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Classical Origin of Quantum Group Symmetries in Wess-Zumino-Witten Conformal Field Theory

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Abstract. We elucidate the way by which the quantum group symmetries of the WZW models arise within the canonical formalism of the classical field theory.

1. Introduction

The quantum group symmetries of the 2-dimensional field theories have attracted considerable attention recently. On one side, they lead to a better understanding of the structure of 2-dimensional integrable models and of their exact solutions. On the other side, they allow us to learn more about the quantum groups [11, 38, 52] themselves and about the non-commutative geometry underlying that concept. In particular, multiple relations to quantum groups were discovered in the study of the exchange algebra of chiral vertex operators of numerous conformal field theories like WZW models, minimal models, Liouville and Toda theories [42, 2, 3, 17, 4, 7] leading finally to the realization that quantum groups should describe the symmetries of the chiral sector of conformal field theories [43, 18, 29, 48, 35–37, 20, 30, 19].

At the very origin of quantum groups lay the fact that they were the integrated version of simpler, infinitesimal structures abstracted from the canonical formalism of classical integrable theories [12, 14]. Hence the interest in exhibiting the classical origin of quantum symmetries of conformal field theories. An important step in this direction was achieved in [5] in the context of the Liouville theory and was used in [29, 30] to explicit the quantum group symmetries of the quantized Liouville theory constructed earlier by Gervais-Neveu [31, 32]. In [13], Faddeev has adapted those observations to the case of WZW model of conformal fields and in [1] Alekseev-Shatashvili have provided still more geometric explanation for the appearance of quantum groups in the chiral WZW models.

The present paper may be viewed as a continuation of [13, 1]. It lays down a more systematic version of the canonical formalism first for the complete WZW model and then for its chiral part. Within this formalism, we show that the left-handed part of the ($SU(2)$) WZW theory possesses, besides the left affine Kac-Moody invariance, a deformed right isotopic symmetry which we realize explicitly. This symmetry is the (semi-)classical counterpart of the quantum group symmetry and it renders the classical origin of the quantum group invariance in the chiral WZW theory quite transparent.

2. Canonical Formalism for the WZW Theories

It may be worth stressing for pedagogical purposes that the canonical formalism is really canonical. What it means is that it may be set up for a vast class of physical systems in a natural way, leaving no room for guesswork. If we deal with a field theory in D space-time dimensions with the action functional given by the integral of a Lagrangian density,

$$S=\int \mathscr{L}(x^\mu,\phi^A,\partial_\nu\phi^A),$$

a convenient approach is to pass to the first order description on the space $\mathscr{P}$ of (x^μ,ϕ^A,ξ^A_ν) or still better of $(x^\mu,\phi^A,\pi^{A\nu})$ with $\pi^{A\nu}=\partial\mathscr{L}/\partial\xi^A_\nu$. Let

$$\alpha=\mathscr{L}dx^0\dots dx^{D-1}+\pi^{A\mu}dx^0\dots\underset{\hat\mu}{(d\phi^A-\xi^A_\nu dx^\nu)}\dots dx^{D-1} \tag{1}$$

be a D-form on $\mathscr{P}$ (we drop the wedge of exterior product, sum over repeated indices and omit dx^μ differential in the second term on the right-hand side). Let $\omega=d\alpha$. The extremal field configurations $\Phi(x)=(\phi^A(x),\xi^A_\mu(x))$ for the variational problem $\delta\int\alpha=0$ annihilate forms $\omega\,\llcorner\,\mathscr{X}$ for arbitrary vector fields $\mathscr{X}$ on $\mathscr{P}$ ($\llcorner$ stands for contraction). This encodes both the Euler-Lagrange equations for $\phi^A(x)$ and gives $\xi^A_\mu(x)=\partial_\mu\phi^A(x)$ (in this sense, the variational problems $\delta\int\alpha=0$ and $\delta\int\mathscr{L}=0$ are equivalent). The space of solutions of the variational equations forms the phase space P of the classical field theory. We usually describe the solutions by the initial conditions $\Phi(x)=(\phi^A(x),\xi^A_\mu(x))$ for x in a hypersurface Σ, with $\phi^A(x)$ and $\xi^A_\mu(x)$ satisfying the restriction of the external equations to the hypersurface. Vectors $X=\delta\Phi$ tangent to P correspond to vector fields $\mathscr{X}$ tangent to $\mathscr{P}$ along Φ and satisfying the linearized equations or to the initial data for them. Expression

$$\langle\Omega(\Phi)\,|\,X,X'\rangle=\int_\Sigma\Phi^*(\omega\,\llcorner\,\mathscr{X}\,\llcorner\,\mathscr{X}') \tag{2}$$

(independent of the hypersurface) defines a closed 2-form on the phase space P. If Ω is degenerate, one has to reduce further the phase space P (this occurs in the presence of gauge invariance). If non-degenerate, Ω defines a symplectic structure on P and allows to associate to functions F on P Hamiltonian vector fields X_F such that

$$dF=\Omega\,\llcorner\,X_F \tag{3}$$

and to define the Poisson bracket by

$$\{F,F'\}=X_F(F'). \tag{4}$$

All of that is an old story which originated in the mathematical work on variational calculus with multiple integrals in the first half of the century [10, 34] and was adapted for the needs of field theories in the seventies [39, 26, 40, 47, 21, 22] and reinvented subsequently [53, 50, 9, 8].

