Commun. Math. Phys. 136, 633–644 (1991)

Communications in
Mathematical
Physics

Correlation Function of Fields in One-Dimensional Bose-Gas

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Received September 10, 1990

Abstract. Correlation function of fields is presented as a Fredholm minor, at finite coupling constant in one-dimensional Bose gas.

1. Introduction

We discuss correlation function of fields in the quantum nonlinear Schrödinger equation model (NS-model). The Hamiltonian of this model is equal to

$$\mathscr{H} = \int_0^L dx(\partial_x\psi^+\partial_x\psi + c\psi^+\psi^+\psi\psi - h\psi^+\psi)\,. \tag{1.1}$$

Here $c > 0$ is a coupling constant, $h > 0$: chemical potential; L: a length of a box; $\psi(x)$: a canonical Bose-field:

$$\begin{aligned}[\psi(x), \psi^+(y)] &= \delta(x-y)\,,\\ \psi(x)\,|0\rangle &= 0\,.\end{aligned} \tag{1.2}$$

In the limit $c = \infty$ (free fermions) the correlator was calculated by Lenard [1] in terms of a Fredholm minor. This representation was used for writing differential equations for the correlator [2, 3]. In the present paper we consider the case of the finite coupling constant c. Using the method of algebraic anzats Bethe we present the correlator as a minor of an integral operator, which depends on auxiliary quantum fields. Such a representation can be used for writing the system of integro-differential equations for the correlation function.

2. Algebraic Anzats Bethe

The main object of algebraic anzats Bethe is the monodromy matrix $T(\lambda)$. In the case of NS-model it is 2×2 matrix:

$$T(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}. \tag{2.1}$$

Matrix elements are quantum operators, which depend on the spectral parameter λ. Commutation relations between these operators are given by the formula:

$$R(\lambda, \mu)\,(T(\lambda) \otimes T(\mu)) = (T(\mu) \otimes T(\lambda))R(\lambda, \mu)\,. \tag{2.2}$$

Here $R(\lambda, \mu)$ is a 4×4 matrix with c-number elements

$$R(\lambda, \mu) = \begin{pmatrix} f(\mu, \lambda) & 0 & 0 & 0 \\ 0 & g(\mu, \lambda) & 1 & 0 \\ 0 & 1 & g(\mu, \lambda) & 0 \\ 0 & 0 & 0 & f(\mu, \lambda) \end{pmatrix}, \tag{2.3}$$

where

$$f(\lambda, \mu) = \frac{\lambda - \mu + ic}{\lambda - \mu}\,; \qquad g(\lambda, \mu) = \frac{ic}{\lambda - \mu}\,. \tag{2.4}$$

The functions

$$h(\lambda, \mu) = \frac{f(\lambda, \mu)}{g(\lambda, \mu)} = \frac{\lambda - \mu + ic}{ic}\,, \tag{2.5}$$

$$t(\lambda, \mu) = \frac{g(\lambda, \mu)}{h(\lambda, \mu)} = -\frac{c^2}{(\lambda - \mu)\,(\lambda - \mu + ic)} \tag{2.6}$$

will also be useful.

Let us write down some of the commutation relations (2.2) explicitly:

$$[B(\lambda), B(\mu)] = [C(\lambda), C(\mu)] = 0\,, \tag{2.7}$$

$$A(\mu)B(\lambda) = f(\mu, \lambda)B(\lambda)A(\mu) + g(\lambda, \mu)B(\mu)A(\lambda)\,, \tag{2.8}$$

$$C(\mu)D(\lambda) = f(\mu, \lambda)D(\lambda)C(\mu) + g(\lambda, \mu)D(\mu)C(\lambda)\,, \tag{2.9}$$

$$[C(\mu), B(\lambda)] = g(\mu, \lambda)\,\{A(\mu)D(\lambda) - A(\lambda)D(\mu)\}\,. \tag{2.10}$$

Other important objects in algebraic anzats Bethe are the pseudovacuum $|0\rangle$ and dual pseudovacuum $\langle 0|$. In the NS-model these vectors coincide with vectors (1.2), so we use the same notations for them. Properties of $|0\rangle$ and $\langle 0|$ are the following:

$$\begin{aligned} &A(\lambda)\,|0\rangle = a(\lambda)\,|0\rangle; \quad && D(\lambda)\,|0\rangle = d(\lambda)\,|0\rangle; \quad && C(\lambda)\,|0\rangle = 0\,, \\ &\langle 0|\,A(\lambda) = a(\lambda)\,\langle 0|; \quad && \langle 0|\,D(\lambda) = d(\lambda)\,\langle 0|; \quad && \langle 0|\,B(\lambda) = 0\,. \end{aligned} \tag{2.11}$$

Here $a(\lambda) = \exp\left\{-\frac{i\lambda L}{2}\right\}$, $d(\lambda) = a^{-1}(\lambda)$. Eigenfunctions of Hamiltonian (1.1) coincide with eigenfunctions of transfer-matrix $\tau(\lambda) = A(\lambda) + D(\lambda)$. They can be written in the form

$$|\Psi_N(\{\lambda\})\rangle = \prod_{j=1}^{N} B(\lambda_j)\,|0\rangle\,, \tag{2.12}$$

where all parameters λ_j are different and satisfy the system

$$\frac{a(\lambda_j)}{d(\lambda_j)} \prod_{k=1}^{N} \frac{h(\lambda_j, \lambda_k)}{h(\lambda_k, \lambda_j)} = (-1)^{N-1}\,. \tag{2.13}$$

The function

$$\langle \tilde{\Psi}_N(\{\lambda\})| = \langle 0| \prod_{j=1}^{N} C(\lambda_j)\,, \tag{2.14}$$

where λ_j also satisfy (2.13), is a dual eigenfunction.

