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Unbounded Elements Affiliated with C^*-Algebras and Non-Compact Quantum Groups**

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Abstract. The affiliation relation that allows to include unbounded elements (operators) into the C^*-algebra framework is introduced, investigated and applied to the quantum group theory. The quantum deformation of (the two-fold covering of) the group of motions of Euclidean plane is constructed. A remarkable radius quantization is discovered. It is also shown that the quantum $SU(1,1)$ group does not exist on the C^*-algebra level for real value of the deformation parameter.

0. Introduction

In practical computations in quantum physics we mostly deal with unbounded physical quantities represented by unbounded operators. On the other hand in the very theoretical approaches (see for example [5, 2]) we consider C^*-algebras consisting of bounded elements only. Therefore it is necessary to investigate the relation between particular unbounded operators and C^*-algebras.

The same problem in a more apparent way arises in the theory of non-compact topological quantum groups, where on the one hand the doctrine [18] says that the C^*-algebra language is the only one to be used and where on the other hand we have to deal with matrix elements of finite-dimensional non-unitary representations which in general are not bounded.

The similar problem was encountered in the von Neumann algebra theory [11] where the affiliation relation $a\eta M$ [where $M \subset B(H)$ is a von Neumann algebra and a is an unbounded operator acting on the Hilbert space H] was invented to describe such situations. We borrow from this theory the name of the relation and its symbol: in what follows we shall speak about unbounded elements a affiliated with a C^*-algebra A and write $a\eta A$. We have however to warn the reader that the affiliation relation that we introduce in the present paper is not a generalization of

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the one used in the von Neumann algebra theory. For example if $a\eta M$ (where M is a von Neumann algebra and η is understood in the C^*-algebra sense) then according to Proposition 1.3 $a \in M$.

As we shall see, elements affiliated with the C^*-algebra A are unbounded multipliers of a special kind. The domain $D(a)$ of an element $a\eta A$ depends on a and need not coincide with the Pedersen ideal of A. It means that the theory presented in this paper goes in the other direction than the one developed in [7] and [13].

The paper is organized in the following way. In the remaining part of the Introduction we remind the necessary definitions and results of the theory of (bounded) multipliers [1].

In Sect. 1 the definition of the affiliation relation is formulated and basic examples are presented. In particular the set of all elements affiliated with the C^*-algebra $CB(H)$ of all compact operators acting on a Hilbert space H coincides with the set of all closed operators acting on H. It shows that the existing theory of unbounded closed Hilbert space operators (that contains such chapters as the theory of symmetric and selfadjoint operators including the Caley transform and extension theory, Friedrichs and Krein extensions of positive operators, Lie group and algebra representations including the Nelson integrability condition, the algebras of unbounded elements and many more ...) is related to a very particular C^*-algebra $CB(H)$ and should be generalized to an arbitrary C^*-algebra A. In an attempt to limit the volume of this paper we decided to shift all this program to a separate paper [20]. We include only a few results concerning normal elements that are used in Sect. 3.

In Sect. 2 we investigate the characteristic properties of the graphs of affiliated elements. The results provide us with a convenient method of introducing particular elements affiliated with C^*-algebras.

Section 3 is devoted to the quantum deformation of the Euclidean plane and its group of motions. This is where the theory of affiliation relation is applied. We discover the remarkable radius quantization that was not seen in the Hopf-algebra framework. The similar quantum group is considered in [3]. The very related material is contained in [17].

Finally in Sect. 4 the quantum group $S_\mu U(1,1)$ is investigated. We show that something is essentially wrong for $\mu \in \mathbb{R}$: on the C^*-algebra level the comultiplication does not exist. In our opinion the only deformation of $SU(1,1) = SL(2,\mathbb{R})$ that may exist is the one corresponding to the value $\mu \in S^1$. This case has not been seriously investigated yet. We have obtained very preliminary results [19] related to the Hilbert space level.

Both Sects. 3 and 4 are divided into three parts corresponding to different levels of the quantum group theory. The most technical Hilbert space level gives the link between a very surface Hopf-algebra level and deep C^*-algebra level. We believe that constructing any non-compact quantum group one has to consider these three levels. In very lucky cases the Hilbert space level may be very easy: all irreducible representations of the considered commutation relations are realized by bounded operators. Then many of the technical difficulties disappear. Such a case is considered in [14].

We have to recall a few facts concerning multipliers on non-unital C^*-algebras. Let $B(A)$ be the algebra of all bounded linear mappings acting on a C^*-algebra A and $a, b \in B(A)$. We say that b is the hermitian adjoint of a and write $b = a^*$ if

$$y^*(ax) = (by)^*x \tag{0.1}$$

for any $x, y \in A$. We have to stress that the existence of the hermitian adjoint is a very restrictive condition. $M(A)$ is by definition the set of all bounded linear mappings that have the hermitian adjoint:

$$M(A) = \{a \in B(A): \text{there exists } a^*\}.$$

$M(A)$ endowed with the natural algebraic operations and with the sup-norm becomes a unital C^*-algebra. There exists the natural embedding $A \hookrightarrow M(A)$; we identify any element $a \in A$ with the left multiplication by a. One can easily verify that A is a closed two-sided ideal in $M(A)$.

Using (0.1) one can prove that

$$(ax_1)x_2 = a(x_1x_2)$$

for any $x_1, x_2 \in A$. If A is unital then inserting $x_1 = I$ we observe that a coincides withe the left multiplication by aI. This way we showed that A is unital if and oly if $A = M(A)$.

It is known that $M(CB(H))$ [where $CB(H)$ denotes the algebra of all compact operators acting on a Hilbert space H] coincides with $B(H)$. Denoting by $C_{\mathrm{bounded}}(\Lambda)[C_\infty(\Lambda)$ respectively] the algebra of all continuous, bounded (vanishing at infinity respectively) functions on a locally compact topological space Λ we have $M(C_\infty(\Lambda)) = C_{\mathrm{bounded}}(\Lambda)$.

We shall prove

Proposition 0.1. *Let A be a C^*-algebra and u be a bijective linear mapping acting on A such that $(uy)^*(ux) = y^*x$ for any $x, y \in A$. Then u is a unitary element of $M(A)$.*

Proof. Inserting in the assumed relation $u^{-1}y$ instead of y we get $y^*(ux) = (u^{-1}y)^*x$ for any $x, y \in A$. It shows that $u^* = u^{-1}$ and the statement follows. Q.E.D.

Proposition 0.2. *Let A be a C^*-algebra and $a \in M(A)$. Assume that aA and a^*A are dense in A. Then there exists unique unitary $u \in M(A)$ such that*

$$a = u|a|, \tag{0.2}$$

where $|a| = (a^*a)^{1/2}$.

Proof. One can easily verify that the mapping

$$u: |a|x \mapsto ax, \tag{0.3}$$

where x runs over A extends by continuity to a linear bijection acting on A satisfying the assumptions of Proposition 0.1. Therefore $u \in M(A)$ and u is unitary. (0.2) follows immediately from (0.3). Q.E.D.

Proposition 0.3. *Let A be a C^*-algebra, $T \in B(A)$ and $a \in M(A)$. Assume that a^*A is dense in A and $aT \in M(A)$. Then $T \in M(A)$.*

Proof. Let $c = aT$. For any $x \in A$ we have

$$\|c^*x\|^2 = \|x^*cc^*x\| = \|x^*aT(c^*x)\| \leqq \|x^*a\| \, \|T\| \, \|c^*x\|.$$

Therefore $\|c^*x\| \leqq \|T\| \cdot \|a^*x\|$. Remembering that a^*A is dense in A we see that there exists $S \in B(A)$ such that

$$Sa^*x = c^*x$$

for any $x \in A$. We shall prove that S is the hermitian adjoint of T. Indeed for any $x, y \in A$ we have

$$(a^*y)^*Tx = y^*aTx = y^*cx = (c^*y)^*x = (Sa^*y)^*x,$$

and using once more the density of a^*A we obtain $y^*Tx = (Sy)^*x$ for any $x, y \in A$. The latter means that $S = T^*$ [cf. (0.1)]. Q.E.D.

At the end of this section we remind the category of C^*-algebras that plays the basic role in the non-commutative topology [18, 16]. All C^*-algebras are objects of this category. For any C^*-algebras A and B, the set of morphisms $\mathrm{Mor}(A, B)$ consists of all *-algebra homomorphisms ϕ acting from A into $M(B)$ such that

$$\phi(A)B \text{ is dense in } B. \tag{0.4}$$

Any $\phi \in \mathrm{Mor}(A, B)$ admits unique extension to the *-algebra homomorphism acting from $M(A)$ into $M(B)$. Due to this fact the compositions of morphisms is well defined. The extension (denoted by the same letter) is introduced in the following way: For any $T \in M(A)$, $\phi(T)$ is a bounded operator acting on B such that

$$\phi(T)(\phi(a)b) = \phi(Ta)b \tag{0.5}$$

for all $a \in A$ and $b \in B$. By virtue of (0.4), $\phi(T)$ is unique and one can show that $\phi(T) \in M(B)$. Clearly $\phi(I_A) = I_B$, where I_A (I_B respectively) denotes the unity of $M(A)$ [$M(B)$ respectively].

1. The Affiliation Relation

In this section we introduce elements affiliated with a C^*-algebra. Heuristic justification of the formal definition given below is the following: In brief, an element T is affiliated with a C^*-algebra A if bounded continuous functions of T belong to $M(A)$. We choose a bounded continuous function

$$\mathbb{C} \ni \lambda \mapsto z_\lambda \in \mathbb{C}$$

defined by a simple algebraic expression such that $z_\lambda \neq z_{\lambda'}$ for $\lambda \neq \lambda'$. Then λ is determined by z_λ. By definition $T\eta A$ if $z_T \in M(A)$. The choice

$$z_\lambda = \lambda(1 + \bar{\lambda}\lambda)^{-1/2} \tag{1.1}$$

leads to Definition 1.1. Indeed in this case formal computations show that

$$T(I - z_T^* z_T)^{1/2} = z_T.$$

Throughout the paper the domain of any (unbounded) linear operator T acting on a C^*-algebra will be denoted by $D(T)$. On the other hand the domain of an operator T acting on a Hilbert space is denoted by $\mathscr{D}(T)$. This distinction is necessary because in some cases we identify operators acting on a Hilbert space H with corresponding operators acting on a C^*-algebra embedded into $B(H)$ (see Examples 3 and 4 in this section).

Definition 1.1. Let A be a C^*-algebra and T be a linear mapping acting on A defined on a linear dense subset $D(T) \subset A$. We say that T is affiliated with A and write $T\eta A$ if and only if there exists $z \in M(A)$ such that $\|z\| \leqq 1$ and

$$\begin{pmatrix} x \in D(T) \\ \text{and } y = Tx \end{pmatrix} \Leftrightarrow \begin{pmatrix} \text{There exists } a \in A \text{ such that} \\ x = (I - z^* z)^{1/2} a \text{ and } y = za \end{pmatrix}$$

for any $x, y \in A$.

Clearly T is determined by z. We say that T is the T-transform of $z: T = T_z$. In the next section we shall prove that z is determined by T. We say that z is the z-transform of $T: z = z_T$. Clearly $z \in M(A)$ is the z-transform of an element affiliated with A if and only if $\|z\| \leqq 1$ and $(I - z^* z)^{1/2} A$ is dense in A.

It follows immediately from the definition that T is a closed linear map, $D(T)$ is a right ideal in A and $T(ab) = (Ta)b$ for any $a \in D(T)$ and $b \in A$. Let $T\eta A$ and D be a dense linear subset in A. We say that D is a core of T if $D \subset D(T)$ and T coincides with the closure of $T|_D$. One can easily check that D is a core of T if and only if $D = (I - z_T^* z_T)^{1/2} D'$, where D' is a dense linear subset in A.