Let us apply this formalism to the 2-dimensional Wess-Zumino-Witten field theory [51, 41, 27]. Its fields $g(x)$ take values in a compact group G which we shall take simple and simply connected. The action of the model is given as

$$S=-(8\pi)^{-1}k\int \mathrm{tr}(g^{-1}\partial_\mu g)(g^{-1}\partial^\mu g)dx^0dx^1+(12\pi)^{-1}k\int d^{-1}\,\mathrm{tr}(g^{-1}dg)^3\,. \quad (5)$$

The first term is the standard Lagrangian term. It may be treated as above leading to a 2-form α_0 on the space $\mathscr{P}$ of (x, g, ξ_μ) with Lie algebra $\mathscr{G}$-valued ξ_μ,

$$\begin{aligned}\alpha_0=&(8\pi)^{-1}k\,\mathrm{tr}\,\xi_\mu\xi^\mu dx^0dx^1-(4\pi)^{-1}k\,\mathrm{tr}\,\xi^0(g^{-1}dg)dx^1\\&-(4\pi)^{-1}k\,\mathrm{tr}\,\xi^1dx^0(g^{-1}dg)\,.\end{aligned} \quad (6)$$

The second (Wess-Zumino) term in the action may be dealt with by either writing it in terms of a Lagrangian density defined for fields avoiding some singular values and proceeding as before or, simpler, just by taking it into account in the 3-form ω by setting

$$\begin{aligned}\omega=&d\alpha_0+(12\pi)^{-1}k\,\mathrm{tr}(g^{-1}dg)^3\\=&(4\pi)^{-1}k\,\mathrm{tr}\,\xi_\mu d\xi^\mu dx^0dx^1-(4\pi)^{-1}k\,\mathrm{tr}\,d\xi^0(g^{-1}dg)dx^1\\&-(4\pi)^{-1}k\,\mathrm{tr}\,d\xi^1dx^0(g^{-1}dg)+(4\pi)^{-1}k\,\mathrm{tr}\,\xi^0(g^{-1}dg)^2dx^1\\&-(4\pi)^{-1}k\,\mathrm{tr}\,\xi^1(g^{-1}dg)^2dx^0+(12\pi)^{-1}k\,\mathrm{tr}(g^{-1}dg)^3\,.\end{aligned} \quad (7)$$

Both procedures lead necessarily to the same classical equations and to the same symplectic form Ω. The classical equations

$$\Phi^*(\Omega\,\llcorner\,\mathscr{X})=0$$

become

$$\xi_\mu=g^{-1}\partial_\mu g \quad \text{and} \quad \partial_+(g^{-1}\partial_-g)=0\,, \quad (8)$$

where $\partial_\pm=(\partial_0\pm\partial_1)/2$. We shall consider the theory on the cylinder, i.e. with the periodic boundary condition

$$g(x^0,x^1+2\pi)=g(x^0,x^1)\,.$$

The general solution of Eqs. (8) is a (non-linear) superposition of left- and right-movers:

$$g(x^0,x^1)=g_L(x^+)g_R(-x^-)\,, \quad (9)$$

where $x^\pm=x^0\pm x^1$ and

$$g_L(x+2\pi)=g_L(x)\gamma\,, \qquad g_R(x+2\pi)=\gamma^{-1}g_R(x) \quad (10)$$

for some $\gamma\in G$. If we parametrize space P of solutions by the initial data

$$g(x)=g(0,x) \quad \text{and} \quad \xi_0(x)=g^{-1}\partial_0 g(0,x)\,, \quad (11)$$

then the 2-form Ω of Eq. (2) becomes

$$\Omega=(4\pi)^{-1}k\int_0^{2\pi}\mathrm{tr}[-d\xi_0(g^{-1}dg)+(\xi_0+g^{-1}\partial_x g)(g^{-1}dg)^2] \quad (12)$$

and is easily seen to be non-degenerate.

We may also parametrize P by pairs of functions (g_L, g_R). (g_L, g_R) and $(g_Lg_0^{-1}, g_0g_R)$ for constant elements $g_0\in G$ give then the same solution. A direct calculation using

$$\begin{aligned} g&=g_L g_R\,,\\ \xi_0&=g_R^{-1}(g_L^{-1}\partial_x g_L)g_R-g_R^{-1}\partial_x g_R\,,\\ g_L^{-1}dg_L(2\pi)&=\mathrm{Ad}_{\gamma^{-1}}(g_L^{-1}dg_L(0)+(d\gamma)\gamma^{-1})\,,\\ (dg_R)g_R^{-1}(2\pi)&=\mathrm{Ad}_{\gamma^{-1}}((dg_R)g_R^{-1}(0)-(d\gamma)\gamma^{-1})\,, \end{aligned} \tag{13}$$

gives

$$\Omega=\Omega_L+\Omega_R\,, \tag{14}$$

where

$$\begin{aligned} \Omega_L=(4\pi)^{-1}k\int_0^{2\pi}\mathrm{tr}(g_L^{-1}dg_L)\partial_x(g_L^{-1}dg_L)\\ +(4\pi)^{-1}k\,\mathrm{tr}(g_L^{-1}dg_L)(0)(d\gamma)\gamma^{-1} \end{aligned} \tag{15}$$

and

$$\begin{aligned} \Omega_R=-(4\pi)^{-1}k\int_0^{2\pi}\mathrm{tr}(dg_R)g_R^{-1}\partial_x((dg_R)g_R^{-1}\\ +(4\pi)^{-1}k\,\mathrm{tr}(dg_R)g_R^{-1}(0)(d\gamma)\gamma^{-1}\,. \end{aligned} \tag{16}$$

Of course, in variables (g_L, g_R) 2-form Ω becomes degenerate, the degeneration corresponding to the orbits $\{(g_L g_0^{-1}, g_0 g_R)\,|\,g_0\in G\}$.

3. Chiral Sector of the WZW Model

Phase space P may be extended to the product space $P_L\times P_R$ of (g_L, g_R) with independent monodromies. More exactly,

$$P=\{(g_L,g_R)\in P_L\times P_R\,|\,g_L(0)^{-1}g_L(2\pi)=g_R(0)g_R(2\pi)^{-1}\}/G\,. \tag{17}$$

We could expect that this allows to consider the left- and the right-movers separately. The real situation is, however, more complicated since a straightforward calculation shows that on P_L

$$d\Omega_L=(12\pi)^{-1}k\,\mathrm{tr}((d\gamma)\gamma^{-1})^3 \tag{18}$$

and on P_R

$$d\Omega_R=-(12\pi)^{-1}k\,\mathrm{tr}((d\gamma)\gamma^{-1})^3\,, \tag{19}$$

where $\gamma=g_L(0)^{-1}g_L(2\pi)$ or $\gamma=g_R(0)g_R(2\pi)^{-1}$, respectively. Ω_L and Ω_R are not closed forms on P_L and P_R and $\Omega_L+\Omega_R$ becomes closed only on the subspace of $P_L\times P_R$, where the monodromies of g_L and g_R coincide. The independent canonical treatment of the left-(right-)movers requires some modifications of the above structures.