To describe commutation relations between fields $\psi^+(x)$, $\psi(x)$ and the operators A, B, C, D we use a lattice approximation of the model. The monodromy matrix (2.1) is given by a product of L-operators

$$T(\lambda) = L_M(\lambda)L_{M-1}(\lambda)\dots L_1(\lambda), \tag{2.15}$$

where

$$L_n(\lambda) = \begin{pmatrix} 1 - \frac{i\lambda\Delta}{2} & -i\sqrt{c}\,\Delta\psi_n^+ \\ i\sqrt{c}\,\Delta\psi_n & 1 + \frac{i\lambda\Delta}{2} \end{pmatrix} + O(\Delta^2). \tag{2.16}$$

Δ is a step of the lattice. Operators ψ_n^+, ψ_n are lattice approximations of the fields $\psi^+(x)$ and $\psi(x)$.

Their commutator is equal to

$$[\psi_n, \psi_m^+] = \frac{1}{\Delta}\delta_{nm}. \tag{2.17}$$

Let us represent $T(\lambda)$ by the following way:

$$T(\lambda) = T_2(\lambda)T_1(\lambda). \tag{2.18}$$

Here

$$T_2(\lambda) = L_M(\lambda)\dots L_n(\lambda); \qquad T_1(\lambda) = L_{n-1}(\lambda)\dots L_1(\lambda). \tag{2.19}$$

n is a fixed site of the lattice and

$$T_i(\lambda) = \begin{pmatrix} A_i(\lambda) & B_i(\lambda) \\ C_i(\lambda) & D_i(\lambda) \end{pmatrix}, \qquad i = 1, 2. \tag{2.20}$$

Using (2.16) one can obtain

$$[T_2(\lambda), \psi_n^+] = i\sqrt{c}\,L_M(\lambda)\dots L_{n+1}(\lambda)\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

In the continuous limit $\Delta \to 0$, $M \to \infty$, $M\Delta = L$ we can rewrite this formula

$$[T_2(\lambda), \psi^+(x)] = i\sqrt{c}T_2(\lambda)\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{2.21}$$

Here the point x corresponds to the n^{th} site of lattice. Finally we have

$$[C_2(\lambda), \psi^+(x)] = i\sqrt{c}\,D_2(\lambda). \tag{2.22}$$

Commutators $\psi^+(x)$ with elements of the matrix $T_1(\lambda)$ are equal to zero. Commutator between $\psi(0)$ and $T(\lambda)$ can be calculated in a similar way. Namely

$$[\psi(0), B(\lambda)] = -i\sqrt{c}\,A(\lambda). \tag{2.23}$$

The properties (2.22), (2.23) are sufficient for calculation of the correlation function of the fields. At the conclusion of the section we'll give some properties of the matrices $T_1(\lambda)$ and $T_2(\lambda)$. Each of them satisfies Eq. (2.2) with the R-matrix (2.3), so commutation relations (2.7)–(2.10) for operators A_i, B_i, C_i, $D_i (i = 1, 2)$ are valid. The elements of the different matrix commute. Matrix $T_i(\lambda)$ has a pseudovacuum $|0\rangle_i$ and dual vector ${}_i\langle 0|$. The pseudovacuum $|0\rangle$ is equal to

$|0\rangle = |0\rangle_2 \otimes |0\rangle_1$. The properties of vectors $|0\rangle_i$ and ${}_i\langle 0|$ are similar to properties of $|0\rangle$ and $\langle 0|$:

$$\begin{aligned} &A_i(\lambda)\,|0\rangle_i = a_i(\lambda)\,|0\rangle_i; \quad D_i(\lambda)\,|0\rangle_i = d_i(\lambda)\,|0\rangle_i; \quad C_i(\lambda)\,|0\rangle_i = 0\,, \\ &{}_i\langle 0|\,A_i(\lambda) = a_i(\lambda)\,{}_i\langle 0|; \quad {}_i\langle 0|\,D_i(\lambda) = d_i(\lambda)\,{}_i\langle 0|; \quad {}_i\langle 0|\,B_i(\lambda) = 0\,. \end{aligned} \tag{2.24}$$

Here $a_1(\lambda) = \exp\left\{-\frac{ix\lambda}{2}\right\}$, $a_2(\lambda) = \exp\left\{\frac{i\lambda}{2}(L-x)\right\}$; $d_i(\lambda) = a_i^{-1}(\lambda)$.

Note that

$$a(\lambda) = a_1(\lambda)a_2(\lambda)\,, \qquad d(\lambda) = d_1(\lambda)d_2(\lambda)\,. \tag{2.25}$$

Finally we'll give a representation for the function $\prod_{j=1}^{N} B(\lambda_j)\,|0\rangle$ in terms of the elements of matrices T_1 and T_2 (see [4]):

$$\begin{aligned} \prod_{j=1}^{N} B(\lambda_j)\,|0\rangle = &\sum_{\{\lambda\}=\{\lambda_{\rm I}\}\cup\{\lambda_{\rm II}\}} \prod_{\rm I} B_1(\lambda_{\rm I})\,|0\rangle_1 \prod_{\rm II} B_2(\lambda_{\rm II})\,|0\rangle_2 \\ &\times \prod_{\rm I} a_2(\lambda_{\rm I}) \prod_{\rm II} d_1(\lambda_{\rm II}) \prod_{\rm I,II} f(\lambda_{\rm I}, \lambda_{\rm II})\,. \end{aligned} \tag{2.26}$$

Here the sum is taken over all partitions of the set $\{\lambda\}$ into two disjoint subsets $\{\lambda_{\rm I}\}$ and $\{\lambda_{\rm II}\}$. The symbol $\prod_{\rm I}$ $\left(\text{or} \prod_{\rm II}\right)$ means the product over all $\lambda \in \{\lambda_{\rm I}\}$ (correspondingly $\lambda \in \{\lambda_{\rm II}\}$).

