat{Q}(x)$ and the most general form allowed for $A(x)$ by Eqs. (7.3) and Eqs. (6.3) is the following:

$$A(x)=1-x^2, \tag{8.2}$$

one finds $\xi=0$. □

We get therefore $\bar{\mathcal{S}}=\{p\in\mathbb{R}^2|(p^2)^{d_1}-(p^1)^2\geqq 0\}$ and $\bar{\mathcal{S}}_1=[-1,1]$.

Case $q=3$. This is the lowest value of q for which the canonical equation is not identically satisfied. To make the formulas more readable, in the rest of this section we shall use the notation (y,x), for the components of the vector $\pi=(\pi^1,\pi^2)$.

For $q=3$ the weight of $A(\pi)$ is bounded below by Eqs. (6.3) and above by Eq. (7.3b):

$$2d_1\leqq w_1(A)\leqq 2d_1+2d_2-2<4d_1; \tag{8.3}$$

therefore, $A(\pi)$ can be written in the following form:

$$\begin{aligned}A(\pi)=1&-w_1(A)(y^2/d_1+x^2/d_2)+y^3s+y^2xt(x)\\&+yx^2u(x)+x^3v(x),\end{aligned} \tag{8.4a}$$

where s is a real constant and the w_1-weights of the polynomials $t(x)$, $u(x)$, and $v(x)$ can only range in the sets of values indicated in the following Eq. (8.4b). There and in the rest of this section a polynomial with negative weight will be identified, conventionally, to the null polynomial,

$$\begin{aligned}&w_1(t)=w_1(A)-2d_1-d_2-2n_t, && n_t=0,1,\ldots;\\&w_1(u)=w_1(A)-d_1-2d_2-2n_u, && n_u=0,1,\ldots;\\&w_1(v)=w_1(A)-3d_2-2n_v, && n_v=0,1,\ldots.\end{aligned} \tag{8.4b}$$

Since $A(\pi)$ is the restriction to $\mathbb{R}^{q-1}$ of a w-homogeneous polynomial of $p\in\mathbb{R}^q$, from Eq. (8.3) it follows that only three values are allowed for $w_1(A)$: $w_1(A)=2d_1$, $2d_1+d_2$, $3d_1$; in the first two cases some of the coefficients in Eq. (8.4a) must vanish for dimensional reasons.

The most general form of $\hat{Q}(\pi)$, satisfying conditions i)–ii) following Eq. (7.1), is the following:

$$\hat{Q}^{\alpha\beta}(\pi)=-d_\alpha d_\beta\pi^\alpha\pi^\beta+\hat{P}^{\alpha\beta}(\pi), \tag{8.5a}$$

where

$$\begin{aligned}&\hat{P}^{11}(\pi)=d_1^2[ya(x)+xb(x)+1];\\&\hat{P}^{12}(\pi)=\hat{P}^{21}(\pi)=d_1d_2[cy+xe(x)];\\&\hat{P}^{22}(\pi)=d_2^2(fy+gx+1).\end{aligned} \tag{8.5b}$$

In Eq. (8.5b), c, f, and g are real parameters and $a(x)$, $b(x)$, and $e(x)$ are polynomial functions of x. The weights of these quantities can only range in the following sets of values:

$$\begin{aligned}&w_1(a)=d_1-2-2n_a, && n_a\in\mathbb{N}_0;\\&w_1(b)=2d_1-d_2-2-2n_b, && n_b\in\mathbb{N}_0;\\&w_1(c)=d_2-2-2n_c, && n_c\in\mathbb{N};\\&w_1(e)=d_1-2-2n_e, && n_e\in\mathbb{N}_0;\\&w_1(f)=2d_2-d_1-2-2n_f, && n_f\in\mathbb{N}_0;\\&w_1(g)=d_2-2-2n_g, && n_g\in\mathbb{N}_0.\end{aligned} \tag{8.5c}$$