One possible way to proceed is to restrict the allowed monodromies of g_L (g_R). This is the procedure used in [31, 32, 29, 13, 1, 33]. For example, if $T\subset G$ is a Cartan subgroup, we may introduce

$$P_L^{\mathrm{res}}=\{g_L\,|\,\gamma=g_L(0)^{-1}g(2\pi)\in T\}$$

(and similarly for the right-movers; below we shall treat only the left-movers and will drop the subscript "L" on g_L). On P_L^{res}, Ω_L becomes closed since $\mathrm{tr}((d\gamma)\gamma^{-1})^3$ vanishes when restricted to T. For general $g \in P_L$, a vector X tangent to P_L at g corresponds to a Lie algebra $\mathscr{G}$-(or $\mathscr{G}^{\mathbf{C}}$-)valued map $\mathscr{X}(x)$ $(= g^{-1}\delta g(x))$ such that

$$\mathscr{X}(x+2\pi) = \mathrm{Ad}_{\gamma^{-1}}(\mathscr{X}(x) + \mathscr{X}_0) \tag{20}$$

and

$$\begin{aligned}(g^{-1}dg)(x) \llcorner X &= \mathscr{X}(x),\\ (d\gamma)\gamma^{-1} \llcorner X &= \mathscr{X}_0.\end{aligned} \tag{21}$$

Consequently,

$$\begin{aligned}\Omega_L \llcorner X &= (4\pi)^{-1}k \int_0^{2\pi} \mathrm{tr}[\mathscr{X}\partial_x(g^{-1}dg) - (g^{-1}dg)\partial_x\mathscr{X}]\\ &\quad + (4\pi)^{-1}k\,\mathrm{tr}[\mathscr{X}(0)(d\gamma)\gamma^{-1} - (g^{-1}dg)(0)\mathscr{X}_0]\\ &= -(2\pi)^{-1}k \int_0^{2\pi} \mathrm{tr}(\partial_x\mathscr{X})g^{-1}dg\\ &\quad + (4\pi)^{-1}k\,\mathrm{tr}(2\mathscr{X}(0) + \mathscr{X}_0)(d\gamma)\gamma^{-1},\end{aligned} \tag{22}$$

where we have integrated by parts and have used (13) and (20).

For $g \in P_L^{\mathrm{res}}$ and X tangent to P_L^{res}, $\mathscr{X}_0 \in \mathscr{T}$, the Cartan subalgebra of $\mathscr{G}$. It is straightforward to see that $\Omega_L \llcorner X = 0$ implies then $X = 0$ if γ is not in the center Z of G. Thus Ω_L is a symplectic form on the subset of P_L^{res} where $\gamma \notin Z$. Consider for $G = SU(2)$ function F on P_L^{res},

$$F(g) = g_{ij}(x)$$

with $x \in [0, 2\pi]$.
Hamiltonian vector field $X_F \equiv X$ corresponding to F satisfies:

$$\begin{aligned}g_{il}(x)\mathscr{X}_{lj}(x) &= -(2\pi)^{-1}k \int_0^{2\pi} \mathrm{tr}(\partial_x\mathscr{X})g^{-1}dg\\ &\quad + (4\pi)^{-1}k\,\mathrm{tr}(2\mathscr{X}(0) + \mathscr{X}_0)(d\gamma)\gamma^{-1}\end{aligned} \tag{23}$$

which gives

$$\mathscr{X}_{mn} = -2\pi k^{-1}(g_{in}(x)\delta_{mj} - g_{ij}(x)\delta_{mn}/2)\Theta(\cdot - x) + \mathscr{X}(0)_{mn} \tag{24}$$

and

$$2\mathscr{X}(0) + \mathscr{X}_0 = 0. \tag{25}$$

Combining (25) with (20) for $x = 0$, we find $\mathscr{X}(0)$ and obtain:

$$\begin{aligned}\mathscr{X}_{11} &= -\pi k^{-1}(g_{i1}(x)\delta_{1j} - g_{ij}(x)/2)(2\Theta(\cdot - x) - 1),\\ \mathscr{X}_{12} &= -2\pi k^{-1}g_{i2}(x)\delta_{1j}(\Theta(\cdot - x) - (1 - e^{-2ip})^{-1}),\\ \mathscr{X}_{21} &= -2\pi k^{-1}g_{i1}(x)\delta_{2j}(\Theta(\cdot - x) - (1 - e^{2ip})^{-1}),\\ \mathscr{X}_{22} &= -\pi k^{-1}(g_{i2}(x)\delta_{2j} - g_{ij}(x)/2)(2\Theta(\cdot - x) - 1),\end{aligned}$$

where $\gamma = \begin{pmatrix} e^{ip} & 0 \\ 0 & e^{-ip} \end{pmatrix}$. This leads to the Poisson brackets

$$\{g_{ij}(x), g_{hl}(y)\} = g_{im}(x)g_{hn}(y)q_{jl}^{\pm mn} \tag{26}$$

or, in a short-hand notation,

$$\{g(x) \overset{\otimes}{,} g(y)\} = (g(x) \otimes g(y)) q^{\pm},$$

where the sign $\pm$ applies according to whether $x_0 \lessgtr y_0$ and

$$q^{\pm} = \pm \pi (2k)^{-1} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 4(e^{\mp 2ip}-1)^{-1} & 0 \\ 0 & 4(e^{\pm 2ip}-1)^{-1} & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{27}$$

in the lexicographic order of the double indices. Notice that the exchange matrix $q^{\pm}$ depends on the "quasi-momentum" p. Relations (26) were already given in [13] (they come basically from [32]) and were obtained in [33] essentially the same way as here. As pointed out in [13], the matrix elements of $q^{\pm}$ may be viewed as the classical counterpart of the quantum deformation of the $SU(2)$ $6j$-symbols.