An analogous formula can be written for the dual function:

$$\begin{aligned} \langle 0| \prod_{j=1}^{N} C(\lambda_j) = &\sum_{\{\lambda\}=\{\lambda_{\rm I}\}\cup\{\lambda_{\rm II}\}} {}_1\langle 0| \prod_{\rm I} C_1(\lambda_{\rm I})\, {}_2\langle 0| \prod_{\rm II} C_2(\lambda_{\rm II}) \\ &\times \prod_{\rm I} d_2(\lambda_{\rm I}) \prod_{\rm II} a_1(\lambda_{\rm II}) \prod_{\rm I,II} f(\lambda_{\rm II}, \lambda_{\rm I})\,. \end{aligned} \tag{2.27}$$

3. Matrix Element of Operators $\psi^+(x)\,\psi(0)$

In this section we'll calculate the matrix element of the operator $\psi^+(x)\psi(0)$:

$$G_N = \langle 0| \prod_{j=1}^{N} C(\lambda_j^C)\psi^+(x)\psi(0) \prod_{j=1}^{N} B(\lambda_j^B)\,|0\rangle\,. \tag{3.1}$$

Here the parameters $\{\lambda^C\}$ and $\{\lambda^B\}$ are arbitrary complex numbers. The only condition is $\lambda_j^B \neq \lambda_K^B$, $\lambda_j^C \neq \lambda_K^C$, $(j, K = 1, \ldots, N)$. Consider the action of the operator $\psi(0)$ on the vector $\prod_{j=1}^{N} B(\lambda_j)\,|0\rangle$. Using (2.23) we have

$$\psi(0) \prod_{j=1}^{N} B(\lambda_j)\,|0\rangle = -i\sqrt{c} \sum_{K=1}^{N} B(\lambda_1) \ldots B(\lambda_{K-1}) A(\lambda_K) B(\lambda_{K+1}) \ldots B(\lambda_N)\,|0\rangle\,. \tag{3.2}$$

One can rewrite this formula as follows [see (2.8), (2.11)]:

$$\psi(0)\prod_{i=1}^{N} B(\lambda_j)\,|0\rangle = -i\sqrt{c}\sum_{K=1}^{N} \Lambda_K a(\lambda_K)\prod_{\substack{m=1\\ m\neq K}}^{N} B(\lambda_m)\,|0\rangle\,, \tag{3.3}$$

where Λ_K is a rational function on λ's, depending on the functions f and g (2.4). Let us calculate this coefficient. Due to (2.7) the right-hand side of (3.3) is symmetric in all λ, so it is sufficient to calculate the coefficient Λ_1. Obviously this term can be obtained only if $K = 1$ in (3.2):

$$-i\sqrt{c}\,A(\lambda_1)B(\lambda_2)\dots B(\lambda_N)\,.$$

Now we must move the operator $A(\lambda_1)$ to the right, using only the first term in formula (2.8). We have

$$\Lambda_1 = \prod_{m=2}^{N} f(\lambda_1, \lambda_m)\,,$$

and so

$$\Lambda_K = \prod_{\substack{m=1\\ m\neq K}}^{N} f(\lambda_K, \lambda_m)\,. \tag{3.4}$$

Consider now the action $\psi^+(x)$ on the vector $\langle 0|\prod_{j=1}^{N} C(\lambda_j)$. To do it, it is necessary to represent this vector in terms of C_1 and C_2 (2.27). After that all calculations are analogous to the case already considered. Formula (2.22) shows that

$$\begin{aligned}\langle 0|\prod_{j=1}^{N} C(\lambda_j)\psi^+(x) = i\sqrt{c} &\sum_{\{\lambda\}=\{\lambda_0\}\cup\{\lambda_{\mathrm{I}}\}\cup\{\lambda_{\mathrm{II}}\}} a_1(\lambda_0)d_2(\lambda_0)\\ &\times {}_1\langle 0|\prod_{\mathrm{I}} C_1(\lambda_{\mathrm{I}})\,{}_2\langle 0|\prod_{\mathrm{II}} C_2(\lambda_{\mathrm{II}})\prod_{\mathrm{I}} d_2(\lambda_{\mathrm{I}})f(\lambda_0, \lambda_{\mathrm{I}})\\ &\times\prod_{\mathrm{II}} a_1(\lambda_{\mathrm{II}})f(\lambda_{\mathrm{II}}, \lambda_0)\prod_{\mathrm{I,II}} f(\lambda_{\mathrm{II}}, \lambda_{\mathrm{I}})\,.\end{aligned} \tag{3.5}$$

Here the sum is taken over three subsets $\{\lambda_0\}$, $\{\lambda_{\mathrm{I}}\}$, $\{\lambda_{\mathrm{II}}\}$. Subsets $\{\lambda_{\mathrm{I}}\}$, $\{\lambda_{\mathrm{II}}\}$ are arbitrary, subset $\{\lambda_0\}$ contains exactly one element.