The functorial properties of the definition are described in the following

Theorem 1.2. *Let A, B be C^*-algebras, $\phi \in \mathrm{Mor}(A, B)$ and $T\eta A$. Then there exists $\phi(T)\eta B$ such that $\phi(D(T))B$ is a core of $\phi(T)$ and*

$$\phi(T)(\phi(a)b) = \phi(Ta)b \tag{1.2}$$

for any $a \in D(T)$ and $b \in B$. The z-transforms of $\phi(T)$ and T are related by the formula

$$z_{\phi(T)} = \phi(z_T). \tag{1.3}$$

Moreover if A, B, C are C^*-algebras, $\phi \in \mathrm{Mor}(A, B)$, $\psi \in \mathrm{Mor}(B, C)$ and $T\eta A$, then

$$\psi(\phi(T)) = (\psi \circ \phi)(T). \tag{1.4}$$

Proof. At first we notice that $\|\phi(z_T)\| \leqq \|z_T\| \leqq 1$ and

$$(I - \phi(z_T)^* \phi(z_T))^{1/2} B = \phi((I - z_T^* z_T)^{1/2}) B \supset \phi((I - z_T^* z_T)^{1/2}) \phi(A) B = \phi(D(T)) B$$

is dense in B [cf. (0.4)]. It shows that $\phi(z_T)$ is the z-transform of an element affiliated with B. Denoting the latter element by $\phi(T)$ we get (1.3).

Let us notice that

$$\phi(D(T))B = \phi(I - z_T^* z_T)^{1/2} A) B = (I - z_{\phi(T)}^* z_{\phi(T)})^{1/2} \phi(A) B.$$

Using once more (0.4) we see that $\phi(D(T))B$ is a core of $\phi(T)$. Let $a \in D(T)$ and $b \in B$. Then $a = (I - z_T^* z_T)^{1/2} x$, $Ta = z_T x$ (where x is an element belonging to A) and

$$\phi(a)b = (I - z_{\phi(T)}^* z_{\phi(T)})^{1/2} \phi(x) b.$$

Therefore

$$\phi(T)\phi(a)b = z_{\phi(T)} \phi(x) b = \phi(z_T) \phi(x) b = \phi(z_T x) b = \phi(Ta) b,$$

and (1.2) follows. (1.4) follows immediately from (1.3). Q.E.D.

Example 1. Let A be a C^*-algebra, $a \in M(A)$ and T be the left multiplication by a. By definition $D(T) = A$. We claim that $T\eta A$. Indeed one can easily check that z-transform of T is given by the formula

$$z_T = a(I + a^* a)^{-1/2}. \tag{1.5}$$

In what follows we shall identify $T\eta A$ with $a\in M(A)$. Let us notice that T is bounded: $\|Tx\|\leqq\|a\|\,\|x\|$ for any $x\in A$. Conversely if $T\eta A$ and T is bounded then $\|z_T\|<1$ and one can easily verify that T is the left multiplication by $a=z_T(I-z_T^*z_T)^{-1/2}\in M(A)$. Therefore

$$M(A)=\{T\eta A:\|T\|<\infty\}. \tag{1.6}$$

If A is unital then any dense right ideal of A coincides with A and (closed graph theorem) any $T\eta A$ is bounded. On the other hand in this case $M(A)=A$. Taking into account (1.6) we get

Proposition 1.3. *If A is unital then any element affiliated with A belongs to A.*

This is a noncommutative version of the known theorem of classical analysis saying that on a compact space any continuous function is bounded.

Finally we notice that for $T\in M(A)$ the definition (1.2) coincides with (0.5).

Example 2. Let Λ be a locally compact topological space, $A=C_\infty(\Lambda)$ be the C^*-algebra of all continuous, vanishing at infinity complex functions on Λ, $a\in C(\Lambda)$ and T be the multiplication by a. By definition $D(T)$ is the set of all $x\in C_\infty(\Lambda)$ such that $\lim\limits_{x\to\infty} a(\lambda)x(\lambda)=0$. Then $T\eta C_\infty(\Lambda)$; the z-transform of T given by $z_T(\lambda)=a(\lambda)(1+|a(\lambda)|^2)^{-1/2}$ is a bounded continuous function on Λ. In what follows we shall identify $T\eta C_\infty(\Lambda)$ with $a\in C(\Lambda)$. Conversely if $T\eta C_\infty(\Lambda)$ then $z_T\in M(C_\infty(\Lambda))=C_{\text{bounded}}(\Lambda)$, $\|z_T\|\leqq 1$, and $|z_T(\lambda)|<1$ for all $\lambda\in\Lambda$ [otherwise $D(T)=(I-z_T^*z_T)^{1/2}C_\infty(\Lambda)$ would not be dense in $C_\infty(\Lambda)$]. Therefore setting

$$a(\lambda)\overset{df}{=}z_T(\lambda)(1-|z_T(\lambda)|^2)^{-1/2}$$

we define an element $a\in C(\Lambda)$ and one can easily verify that T coincides with the multiplication by a. It shows that the set of all elements affiliated with $C_\infty(\Lambda)$ coincides with $C(\Lambda)$.

Example 3. Let H be a Hilbert space ($\dim H=\infty$), $A=CB(H)$ be the C^*-algebra of all compact operators acting on H, a be a closed operator acting on H with a dense domain $\mathscr{D}(a)\subset H$ and T be the left multiplication by a. By definition $D(T)$ is the set of all $x\in CB(H)$ such that the product ax is well defined [i.e. $xH\subset\mathscr{D}(a)$] and belongs to $CB(H)$. Then $T\eta CB(H)$; the z-transform of T given by (1.5) is a bounded operator acting on H [in the considered case $M(A)=B(H)$]. In what follows we shall identify $T\eta CB(H)$ with the closed operator a acting on H.

Conversely if $T\eta CB(H)$ then $z_T\in M(CB(H))=B(H)$, $\|z_T\|\leqq 1$, and 1 is not an eigenvalue of $z_T^*z_T$ [otherwise $D(T)=(I-z_T^*z_T)^{1/2}CB(H)$ would not be dense in $CB(H)$]. Therefore $a=z_T(I-z_T^*z_T)^{-1/2}$ is a well defined closed operator and one can easily verify that T coincides with the left multiplication by a. It shows that the set of all elements affiliated with $CB(H)$ coincides with the set of all closed operators acting on H.

Example 4. Let H be a Hilbert space and $A\subset B(H)$ be a C^*-algebra of operators acting on H. As usual we assume that A is nondegenerate, i.e. for any non-zero $\psi\in H$ there exists $a\in A$ such that $a\psi\neq 0$. Then the embedding

$$i: A\ni a\mapsto a\in B(H)$$

belongs to $\operatorname{Mor}(A, CB(H))$. Therefore for any $T\eta A$, $i(T)\eta CB(H)$ and (cf. Example 3) $i(T)$ is a closed operator acting on H. In what follows we shall identify T with $i(T)$.

For any closed operator T acting on H we have

$$(T\eta A) \Leftrightarrow \begin{pmatrix} T(I+T^*T)^{-1/2} \in M(A) \text{ and} \\ (I+T^*T)^{-1/2}A \text{ is dense in } A \end{pmatrix}.$$

Example 5. Let $\{A_n\}_{n\in N}$ be a family of C^*-algebras labeled by a set N (in most cases N is denumerable) and

$$A = \sum_{n\in N}^{\oplus} A_n .$$

By definition the elements of A are sequences $a=(a_n)_{n\in N}$ such that $a_n \in A_n$ for all $n\in N$ and $\lim_{n\to\infty} \|a_n\| = 0$. Let $\pi_n : A \to A_n$ be the canonical projection. Then $\pi_n \in \mathrm{Mor}(A, A_n)$.

Let $T\eta A$. Then $T_n = \pi_n(T)\eta A_n$. One can easily show that T is determined by the sequence $(T_n)_{n\in N}$ and that any sequence $(T_n)_{n\in N}$ (where $T_n \eta A_n$ for any $n\in N$) can be obtained in this way. If all A_n are unital, then (cf. Proposition 1.3) $T_n \eta A_n$ means $T_n \in A_n$ and the set of all elements affiliated with $\Sigma^{\oplus} A_n$ coincides with the cartesian product $\times_n A_n$. This fact is very much used in [14].

For the C^*-algebras A considered in Examples 2 and 5 (with unital A_n) the set of all elements affiliated with A is endowed with the natural $*$-algebra structure. Example 3 shows that this is not always the case: In general even the sum of two closed operators is not well defined. We have however

Theorem 1.4. *Let A be a C^*-algebra and $T\eta A$. Then there exists $T^*\eta A$ such that for any $a, b \in A$*

$$\begin{pmatrix} a\in D(T^*) \textit{ and} \\ b = T^*a \end{pmatrix} \Leftrightarrow \begin{pmatrix} \textit{For any } x \in D(T) \\ a^*(Tx) = b^*x \end{pmatrix}. \tag{1.7}$$

The z-transforms of T^* and T are related by the formula

$$z_{T^*} = z_T^* . \tag{1.8}$$

If A, B are C^*-algebras, $\phi \in \mathrm{Mor}(A,B)$ and $T\eta A$, then

$$\phi(T^*) = \phi(T)^* . \tag{1.9}$$

Proof. Clearly $\|z_T^*\| = \|z_T\| \leqq 1$. Assume for the moment that $(I - z_T z_T^*)^{1/2} A$ is not dense in A. Then (cf. [4, Theorem 2.9.5]) there exists a state ω on A such that

$$\omega((I - z_T z_T^*)^{1/2} a) = 0$$

for all $a\in A$. Using the GNS procedure we construct a representation π of A acting on a Hilbert space H and a cyclic vector $\Omega \in H$ such that

$$\omega(a) = (\Omega | \pi(a)\Omega)$$

for all $a \in A$. Comparing the last two formulae and remembering that Ω is cyclic we get

$$\pi(z_T)\pi(z_T)^*\Omega = \Omega . \tag{1.10}$$

Let $\Omega' = \pi(z_T)^*\Omega$ and for any $a\in A$

$$\omega'(a) = (\Omega' | \pi(a)\Omega') .$$

Clearly ω' is a state on A. Applying $\pi(z_T)^*$ to the both sides of (1.10) we get

$$\pi(z_T)^*\pi(z_T)\Omega'=\Omega'.$$

Therefore for any $a\in A$

$$\omega'((I-z_T^*z_T)^{1/2}a)=0.$$

On the other hand $D(T)=(I-z_T^*z_T)^{1/2}A$ is dense in A. This contradiction shows that $(I-z_Tz_T^*)^{1/2}A$ is dense in A. It means that z_T^* is a z-transform of an element affiliated with A. Denoting the latter element by T^* we get (1.8).

Let us notice that the right-hand side of (1.7) is equivalent to

$$a^*z_T=b^*(I-z_T^*z_T)^{1/2}. \tag{1.11}$$

If $a\in D(T^*)$ and $b=T^*a$ then $a=(I-z_Tz_T^*)^{1/2}y$, $b=z_T^*y$ (where $y\in A$) and using the equality

$$(I-z_Tz_T^*)^{1/2}z_T=z_T(I-z_T^*z_T)^{1/2} \tag{1.12}$$

one can easily verify (1.11). Conversely if (1.11) holds then setting

$$y=(I-z_Tz_T^*)^{1/2}a+z_Tb$$

and using (1.12) we obtain

$$\begin{aligned}(I-z_Tz_T^*)^{1/2}y&=(I-z_Tz_T^*)a+(I-z_Tz_T^*)^{1/2}z_Tb\\&=(I-z_Tz_T^*)a+z_T(I-z_T^*z_T)^{1/2}b\\&=(I-z_Tz_T^*)a+z_Tz_T^*a=a\end{aligned}$$

and

$$\begin{aligned}z_T^*y&=z_T^*(I-z_Tz_T^*)^{1/2}a+z_T^*z_Tb\\&=(I-z_T^*z_T)^{1/2}z_T^*a+z_T^*z_Tb\\&=(I-z_T^*z_T)b+z_T^*z_Tb=b.\end{aligned}$$

It shows that $a\in D(T^*)$ and $b=Ta$. The equivalence (1.7) is proved. Relation (1.9) follows immediately from (1.8) and (1.3). Q.E.D.