The appearance of exchange relation (26) is a clear signal that the quantum group enters in the game. To obtain a monodromy-independent exchange relation that we would expect for quantum group symmetric fields, Faddeev [13], adapting to the WZW case the observation of Babelon [5] developed further by Gervais [29, 30], pointed out that one may twist the original fields

$$g_{ij} \mapsto g_{il} \chi_{lj}(p),$$

choosing $\chi(p)$ so that after the twist

$$\{g(x) \overset{\otimes}{,} g(y)\} = (g(x) \otimes g(y)) r^{\pm}, \tag{28}$$

where $r^{\pm}$ are the constant matrices solving the classical Yang-Baxter equation,

$$r^{\pm} = \pm \pi (2k)^{-1} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & -2 \mp 2 & 0 \\ 0 & -2 \pm 2 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{29}$$

It is easy to find that

$$\chi(p) = \begin{pmatrix} (2id \sin p)^{-1} e^{ip} & (2iDd \sin p)^{-1} \\ Dde^{-ip} & d \end{pmatrix}$$

with D an arbitrary constant and d an arbitrary function of p. The twisting of fields $g(x)$ may be viewed as the classical field theory analogue of the transformation from the IRF to the vertex form in the lattice exactly solvable models [46]. It seems, however, somewhat *ad hoc* (hence the ambiguity in the choice of χ, the loss of unitarity for twisted g's, their unnatural monodromy).

We propose to proceed somewhat differently in order to build the canonical formalism for the left-handed part of the WZW theory. We shall work with the complete phase space P_L changing instead form Ω_L by a monodromy-dependent term

$$\Omega_L \mapsto \tilde{\Omega}_L = \Omega_L - (4\pi)^{-1} k \varrho(\gamma)$$

so that $\tilde{\Omega}_L$ becomes closed. In view of (18), this requires that

$$d\varrho(\gamma) = \operatorname{tr}((d\gamma)\gamma^{-1})^3/3. \tag{30}$$

Of course, ϱ cannot be globally non-singular 2-form on the group so we shall require that it be such on an open dense set in G containing 1. We shall show that for $G=SU(2)$ there is a unique choice of ϱ which puts the classical exchange relation for the original fields $g_{ij}(x)$ into the form (28) (in general such ϱ's are in close relation to the constant solutions of the classical Yang-Baxter equation; we shall discuss the case of the general group in a separate publication). Notice that if, at the same time, we modify

$$\Omega_R \mapsto \tilde{\Omega}_R = \Omega_R + (4\pi)^{-1} k\varrho(\gamma),$$

then the symplectic structure induced by $\tilde{\Omega}_L + \tilde{\Omega}_R$ on P as given by (17) will coincide with the original one. The modifications do not effect the complete WZW theory but allow to build the canonical formalism for its chiral parts.

Let us write

$$\varrho = \operatorname{tr} \eta((d\gamma)\gamma^{-1} \otimes (d\gamma)\gamma^{-1}), \tag{31}$$

where η is a function on G with values in $\wedge^2 \mathscr{G}^{\mathbf{C}}$ and "tr" contracts in both tensor factors separately. For $F(g)=g_{ij}(x)$, $x \in [0, 2\pi]$, $dF = \tilde{\Omega} \llcorner \tilde{X}$ implies that

$$\begin{aligned} \tilde{\mathscr{X}}_{mn} &= -2\pi k^{-1}(g_{in}(x)\delta_{mj} - g_{ij}(x)\delta_{mn}/2)\Theta(\cdot - x) + \tilde{\mathscr{X}}(0)_{mn} \\ &\equiv \mathscr{Z}_{mn}\Theta(\cdot - x) + \tilde{\mathscr{X}}(0)_{mn}, \end{aligned} \tag{32}$$

where now

$$2\tilde{\mathscr{X}}(0) + \tilde{\mathscr{X}}_0 = \eta \llcorner \tilde{\mathscr{X}}_0, \tag{33}$$

compare (24) and (25). $\tilde{\mathscr{X}}_0$ is related to $\mathscr{X}(0)$ by

$$\tilde{\mathscr{X}}(0) + \mathscr{Z} = \operatorname{Ad}_{\gamma^{-1}}(\tilde{\mathscr{X}}(0) + \tilde{\mathscr{X}}_0) \tag{34}$$

which allows to eliminate $\tilde{\mathscr{X}}_0$ from (34) leading to

$$\tilde{\mathscr{X}}(0) + \operatorname{Ad}_\gamma(\tilde{\mathscr{X}}(0) + \mathscr{Z}) = \eta \llcorner (\operatorname{Ad}_\gamma(\tilde{\mathscr{X}}(0) + \mathscr{Z}) - \tilde{\mathscr{X}}(0)). \tag{35}$$

At $\gamma = 1$, we obtain

$$\tilde{\mathscr{X}}(0) = (\eta(1) \llcorner - 1)\mathscr{Z}/2. \tag{36}$$

Inserting (36) to (32) gives the Hamiltonian vector field $\tilde{X}$ corresponding to $g_{ij}(x_0)$ for $\gamma = 1$:

$$\tilde{\mathscr{X}} = (\eta(1) \llcorner + 2\Theta(y - x) - 1)\mathscr{Z}/2. \tag{37}$$

Consequently, at $\gamma = 1$,

$$\{g_{ij}(x), g_{hl}(y)\} = \tfrac{1}{2}(g(y)(\eta(1) \llcorner + 2\Theta(y-x) - 1)\mathscr{Z})_{hl}. \tag{38}$$

The requirement that this agrees with (28) fixes $\eta(1)$:

$$\eta(1) = (\sigma^+ \otimes \sigma^- - \sigma^- \otimes \sigma^+)/2, \tag{39}$$

where $\sigma^{\pm}$ are the Pauli matrices.