Formula (3.3) also can be written in terms of B_1 and B_2. Combining this formula with (3.5) we find matrix element G_N:

$$\begin{aligned}c^{-1}G_N = &\sum_{\substack{\{\lambda^C\}=\{\lambda_0^C\}\cup\{\lambda_{\mathrm{I}}^C\}\cup\{\lambda_{\mathrm{II}}^C\}\\ \{\lambda^B\}=\{\lambda_0^B\}\cup\{\lambda_{\mathrm{I}}^B\}\cup\{\lambda_{\mathrm{II}}^B\}}} {}_1\langle 0|\prod_{\mathrm{I}} C_1(\lambda_{\mathrm{I}}^C)\prod_{\mathrm{I}} B_1(\lambda_{\mathrm{I}}^B)\,|0\rangle_1\\ &\times {}_2\langle 0|\prod_{\mathrm{II}} C_2(\lambda_{\mathrm{II}}^C)\prod_{\mathrm{II}} B_2(\lambda_{\mathrm{II}}^B)\,|0\rangle_2 a_1(\lambda_0^C)d_2(\lambda_0^C)a_1(\lambda_0^B)a_2(\lambda_0^B)\\ &\times\prod_{\mathrm{I}}\{a_2(\lambda_{\mathrm{I}}^B)d_2(\lambda_{\mathrm{I}}^C)f_{0\mathrm{I}}^{CC}f_{0\mathrm{I}}^{BB}\}\prod_{\mathrm{II}}\{d_1(\lambda_{\mathrm{II}}^B)a_1(\lambda_{\mathrm{II}}^C)f_{\mathrm{II}0}^{CC}f_{0\mathrm{II}}^{BB}\}\\ &\times\prod_{\mathrm{I,II}}\{f_{\mathrm{I,II}}^{BB}f_{\mathrm{II,I}}^{CC}\}\,.\end{aligned} \tag{3.6}$$

Here partitions of the sets $\{\lambda^C\}$ and $\{\lambda^B\}$ are independent except that $\mathrm{card}\{\lambda_{\mathrm{I}}^C\} = \mathrm{card}\{\lambda_{\mathrm{I}}^B\}$, $\mathrm{card}\{\lambda_0^C\} = \mathrm{card}\{\lambda_0^B\} = 1$. In (3.6) we also use abbreviated notations:

$$f_{\mathrm{I,II}}^{BB} = f(\lambda_{\mathrm{I}}^B, \lambda_{\mathrm{II}}^B), \qquad f_{0\mathrm{I}}^{CC} = f(\lambda_0^C, \lambda_{\mathrm{I}}^C)$$

etc. Such notations as

$$h_{\mathrm{I,II}}^{CB} = h(\lambda_{\mathrm{I}}^C, \lambda_{\mathrm{II}}^B), \qquad g_{jK}^{CC} = g(\lambda_j^C, \lambda_K^C), \qquad t_{jK} = t(\lambda_j, \lambda_K)$$

and others we'll use in the next section.

So we expressed the matrix element G_N in terms of scalar products

$$\langle 0| \prod C(\lambda^C) \prod B(\lambda^B) |0\rangle$$

4. Dual Fields

To write formula (3.6) as a determinant of $N \times N$ matrix we'll use the technique of dual fields. This approach was developed in [5]. Let us introduce 10 new fields. Each of them is the sum of operator "coordinate" and operator "momentum":

$$\begin{aligned}
&\Phi_{A_K}(\lambda) = Q_{A_K}(\lambda) + P_{D_K}(\lambda); && \Phi_{D_K}(\lambda) = Q_{D_K}(\lambda) + P_{A_K}(\lambda); \quad K = 1, 2, \\
&\varphi_{A_1}(\lambda) = q_{A_1}(\lambda) + p_{D_2}(\lambda); && \varphi_{D_1}(\lambda) = q_{D_1}(\lambda) + p_{A_2}(\lambda), \\
&\varphi_{A_2}(\lambda) = q_{A_2}(\lambda) + p_{D_1}(\lambda); && \varphi_{D_2}(\lambda) = q_{D_2}(\lambda) + p_{A_1}(\lambda), \\
&\varphi_{A_3}(\lambda) = q_{A_3}(\lambda) + p_{D_3}(\lambda); && \varphi_{D_3}(\lambda) = q_{D_3}(\lambda) + p_{A_3}(\lambda).
\end{aligned} \tag{4.1}$$

These are Bose fields.

The fields Φ and φ act in auxiliary Fock space. Vacuum in this space also will be denoted by $|0\rangle$. All "momenta" annihilate it:

$$P(\lambda)\,|0\rangle = p(\lambda)\,|0\rangle = 0. \tag{4.2}$$

The dual vacuum $\langle 0|$ is the eigenvector for "coordinates":

$$\langle 0|\, Q_{A_K}(\lambda) = \langle 0|\, q_{A_K}(\lambda) = \ln a_K(\lambda)\, \langle 0|, \qquad K = 1, 2, \tag{4.3}$$

$$\langle 0|\, Q_{D_K}(\lambda) = \langle 0|\, q_{D_K}(\lambda) = \ln d_K(\lambda)\, \langle 0|, \qquad K = 1, 2, \tag{4.4}$$

$$\langle 0|\, q_{A_3}(\lambda) = \langle 0|\, q_{D_3}(\lambda) = 0, \tag{4.5}$$

$$\langle 0 \,|\, 0\rangle = 1.$$

Nonzero commutators are

$$\begin{aligned}
[P_{A_j}(\lambda), Q_{A_K}(\mu)] &= \delta_{jK} \ln h(\mu, \lambda) \\
[P_{D_j}(\lambda), Q_{D_K}(\mu)] &= \delta_{jK} \ln h(\lambda, \mu)
\end{aligned}; \qquad j, K = 1, 2, \tag{4.6}$$

$$\begin{aligned}
[p_{A_j}(\lambda), q_{A_K}(\mu)] &= \delta_{jK} \ln h(\mu, \lambda) \\
[p_{D_j}(\lambda), q_{D_K}(\mu)] &= \delta_{jK} \ln h(\lambda, \mu)
\end{aligned}; \qquad j, K = 1, 2, 3. \tag{4.7}$$

The remarkable property of the fields Φ, φ is that they all commute:

$$[\Phi_\alpha(\lambda), \Phi_\beta(\lambda)] = [\Phi_\alpha(\lambda), \varphi_\beta(\mu)] = [\varphi_\alpha(\lambda), \varphi_\beta(\mu)] = 0. \tag{4.8}$$

Here α, β run throw all the possible indices.