It is possible to get some further simplifications of the general form of $\hat{P}(\pi)$ with a convenient choice of the coordinate system, as explained under item 2., only in the following two cases A) and B):

A) $d_1 = d_2$.
In this case it will be advantageous to write the matrix elements $\hat{P}^{\alpha\beta}(\pi)$ in the following form:

$$\hat{P}^{\alpha\beta}(\pi) = d_\alpha d_\beta(\delta^{\alpha\beta} + yA_1^{\alpha\beta} + xA_2^{\alpha\beta}), \qquad \alpha, \beta = 1, 2, \tag{8.6}$$

where A_1 and A_2 are real symmetric constant matrices and we are still free to make the following linear MIBT's, without modifying the parameters which have already been fixed in $\hat{P}(\pi)$:

$$\pi' = DOD^{-1}\pi, \qquad O \in O_2(\mathbb{R}).$$

Transformations of this kind lead to

$$\hat{P}'^{\alpha\beta}(\pi') = d_\alpha d_\beta(\delta^{\alpha\beta} + y'A_1'^{\alpha\beta} + x'A_2'^{\alpha\beta}), \qquad \alpha, \beta = 1, 2, \tag{8.7a}$$

where

$$A'_\alpha = \sum_1^2{}_\beta\, O_\alpha^\beta O A_\beta O, \qquad \alpha = 1, 2. \tag{8.7b}$$

For

$$O = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix},$$

one finds, in particular,

$$\begin{aligned} A_1'^{12} = \sin^3\phi[&A_1^{12}\cot^3\phi + (A_1^{11} - A_2^{12} - A_1^{22})\cot^2\phi \\ &-(A_2^{11} + A_1^{12} - A_2^{22})\cot\phi + A_2^{12}]. \end{aligned} \tag{8.8}$$

Since there is always a value of ϕ for which $A_1'^{12} = 0$, it is not restrictive to assume that the coordinate system has been chosen, from the very beginning, so that $A_1^{12} = 0$.

B) $d_1 = (m+1)d_2$, $m \in \mathbb{N}$.
In this case, owing to the restrictions of Eq. (8.5c), we can set in Eq. (8.5b),

$$\begin{aligned} e(x) &= \sum_0^m{}_j\, e_j x^j; \\ f &= 0, \end{aligned} \tag{8.9}$$

and we are still free to make the following MIBT's, without modifying the values of those parameters which have already been fixed in $\hat{P}(\pi)$:

$$\begin{aligned} y' &= y + \xi x^{m+1}, \qquad \xi \in \mathbb{R}; \\ x' &= x. \end{aligned} \tag{8.10}$$

A straightforward calculation leads to

$$\begin{aligned} \hat{P}'^{12}(\pi') &= \hat{P}^{12}(\pi(\pi')) + m\xi x'^m \hat{P}^{22}(\pi(\pi')) \\ &= (m+1)d_2^2[\xi(-cx' + gx' + 1)x'^m + x'e(x') + cy']. \end{aligned} \tag{8.11}$$

The coefficient of x'^m in (8.11) vanishes for $\xi = -e_{m-1}$. Therefore, it is not restrictive to assume that the MIB has been chosen so that $e_{m-1}=0$, from the very beginning.

The solution of the canonical equation determines completely the remaining unknown polynomials and constants appearing in Eq. (8.5b) and, after a convenient rescaling of x and y, leads to the following classes of solutions:

I) $w_1(A)=2d_1$; $d_1=(n_1+n_2)d_2/2$; $n_1, n_2 \in \mathbb{N}$; $n_1 \geqq n_2$; $n_1=n_2$, for d_2 odd.