The reader easily examines how the *-operation introduced by (1.7) acts in particular examples. In Example 1 it coincides with the hermitian conjugation in $M(A)$, in Examples 3 and 4 T^* is the adjoint of T in the sense of the theory of closed operators acting on a Hilbert space (see e.g. [10]). In Example 2

$$(T^*)(\lambda)=\overline{T(\lambda)} \tag{1.13}$$

for any $T\eta C(\Lambda)$ and $\lambda\in\Lambda$.

It follows immediately from (1.8) that

$$T^{**}=T$$

for any $T\eta A$. We say that an element T affiliated with a C^*-algebra A is normal if $D(T)=D(T^*)$ and

$$(Ta)^*(Ta)=(T^*a)^*(T^*a) \tag{1.14}$$

for any $a\in D(T)$. We shall prove that

$$(T\text{ is normal})\Leftrightarrow(z_T^*z_T=z_Tz_T^*). \tag{1.15}$$

In particular any element affiliated with a commutative C^*-algebra is normal. The implication "$\Rightarrow$" is obvious. To prove the converse we notice that the relation $D(T) = D(T^*)$ means that there exists a bijective mapping

$$A \ni x \rightarrow ux \in A$$

such that

$$(I - z_T^* z_T)^{1/2} ux = (I - z_T z_T^*)^{1/2} x \tag{1.16}$$

for any $x \in A$. Let $a = (I - z_T z_T^*)^{1/2} x$. Then $T^* a = z_T^* x$, $Ta = z_T ux$ and (1.14) shows that $(ux)^* z_T^* z_T (ux) = x^* z_T z_T^* x$. Combining this result with (1.16) we obtain $(ux)^*(ux) = x^* x$ for any $x \in A$ and by polarization $(uy)^*(ux) = y^* x$ for any $x, y \in A$. Therefore (cf. Proposition 0.1) $u \in M(A)$ and u is unitary. Equation (1.16) means that $(I - z_T^* z_T)^{1/2} u = (I - z_T z_T^*)^{1/2}$ and remembering that the polar decomposition is unique we see that $u = I$, $(I - z_T^* z_T)^{1/2} = (I - z_T z_T^*)^{1/2}$ and the relation $z_T^* z_T = z_T z_T^*$ follows. This way the equivalence (1.15) is proved.

It turns out (see Theorem 1.5 below) that there exists an universal normal element. Let ζ be an element affiliated with $C_\infty(\mathbb{C})$ introduced by the formula $\zeta(\lambda) = \lambda$ for any $\lambda \in C$. Clearly ζ is normal [$C_\infty(\mathbb{C})$ is commutative!]. We have

Theorem 1.5. *Let A be a C^*-algebra and T be a normal element affiliated with A. Then there exists unique $\varphi_T \in \operatorname{Mor}(C_\infty(\mathbb{C}), A)$ such that*

$$\varphi_T(\zeta) = T. \tag{1.17}$$

Proof. Let

$$D^1 = \{\lambda \in C : |\lambda| \leqq 1\},$$
$$S^1 = \partial D^1 = \{\lambda \in C : |\lambda| = 1\}.$$

We know that z_T is normal and $\|z_T\| \leqq 1$. Therefore $\operatorname{Sp} z_T \subset D^1$. The same relation holds for z_ζ. We shall use the continuous function calculus of normal elements of $M(A)$ and $M(C_\infty(\mathbb{C})) = C_{\text{bounded}}(\mathbb{C})$. Relation (1.17) means that

$$\varphi_T(z_\zeta) = z_T. \tag{1.18}$$

Therefore

$$\varphi_T(f(z_\zeta)) = f(z_T) \tag{1.19}$$

for any $f \in C(D^1)$. One can easily check that any function belonging to $C_\infty(\mathbb{C})$ is of the form $f(z_\zeta)$, where $f(z_\zeta)$, where $f \in C(D^1)$ and $f|_{S^1} = 0$ and the uniqueness of φ_T follows.

On the other hand the formula (1.19) defines a mapping φ_T acting from $C_\infty(\mathbb{C})$ into $M(A)$. One can easily check that φ_T is a *-algebra homomorphism. Let x be the element of $C_\infty(\mathbb{C})$ such that $x(\lambda) = (1 + \bar{\lambda}\lambda)^{-1/2}$ for all $\lambda \in C$. By virtue of (1.19) $\varphi_T(x) = (I - z_T^* z_T)^{1/2}$ and $\varphi_T(C_\infty(\mathbb{C}))A$ contains $\varphi_T(x)A = (I - z_T^* z_T)^{1/2} A = D(T)$. Remembering that $D(T)$ is dense in A we obtain $\varphi_T \in \operatorname{Mor}(C_\infty(\mathbb{C}), A)$. (1.18) is the special case of (1.19) and (1.17) follows. Q.E.D.

In the general φ_T is not an embedding. This fact is related to the spectral properties of T. The general notion of spectrum of any (not necessary normal) element affiliated with a C^*-algebra is introduced in [20]. For normal elements

$$\operatorname{Sp} T = \left\{ \lambda \in \mathbb{C} : \begin{matrix} f(\lambda) = 0 \text{ for} \\ \text{all } f \in \operatorname{Ker} \varphi_T \end{matrix} \right\}, \tag{1.20}$$

where

$$\operatorname{Ker}\varphi_T = \{f \in C_\infty(\mathbb{C}) : \varphi_T(f) = 0\}.$$

Clearly $\operatorname{Sp} T$ is a closed subset of $\mathbb{C}$. It is never empty. Let $\phi \in \operatorname{Mor}(A, A')$, where A' is another C^*-algebra. Then $\phi(T)$ is normal [this fact follows immediately from (1.3) and (1.15)] and

$$\operatorname{Sp}\phi(T) \subset \operatorname{Sp} T. \tag{1.21}$$

Indeed $(\phi \circ \varphi_T)(\zeta) = \phi(\varphi_T(\zeta)) = \phi(T)$. Therefore $\varphi_{\phi(T)} = \phi \circ \varphi_T$, $\operatorname{Ker}\varphi_T \subset \operatorname{Ker}\varphi_{\phi(T)}$ and (1.21) follows. We shall prove

Theorem 1.6. *Let A be a C^*-algebra, T be a normal element affiliated with A and $\zeta_{\operatorname{Sp} T}$ be the restriction of ζ to $\operatorname{Sp} T$. Then there exists a unique embedding $\psi_T \in \operatorname{Mor}(C_\infty(\operatorname{Sp} T), A)$ such that*

$$\psi_T(\zeta_{\operatorname{Sp} T}) = T.$$

Proof. $\operatorname{Ker}\varphi_T$ is a closed ideal in $C_\infty(\mathbb{C})$. Let $\pi_{\operatorname{Sp} T} \in \operatorname{Mor}(C_\infty(\mathbb{C}), C_\infty(\mathbb{C})/\operatorname{Ker}\varphi_T)$ be the canonical epimorphism and ψ_T be the element of $\operatorname{Mor}(C_\infty(\mathbb{C})/\operatorname{Ker}\varphi_T, A)$ that makes the diagram

$$\begin{tikzcd} & C_\infty(\mathbb{C}) \arrow[dl, "\pi_{\operatorname{Sp} T}"'] \arrow[dr, "\varphi_T"] & \\ C_\infty(\mathbb{C})/\operatorname{Ker}\varphi_T \arrow[rr, "\psi_T"'] & & A \end{tikzcd}$$

commutative. To complete the proof we notice that

$$\operatorname{Ker}\varphi_T = \{f \in C_\infty(\mathbb{C}) : f|_{\operatorname{Sp} T} = 0\},$$

$$C_\infty(\mathbb{C})/\operatorname{Ker}\varphi_T = C_\infty(\operatorname{Sp} T),$$

and $\pi_{\operatorname{Sp} T} \in \operatorname{Mor}(C_\infty(\mathbb{C}), C_\infty(\operatorname{Sp} T))$ is the restriction map. The uniqueness of ψ_T is obvious. Q.E.D.

The reader easily examines how the morphism φ_T acts in the particular examples. In Example 3 and 4 T is normal if and only if it is normal in the operator theory sense (i.e. $T^*T = TT^*$). In this case

$$\varphi_T(f) = \int_{\mathbb{C}} f(\lambda) dE(\lambda),$$

where $dE(\cdot)$ is the spectral measure associated with the normal operator T:

$$T = \int_{\mathbb{C}} \lambda dE(\lambda).$$

Taking into account (1.20) we see that $\operatorname{Sp} T$ coincides with the support of the spectral measure and the definition (1.20) agrees with the one known from the operator theory.

We know [cf. (1.15)] that the normality of an element $T\eta A$ can be expressed in terms of its z-transform. One can also check that

$$\operatorname{Sp} T = \{\lambda(1 - \bar{\lambda}\lambda)^{-1/2} : \lambda \in \operatorname{Sp}(z_T), |\lambda| < 1\}$$

Remembering that any C^*-algebra A admits a non-degenerate embedding into $B(H)$, where H is a Hilbert space [the embedding belongs to $\operatorname{Mor}(A, CB(H))$] and that any representation of A can be decomposed into direct integral of irreducible ones we get

Proposition 1.7. *Let A be a C^*-algebra, $T\eta A$ and Λ be a closed subset of $\mathbb{C}$. Then T is normal and* $\operatorname{Sp} T \subset \Lambda$ *if and only if $\pi(T)$ is normal and* $\operatorname{Sp} \pi(T) \subset \Lambda$ *for any irreducible representation π of A.*

2. The Graphs of Affiliated Elements

The definition of the affiliation relation given in Sect. 1 is nice if one is going to investigate properties of affiliated elements. It is however less convenient if one has to show that a given linear operator acting on a C^*-algebra is affiliated with it. In the present section we develop a method that in many cases allows us to prove that the affiliation relation holds. To this end we give new conditions characterizing elements affiliated with a C^*-algebra in terms of their graphs. These conditions make no use of any particular z-transform [like (1.1) used in Definition 1.1].

For any C^*-algebra A we set $A_2 = A \oplus A$. We endow A_2 with its canonical Hilbert right A-module structure [12]:

$$\begin{pmatrix} a \\ b \end{pmatrix} x \stackrel{df}{=} \begin{pmatrix} ax \\ bx \end{pmatrix},$$

$$\left(\begin{pmatrix} a \\ b \end{pmatrix} \middle| \begin{pmatrix} a' \\ b' \end{pmatrix} \right)_A \stackrel{df}{=} a^*a' + b^*b'$$

for any $a, b, a', b', x \in A$. For any $G \subset A_2$ we put

$$G^\perp = \{k \in A_2 : (k|l)_A = 0 \text{ for all } l \in G\}.$$

$G^\perp$ is always a submodule of A_2. In the great contrast with the Hilbert space theory, even if G is a closed submodule of A_2 then in general $G \subsetneqq G^{\perp\perp}$ and $G \oplus G^\perp \subsetneqq A_2$. The canonical projections $A_2 \to A$ will be denoted by p_1 and p_2:

$$p_1 \begin{pmatrix} a \\ b \end{pmatrix} = a, \quad p_2 \begin{pmatrix} a \\ b \end{pmatrix} = b$$

for any $a, b \in A$.