One of the results of paper [5] is the representation of scalar product in terms of dual fields Φ:

$$
{}_m\langle 0| \prod_{j=1}^{N} C_m(\lambda_j^C) \prod_{j=1}^{N} B_m(\lambda_j^B) |0\rangle_m
= \prod_{j>K}^{N} g_{jK}^{CC} g_{Kj}^{BB} \langle 0| \det{}_N S^{(m)}(\lambda^C, \lambda^B) |0\rangle , \qquad m = 1, 2. \tag{4.9}
$$

Here $S^{(m)}(\lambda^C, \lambda^B)$ is an $N \times N$ matrix with elements

$$
S_{jK}^{(m)}(\lambda^C, \lambda^B) = t_{jK}^{CB} \exp\{\Phi_{A_m}(\lambda_j^C) + \Phi_{D_m}(\lambda_K^B)\}
+ t_{Kj}^{BC} \exp\{\Phi_{D_m}(\lambda_j^C) + \Phi_{A_m}(\lambda_K^B)\}. \tag{4.10}
$$

Recall that the notations g_{jK}^{CC}, t_{jK}^{CB} mean $g(\lambda_j^C, \lambda_K^C)$, $t(\lambda_j^C, \lambda_K^B)$ correspondingly. Using (4.9) one can write (3.6) in such a way

$$
\begin{aligned}
c^{-1} G_N = \prod_{j>K}^{N} g_{jK}^{CC} g_{Kj}^{BB} & \sum_{\substack{\{\lambda^C\}=\{\lambda_0^C\}\cup\{\lambda_{\mathrm{I}}^C\}\cup\{\lambda_{\mathrm{II}}^C\} \\ \{\lambda^B\}=\{\lambda_0^B\}\cup\{\lambda_{\mathrm{I}}^B\}\cup\{\lambda_{\mathrm{II}}^B\}}} (-1)^{[P_C+P_B]+N_1} \\
& \times \langle 0| \det{}_{N_1} S^{(1)}(\lambda_{\mathrm{I}}^C, \lambda_{\mathrm{I}}^B) \det{}_{N_2} S^{(2)}(\lambda_{\mathrm{II}}^C, \lambda_{\mathrm{II}}^B) |0\rangle \\
& \times a_1(\lambda_0^C) d_2(\lambda_0^C) a_1(\lambda_0^B) a_2(\lambda_0^B) \prod_{\mathrm{I}} \{a_2(\lambda_{\mathrm{I}}^B) d_2(\lambda_{\mathrm{I}}^C) h_{0\mathrm{I}}^{CC} h_{0\mathrm{I}}^{BB}\} \\
& \times \prod_{\mathrm{II}} \{d_1(\lambda_{\mathrm{II}}^B) a_1(\lambda_{\mathrm{II}}^C) h_{\mathrm{II}0}^{CC} h_{0\mathrm{II}}^{BB}\} \prod_{\mathrm{I},\mathrm{II}} \{h_{\mathrm{II},\mathrm{I}}^{CC} h_{\mathrm{I},\mathrm{II}}^{BB}\}.
\end{aligned} \tag{4.11}
$$

Here we write all functions $f(\lambda, \mu)$ as a product $g(\lambda, \mu) h(\lambda, \mu)$ [see (2.5)] and use property of antisymmetry of functions $g(\lambda, \mu)$. $P_{C,B}$ is permutation, which transforms sequence $\{\lambda_0^{C,B}\}$, $\{\lambda_{\mathrm{I}}^{C,B}\}$, $\{\lambda_{\mathrm{II}}^{C,B}\}$ into sequence $\{\lambda_1^{C,B}, \ldots, \lambda_N^{C,B}\}$,

$$
N_1 = \mathrm{card}\{\lambda_{\mathrm{I}}^C\}, \qquad N_2 = \mathrm{card}\{\lambda_{\mathrm{II}}^C\} = N - N_1 - 1.
$$

Now everything is ready to prove the following theorem.

Theorem. *The matrix element G_N of operator $\psi^+(x)\psi(0)$ is equal to*

$$
G_N = \prod_{j>K} g_{jK}^{CC} g_{Kj}^{BB} \frac{\partial}{\partial \alpha} \langle 0| \det{}_N M |0\rangle|_{\alpha=0}. \tag{4.12}
$$

Here M is an $N \times N$ matrix:

$$
\begin{aligned}
M_{jK} = {} & S_{jK}^{(2)}(\lambda_j^C, \lambda_K^B) \exp\{\varphi_{A_1}(\lambda_j^C) + \varphi_{D_1}(\lambda_K^B)\} \\
& - S_{jK}^{(1)}(\lambda_j^C, \lambda_K^B) \exp\{\varphi_{D_2}(\lambda_j^C) + \varphi_{A_2}(\lambda_K^B) + \varphi_{D_3}(\lambda_K^B)\} \\
& + c\alpha\, a_1(\lambda_K^B) \exp\{\varphi_{A_3}(\lambda_K^B) + \varphi_{A_2}(\lambda_K^B) \\
& + \varphi_{A_1}(\lambda_j^C) + \varphi_{D_2}(\lambda_j^C)\}.
\end{aligned} \tag{4.13}
$$