$$\begin{aligned} \hat{P}^{11}(\pi) &= d_1^2[(n_1-n_2)x+n_1n_2](n_1-x)^{n_1-1}(n_2+x)^{n_2-1};\\ \hat{P}^{12}(\pi) &= 0;\\ \hat{P}^{22}(\pi) &= d_2^2[n_1n_2+(n_1-n_2)x];\\ A(\pi) &= n_1^{-n_1}n_2^{-n_2}[(n_1-x)^{n_1}(n_2+x)^{n_2}-y^2];\\ \det\hat{Q}(\pi) &= n_1^{n_1}n_2^{n_2}d_1^2d_2^2[n_1n_2+(n_1-n_2)x]A(\pi). \end{aligned} \tag{8.12}$$

The number of non-equivalent classes of solutions for each fixed choice of (d_1, d_2) is equal to the maximum integer $[d_1/d_2]$ contained in d_1/d_2, for d_2 even; it is equal to one for d_2 odd.

II) $w_1(A)=2d_1+d_2$; $d_1=(m+1)k$; $d_2=2k$; $m, k \in \mathbb{N}$.

$$\begin{aligned} \hat{P}^{11}(\pi) &= d_1^2(m+1-x)^m;\\ \hat{P}^{12}(\pi) &= -d_1d_2y;\\ \hat{P}^{22}(\pi) &= d_2^2(m+1+mx);\\ A(\pi) &= (m+1)^{-m-1}(1+x)[(m+1-x)^{m+1}-(m+2)y^2];\\ \det\hat{Q}(\pi) &= (m+1)^{m+1}d_1^2d_2^2A(\pi). \end{aligned} \tag{8.13}$$

For each fixed choice of (d_1, d_2) there is only one class of non-equivalent solutions.

III.1) $w_1(A)=3d_1$; $d_1=4m$; $d_2=3m$; $m \in \mathbb{N}$.

$$\begin{aligned} \hat{P}^{11}(\pi) &= d_1^2(2-y+x^2);\\ \hat{P}^{12}(\pi) &= 2d_1d_2x;\\ \hat{P}^{22}(\pi) &= d_2^2(2+y);\\ A(\pi) &= \tfrac{1}{4}(4-3y^2-4x^2-y^3+6yx^2-x^4);\\ \det\hat{Q}(\pi) &= 4d_1^2d_2^2A(\pi). \end{aligned} \tag{8.14}$$

For each fixed choice of (d_1, d_2) there is only one class of non-equivalent solutions.

III.2) $w_1(A)=3d_1$; $d_1=6m$; $d_2=4m$; $m \in \mathbb{N}$.

$$\begin{aligned} \hat{P}^{11}(\pi) &= d_1^2(64-4y-16x-yx+8x^2);\\ \hat{P}^{12}(\pi) &= d_1d_2(-2y+12x+x^2);\\ \hat{P}^{22}(\pi) &= d_2^2(16+y+4x);\\ A(\pi) &= \tfrac{1}{1024}(8+y-3x)(128-16y+48x-y^2-12yx+3x^3);\\ \det\hat{Q}(\pi) &= 1024d_1^2d_2^2A(\pi). \end{aligned} \tag{8.15}$$

For each fixed choice of (d_1, d_2) there is only one class of non-equivalent solutions.

III.3) $w_1(A) = 3d_1$; $d_1 = 10m$; $d_2 = 6m$; $m \in \mathbb{N}$.

$$\begin{aligned}
&\hat{P}^{11}(\pi) = d_1^2(1152 - 12y - 168x - 4yx + 44x^2 + x^3);\\
&\hat{P}^{12}(\pi) = d_1 d_2(-6y + 60x + 5x^2);\\
&\hat{P}^{22}(\pi) = d_2^2(96 + y + 14x);\\
&\quad A(\pi) = \tfrac{1}{110592}(110592 - 144y^2 - 2880x^2 - y^3 - 30y^2x + 180yx^2\\
&\qquad + 280x^3 + 15yx^3 - 55x^4 - x^5);\\
&\det \hat{Q}(\pi) = 110592 d_1^2 d_2^2 A(\pi).
\end{aligned} \tag{8.16}$$

For each fixed choice of (d_1, d_2) there is only one class of non-equivalent solutions.