Proposition 2.1. *Let T be an element affiliated with a C^*-algebra A and G be the graph of T:*

$$G = \left\{ \begin{pmatrix} x \\ Tx \end{pmatrix} : x \in D(T) \right\}. \tag{2.1}$$

Then G is a closed submodule of A_2,

$$A_2 = G \oplus G^\perp \tag{2.2}$$

and the orthogonal complement is given by

$$G^\perp = \left\{ \begin{pmatrix} -T^*y \\ y \end{pmatrix} : y \in D(T^*) \right\}. \tag{2.3}$$

Proof. Let $z = z_T$ be the z-transform of T. One can easily verify that

$$E = \begin{pmatrix} I - z^*z, & (I - z^*z)^{1/2} z^* \\ z(I - z^*z)^{1/2}, & zz^* \end{pmatrix} \tag{2.4}$$

is an orthogonal projection onto (2.1) and (2.2) follows. (2.3) follows immediately from definition (1.7). Q.E.D.

Let us notice that in the situation described in Proposition 2.1, $p_1 G = D(T)$ and $p_2 G^\perp = D(T^*)$ are dense in A.

Proposition 2.2. *Let A be a C^*-algebra and $G \subset A_2$ be a closed submodule such that $p_1 G$ and $p_2 G^\perp$ are dense in A. Assume that the decomposition* (2.2) *holds. Then G is the graph of an element $T \eta A$.*

Proof. Any bounded linear mapping E acting on A_2 can be represented by a matrix

$$E = \begin{pmatrix} p, & q^* \\ q, & r \end{pmatrix},$$

where p, q, q^*, r are bounded linear mappings acting on A. In what follows E will denote the projection onto G along $G^\perp$: $E(\psi + \psi^\perp) = \psi$ for any $\psi \in G$ and $\psi^\perp \in G^\perp$. One can easily check that p and r are self adjoint and that q^* is the adjoint of q. Therefore (cf. Sect. 0) $p, q, q^*, r \in M(A)$. We have to show that E is of the form (2.4), i.e. that

$$p = I - z^* z\,, \tag{2.5}$$

$$q = z(I - z^* z)^{1/2}\,, \tag{2.6}$$

$$r = z z^*\,, \tag{2.7}$$

where $z \in M(A)$, $\|z\| \leqq 1$, and $(I - z^* z)^{1/2} A$ is dense in A. Remembering that $E^2 = E$ we obtain

$$p = p^2 + q^* q\,, \tag{2.8}$$

$$(I - r) q = q p\,, \tag{2.9}$$

$$I - r = (I - r)^2 + q q^*\,. \tag{2.10}$$

Equations (2.8) and (2.10) show that p and $I - r$ are positive. Equation (2.9) implies that

$$(I - r)^{1/2} q = q p^{1/2}\,. \tag{2.11}$$

Assume for the moment pA is not dense in A. Then there exists a state ω on A such that $\omega(p) = 0$. By virtue of (2.8) $\omega(p^2) = \omega(q^* q) = 0$ and (Schwarz inequality) $\omega(px + q^* y) = 0$ for any $x, y \in A$. On the other hand if $\begin{pmatrix} x \\ y \end{pmatrix} \in G$ then $\begin{pmatrix} x \\ y \end{pmatrix} = E \begin{pmatrix} x \\ y \end{pmatrix}$, $x = px + q^* y$, and $\omega(x) = 0$. It shows that ω kills all elements of $p_1 G$ and we obtain contradiction with the assumed density of $p_1 G$. Therefore pA is dense in A, so is $p^{1/2} A$. Similarly using (2.10) and the density of $p_1 G^\perp$ one can show that $(I - r)^{1/2} A$ is dense in A.

Using (2.8) one can easily check that

$$\|qx\| \leqq \|p^{1/2} x\|.$$

for any $x \in A$. Therefore there exists bounded linear mapping z acting on A such that $D(z) =$ closure of $p^{1/2} A = A$, $\|z\| \leqq 1$, and $z p^{1/2} x = qx$ for any $x \in A$. According to this definition

$$z p^{1/2} = q\,. \tag{2.12}$$

By virtue of (2.11) $(I-r)^{1/2}zp^{1/2}=(I-r)^{1/2}q=qp^{1/2}$ and remembering that $p^{1/2}A$ is dense in A we get

$$(I-r)^{1/2}z=q\,. \tag{2.13}$$

Using now Proposition 0.3 [with T and a replaced by z and $(I-r)^{1/2}$ respectively] we conclude that $z\in M(A)$.

Inserting (2.12) into (2.8) we obtain $p=p^2+p^{1/2}z^*zp^{1/2}$. Therefore

$$p^2=p^{1/2}(I-z^*z)p^{1/2}$$

and (2.5) follows. Similarly using (2.13) and (2.10) we get (2.7). Finally combining (2.5) and (2.12) we prove (2.6). Q.E.D.

Remark. In the situation described in Proposition 2.2 for given p, q, r there exists at most one z satisfying relations (2.5)–(2.7). Indeed (2.12) is implied by (2.5) and (2.6). It means that the z-transform of any element $T\eta A$ is determined uniquely.

Theorem 2.3. *Let A be a C^*-algebra; $a, b, c, d\in M(A)$ and $Q=\begin{pmatrix} d, & -c^* \\ b, & a^* \end{pmatrix}$. Assume that*

$$ab=cd\,, \tag{2.14}$$

$$a^*A \textit{ is dense in } A\,, \tag{2.15}$$

$$dA \textit{ is dense in } A\,, \tag{2.16}$$

$$QA_2 \textit{ is dense in } A_2\,. \tag{2.17}$$

Then there exists $T\eta A$ such that

1° *dA is a core for T and*

$$Tdx=bx \tag{2.18}$$

for any $x\in A$.

2° *For any $x, y\in A$*

$$\begin{pmatrix} x\in D(T) \textit{ and} \\ y=Tx \end{pmatrix} \Leftrightarrow (ay=cx)\,. \tag{2.19}$$

If Q is invertible (this assumption is stronger than (2.17)) then $D(T)=dA$.

Proof. We consider the following submodules of A_2:

$$\tilde{G}=\left\{\begin{pmatrix} dx \\ bx \end{pmatrix} : x\in A\right\},$$

$$G=\text{the closure of } \tilde{G}\,, \tag{2.20}$$

$$\tilde{G}_1=\left\{\begin{pmatrix} -c^*x \\ a^*x \end{pmatrix} : x\in A\right\}.$$

One can easily verify that

$$\tilde{G}_1^\perp=\left\{\begin{pmatrix} x \\ y \end{pmatrix}\in A_2 : cx=ay\right\}. \tag{2.21}$$

The relation (2.14) implies that

$$\tilde{G}\subset\tilde{G}_1^\perp\,.$$

It means that submodules $\tilde{G}$ and $\tilde{G}_1$ are mutually orthogonal. We have assumed that $\tilde{G}+\tilde{G}_1=QA_2$ is dense in A_2. Therefore $G^\perp=$the closure of $\tilde{G}_1$, $G=\tilde{G}_1^\perp$, and $A_2=G\oplus G^\perp$. Let us notice that $p_1G\supset p_1\tilde{G}=dA$ and $p_2G^\perp\supset p_2\tilde{G}_1=a^*A$ are assumed to be dense in A [cf. (2.15) and (2.16)]. According to Proposition 2.2, G is a graph of an element $T\eta A$. Equation (2.20) shows that dA is a core for T and that (2.18) holds. Remembering that $G=\tilde{G}_1^\perp$ and using (2.21) we obtain (2.19).

One can easily check that $\tilde{G}=Q(\ker p_2)$. If Q is invertible then $\tilde{G}$ is closed: $\tilde{G}=G$ and $D(T)=p_1G=p_1\tilde{G}=dA$. Q.E.D.

In the examples illustrating Theorem 2.3 we shall use the following

Lemma 2.4. *Let B be a unital C^*-algebra, V be an invertible element of B, E_1, E_2 be orthogonal projections belonging to B such that $E_1+E_2=I$ and $Q=VE_1+V^{*-1}E_2$. Then Q is invertible.*

Proof. At first we notice that

$$Q^*Q=E_1V^*VE_1+E_2(V^*V)^{-1}E_2\geqq cI,$$

where $c=\inf\{\mathrm{Sp}(V^*V)\cup\mathrm{Sp}(V^*V)^{-1}\}>0$. Therefore $\mathrm{Sp}(Q^*Q)\subset[c,\infty[$ and (cf. [6, Problem 61]) $\mathrm{Sp}(QQ^*)\subset\{0\}\cup[c,\infty[$. Let f be a continuous function on $\mathbb{R}$ such that

$$f(\lambda)=\begin{cases}1 & \text{for} \quad \lambda=0\\ 0 & \text{for} \quad \lambda\geqq c\end{cases}.$$

Then $f(Q^*Q)=0$, $F=f(QQ^*)$ is an orthogonal projection and

$$Q^*F=Q^*f(QQ^*)=f(Q^*Q)Q^*=0.$$

Therefore $E_1V^*F=E_1Q^*F=0$, $F^*V^{*-1}E_2=F^*QE_2=0$, and

$$F=F^*F=F^*V^{*-1}(E_1+E_2)V^*F=0.$$

It shows that $0\notin\mathrm{Sp}(QQ^*)$. This way we showed that both Q^*Q and QQ^* are invertible, so is Q. Q.E.D.

In the following examples A is a C^*-algebra, S is an element affiliated with A and $z=z_S$ is the z-transform of S. We shall also use the unitary matrix

$$Q_0=\begin{pmatrix}(I-z^*z)^{1/2}, & -z^*\\ z, & (I-zz^*)^{1/2}\end{pmatrix}.$$

Example 1. Let $q\in M(A)$ and

$$a=(I-zz^*)^{1/2},\quad b=z+q(I-z^*z)^{1/2},\quad c=z+(I-zz^*)^{1/2}q,\quad d=(I-z^*z)^{1/2}.$$

The relations (2.14)–(2.16) are obviously satisfied. One can easily check that

$$Q=\begin{pmatrix}d, & -c^*\\ b, & a^*\end{pmatrix}=V\begin{pmatrix}I, & 0\\ 0, & 0\end{pmatrix}+V^{*-1}\begin{pmatrix}0, & 0\\ 0, & I\end{pmatrix},$$

where

$$V=\begin{pmatrix}I, & 0\\ q, & I\end{pmatrix}Q_0.$$

Using Lemma 2.4 we conclude that Q is invertible and all assumptions of Theorem 2.3 are verified. Let T be the element affiliated with A introduced in this theorem. Then $D(T)=dA=D(S)$ and

$$Tx=Sx+qx$$

for any $x\in D(S)$. In what follows the element T considered in this example will be denoted by $S+q$.

Example 2. Let v be an invertible element of $M(A)$ and

$$a=(I-zz^*)^{1/2}v^{-1}, \quad b=vz, \quad c=z, \quad d=(I-z^*z)^{1/2}.$$

Again the relations (2.14)–(2.16) are obviously satisfied. Moreover in this case

$$Q=\begin{pmatrix} d, & -c^* \\ b, & a^* \end{pmatrix}=V\begin{pmatrix} I, & 0 \\ 0, & 0 \end{pmatrix}+V^{*-1}\begin{pmatrix} 0, & 0 \\ 0, & I \end{pmatrix},$$

where

$$V=\begin{pmatrix} I & 0 \\ 0 & v \end{pmatrix}Q_0$$

and (cf. Lemma 2.4) Q is invertible. Let $T\eta A$ be the element introduced in Theorem 2.3. Then $D(T)=dA=D(S)$ and

$$Tx=vSx$$

for any $x\in D(S)$. In what follows the element T considered in this example will be denoted by vS.