We would like the exchange relation to have the monodromy-independent form (28) for all values of γ. This implies that the Hamiltonian vector field $\tilde{X}$ of $g_{ij}(x)$ has to be given by (37) everywhere and not only at $\gamma = 1$. In other words, we should have

$$\Delta\tilde{\mathscr{X}}(0) \equiv \tilde{\mathscr{X}}(0) - (\eta(1) \llcorner - 1)\mathscr{Z} = 0.$$

But (35) implies that

$$\begin{aligned}&2\Delta\tilde{\mathscr{X}}(0)+(\mathrm{Ad}_\gamma-1)((\eta(1)\llcorner+1)/2+\Delta\tilde{\mathscr{X}}(0))\\&\quad=\eta\llcorner((\mathrm{Ad}_\gamma-1)((\eta(1)\llcorner+1)\mathscr{Z}/2+\Delta\tilde{\mathscr{X}}(0))+(\eta-\eta(1))\llcorner\mathscr{Z}\,.\end{aligned}$$

Since we want $\Delta\tilde{\mathscr{X}}(0)$ to vanish for any value of $\mathscr{Z}$, it follows that

$$(\eta\llcorner-1)(\mathrm{Ad}_\gamma-1)(\eta\llcorner(1)+1)/2+(\eta-\eta(1))\llcorner=0 \tag{40}$$

or that

$$\eta\llcorner=1+(\eta(1)\llcorner-1)[1+(\mathrm{Ad}_\gamma-1)(\eta(1)\llcorner+1)/2]^{-1} \tag{41}$$

wherever the operator $1+(\mathrm{Ad}_\gamma-1)(\eta(1)\llcorner+1)/2$ on $\mathscr{G}^{\mathbf{C}}$ is invertible (it certainly is around $\gamma=1$). We see that $\eta(1)$ fixes η, at least locally. It is not difficult to solve (41) explicitly in the Gauss parametrization of $SL(2,\mathbf{C})$

$$\gamma=\begin{pmatrix}1 & v\\ 0 & 1\end{pmatrix}\begin{pmatrix}e^{\phi} & 0\\ 0 & e^{-\phi}\end{pmatrix}\begin{pmatrix}1 & 0\\ w & 0\end{pmatrix}. \tag{42}$$

An easy calculation gives

$$\eta=(\sigma^+\otimes\sigma^- -\sigma^-\otimes\sigma^+)/2+v(\sigma^+\otimes\sigma^3-\sigma^3\otimes\sigma^+)/2 \tag{43}$$

and

$$\varrho=-e^{-2\phi}dvdw \tag{44}$$

so that

$$d\varrho=2e^{-2\phi}d\phi dvdw=\mathrm{tr}((d\gamma)\gamma^{-1})^3/3\,. \tag{45}$$

More exactly, Eq. (44) gives the extension of ϱ to a holomorphic 2-form on the open dense stratum of $SL(2,\mathbf{C})$ parametrized by (42). By restriction, it produces the desired 2-form on $\left\{\gamma\in SU(2)\,\middle|\,\mathrm{tr}\gamma\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\neq 0\right\}$. Form $\tilde{\Omega}_L$ on the corresponding subset P_{L0} of P_L is non-degenerate and defines the symplectic structure which leads to the classical exchange relation (28). Notice that ϱ (and $\tilde{\Omega}_L$) is a complex form. This modifies the natural unitarity properties of chiral fields [29, 30]. η of (43) lives effectively on $G^{\mathbf{C}}/B^-$, where B^- is a Borel subgroup of $G^{\mathbf{C}}$. This will still hold for other groups.

4. Quantum Group Symmetries of the Chiral $SU(2)$ WZW Model

Let us discuss the symmetries of the left-handed theory, i.e. of symplectic space $(P_{L0},\tilde{\Omega}_L)$. $\tilde{\Omega}_L$ is invariant under the left action of the loop group LG of G-valued maps $h(x)$, $h(x+2\pi)=h(x)$,

$$g\mapsto hg\,.$$

The infinitesimal version of this action is generated by the Hamiltonian vector fields of the currents

$$J^a=\frac{k}{2i}\,\mathrm{tr}\,\sigma^a g\partial_x g^{-1} \tag{46}$$

which have the affine Kac-Moody type of the Poisson brackets

$$\{J^a(x),J^b(y)\}=2\pi\varepsilon^{abc}\delta(x-y)J^c(y)+\pi k\delta^{ab}\partial_x\delta(x-y)\,. \tag{47}$$

We also have

$$\{J^a(x), g_{ij}(y)\} = \pi i\delta(x-y)(\sigma^a)_{il} g_{lj}(y). \tag{48}$$

As usually, the current algebra invariance implies conformal invariance with the Sugawara form of the energy-momentum tensor $T=(J^a)^2/k$,

$$\{T(x), T(y)\} = -2\pi\delta(x-y)\partial_y T(y) + 4\pi\partial_x\delta(x-y)T(y), \tag{49}$$

$$\{T(x), J^a(y)\} = -2\pi\delta(x-y)\partial_y J^a(y) + 2\pi\partial_x\delta(x-y)J^a(y), \tag{50}$$

$$\{T(x), g_{ij}(y)\} = -2\pi\delta(x-y)\partial_y g_{ij}(y). \tag{51}$$

Form Ω_L on P_L is also invariant under the right action of G

$$g \mapsto g g_0^{-1}.$$

inducing the adjoint action of G on monodromies γ. This invariance is broken by $\tilde{\Omega}_L$ to the Cartan subgroup of diagonal matrices in $G=SU(2)$ since 2-form ϱ of (44) is invariant only under the adjoint action of the Cartan subgroup. The infinitesimal action of the preserved $U(1)$ is generated by the Hamiltonian vector field of the (multivalued) function h of γ,

$$h(\gamma) = (2\pi i)^{-1} k\phi \tag{52}$$

in parametrization (42).