9. Conclusions

In this concluding section we shall briefly stress, under items a)–e), some relevant features of the solutions of the canonical equation (7.1) listed in the last section and make a few comments.

a) For $q = 3$ and for each choice of the integer numbers $d_1 \geqq d_2 \geqq d_3 = 2$, there are only finitely many non-equivalent (with respect to MIBT's) solutions of Eqs. (7.1). The numbers and types of these solutions are reported in Table I, for $d_1 \leqq 7$. In the table, class I solutions are distinguished by the values of the integer parameters (n_1, n_2) and the missing cases correspond to no solution.

b) The contravariant metric matrices $\hat{P}(p)$, obtained from the solutions we have found, are positive semi-definite *only* in a semi-algebraic connected 3-dimensional subset of $\mathbb{R}^3$, whose intersection with the plane $\Pi = \{p \in \mathbb{R}^3 \mid p^3 = 1\}$ is projected onto a compact connected semi-algebraic subset $\bar{\mathscr{R}}_1 \subset \mathbb{R}^2$. All the solutions for $q = 3$ are therefore *allowable* solutions. The graphs of some of the sets $\bar{\mathscr{R}}_1$ are plotted in Fig. 1. All of them, but possibly the second, forth and sixth, correspond to the images of orbit spaces of well known CCLG's.

c) Irreducible factors of $\det \hat{Q}(\pi)$ which are not solutions of the canonical equation, determine algebraic subsets of $\mathbb{R}^2$, whose intersection with $\bar{\mathscr{S}}_1$ has dimension 0. This fact suggest that the assumption iv) in Sect. 7 might be a consequence of i)–iii) and of the structure of the canonical equation.

d) The coordinate system can be so chosen that all the coefficients in the elements of the contravariant metric matrix $\hat{P}(p)$ are integer numbers.

e) There are no solutions in correspondence with the integers $d_1 \geqq d_2 \geqq 2$ for which the quotient d_1/d_2 does not equal an integer or semi-integer number, or 4/3, or 5/3 for even d_2.

The solutions of the canonical equation we have found for $q = 4$ and $d_1 \leqq 5$ also agree with the statements in items a)–d) above. Even if we have no proof that our results are independent of the particular range of values chosen for q and the d_i's, we believe they yield a reasonable support to the following conjecture [Sa 2]:

Table I

d_1	d_2	Classes	n*
2	2	I(1, 1), II	2
3	2	I(2, 1), II	2
3	3	I(1, 1)	1
4	2	I(3, 1), I(2, 2), II	3
4	3	III	1
4	4	I(1, 1), II	2
5	2	I(4, 1), I(3, 2), II	3
5	5	I(1, 1)	1
6	2	I(5, 1), I(4, 2), I(3, 3), II	4
6	3	I(2, 2)	1
6	4	I(2, 1), II, III	3
6	6	I(1, 1), II	2
7	2	I(6, 1), I(5, 2), I(4, 3), II	4
7	7	I(1, 1)	1

* (n = number of non-equivalent solutions)

Conjecture. *For each choice of the integer numbers q and $d_1 \geqq \ldots \geqq d_{q-1} \geqq 2$ there is only a finite number of equivalence classes of allowable matrices $\hat{P}(p)$. Moreover, the* MIB *can always be chosen so that all the coefficients in the polynomial matrix elements $\hat{P}^{ab}(p)$ are integer numbers.*

We have not proved and cannot do it by now, that every allowable matrix $\hat{P}(p)$ is the contravariant metric matrix of at least one CCLG; it is clear, however, that the contravariant metric matrices of all CCLG's are allowable matrices $\hat{P}(p)$. Therefore, there are no CCLG's whose MIB's have homogeneity degrees $d_1 \geqq d_2 \geqq d_3 = 2$ if d_1 and d_2 satisfy the conditions of item e); moreover if the conjecture holds true, the orbit spaces of all the compact linear groups whose MIB's are formed by the same number q of elements with the same weights $(d_1, \ldots, d_q)$ can be classified in a finite (and small, if q and d_1 is small) number of isomorphism classes and, for each CCLG, the MIB can be chosen so that the contravariant metric matrix is a w-homogeneous polynomial tensor with integer coefficients.