Example 3. Let $v\in M(A)$ be invertible and

$$a=(I-zz^*)^{1/2}, \quad b=z, \quad c=zv, \quad d=v^{-1}(I-z^*z)^{1/2}.$$

Also in this example the relations (2.14)–(2.16) are obvious. Moreover

$$Q=\begin{pmatrix} d, & -c^* \\ b, & a^* \end{pmatrix}=V\begin{pmatrix} I, & 0 \\ 0, & 0 \end{pmatrix}+V^{*-1}\begin{pmatrix} 0, & 0 \\ 0, & I \end{pmatrix},$$

where

$$V=\begin{pmatrix} v^{-1}, & 0 \\ 0, & I \end{pmatrix}Q_0$$

and (cf. Lemma 2.4) Q is invertible. Let $T\eta A$ be the element introduced in Theorem 2.3. Then $D(T)=dA=v^{-1}D(S)$ and

$$Tx=S(vx)$$

for any $x\in v^{-1}D(S)$. In what follows the element T considered in this example will be denoted by Sv.

In concrete application of Theorem 2.3 the most difficult assumption to verify is the one saying that QA_2 is dense in A. The following proposition shows that the problem may be reduced to the similar problem in the Hilbert space theory.

Proposition 2.5.
1° *Let A be a C^*-algebra and $d \in M(A)$. Then dA is dense in A if and only if for any irreducible representation π of A acting on a Hilbert space H_π the range of $\pi(d)$ is dense in H_π.*
2° *Let A be a C^*-algebra, $Q \in M(M_2(\mathbb{C}) \otimes A)$ and q denote the canonical representation of $M_2(\mathbb{C})$ on $\mathbb{C}^2$. Then QA_2 is dense in A_2 if and only if for any irreducible representation π of A acting on a Hilbert space H_π the range of $(q \otimes \pi)(Q)$ is dense in $\mathbb{C}^2 \otimes H_\pi = H_\pi \oplus H_\pi$.*

Proof.
Ad 1° If dA is dense in A then $\pi(d)\pi(A)H$ is dense in $\pi(A)H$ which in turn is dense in H. Therefore the range of $\pi(d)$ is dense.

If dA is not dense in A then (cf. [4, Theorem 2.9.5]) there exists a pure state ω on A such that

$$\omega(dx) = 0$$

for all $x \in A$. Let π be the GNS representation of A acting on a Hilbert space H_π and $\Omega \in H_\pi$ be the corresponding cyclic vector:

$$\omega(x) = (\Omega \mid \pi(x)\Omega)$$

for any $x \in A$. Combining the two formulae we see that the range of $\pi(d)$ is contained in $\Omega^\perp = \{\varphi \in H_\pi : (\varphi \mid \Omega) = 0\}$. Therefore it is not dense and Statement 1° is proved.
Ad 2° This case can be easily reduced to the previous one. Indeed QA_2 is dense in A_2 if and only if $Q(M_2(\mathbb{C}) \otimes A)$ is dense in $M_2(\mathbb{C}) \otimes A$. On the other hand any irreducible representation of $M_2(\mathbb{C}) \otimes A$ is of the form $q \otimes \pi$ when π is an irreducible representation of A. Q.E.D.

We shall also use

Proposition 2.6. *Let A be a C^*-algebra and $a, b, c, d \in M(A)$. Assume that for any irreducible representation π of A acting on a Hilbert space H_π there exists a closed operator T_π acting on H_π such that*
1° *$\pi(d)H_\pi$ is a core for T_π and*

$$T_\pi \pi(d)\varphi = \pi(b)\varphi$$

for any $\varphi \in H_\pi$.
2° *$\pi(a)^* H_\pi$ is a core for T_π^* and*

$$T_\pi^* \pi(a)^* \psi = \pi(c)^* \psi$$

for any $\psi \in H_\pi$.

Then the assumptions (2.14)–(2.17) *of Theorem 2.3 are satisfied and denoting by T the element affiliated with A introduced in this theorem we have*

$$\pi(T) = T_\pi \tag{2.22}$$

for any irreducible representation π of A.

Proof. For any π and any $\varphi, \psi \in H_\pi$ we have

$$\begin{aligned}(\psi \mid \pi(ab)\varphi) &= (\pi(a)^*\psi \mid \pi(b)\varphi) = (\pi(a)^*\psi \mid T_\pi \pi(d)\varphi) = (T_\pi^* \pi(a)^*\psi \mid \pi(d)\varphi) \\ &= (\pi(c)^*\psi \mid \pi(d)\varphi) = (\psi \mid \pi(cd)\varphi)\end{aligned}$$

and (2.14) follows. We assumed that $\pi(d)H_\pi$ and $\pi(a)^*H_\pi$ are cores of closed operators. Therefore these sets are dense in H and using Proposition 2.5.1° we obtain (2.15) and (2.16). Let Q be the element of $M_2(\mathbb{C})\otimes A$ introduced in Theorem 2.3. Then

$$(q\otimes\pi)(Q)=\begin{pmatrix}\pi(d), & -\pi(c)^* \\ \pi(b), & \pi(a)^*\end{pmatrix}$$

and the range of $(q\otimes\pi)(Q)$ equals to $\tilde{K}_1+\tilde{K}_2$, where

$$\tilde{K}_1=\left\{\begin{pmatrix}\pi(d)\varphi \\ \pi(b)\varphi\end{pmatrix}:\varphi\in H_\pi\right\},$$

$$\tilde{K}_2=\left\{\begin{pmatrix}-\pi(c)^*\psi \\ \pi(a)^*\psi\end{pmatrix}:\psi\in H_\pi\right\}.$$

Let

$$K_1=\left\{\begin{pmatrix}\varphi \\ T_\pi\varphi\end{pmatrix}:\varphi\in\mathscr{D}(T_\pi)\right\},$$

$$K_2=\left\{\begin{pmatrix}-T_\pi^*\psi \\ \psi\end{pmatrix}:\psi\in\mathscr{D}(T_\pi^*)\right\}.$$

Assumption 1° (Assumption 2° respectively) means that $\tilde{K}_1$ ($\tilde{K}_2$ respectively) is dense in K_1 (K_2 respectively). On the other hand for any closed operator T_π, K_1 is orthogonal to K_2 and $K_1\oplus K_2=\mathbb{C}^2\otimes H_\pi$. This way we showed that the range of $(q\otimes\pi)(Q)$ is dense and using Proposition 2.5.2° we obtain (2.17).

Formula (2.22) follows immediately from assumption 1° and the following lemma.

Lemma 2.7. *Let A be a C^*-algebra, $T\eta A$, $b, d\in M(A)$, and π be a representation of A acting on a Hilbert space H_π. Assume that the statement* 1° *of Theorem 2.3 holds. Then $\pi(d)H_\pi$ is a core for $\pi(T)$ and*

$$\pi(T)\pi(d)\varphi=\pi(b)\varphi$$

for any $\varphi\in H_\pi$.

Proof. We recall that the element $\pi(T)$ affiliated with the algebra $CB(H_\pi)$ of all compact operators acting on H_π is introduced in Theorem 1.2 [with B and ϕ replaced by $CB(H_\pi)$ and π respectively]. According to Example 3 of Sect. 1, $\pi(T)$ may be identified with a closed operator acting on H_π. In the proof we shall frequently refer to these parts of the paper without any further notice. We have however to distinguish carefully the domains $\mathscr{D}(\pi(T))\subset H_\pi$ and $D(\pi(T))\subset CB(H_\pi)$. Similarly the notion of core of $\pi(T)$ will be used in the two meanings (which one is used will be clear from the context).

We have to show that:

ⓐ $$\pi(d)\varphi\in\mathscr{D}(\pi(T))\quad\text{and}\quad\pi(T)\pi(d)\varphi=\pi(b)\varphi$$

for any $\varphi\in H_\pi$ and

ⓑ For any $\psi\in\mathscr{D}(\pi(T))$ there exists a sequence $\{\varphi_n\}$ of elements of H_π such that

$$\left.\begin{aligned}\psi&=\lim\pi(d)\varphi_n\\ \pi(T)\psi&=\lim\pi(b)\varphi_n\end{aligned}\right\}. \tag{2.23}$$

Ad ⓐ. Any $\varphi \in H_\pi$ is of the form $\varphi = \pi(x)\varphi'$, where $x \in A$ and $\varphi' \in H_\pi$. Therefore $\pi(d)\varphi = \pi(dx)\varphi' = \pi(dx)E_{\varphi'}\varphi'$, where $E_{\varphi'} \in CB(H_\pi)$ is the one-dimensional projection onto $\mathbb{C}\varphi'$. On the other hand $\pi(dx)E_{\varphi'} \in D(\pi(T))$ and

$$\pi(T)\pi(dx)E_{\varphi'} = \pi(Tdx)E_{\varphi'} = \pi(bx)E_{\varphi'}.$$

Therefore $\pi(d)\varphi = \pi(dx)E_{\varphi'}\varphi' \in \mathscr{D}(\pi(T))$ and

$$\pi(T)\pi(d)\varphi = \pi(bx)E_{\varphi'}\varphi' = \pi(b)\pi(x)\varphi' = \pi(b)\varphi.$$

Ad ⓑ. We assumed that dA is a core for T. Therefore $\pi(dA)CB(H_\pi)$ is a core for $\pi(T)$. Let $\psi \in \mathscr{D}(\pi(T))$. Then the one-dimensional projection E onto $\mathbb{C}\psi$ belongs to $D(\pi(T))$ and for any natural n one can find $x_1, x_2, \ldots, x_k \in A$ and $F_1, F_2, \ldots, F_k \in CB(H_\pi)$ such that

$$\left\|\sum_i \pi(dx_i)F_i - E\psi\right\| < \frac{1}{n}, \tag{2.24}$$

$$\left\|\pi(T)\left(\sum_i \pi(dx_i)F_i\right) - \pi(T)E\psi\right\| < \frac{1}{n}.$$

We know that $\pi(T)\pi(dx_i)F_i = \pi(T(dx_i))F_i = \pi(bx_i)F_i$ and the second estimate may be rewritten in the following way:

$$\left\|\sum_i \pi(bx_i)F_i - \pi(T)E\psi\right\| < \frac{1}{n}. \tag{2.25}$$

Let $\varphi_n = \sum_i \pi(x_i)F_i\psi$. Then (2.24) and (2.25) imply that

$$\|\pi(d)\varphi_n - \psi\| \leqq \frac{1}{n},$$

$$\|\pi(b)\varphi_n - \pi(T)\psi\| \leqq \frac{1}{n}$$

and (2.23) follows. Q.E.D.

3. The Group of Motions of the Euclidean Quantum Plane

Let G be the set of all matrices of the form

$$g = \left(\begin{pmatrix} v, & n \\ 0, & v^{-1} \end{pmatrix}\right), \tag{3.1}$$

where $v, n \in \mathbb{C}$, $|v| = 1$. Then G is a three-dimensional solvable Lie group. For any g of the form (3.1) and $\zeta \in \mathbb{C}$ we set

$$g\zeta = v^2\zeta + vn. \tag{3.2}$$

One can easily check that this formula defines a homomorphism of G onto the group of all transformations of $\mathbb{C}$ preserving the orientation and the Euclidean distance. The kernel of this homomorphism is a normal subgroup of G isomorphic to $\mathbb{Z}_2$. Therefore, G is the two-fold covering of the group of motions of two-dimensional Euclidean plane.