Proof. One should write $\det_N M$ as a determinant of the sum of three matrices. Calculating the derivative of α we have

$$\frac{\partial}{\partial\alpha}\langle 0|\det_N M|0\rangle = \sum_{\substack{\{\lambda^C\}=\{\lambda_0^C\}\cup\{\lambda_{\mathrm{I}}^C\}\cup\{\lambda_{\mathrm{II}}^C\}\\ \{\lambda^B\}=\{\lambda_0^B\}\cup\{\lambda_{\mathrm{I}}^B\}\cup\{\lambda_{\mathrm{II}}^B\}}} (-1)^{[P_C+P_B]+N_1}$$
$$\times\langle 0|\det_{N_1} S^{(1)}(\lambda_{\mathrm{I}}^C,\lambda_{\mathrm{I}}^B)\det_{N_2} S^{(2)}(\lambda_{\mathrm{II}}^C,\lambda_{\mathrm{II}}^B)\cdot ca_1(\lambda_0^B)$$
$$\times\exp\{\varphi_{A_3}(\lambda_0^B)+\varphi_{A_2}(\lambda_0^B)+\varphi_{A_1}(\lambda_0^C)+\varphi_{D_2}(\lambda_0^C)\}$$
$$\times\prod_{\mathrm{II}}\exp\{\varphi_{A_1}(\lambda_{\mathrm{II}}^C)+\varphi_{D_1}(\lambda_{\mathrm{II}}^B)\}\prod_{\mathrm{II}}\exp\{\varphi_{D_2}(\lambda_{\mathrm{II}}^C)$$
$$+\varphi_{A_2}(\lambda_{\mathrm{II}}^B)+\varphi_{D_3}(\lambda_{\mathrm{II}}^B)\}|0\rangle\,. \tag{4.14}$$

Calculating the vacuum mean value of the products e^{φ} we obtain formula (4.11), which completes the proof.

So the matrix element $\psi^+(x)\psi(0)$ is represented as a determinant of the $N\times N$ matrix.

5. Correlation Function of Fields

Now let us use (4.12) to calculate the correlator in the N-particle state. To do it one should put $\lambda_j^C=\lambda_j^B=\lambda_j$ in (4.12), (4.13) and demand the $\{\lambda\}$ satisfy system (2.13). First of all we transform the determinant:

$$\langle 0|\det_N M|0\rangle$$
$$=\langle 0|\prod_{m=1}^{N}\exp\{\Phi_{A_2}(\lambda_m^C)+\Phi_{D_2}(\lambda_m^B)+\varphi_{A_1}(\lambda_m^C)+\varphi_{D_1}(\lambda_m^B)\det_N\tilde M|0\rangle$$
$$=\prod_{m=1}^{N}a(\lambda_m^C)\,d(\lambda_m^B)\prod_{m,e=1}^{N}h_{me}^{CB}\langle\tilde 0|\det_N\tilde M|0\rangle\,, \tag{5.1}$$

where

$$\langle\tilde 0|=\langle 0|\prod_{m=1}^{N}\exp\{P_{D_2}(\lambda_m^C)+P_{A_2}(\lambda_m^B)+p_{D_2}(\lambda_m^C)+p_{A_2}(\lambda_m^B)\}\,, \tag{5.2}$$

$$\tilde M_{jK}=M_{jK}\exp\{-\Phi_{A_2}(\lambda_j^C)-\Phi_{D_2}(\lambda_K^B)-\varphi_{A_1}(\lambda_j^C)-\varphi_{D_1}(\lambda_K^B)\}\,. \tag{5.3}$$

Then we'll construct new fields $\tilde\Phi_\alpha$, $\tilde\varphi_\alpha$ which are equal to

$$\begin{aligned}\tilde\Phi_\alpha(\lambda)&=\Phi_\alpha(\lambda)-\langle\tilde 0|\Phi_\alpha(\lambda)|0\rangle\,,\\ \tilde\varphi_\alpha(\lambda)&=\varphi_\alpha(\lambda)-\langle\tilde 0|\varphi_\alpha(\lambda)|0\rangle\,.\end{aligned} \tag{5.4}$$

Each of the new fields can be, as before, expressed in terms of "coordinate" and "momentum" by formulae (4.1), in which Φ_α and φ_α must be replaced by $\tilde\Phi_\alpha$ and $\tilde\varphi_\alpha$ correspondingly. Commutation relations (4.6), (4.7) are also valid. The only difference is that now all "coordinates" annihilate the dual vacuum $\langle\tilde{0}|$. In terms of $\tilde{\Phi}_\alpha$ and $\tilde{\varphi}_\alpha$ the matrix $\tilde{M}$ looks as follows:

$$\begin{aligned}\tilde{M}_{jK} &= t_{jK}^{CB} + t_{Kj}^{BC}\exp\{\tilde{\Phi}_{A_2}(\lambda_K^B) - \tilde{\Phi}_{D_2}(\lambda_K^B) + \tilde{\Phi}_{D_2}(\lambda_j^C) - \tilde{\Phi}_{A_2}(\lambda_j^C)\} \times Z(\lambda_K^B)Z^{-1}(\lambda_j^C)\\ &\quad - \Bigg[t_{jK}^{CB}\exp\{\tilde{\Phi}_{A_1}(\lambda_j^C) + \tilde{\Phi}_{D_1}(\lambda_K^B)\}\\ &\quad + t_{Kj}^{BC}\exp\{\tilde{\Phi}_{A_1}(\lambda_K^B) + \tilde{\Phi}_{D_1}(\lambda_j^C)\}\frac{a_1(\lambda_K^B)d_1(\lambda_j^C)}{d_1(\lambda_K^B)a_1(\lambda_j^C)}\Bigg]\\ &\quad \times Z(\lambda_K^B)Z^{-1}(\lambda_j^C)\exp\{\tilde{\varphi}_{D_2}(\lambda_j^C) - \tilde{\Phi}_{A_2}(\lambda_j^C) - \tilde{\varphi}_{A_1}(\lambda_j^C)\\ &\quad + \tilde{\varphi}_{A_2}(\lambda_K^B) + \tilde{\varphi}_{D_3}(\lambda_K^B) - \tilde{\Phi}_{D_2}(\lambda_K^B) - \tilde{\varphi}_{D_1}(\lambda_K^B)\}\\ &\quad + c\alpha\frac{a_1(\lambda_K^B)}{d_1(\lambda_K^b)}Z(\lambda_K^B)Z^{-1}(\lambda_j^C)\exp\{\tilde{\varphi}_{D_2}(\lambda_j^C) - \tilde{\Phi}_{A_2}(\lambda_j^C) - \tilde{\varphi}_{A_1}(\lambda_j^C)\\ &\quad + \tilde{\varphi}_{A_2}(\lambda_K^B) + \tilde{\varphi}_{D_3}(\lambda_K^B) - \tilde{\Phi}_{D_2}(\lambda_K^B) - \tilde{\varphi}_{D_1}(\lambda_K^B)\}\,. \end{aligned} \tag{5.5}$$