The class of finite coregular linear groups is formed by all the finite groups generated by pseudo-reflections. All these groups have been classified and their minimal integrity bases determined [ST]. The contravariant metric matrices of the images of the 2- and 3-dimensional orbit spaces of all these groups can be found among the solutions of the canonical equation we have listed in the preceding

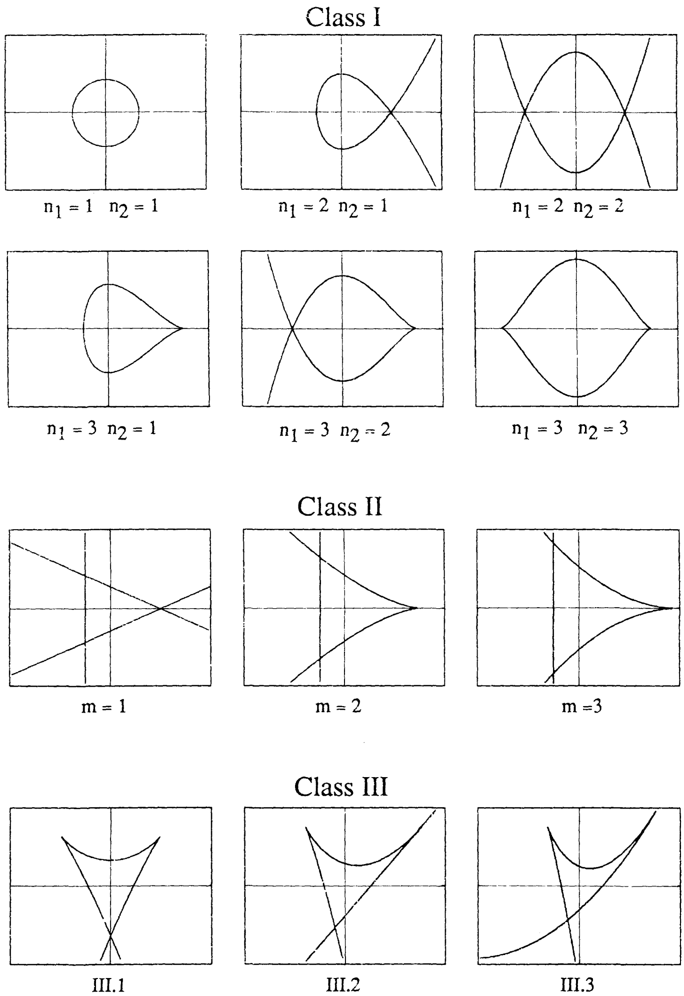

Fig. 1. Images in the plane (p^2, p^1) of some possible 3-dimensional orbit spaces

section. To our knowledge, in the literature there is no complete classification of general compact coregular linear Lie groups (for simple linear Lie groups see [Sc 2]).

For *non*-coregular CLG's, the polynomial relations among the elements of the MIB $\{p\}$ define an algebraic variety $Z \subset \mathbb{R}^q$. The set $\bar{\mathcal{S}}$ is the unique connected semi-algebraic subset of Z where the matrix $\hat{P}(p)$ is positive semi-definite [PS 1]. The approach to the determination of the orbit spaces of CCLG's we have followed can lead to useful results also in this case. In fact, it can be shown that, if the maximal ideal $\mathcal{I}_Z$, associated to Z, is generated by a single irreducible polynomial $\imath(p)$, then $\imath(p)$ is a factor of $\det \hat{P}(p)$ and must satisfy the canonical equation. Therefore, some of the solutions we have classified in the last section might correspond to non-coregular CLG's. If $\mathcal{I}_Z$ has more than one independent generator, however, the only thing one can say is that the contravariant derivative of any irreducible element of $\mathscr{I}_Z$ must belong to $\mathscr{I}_Z$. Thus, the canonical equation assumes a more complicated form and its solutions are not so easy to determine. Work in this direction is in progress.