Now we shall use the mathematical tools developed in Sects. 1 and 2 to analyse the quantum deformation of the group G. To clarify the exposition we divide this section into three parts. In part A we introduce the *-Hopf algebra $\mathscr{A}$ of all polynomials on G_μ (G_μ is the quantum version of G) and the *-algebra $\mathscr{B}$ of all polynomials on $\mathbb{C}_\mu$ ($\mathbb{C}_\mu$ is the quantum version of $\mathbb{C}$). In part B we investigate the Hilbert space representations of $\mathscr{A}$ discovering some unexpected unpleasant features. To remove them we have to complete the list of commutation relations defining the algebra $\mathscr{A}$ by adding a relation of a non-algebraic nature. In the last part we construct the C^*-algebra A of all "continuous vanishing at infinity" functions on G_μ and show that there exists the natural comultiplication $\phi \in \mathrm{Mor}(A, A \otimes A)$. The C^*-algebra B of all "vanishing at infinity continuous" functions on $\mathbb{C}_\mu$ and the morphism $\psi \in \mathrm{Mor}(B, A \otimes B)$ describing the natural action of G_μ on $\mathbb{C}_\mu$ will also be discussed.

The reader should notice that the meaning of symbols v and n varies throughout this section. In the introductory part v and n are complex numbers; in part A they are elements of $\mathscr{A}$, in part B – operators acting on a Hilbert space and finally in part C – elements affiliated with the C^*-algebra A. Similarly, "$\otimes$" denotes in part A the algebraic tensor product, in part B – the tensor product known from the theory of Hilbert spaces and in part C – the algebraic tensor product followed by the largest C^*-norm completion. This remark applies also to the next section as well as to other symbols (ϕ, ψ, etc.) used in this section.

A. Hopf-Algebra Level

Let us fix a real number $\mu \neq 0$. We shall assume that $|\mu| > 1$. For $|\mu| < 1$ we obtain isomorphic objects; $\mu = 1$ corresponds to the classical (i.e. non-quantum) case. The very interesting case $\mu = -1$ will not be considered in this paper.

Let $\mathscr{B}$ be the *-algebra generated by a single element ζ such that

$$\zeta^*\zeta = \mu^2 \zeta\zeta^* .$$

Elements of $\mathscr{B}$ are called polynomials on the Euclidean quantum plane $\mathbb{C}_\mu$.

Let $\mathscr{A}$ be the *-algebra generated by two elements v and n such that

$$\left.\begin{aligned} v^*v &= vv^* = I , \\ n^*n &= nn^* , \\ v^*nv &= \mu n . \end{aligned}\right\} \tag{3.3}$$

Elements of $\mathscr{A}$ are called polynomials on G_μ. We endow $\mathscr{A}$ with the Hopf-algebra structure introducing comultiplication ϕ, counit e and coinverse (antipode) κ in the following way

$$\phi(v) = v \otimes v , \tag{3.4}$$

$$\phi(n) = v \otimes n + n \otimes v^* , \tag{3.5}$$

$$e(v) = 1 , \qquad e(n) = 0 ,$$

$$\kappa(v) = v^* , \qquad \kappa(v^*) = v ,$$

$$\kappa(n) = -\mu n , \qquad \kappa(n^*) = -\frac{1}{\mu} n^* .$$

We remind that ϕ and e are *-algebra homomorphisms whereas κ is linear and antimultiplicative. One can easily verify that all axioms (cf. [8]) of the *-Hopf algebra theory are satisfied.

The action of G_μ on $\mathbb{C}_\mu$ is described by the *-algebra homomorphism $\psi: \mathscr{B} \to \mathscr{A} \otimes \mathscr{B}$ introduced by the formula [cf. (3.2)]

$$\psi(\zeta) = v^2 \otimes \zeta + vn \otimes I. \tag{3.6}$$

One can easily check that the diagram

$$\begin{tikzcd} \mathscr{B} \arrow[r, "\psi"] \arrow[d, "\psi"'] & \mathscr{A} \otimes \mathscr{B} \arrow[d, "\phi \otimes \mathrm{id}"] \\ \mathscr{A} \otimes \mathscr{B} \arrow[r, "\mathrm{id} \otimes \psi"'] & \mathscr{A} \otimes \mathscr{A} \otimes \mathscr{B} \end{tikzcd}$$

is commutative.

Let us notice that identifying ζ with vn we embed $\mathscr{B}$ into $\mathscr{A}$. With this embedding ψ coincides with $\phi|_{\mathscr{B}}$.

B. Hilbert Space Level

Let v and n be operators acting on a Hilbert space H. We say that the pair (v, n) is a representation of commutation relations (3.3) if v is unitary, n is normal and $v^* n v = \mu n$. We recall that the normality of n means that $\mathscr{D}(n^*) = \mathscr{D}(n)$ and $\|n^* \psi\| = \|n \psi\|$ for any $\psi \in \mathscr{D}(n)$.

In the considered case the representation theory is relatively simple. One can easily verify the following facts: As usual any representation is a direct integral of irreducible ones. Any representation is either infinite or one-dimensional. All one-dimensional representations are of the form $(cI, 0)$, where $c \in \mathbb{C}$, $|c| = 1$.

If (v, n) is an irreducible infinite-dimensional representation then $\mathrm{Sp}\, n = \{0\} \cup \Lambda$, where Λ is of the form

$$\Lambda = \{t_0 \mu^k : k \in \mathbb{Z}\} \tag{3.7}$$

and $t_0 \in \mathbb{C} - \{0\}$. Moreover, one can find an orthonormal basis $\{|t\rangle : t \in \Lambda\}$ such that

$$v|t\rangle = |\mu t\rangle, \tag{3.8}$$

$$n|t\rangle = t|t\rangle \tag{3.9}$$

for any $t \in \Lambda$. It means that (v, n) is uniquely determined (up to a unitary equivalence) by $\mathrm{Sp}\, n$.

For any subset $\Theta \subset \mathbb{C}$ we set

$$|\Theta| = \{ct : t \in \Theta,\ c \in \mathbb{C}, \text{ and } |c| = 1\}.$$

Theorem 3.1. *Let (v_1, n_1) and (v_2, n_2) be infinite-dimensional irreducible representations of* (3.3) *and*

$$N = v_1 \otimes n_2 + n_1 \otimes v_2^*. \tag{3.10}$$

By definition $\mathscr{D}(N) = \mathscr{D}(v_1 \otimes n_2) \cap \mathscr{D}(n_1 \otimes v_2^*) = \mathscr{D}(I \otimes n_2) \cap \mathscr{D}(n_1 \otimes I)$. Then

1° *If $|\mathrm{Sp}\, n_1| \neq |\mathrm{Sp}\, n_2|$ then N is closed, N is not normal and has no normal extension.*

2° *If $|\mathrm{Sp}\, n_1| = |\mathrm{Sp}\, n_2|$ then N is closeable, its closure $\bar{N}$ is normal and*

$$\mathrm{Sp}\, \bar{N} = |\mathrm{Sp}\, n_2|. \tag{3.11}$$

Proof. According to (3.7) and (3.9) operators n_1 and n_2 are up to numeric phase factors positive selfadjoint. On the other hand [cf. (3.10)], the phase factor of n_2 can be absorbed into v_1 and that of n_1 into v_2^*. Therefore, we may (and shall) assume that n_1, n_2 are positive selfadjoint.

Let H_1 (H_2 respectively) be the Hilbert space where operators v_1, n_1 (v_2, n_2 respectively) act. We shall identify $H_1 \otimes H_2$ with $L^2(\mathbb{Z} \times S^1)$:

$$H_1 \otimes H_2 = L^2(\mathbb{Z} \times S^1), \tag{3.12}$$

where $\mathbb{Z}$ is the set of all integers endowed with the counting measure ν_1 [$\nu_1(\Delta)$ = the number of elements of Δ for any $\Delta \subset \mathbb{Z}$)], S^1 is the unit circle

$$S^1 = \{z \in \mathbb{C} : |z| = 1\}$$

endowed with the normalized Lebesgue measure $d\nu(z) = \frac{1}{2\pi i} \frac{dz}{z}$ and the cartesian product $\mathbb{Z} \times S^1$ is endowed with the product measure $\nu_1 \otimes \nu$.

Let us fix $t_1 \in \operatorname{Sp} n_1$ and $t_2 \in \operatorname{Sp} n_2$ such that

$$t_1 \leqq t_2 < \mu t_1 .$$

Then $\{|\mu^k t_1\rangle \otimes |\mu^l t_2\rangle : k, l \in \mathbb{Z}\}$ is an orthonormal basis in $H_1 \otimes H_2$. To give the meaning to (3.12), for any $\varphi \in H_1 \otimes H_2$ and any $(m, z) \in \mathbb{Z} \times S^1$ we set

$$\varphi(m, z) = \sum_{l=-\infty}^{+\infty} \varphi_{m+l,l} z^l , \tag{3.13}$$

where φ_{kl} are Fourier coefficients of φ:

$$\varphi = \sum_{kl} \varphi_{kl} |\mu^k t_1\rangle \otimes |\mu^l t_2\rangle .$$

One can easily verify that the series (3.13) is convergent in the sense of $L^2(\mathbb{Z} \times S^1)$-norm and that the correspondence $H_1 \otimes H_2 \ni \varphi \leftrightarrow \varphi(\cdot,\cdot) \in L^2(\mathbb{Z} \times S^1)$ is bijective and respects the Hilbert space structures of $H_1 \otimes H_2$ and $L^2(\mathbb{Z} \times S^1)$.

Let R be the ring

$$R = \{\zeta \in \mathbb{C} : 1 \leqq |\zeta| \leqq \mu\} .$$

We say that a function $\psi(\cdot) \in L^2(S^1)$ admits a continuous extension on R holomorphic (meromorphic respectively) inside R if there exists a holomorphic (meromorphic with a finite number of poles respectively) function $\tilde{\psi}$ defined on the interior of R and $\psi' \in L^2(S^1)$ such that

$$\lim_{r \to 1+0} \tilde{\psi}(rz) = \psi(z),$$

$$\lim_{r \to \mu-0} \tilde{\psi}(rz) = \psi'(z),$$

where limits are understood in the sense of $L^2(S^1)$-norm. In this case we write $\psi(\zeta)$ and $\psi(\mu z)$ instead of $\tilde{\psi}(\zeta)$ and $\psi'(z)$. The reader should notice that $\tilde{\psi}$ and ψ' are uniquely determined by ψ.

Let $\varphi \in H_1 \otimes H_2$. Using (3.8), (3.9), and (3.13) one can verify that

$$((v_1 \otimes I)\varphi)(m, z) = \varphi(m-1, z), \tag{3.14}$$

$$((I \otimes v_2)\varphi)(m, z) = z\varphi(m+1, z) .$$

Moreover, $\varphi \in \mathscr{D}(n_1 \otimes I)$ if and only if all functions $\varphi(m, \cdot)$ admit continuous extensions on R holomorphic inside R and

$$\sum_{m=-\infty}^{+\infty} \int_{S^1} \mu^{2m} |\varphi(m, \mu z)|^2 d\nu(z) < \infty .$$

In this case

$$((n_1 \otimes I)\varphi)(m, z) = t_1 \mu^m \varphi(m, \mu z) . \tag{3.15}$$

Similarly, $\varphi \in \mathscr{D}(I \otimes n_2)$ if and only if all functions $\varphi(m, \cdot)$ admit continuous extensions on R holomorphic inside R and

$$\sum_{m=-\infty}^{+\infty} \int_{S^1} |\varphi(m, \mu z)|^2 d\nu(z) < \infty . \tag{3.16}$$

In this case

$$((I \otimes n_2)\varphi)(m, z) = t_2 \varphi(m, \mu z) . \tag{3.17}$$

Consequently, $\varphi \in \mathscr{D}(N)$ if and only if all functions $\varphi(m, \cdot)$ admit continuous extensions on R holomorphic inside R and

$$\sum_{m=-\infty}^{+\infty} \int_{S^1} (1 + \mu^{2m}) |\varphi(m, \mu z)|^2 d\nu(z) < \infty .$$

In this case

$$(N\varphi)(m, z) = (t_2 + t_1 \mu^{m-1} \bar{z}) \varphi(m-1, \mu z) . \tag{3.18}$$

Comparing (3.17) and (3.18) we get

$$(N\varphi)(m, z) = (1 + (t_1/t_2)\mu^{m-1}\bar{z})((I \otimes n_2)\varphi)(m-1, z) \tag{3.19}$$

for any $\varphi \in \mathscr{D}(N)$.