Here

$$\begin{aligned} Z(\lambda_K^B) &= \frac{a_2(\lambda_K^B)}{d_2(\lambda_K^B)}\prod_{m=1}^N \frac{h_{Km}^{BB}}{h_{mK}^{CB}}\,,\\ Z(\lambda_j^C) &= \frac{a_2(\lambda_j^C)}{d_2(\lambda_j^C)}\prod_{m=1}^N \frac{h_{jm}^{CB}}{h_{mj}^{CC}}\,. \end{aligned} \tag{5.6}$$

The last step is to put $\lambda_j^C = \lambda_j^B = \lambda_j$ and use (2.13). In this case

$$Z(\lambda) = \frac{d_1(\lambda)}{a_1(\lambda)} = e^{ix\lambda}\,,$$

and we have

$$\langle 0|\prod_{j=1}^N C(\lambda_j)\psi^+(x)\psi(0)\prod_{j=1}^N B(\lambda_j)|0\rangle = \prod_{\substack{j,K=1\\ j\neq K}}^N f_{jK}\frac{\partial}{\partial\alpha}\langle\tilde{0}|\det{}_N V|0\rangle\Bigg|_{\alpha=0}\,, \tag{5.7}$$

$$\begin{aligned} V_{jK} &= c\delta_{jK}\left(L + \sum_{m=1}^N K_{jm}\right) + t_{jK} + t_{Kj}\exp\{ix\lambda_{Kj}\\ &\quad + \tilde{\Phi}_{A_2}(\lambda_K) - \tilde{\Phi}_{D_2}(\lambda_K) - \tilde{\Phi}_{A_2}(\lambda_j) - \tilde{\Phi}_{D_2}(\lambda_j)\}\\ &\quad - [t_{jK}\exp\{ix\lambda_{Kj} + \tilde{\Phi}_{A_1}(\lambda_j) + \tilde{\Phi}_{D_1}(\lambda_K)\} + t_{Kj}\exp\{\tilde{\Phi}_{A_1}(\lambda_K) + \tilde{\Phi}_{D_1}(\lambda_j)\}]\\ &\quad \times \exp\{\tilde{\varphi}_{D_2}(\lambda_j) - \tilde{\Phi}_{A_2}(\lambda_j) - \tilde{\varphi}_{A_1}(\lambda_j) + \tilde{\varphi}_{A_2}(\lambda_K)\\ &\quad + \tilde{\varphi}_{D_3}(\lambda_K) - \tilde{\Phi}_{D_2}(\lambda_K) - \tilde{\varphi}_{D_1}(\lambda_K)\}\\ &\quad + c\alpha\exp_1\{-ix\lambda_j + \tilde{\varphi}_{A_3}(\lambda_K) + \tilde{\varphi}_{A_2}(\lambda_K) - \tilde{\varphi}_{D_1}(\lambda_K)\\ &\quad - \tilde{\Phi}_{D_2}(\lambda_K) + \tilde{\varphi}_{D_2}(\lambda_j) - \tilde{\Phi}_{A_2}(\lambda_j)\}\,, \end{aligned} \tag{5.8}$$

where $\lambda_{Kj} = \lambda_K - \lambda_j$ and

$$K_{jm} = K(\lambda_j, \lambda_m) = \frac{2c}{(\lambda_j - \lambda_m)^2 + c^2}\,. \tag{5.9}$$

Formula (5.8) gives us an expression for the correlator of fields in the N-particle state.

Consider now the correlator of fields in the ground state of the Hamiltonian: $|\Phi\rangle$

$$\langle\psi^+(x)\psi(0)\rangle = \frac{\langle\Phi|\,\psi^+(x)\psi(0)\,|\Phi\rangle}{\langle\Phi\,|\,\Phi\rangle}\,. \tag{5.10}$$

This correlator can be obtained from (5.7) in the thermodynamic limit: $N\to\infty$, $L\to\infty$, $N/L=\text{const}$. The eigenstate $|\Omega\rangle$ is a Dirac sea. Momenta of particals λ_j are bounded by the Fermi momentum $q : |\lambda_j| < q$. They are described by the distribution density $\varrho(\lambda)$ which satisfies the equation

$$\left(1-\frac{1}{2\pi}\hat{K}\right)\varrho(\lambda) = \frac{1}{2\pi}\,. \tag{5.11}$$

Here $\hat{K}$ is the integral operator, acting in the interval $[-q, q]$ with the kernel $K(\lambda, \mu)$ [see (5.9)]. The square of the norm of the eigenfunction $|\Phi\rangle$ was calculated in [7]:

$$\langle\Phi\,|\,\Phi\rangle = \prod_{j=1}^{N}(2\pi c L\varrho(\lambda_j)) \prod_{\substack{j,K=1\\ j\neq K}}^{N} f_{jK}\,\det\left(1-\frac{1}{2\pi}\hat{K}\right). \tag{5.12}$$