The results stated above have an immediate impact on the physical theories mentioned in the Introduction. In fact, in these theories the potential V is usually written as the most general polynomial function of given degree which is invariant by G. When V is expressed as a function $\hat{V}$ of the elements of a MIB, it has the same form for all the CCLG's whose MIB's share the same weights $d_1, \ldots, d_q$. Since we have proved that there are only finitely many non-isomorphic orbit spaces associated to these groups, for all the compact symmetry groups whose MIB's have the same weights $d_1, \ldots, d_q$, the problems of determining the ground state, the possible phases, their stability properties and the allowed transitions to other phases are reduced to a small number of identical analytical problems: the universality properties discovered in the orbit spaces of the CCLG's give rise to universality properties in the patterns of spontaneous symmetry breaking.

References

[AS 1] Abud, M., Sartori, G.: The geometry of orbit space and natural minima of Higgs potentials. Phys. Lett. **104B**, 147–152 (1981)

[AS 2] Abud, M., Sartori, G.: The geometry of spontaneous symmetry breaking. Ann. Phys. **150**, 307–372 (1983)

[Br] Bredon, G.: Introduction to compact transformation groups. New York: Academic Press 1972

[G] Gufan, Yu.M.: Theory of phase transitions characterized by a many-component order parameter. Soviet Phys. Solid State **13**, 175–184 (1971)

[H] Hilbert, D.: Über die Theorie der algebraischen Formen. Math. Ann. **36**, 473–534 (1890) and Über die vollen Invariantensysteme. Math. Ann. **42**, 313–373 (1893)

[J 1] Jarić, M.V.: Landau theory, symmetry breaking and the chain criterium. Lecture Notes in Physics, Vol. 135, pp. 12–16. Berlin, Heidelberg, New York: Springer 1980)

[J 2] Jarić, M.V.: Orbit space and minima of Landau-Higgs potentials. J. Math. Phys. **24**, 917–921 (1983)

[JMS] Jarić, M.V., Michel, L., Sharp, R.T.: Zeroes of covariant vector fields for the point groups: Invariant formulation. J. Phys. **45**, 1–27 (1984)

[N] Noether, E.: Math. Ann. **77**, 89–92 (1916)

[PS 1] Procesi, C., Schwarz, G.W.: Inequalities defining orbit spaces. Inv. Math. **81**, 539–554 (1985)

[PS 2] Procesi, C., Schwarz, G.W.: The geometry of orbit spaces and gauge symmetry breaking in supersymmetric gauge theories. Phys. Lett. **161B**, 117–121 (1985)

[Sa 1] Sartori, G.: A theorem on orbit structures (strata) of compact linear Lie groups. J. Math. Phys. **24**, 765–768 (1983)

[Sa 2] Sartori, G.: Universality in orbit spaces of compact linear groups. Mod. Phys. Lett. A **4**, 91–98 (1989)

[Sc 1] Schwarz, G.W.: Smooth functions invariant under the action of a compact Lie group. Topology **14**, 63–68 (1975)

[Sc 2] Schwarz, G.W.: Representations of simple Lie groups with regular rings of invariants. Invent. Math. **49**, 167–191 (1978)

[Sc 3] Schwarz, G.W.: Lifting smooth homotopies of orbit spaces. I.H.E.S. Publ. **51**, 37–135 (1980)

[ST] Shephard, G.C., Todd, J.A.: Finite unitary reflection groups. Can. J. Math. **6**, 274–304 (1954)

Communicated by H. Araki