$$\{h, g_{ij}(x)\} = -\frac{i}{2} g_{il}(x)(\sigma^3)_{lj}. \tag{53}$$

Although the rest of right $SU(2)$ symmetry is broken by $\tilde{\Omega}_L$, it does not disappear altogether but becomes deformed to a semi-classical form of the quantum group symmetry. Indeed, consider functions e and f of γ given by

$$e = (2\pi i)^{-1} k e^{-\phi} w, \tag{54}$$

$$f = (2\pi i)^{-1} k e^{-\phi} v. \tag{55}$$

Denoting $q \equiv e^{-\pi i/k}$, we have

$$\{q^{\pm 2h}, e\} = \mp 2\pi k^{-1} e q^{\pm 2h}, \tag{56}$$

$$\{q^{\pm 2h}, f\} = \pm 2\pi k^{-1} f q^{\pm 2h}, \tag{57}$$

$$\{e, f\} = (2\pi)^{-1} k (q^{4h} - 1) + 2\pi k^{-1} e f. \tag{58}$$

The Poisson brackets of e, f, and $q^{\pm 2h}$ with $g_{ij}(x)$ are

$$\{e, g_{ij}(x)\} = -i g_{il}(x)(\sigma^+)_{lj} q^{2h}, \tag{59}$$

$$\{f, g_{ij}(x)\} = -i g_{il}(x)(\sigma^-)_{lj} q^{2h}, \tag{60}$$

$$\{q^{\pm 2h}, g_{ij}(x)\} = \mp \frac{\pi}{k} g_{il}(x)(\sigma^3)_{lj} q^{\pm 2h}. \tag{61}$$

Besides, the Poisson brackets of e, f, and $q^{\pm 2h}$ with J^a and T vanish. Functions e, f may be reexpressed as integrals

$$e = \int_0^{2\pi} \operatorname{tr}\sigma^+ j, \qquad f = \int_0^{2\pi} \operatorname{tr}\sigma^- j,$$

where

$$j(x)=(2\pi i)^{-1}k\sum_{n=0}^{\infty}\int_0^x dx_1\dots\int_0^{x_{n-1}}dx_n(g^{-1}\partial_x g)(x_n)\dots(g^{-1}\partial_x g)(x_1)$$

plays the role of the (non-local) current generating the right-hand symmetry.

It would be nice to formulate the deformed symmetry relations (56) to (61) in an abstract way. Notice that they are the semi-classical version of the quantum group $SU(2)_q$ relations

$$q^{\pm 2h}eq^{\mp 2h}=q^{\pm 2}e, \tag{62}$$

$$q^{\pm 2h}fq^{\mp 2h}=q^{\mp 2}f, \tag{63}$$

$$qef-q^{-1}fe=(q^{4h}-1)/(q-q^{-1}) \tag{64}$$

together with the quantum group symmetry of the vertex operators $g_{ij}(x)$,

$$eg_{ij}(x)=g_{ij}(x)e+g_{il}(x)(\sigma^+q^{\sigma^3/2})_{lj}q^{2h}, \tag{65}$$

$$fg_{ij}(x)=g_{ij}(x)f+g_{il}(x)(\sigma^-q^{\sigma^3/2})_{lj}q^{2h}, \tag{66}$$

$$q^{\pm 2h}g_{ij}(x)q^{\mp 2h}=g_{il}(x)(q^{\pm\sigma^3})_{lj} \tag{67}$$

(in more standard notation, $e=J^+q^h$, $f=J^-q^h$). Equations (56) to (61) may be obtained from (62) to (68) by expanding to the leading order in $1/k$, replacing the commutators by $i\{\cdot,\cdot\}$ (which is of order $1/k$) and taking into account the fact that classical e, f, and h are of order k (realization of such a semi-classical limit on matrix elements would require considering representations with spins going to infinity[1]). In fact, on the quantum level one should take $q=e^{-\pi i/(k+2)}$. Notice that the commutation relations (63) to (65) may be rephrased by saying that the quantum group generators e, f, and $q^{\pm 2h}$ act on states

$$g_{i_1j_1}(x_1)\dots g_{i_nj_n}(x_n)|0\rangle$$

by the coproduct formulae

$$\Delta^n(e)=\sum_{s=1}^n 1\otimes\dots\otimes 1\otimes\underset{\hat{s}}{e}\otimes q^{2h}\otimes\dots\otimes q^{2h},$$

$$\Delta^n(f)=\sum_{s=1}^n 1\otimes\dots\otimes 1\otimes\underset{\hat{s}}{f}\otimes q^{2h}\otimes\dots\otimes q^{2h},$$

$$\Delta^n(q^{\pm 2h})=q^{\pm 2h}\otimes\dots\otimes q^{\pm 2h}.$$

Upon quantization, the left-handed symmetries (47) to (51) should become:

$$[J^a(x),J^b(y)]=2\pi i\varepsilon^{abc}\delta(x-y)J^c(y)+\pi ik\delta^{ab}\partial_x\delta(x-y), \tag{68}$$

$$[J^a(x),g_{ij}(y)]=-\pi\delta(x-y)(\sigma^a)_{il}g_{lj}(y), \tag{69}$$

$$\begin{aligned}[T(x),T(y)]=&-2\pi i\delta(x-y)\partial_yT(y)+4\pi i\partial_x\delta(x-y)T(y)\\&-\pi ik(2(k+2))^{-1}(\partial_x^3+\partial_x)\delta(x-y),\end{aligned} \tag{70}$$

$$[T(x),J^a(y)]=-2\pi i\delta(x-y)\partial_yJ^a(y)+2\pi i\partial_x\delta(x-y)J^a(y), \tag{71}$$

$$[T(x),g_{ij}(y)]=-2\pi i\delta(x-y)\partial_yg_{ij}(y)+3\pi i(2(k+2))^{-1}\partial_x\delta(x-y)g_{ij}(x). \tag{72}$$