To write down the expression for the correlator in thermodynamic limit it is sufficient to replace the sum $\sum_{m=1}^{N} K_{jm}$ in (5.8) by the correspondent integral. Using (5.11) one have

$$L+\sum_{m=1}^{N} K_{jm} \to L\left(1+\int_{-q}^{q} K(\lambda_j, \mu)\varrho(\mu)d\mu\right) = 2\pi L\varrho(\lambda_j)\,, \tag{5.13}$$

so we obtain

$$\langle\psi^+(x)\psi(0)\rangle = \frac{\partial}{\partial\alpha}\left.\frac{\langle\tilde{0}|\det\left(1+\frac{1}{2\pi c}\hat{V}_0\right)|0\rangle}{\det\left(1-\frac{1}{2\pi}\hat{K}\right)}\right|_{\alpha=0}, \tag{5.14}$$

where $\hat{V}_0$ is the integral operator, acting in the interval $[-q, q]$ with the kernel:

$$\begin{aligned} V_0(\lambda, \mu) &= t(\lambda, \mu)\, e^{\frac{ix}{2}(\lambda-\mu)} + t(\mu, \lambda) \\ &\quad\times \exp\left\{\frac{ix(\mu-\lambda)}{2} + \tilde{\Phi}_{A_2}(\lambda) - \tilde{\Phi}_{D_2}(\lambda) + \tilde{\Phi}_{D_2}(\mu) - \tilde{\Phi}_{A_2}(\mu)\right\} \\ &\quad - \left[t(\lambda, \mu)\exp\left\{\frac{ix(\mu-\lambda)}{2} + \tilde{\Phi}_{A_1}(\lambda) + \tilde{\Phi}_{D_1}(\mu)\right\}\right. \\ &\quad \left. + t(\mu, \lambda)\exp\left\{\frac{ix(\lambda-\mu)}{2} + \tilde{\Phi}_{A_1}(\mu) + \tilde{\Phi}_{D_1}(\lambda)\right\}\right] \end{aligned}$$

$$\times \exp\{\tilde{\varphi}_{D_2}(\lambda) - \tilde{\varphi}_{A_1}(\lambda) - \tilde{\Phi}_{A_2}(\lambda) + \tilde{\varphi}_{A_2}(\mu) + \tilde{\varphi}_{D_3}(\mu) - \tilde{\varphi}_{D_1}(\mu) - \tilde{\Phi}_{D_2}(\mu)\}$$
$$+ \alpha c \exp\left\{ -\frac{ix}{2}(\lambda+\mu) + \tilde{\varphi}_{D_2}(\lambda) - \tilde{\Phi}_{A_2}(\lambda) + \tilde{\varphi}_{A_3}(\lambda) + \tilde{\varphi}_{A_2}(\mu) - \tilde{\varphi}_{D_1}(\mu) - \tilde{\Phi}_{D_2}(\mu)\right\}. \tag{5.15}$$

In the case of finite temperature the correlation function of fields is equal to [8]

$$\langle \psi^+(x)\psi(0)\rangle_T = \frac{\langle \Phi_T | \psi^+(x)\psi(0) | \Phi_T\rangle}{\langle \Phi_T | \Phi_T\rangle}, \tag{5.16}$$

where $|\Phi_T\rangle$ is one of the eigenfunctions, describing the state of thermodynamic equilibrium. The distribution density is equal to

$$2\pi\varrho(\lambda)\theta^{-1}(\lambda) = 1 + \int_{-\infty}^{\infty} K(\lambda,\mu)\varrho(\mu)d\mu, \tag{5.17}$$

$$\theta^{-1}(\lambda) = 1 + \exp\left[\frac{\varepsilon(\lambda)}{T}\right]. \tag{5.18}$$

T is the temperature, and $\varepsilon(\lambda)$ – density of energy:

$$\varepsilon(\lambda) = \lambda^2 - h + \frac{T}{2\pi}\int_{-\infty}^{\infty} K(\lambda,\mu)\ln\left(1+\exp\left[-\frac{\varepsilon(\mu)}{T}\right]\right)d\mu. \tag{5.19}$$

It is easy to see that in the case of finite temperature the correlator is equal to

$$\langle \psi^+(x)\psi(0)\rangle_T = \frac{\partial}{\partial\alpha} \left.\frac{\langle\tilde{0}|\det\left(1+\frac{1}{2\pi c}\hat{V}_T\right)|0\rangle}{\det\left(1-\frac{1}{2\pi}\hat{K}_T\right)}\right|_{\alpha=0}. \tag{5.20}$$

Here

$$V_T(\lambda,\mu) = V_0(\lambda,\mu)\sqrt{\theta(\lambda)\theta(\mu)}, \tag{5.21}$$
$$V_T(\lambda,\mu) = K(\lambda,\mu)\sqrt{\theta(\lambda)\theta(\mu)}. \tag{5.22}$$

Note that in the point of free fermions ($c=\infty$) all dual fields can be put equal to zero, because all "coordinates" and "momenta" commute. The kernel V_0 symplifies:

$$c^{-1}V_0|_{c=\infty} = -\frac{2}{\pi}\frac{\sin\frac{\mathrm{x}}{2}(\lambda-\mu)}{\lambda-\mu} + \frac{\alpha}{2\pi}e^{-\frac{ix}{2}(\lambda+\mu)}. \tag{5.23}$$

In such a form this answer was obtained before in [1].

In conclusion let us notice that the method of dual fields, described in this paper and before in [5, 6], can be easily generalized for calculation of multipoint correlation functions. It also gives us the possibility to calculate correlators in models with an R-matrix of XXZ type.

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Communicated by N.Yu. Reshetikhin