Assume now that

$$|\mathrm{Sp}\, n_1| \neq |\mathrm{Sp}\, n_2| .$$

Then $t_1 < t_2 < \mu t_1$ and $t_2 + t_1 \mu^{m-1} \bar{z} \neq 0$ for any $(m, z) \in \mathbb{Z} \times S^1$. In fact, one can easily show that there exist positive constants c_1 and c_2 such that

$$t_1 \mu^m \leqq c_1 |t_2 + t_1 \mu^m \bar{z}| , \tag{3.20}$$

$$t_2 \leqq c_2 |t_2 + t_1 \mu^m \bar{z}| \tag{3.21}$$

for any $(m, z) \in \mathbb{Z} \times S^1$. Taking into account (3.15), (3.17), and (3.18) we see that

$$\|(n_1 \otimes I)\varphi\| \leqq c_1 \|N\varphi\| , \qquad \|(I \otimes n_2)\varphi\| \leqq c_2 \|N\varphi\|$$

for any $\varphi \in \mathscr{D}(N)$. Therefore, N is closed. Indeed, if $\{\varphi_n\}$ is a converging sequence of elements of $\mathscr{D}(N)$ such that $\{N\varphi_n\}$ is converging, then using the above estimates we see that $\{(n_1 \otimes I)\varphi_n\}$ and $\{(I \otimes n_2)\varphi_n\}$ are also converging. Remembering that $(n_1 \otimes I)$ and $(I \otimes n_2)$ are closed we conclude that

$$\lim \varphi_n \in \mathscr{D}(n_1 \otimes I) \cap \mathscr{D}(I \otimes n_2) = \mathscr{D}(N) ,$$

$$\begin{aligned} N(\lim \varphi_n) &= (v_1 \otimes n_2)(\lim \varphi_n) + (n_1 \otimes v_2^*)(\lim \varphi_n) \\ &= \lim (v_1 \otimes n_2 + n_1 \otimes v_2^*)\varphi_n = \lim N\varphi_n , \end{aligned}$$

and the statement follows.

Let $\psi \in H_1 \otimes H_2$. We shall prove that $\psi \in \mathscr{D}(N^*)$ if and only if the following three conditions are satisfied:

ⓐ For all $m \neq 1$, $\psi(m, \cdot)$ admits a continuous extension on R holomorphic inside R.
ⓑ $\psi(1, \cdot)$ admits a continuous extension on R meromorphic inside R with the only singularity of the type of simple pole located at the point $\zeta = -t_2/t_1$.

ⓒ
$$\sum_{m=-\infty}^{+\infty} \int_{S^1} (1+\mu^{2m})|\psi(m, \mu z)|^2 dv(z) < \infty . \tag{3.22}$$

To this end we choose a bounded sequence of strictly positive numbers $\{r_m\}_{m \in \mathbb{Z}}$ such that

$$\sup_{(m,z) \in \mathbb{Z} \times S^1} |r_m(1+(t_1/t_2)\mu^m z)| < \infty ,$$

and for any $\psi \in H_1 \otimes H_2$ we set

$$\begin{aligned} (r\psi)(m,z) &= r_m \psi(m,z) , \\ (\tilde{r}\psi)(m,z) &= r_m(1+(t_1/t_2)\mu^m z)\psi(m+1,z) . \end{aligned} \tag{3.23}$$

Clearly, r and $\tilde{r}$ are bounded operators acting on $H_1 \otimes H_2$. Let us notice that $r^* = r$ and $r\mathscr{D}(N) \subset \mathscr{D}(N)$.

Assume that $\psi \in \mathscr{D}(N^*)$. Then for all $\varphi \in \mathscr{D}(N)$

$$\begin{aligned} (\varphi | rN^*\psi) &= (r\varphi \mid N^*\psi) \\ &= (Nr\varphi \mid \psi) = ((I \otimes n_2)\varphi \mid \tilde{r}\psi) , \end{aligned}$$

where in the last step we used (3.19). Since $\mathscr{D}(N) = \mathscr{D}(n_1 \otimes I) \cap \mathscr{D}(I \otimes n_2)$ is a core for $I \otimes n_2$, the above relation holds for all $\varphi \in \mathscr{D}(I \otimes n_2)$. It shows that

$$\tilde{r}\psi \in \mathscr{D}((I \otimes n_2)) \tag{3.24}$$

and

$$rN^*\psi = (I \otimes n_2)(\tilde{r}\psi) . \tag{3.25}$$

Let us notice that the function

$$\mathbb{Z} \times R \ni (m, \zeta) \to r_m(1+(t_1/t_2)\mu^m \zeta) \in C$$

vanishes only at one point $(m, \zeta) = (0, -t_2/t_1)$. Therefore, (3.23) and (3.24) imply [cf. the description of $\mathscr{D}(I \otimes n_2)$ given earlier in this proof] that ψ satisfies the conditions ⓐ and ⓑ. Moreover, using (3.25), (3.23), and (3.17) we get

$$(N^*\psi)(m,z) = (t_2 + t_1 \mu^{m+1} z)\psi(m+1, \mu z) .$$

Therefore,

$$\sum_{m=-\infty}^{+\infty} \int_{S^1} |t_2 + t_1 \mu^m z|^2 |\psi(m, \mu z) dv(z) = \|N^*\varphi\| < \infty ,$$

and using the estimates (3.20) and (3.21) (with z replaced by $\bar{z}$) we get (3.22).

Conversely assume that ψ satisfies the conditions a, b, and c. Let

$$\psi'(m,z) = (1+(t_1/t_2)\mu^m z)\psi(m+1,z) .$$

Then all $\psi'(m,\cdot)$ admit continuous extensions on R holomorphic inside R [the simple pole of $\psi(1,\zeta)$ at the point $\zeta=-t_2/t_1$ meets the zero of the first factor placed at the same point]. Moreover, (3.22) implies (3.16) with φ replaced by ψ'. Therefore, $\psi'\in\mathscr{D}((I\otimes n_2))$. Using (3.19) we get

$$(N\varphi\,|\,\psi)=((I\otimes n_2)\varphi\,|\,\psi')=(\varphi\,|\,(I\otimes n_2)\psi')$$

for any $\varphi\in\mathscr{D}(N)$. It shows that $\psi\in\mathscr{D}(N^*)$ [and $N^*\psi=(I\otimes n_2)\psi'$]. This way we proved that

$$\begin{pmatrix} ⓐ \text{ and } ⓑ \\ \text{and } ⓒ \end{pmatrix} \Leftrightarrow (\psi\in\mathscr{D}(N^*)).$$

For any $\psi\in\mathscr{D}(N^*)$ we set

$$l(\psi)=\operatorname*{res}_{\zeta=-t_2/t_1}\psi(1,\zeta)=\lim_{\zeta\to-t_2/t_1}(\zeta+t_2/t_1)\psi(1,\zeta).$$

Then l is a linear functional defined on $\mathscr{D}(N^*)$.

Comparing the description of $\mathscr{D}(N)$ and $\mathscr{D}(N^*)$ we see that $\mathscr{D}(N)\subset\mathscr{D}(N^*)$ and that $\psi\in\mathscr{D}(N^*)$ belongs to $\mathscr{D}(N)$ if and only if $l(\psi)=0$. Therefore,

$$\dim(\mathscr{D}(N^*)/\mathscr{D}(N))=1. \tag{3.26}$$

Let $\tilde{N}$ be a closed extension of N. Then

$$\dim(\mathscr{D}(\tilde{N})/\mathscr{D}(N))=\dim(\mathscr{D}(N^*)/\mathscr{D}(\tilde{N}^*)).$$

If $\tilde{N}$ is normal then $\mathscr{D}(\tilde{N})=\mathscr{D}(\tilde{N}^*)$ and

$$\dim(\mathscr{D}(N^*)/\mathscr{D}(N))=\dim(\mathscr{D}(N^*)/(\mathscr{D}(\tilde{N}^*))+\dim(\mathscr{D}(\tilde{N})/\mathscr{D}(N))$$

would be even. The obvious contradiction with (3.26) shows that N admits no normal extensions. This ends the proof of the first part of the theorem.

Assume now that

$$|\mathrm{Sp}\,n_1|=|\mathrm{Sp}\,n_2|.$$

Then $t_1=t_2$ and combining (3.19) and (3.14) we get

$$(N\varphi)(m,z)=(1+\mu^{m-1}z^{-1})((v_1\otimes n_2)\varphi)(m,z). \tag{3.27}$$

For any $m\in\mathbb{Z}$ and $\zeta\in\mathbb{C}-\{0\}$ we set

$$U(m,\zeta)=\prod_{k=0}^{\infty}\frac{1+\mu^{m-1-2k}\zeta}{1+\mu^{m-1-2k}\zeta^{-1}}.$$

One can easily check that this formula defines the denumerable family of functions $U(m,\cdot)$ meromorphic on $\mathbb{C}-\{0\}$. For each m, $U(m,\zeta)$ has simple zeroes at points $\zeta=-\mu^{|m|+q}$, where $q=1,3,5,\dots$ and simple poles at points $\zeta=-\mu^{-|m|-q}$ (q as above). In particular, all $U(m,\zeta)$ are holomorphic in a neighbourhood of R and only $U(0,\zeta)$ has a zero in R. Moreover,

$$\overline{U(m,\zeta)}=U(m,\bar{\zeta}), \tag{3.28}$$

$$U(m,\zeta^{-1})=U(m,\zeta)^{-1}, \tag{3.29}$$

$$U(m-1,\mu\zeta)=(1+\mu^{m-1}\zeta^{-1})U(m,\zeta) \tag{3.30}$$

for any $m\in\mathbb{Z}$ and $\zeta\in\mathbb{C}-\{0\}$.

For any $\varphi \in H_1 \otimes H_2$ we set

$$(U\varphi)(m,z) = U(m,z)\varphi(m,z). \tag{3.31}$$

By virtue of (3.28) and (3.29), $|U(m,z)|=1$ for all $(m,z)\in \mathbb{Z}\times S^1$. Therefore, U introduced by (3.31) is a unitary operator acting on $H\otimes H'$.

Inserting in (3.30) z instead of ζ we see that

$$|U(m,\mu z)| = |1+\mu^m \bar{z}|.$$

Having in mind the descriptions of $\mathscr{D}(N)$ and $\mathscr{D}(I\otimes n_2)$ given in the introductory part of this proof, using analytical properties of $U(m,\cdot)$ and the above relation one can easily check that $U\varphi \in \mathscr{D}(I\otimes n_2)$ for any $\varphi \in \mathscr{D}(N)$. Moreover,

$$(v_1 \otimes n_2)U\varphi = UN\varphi. \tag{3.32}$$

Indeed, using (3.14), (3.17), (3.31), (3.30), and (3.27) we have

$$\begin{aligned}((v_1\otimes n_2)U\varphi)(m,z) &= t_2(U\varphi)(m-1,\mu z)\\ &= t_2 U(m-1,\mu z)\varphi(m-1,\mu z)\\ &= U(m,z)(1+\mu^{m-1}z^{-1})((v_1\otimes n_2)\varphi)(m,z)\\ &= U(m,z)(N\varphi)(m,z) = (UN\varphi)(m,z).\end{aligned}$$

Let $\mathscr{D}_0 = U\mathscr{D}(N)$. Formula (3.32) means that

$$N = U^*(v_1\otimes n_2)|_{\mathscr{D}_0} U. \tag{3.33}$$

An element $\varphi \in H_1\otimes H_2$ is said to be a polynomial if all $\varphi(m,z)$ are polynomials in z and z^{-1} and only finite number of them are not zero. One can easily verify that the set of all polynomials is a core for $I\otimes n_2$ (for it contains the complete set of eigenvectors) and that a polynomial $\varphi \in H_1\otimes H_2$ belongs to $\mathscr{D}_0$ if $\varphi(0,-\mu)=0$.