That the structure with properties (62) to (72), realizing the $SU(2)_q$ symmetries of the quantum WZW theory (or the related structures for the minimal or Toda

[1] We thank J.-L. Gervais for pointing this out

conformal field theories) should exist was indicated by the study of the braiding and modular properties of chiral vertex operators [49, 44, 45, 42, 2, 3, 17, 4, 43, 18, 29, 48]. Recently, it was observed [35, 36, 19] that such structures may be realized explicitly for generic values of q with the help of the free field representations of the conformal field theories [16, 28, 15, 6, 23]. For q a root of unity (so for integer level k in the WZW model, the values we are really interested in), existence of chiral fields with the properties (65) to (67) has not yet been demonstrated (partial results were discussed in [36, 37, 20, 24, 25]). It would be interesting to construct the chiral WZW theory by a direct quantization of the phase space $(P_{L0}, \tilde{\Omega}_L)$ rather than of $(P_L^{\mathrm{res}}, \Omega_L)$.

Let us finish the paper by few comments about the relation of this work to the recent paper by Alekseev-Shatashvili [1] who related the monodromy independent classical exchange relation (28) to the choice of boundary conditions

$$g(0)\in B^{+}\,, \qquad g(2\pi)\in B^{-} \tag{73}$$

for the left-handed field ($B^{\pm}$ are the Borel subgroups of upper-/lower-triangular matrices). Of course, such a boundary condition makes sense only for fields taking values in the complexified group so what follows applies to the holomorphic version of the canonical formalism developed above. Let $P_L^{\pm}$ denote the subspace of the complex phase space $P_{L0}^{\mathbf{C}}$ defined by the conditions (73). This is the subspace which was considered in [1] (besides P_L^{res}). It is easy to see that the restriction of $\tilde{\Omega}_L$ to $P_L^{\pm}$ is non-degenerate and that the Hamiltonian vector fields of functions $F(g) = g_{ij}(x)$ on $P_{L0}^{\mathbf{C}}$ are at $g\in P_L^{\pm}$ tangent to $P_L^{\pm}$. Consequently, the Poisson bracket $\{g(x)^{\otimes}, g(x)\}$ computed on $P_L^{\pm}$ using the restriction of $\tilde{\Omega}_L$ is the same as the one computed in the big space $P_{L0}^{\mathbf{C}}$, i.e. it is given by (28), as was postulated in [1]. Notice, however, that this is not the case for the currents J^a nor for the generators e, f which limits usefulness of the boundary conditions (73).

The right symmetry of the chiral WZW theory under the deformed $SU(2)$ may be also formulated in the dual way, as was suggested in Sect. 2.c of [1]. Consider the special Poisson structure [1] on G, leading to the Poisson brackets

$$\{g_0{}^{\otimes}, g_0\} = [g_0 \otimes g_0, r^{\pm}]\,.$$

Then the map m

$$P_L \times G \ni (g, g_0) \overset{m}{\longmapsto} g g_0 \in P_L \tag{74}$$

preserves the Poisson structures. In other words, the Poisson brackets of the pullbacks by m of functions $F(g) = g_{ij}(x)$ on P_L, computed with the use of the product symplectic structure, is still given by (28).

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References

1. Alekseev, A., Shatashvili, S.: Quantum groups and WZNW models. Commun. Math. Phys. **133**, 353–368 (1990)
2. Alvarez-Gaumé, L., Goméz, C., Sierra, G.: Hidden quantum symmetries in rational conformal field theories. Nucl. Phys. B **319**, 155–186 (1989)
3. Alvarez-Gaumé, L., Gómez, C., Sierra, G.: Quantum group interpretation of some conformal field theories. Phys. Lett. B **220**, 142–152 (1989)
4. Alvarez-Gaumé, L., Gómez, C., Sierra, G.: Duality and quantum groups. Nucl. Phys. B **330**, 347–398 (1990)
5. Babelon, O.: Extended conformal algebra and Yang-Baxter equation. Phys. Lett. B **215**, 523–529 (1988)
6. Bernard, D., Felder, G.: Fock representations and BRST cohomology in $SL(2)$ current algebra. Commun. Math. Phys. **127**, 145–168 (1990)
7. Bouwknegt, P., McCarthy, J., Pilch, K.: Quantum group structure in the Fock space resolutions of $\hat{sl}(n)$ representations. Commun. Math. Phys. **131**, 125–155 (1990)
8. Crnković, Č.: Symplectic geometry and (super-) Poincaré algebra in geometrical theories. Nucl. Phys. B **288**, 419–430 (1987)
9. Crnković, Č., Witten, E.: Covariant description of canonical formalism in geometrical theories. In: Three Hundred Years of Gravitation. Hawking, S.W., Israel, W. (eds.) pp. 676–684. Cambridge: Cambridge University Press 1987
10. Dedecker, P.: Calcul des variations, formes différentielles et champs géodésiques. In: Coll. Intern. Géométrie Diff. Strasbourg, Publications C.N.R.S. 1953
11. Drinfel'd, V.G.: Quantum groups. In: Proceedings of the International Congress of Mathematicians, pp. 798–820. Berkeley, California, USA, 1986
12. Faddeev, L.D.: Integrable models in $(1+1)$-dimensional quantum field theory. In: Recent Advances in Field Theory and Statistical Mechanics. Zuber, J.-B., Stora, R. (eds.) Amsterdam: North Holland 1984
13. Faddeev, L.D.: On the exchange matrix for WZNW model. Commun. Math. Phys. **132**, 131–138 (1990)
14. Faddeev, L.D., Reshetikhin, N.Yu., Takhtajan, L.A.: Quantization of Lie groups and Lie algebras. Algebra and Analysis **1**, 178–206 (1989) (in Russian)
15. Feigin, B.L., Frenkel, E.V.: Representations of affine Lie Kac-Moody algebras and bosonization. In: V.G. Knizhnik Memorial Volume. Singapore: World Scientific 1989
16. Felder, G.: BRST approach to minimal models. Nucl. Phys. B **317**, 215–236 (1989)
17. Felder, G., Fröhlich, J., Keller, G.: Braid Matrices and structure constants for minimal conformal models. Commun. Math. Phys. **124**, 647–664 (1989)
18. Felder, G., Fröhlich, J., Keller, G.: On the structure of unitary conformal field theory II: Representation theoretic approach. Commun. Math. Phys. **130**, 1–49 (1990)
19. Felder, G., Wieczerkowski, C.: Topological representations of the quantum group $U_q(sl_2)$. Zürich preprint 1990
20. Fröhlich, J., Kerler, T.: On the role of quantum groups in low dimensional local quantum field theories. Zürich preprint 1990
21. Garcia, P.L.: Reducibility of the symplectic structure of classical fields with gauge symmetry. In: Differential Geometric Methods in Mathematical Physics. Bleuler, K., Reetz, A. (eds.) Lecture Notes in Mathematics Vol. **570**. Berlin Heidelberg New York: Springer 1977
22. Gawędzki, K.: Gauge field periodic vacua and geometric quantization. Warsaw preprint 1978
23. Gawędzki, K.: Quadrature of conformal field theories. Nucl. Phys. B **328**, 733–752 (1989)
24. Gawędzki, K.: Geometry of Wess-Zumino-Witten models of conformal field theory. Bures-sur-Yvette preprint 1990
25. Gawędzki, K.: Edda of WZW, lectures given at Nordisk Forskakurs, Laugarvatn, Iceland, July 9–18, 1990
26. Gawędzki, K., Kondracki, W.: Canonical formalism for the local-type functionals in the classical field theory. Rep. Math. Phys. **6**, 465–476 (1974)
27. Gepner, D., Witten, E.: String theory on group manifolds. Nucl. Phys. B **278**, 493–549 (1986)
28. Gerasimov, A., Marshakov, A., Morozov, A., Olshanetsky, M., Shatashvili, S.: Wess-Zumino-Witten model as theory of free fields. Moscow preprint 1989
29. Gervais, J.-L.: The quantum group structure of $2d$ gravity and minimal models I. Commun. Math. Phys. **130**, 257–283 (1990)

30. Gervais, J.-L.: Solving strongly coupled 2*D*-gravity: 1. Unitary truncation and quantum group structure. Paris preprint 1990
31. Gervais, J.-L., Neveu, A.: New quantum treatment of Liouville field theory. Nucl. Phys. B **224**, 329–348 (1983)
32. Gervais, J.-L., Neveu, A.: Novel triangle relation and absence of tachyons in Liouville string field theory. Nucl. Phys. B **238**, 125–141 (1984)
33. Goddard, P., Olive, D., Schwimmer, A.: In preparation
34. Goldschmidt, H., Sternberg, S.: The Hamilton-Cartan formalism in the calculus of variations. Ann. Inst. Fourier (Grenoble) **23**, 203–267 (1973)
35. Gómez, C., Sierra, G.: Quantum group meaning of the Coulomb gas. Phys. Lett. B **240**, 149–157 (1990)
36. Gómez, C., Sierra, G.: The quantum symmetry of rational conformal field theories. Geneva preprint 1990
37. Hadjivanov, L.K., Paunov, R.R., Todorov, I.T.: Extended chiral conformal theories with a quantum symmetry. Orsay preprint 1990
38. Jimbo, M.: A q-difference analogue of $U(g)$ and Yang-Baxter equation. Lett. Math. Phys. **10**, 63–69 (1985)
39. Kijowski, J.: A finite-dimensional canonical formalism in the classical field theory. Commun. Math. Phys. **30**, 99–128 (1973)
40. Kijowski, J., Szczyrba, W.: A canonical structure for classical field theories. Commun. Math. Phys. **46**, 183–206 (1976)
41. Knizhnik, V., Zamolodchikov, A.B.: Current algebra and Wess-Zumino model in two dimensions. Nucl. Phys. B **247**, 83–103 (1984)
42. Kohno, T.: Monodromy representations of braid groups and Yang-Baxter equations. Ann. Inst. Fourier (Grenoble) **37** (4), 139–160 (1987)
43. Moore, G., Reshetikhin, K.: A comment on quantum group symmetry in conformal field theory. Nucl. Phys. B **328**, 557–574 (1989)
44. Moore, G., Seiberg, N.: Polynomial equations for rational conformal field theories. Phys. Lett. B **212**, 451–460 (1988)
45. Moore, G., Seiberg, N.: Classical and quantum conformal field theory. Commun. Math. Phys. **123**, 177–254 (1989)
46. Pasquier, V.: Etiology of IRF Models. Commun. Math. Phys. **118**, 355–364 (1988)
47. Szczyrba, W.: A symplectic structure on the set of Einstein metrics. Commun. Math. Phys. **51**, 163–182 (1976)
48. Todorov, I.T.: Quantum groups as symmetries of chiral conformal algebras. Orsay preprint 1989
49. Tsuchiya, A., Kanie, Y.: Vertex operators in the conformal field theory on $P1$ and monodromy representations of the braid group. In: Conformal Field Theory and Solvable Lattice Models. Adv. Stud. Pure Math. **16**, 297–372 (1988)
50. Witten, E.: Interacting field theory of open superstrings. Nucl. Phys. B **276**, 291–324 (1986)
51. Witten, E.: Non-abelian bosonization in two dimensions. Commun. Math. Phys. **92**, 455–472 (1984)
52. Woronowicz, S.L.: Twisted $SU(2)$ group. An example of a non-commutative differential calculus. Publ. RIMS, Kyoto University **23**, 117–181 (1987)
53. Zuckerman, G.: Action functionals and global geometry. Yale preprint 1986

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