Let $\varphi \in H_1 \otimes H_2$ be a polynomial. For any natural k we set

$$\varphi_k(m,z) = \begin{cases} \varphi(m,z) & \text{for} \quad m \neq 0, \\ \varphi(0,z) - \varphi(0,-\mu)t(z)^k & \text{for} \quad m = 0, \end{cases}$$

where $t(z) = (1-z)/(1+\mu)$. Then $\varphi_k(0,-\mu)=0$ and $\varphi_k \in \mathscr{D}_0$. Moreover,

$$\|\varphi - \varphi_k\| = |\varphi(0,-\mu)| \Big(\int_{S^1} |t(z)|^{2k} dv(z)\Big)^{1/2}$$

and

$$\|(I\otimes n_2)(\varphi-\varphi_k)\| = t_2|\varphi(0,-\mu)|\Big(\int_{S^1} |t(\mu z)|^{2k} dv(z)\Big)^{1/2}$$

both converge to 0 as $k\to\infty$. It shows that $\mathscr{D}_0$ is a core for $I\otimes n_2$.

Passing to the closures on the both sides of (3.33) we obtain

$$\bar{N} = U^*(v_1\otimes n_2)U.$$

Therefore, $\bar{N}$ has the same analytical properties as $v_1\otimes n_2$. In particular, $\bar{N}$ is normal and (3.11) holds. Q.E.D.

The possibility indicated by the first part of Theorem 3.1 is very disturbing. It shows that the operator $v_1\otimes n_2 + n_1 \otimes v_2^*$ may have completely different analytical properties than n itself. It means that we should not expect to have any formula like (3.5).

Fortunately, we can eliminate this embarassing possibility by adding to the relations (3.3) the following new condition:

$$\mathrm{Sp}\, n \subset \mathbb{C}_{(\mu)}, \tag{3.34}$$

where $\mathbb{C}_{(\mu)} = \{\zeta \in \mathbb{C}\colon\ \zeta = 0$ or $|\zeta| = \mu^k$, where $k \in \mathbb{Z}\}$ and saying that only the representations (v, n) satisfying this additional condition will be considered. We call (3.34) the spectral condition.

C. C^*-Algebra Level

We look for a C^*-algebra A and two elements $v, n \eta A$ such that v is unitary, n is normal, $\mathrm{Sp}\, n \subset \mathbb{C}_{(\mu)}$ and $v^* n v = \mu n$ having the following:

Universality property: For any C^*-algebra A' and any V, $N \eta A'$ such that V is unitary, N is normal, $\mathrm{Sp}\, N \subset \mathbb{C}_{(\mu)}$ and $V^* N V = \mu N$ there exists unique $\phi \in \mathrm{Mor}(A, A')$ such that $\phi(v) = V$ and $\phi(n) = N$.

The (obviously unique up to a C^*-algebra isomorphism) solution of this problem is provided by the crossed product construction [15].

Let $C_\infty(\mathbb{C}_{(\mu)})$ be the algebra of all continuous, vanishing at infinity functions on $\mathbb{C}_{(\mu)}$. There is a natural action of $\mathbb{Z}$ on $C_\infty(\mathbb{C}_{(\mu)})$: For any $k \in \mathbb{Z}$ and $f \in C_\infty(\mathbb{C}_{(\mu)})$ we set

$$(\mu_k f)(\zeta) = f(\mu^{-k}\zeta)$$

for all $\zeta \in \mathbb{C}_{(\mu)}$. Let

$$A = C_\infty(\mathbb{C}_{(\mu)}) \ltimes_\mu \mathbb{Z}$$

be the corresponding crossed product. Then A contains $C_\infty(\mathbb{C}_{(\mu)})$ in a nondegenerate way [i.e. the embedding $C_\infty(\mathbb{C}_{(\mu)}) \hookrightarrow A$ belongs to $\mathrm{Mor}(C_\infty(\mathbb{C}_{(\mu)}), A)$] and there exists a unitary $v \in M(A)$ such that

$$v^* f v = \mu_1 f$$

for any $f \in C_\infty(\mathbb{C}_{(\mu)})$. Let n be the function on $\mathbb{C}_{(\mu)}$ such that $n(\zeta) = \zeta$ for all $\zeta \in \mathbb{C}_{(\mu)}$. Then $n \in C(\mathbb{C}_{(\mu)})$ and making use of the affiliation relation (see Example 2 of Sect. 1) $n \eta C_\infty(\mathbb{C}_{(\mu)})$. Having in mind the nondegenerate inclusion $C_\infty(\mathbb{C}_{(\mu)}) \subset A$ we get $n \eta A$. Obviously, n is normal, $\mathrm{Sp}\, n = \mathbb{C}_{(\mu)}$ and

$$v^* n v = \mu n.$$

Using Theorem 1.6 and the universality of the crossed product one can easily prove the universality properly formulated above.

Now we shall give the meaning to the expression

$$N = v \otimes n + n \otimes v^*.$$

Theorem 3.2. *There exists an element $N \eta A \otimes A$ such that $D(n) \otimes_{\mathrm{alg}} D(n)$ is a core for N and*

$$N y_1 \otimes y_2 = v y_1 \otimes n y_2 + n y_1 \otimes v^* y_2 \tag{3.35}$$

for any $y_1, y_2 \in \mathcal{D}(n)$. N is normal and $\mathrm{Sp}\, N = \mathbb{C}_{(\mu)}$.

Proof. Let

$$a = d = (I + n^*n)^{-1/2} \otimes (I + n^*n)^{-1/2}, \tag{3.36}$$

$$b = v(I + n^*n)^{-1/2} \otimes n(I + n^*n)^{-1/2} + n(I + n^*n)^{-1/2} \otimes v^*(I + n^*n)^{-1/2}, \tag{3.37}$$

$$c = (I + n^*n)^{-1/2} v \otimes n(I + n^*n)^{-1/2} + n(I + n^*n)^{-1/2} \otimes (I + n^*n)^{-1/2} v^*. \tag{3.38}$$

Then $a, b, c, d \in M(A \otimes A)$. We shall use Theorem 2.3 and Proposition 2.6.

Let π be an irreducible representation of $A \otimes A$ acting on a Hilbert space H_π. The algebra A is of type I. Therefore, $H_\pi = H_1 \otimes H_2$ and

$$\pi = \pi_1 \otimes \pi_2, \tag{3.39}$$

where π_1 and π_2 are irreducible representations of A acting on Hilbert spaces H_1 and H_2, respectively. We set

$$v_1 = \pi_1(v), \quad n_1 = \pi_1(n),$$

$$v_2 = \pi_2(v), \quad n_2 = \pi_2(n).$$

Then (v_1, n_2) and (v_2, n_2) are irreducible representations of (3.3) satisfying the spectral condition (3,34). We shall assume that both representations are infinite-dimensional. The reader himself should consider the much simpler case when at least one of the representations is one-dimensional.

Let N_π be the closure of the operator introduced by (3.10). Taking into account definitions (3.36) and (3.37) one can easily check that $\pi(d)H_\pi$ is a core for N_π and

$$N_\pi \pi(d)\varphi = \pi(b)\varphi$$

for any $\varphi \in H_\pi$. According to Theorem 3.1.2° operator N_π is normal. Therefore, any core for N_π is a core for N_π^*. In particular, $\pi(a^*)H_\pi = \pi(d)H_\pi$ is a core for N_π^* and using (3.38) one can verify that

$$N_\pi^* \pi(a^*)\psi = \pi(c^*)\psi$$

for any $\psi \in H_\pi$. This way we showed that the assumptions of Proposition 2.6 are satisfied.

Using now Theorem 2.3 (with A replaced by $A \otimes A$) we see that there exists an element $N \eta A \otimes A$ such that $d(A \otimes A)$ is a core for N and $Ndx = bx$ for any $x \in A \otimes A$. Remembering that $A \otimes_{\mathrm{alg}} A$ is dense in $A \otimes A$, one can easily show that $d(A \otimes_{\mathrm{alg}} A) = (I + n^*n)^{-1/2} A \otimes_{\mathrm{alg}} (I + n^*n)^{-1/2} A = D(n) \otimes_{\mathrm{alg}} D(n)$ is a core for N. Moreover, for any $x_1, x_2 \in A$ we have

$$\begin{aligned} N[(I + n^*n)^{-1/2} x_1 \otimes (I + n^*n)^{-1/2} x_2] &= Nd(x_1 \otimes x_2) = b(x_1 \otimes x_2) \\ = v(I + n^*n)^{-1/2} x_1 \otimes n(I + n^*n)^{-1/2} x_2 & \\ + n(I + n^*n)^{-1/2} x_1 \otimes v^*(I + n^*n)^{-1/2} x_2, & \end{aligned}$$

and setting $(I + n^*n)^{-1/2} x_1 = y_1$ and $(I + n^*n)^{-1/2} x_2 = y_2$ we get (3.35).

We shall use Proposition 1.7 to prove that N is normal and that $\mathrm{Sp}\, N \subset \mathbb{C}_{(\mu)}$. To this end it is sufficient to show that for any irreducible representation π of $A \otimes A$, $\pi(N)$ is normal and $\mathrm{Sp}\, \pi(N) \subset \mathbb{C}_{(\mu)}$. If π is given by (3.39), where both π_1 and π_2 are infinite-dimensional, then these statements follow directly from the formula (2.22) with T and T_π replaced by N and N_π, respectively and Theorem 3.1.2° [cf. (3.11)]. The case when at least one of the representations π_1 and π_2 is one-dimensional, is again left to the reader. Q.E.D.

In what follows, the element N introduced by (3.35) will be denoted by $v\otimes n+n\otimes v^*$. One can easily show that

$$(v\otimes v)^*(v\otimes n+n\otimes v^*)(v\otimes v)=\mu(v\otimes n+n\otimes v^*),$$

and using the Universality property we obtain:

Theorem 3.3 *There exists unique $\phi\in\mathrm{Mor}(A,A\otimes A)$ such that*

$$\phi(v)=v\otimes v,$$
$$\phi(n)=v\otimes n+n\otimes v^*.$$

We also have

Theorem 3.4. *Let ϕ be the morphism introduced in Theorem 3.3. Then the diagram*

$$\begin{array}{ccc} A & \xrightarrow{\phi} & A\otimes A \\ \downarrow & & \downarrow{\scriptstyle \mathrm{id}\otimes\phi} \\ A\otimes A & \xrightarrow[\phi\otimes\mathrm{id}]{} & A\otimes A\otimes A \end{array}$$

is commutative.

Proof. Obviously,

$$(\phi\otimes\mathrm{id})\phi(v)=v\otimes v\otimes v=(\mathrm{id}\otimes\phi)\phi(v).$$

Moreover, denoting by $\tilde{N}$ any of the two elements $(\phi\otimes\mathrm{id})\phi(n)$ and $(\mathrm{id}\otimes\phi)\phi(n)$ affiliated with $A\otimes A\otimes A$, one can easily show that $D(n)\otimes_{\mathrm{alg}}D(n)\otimes_{\mathrm{alg}}D(n)$ is a core for $\tilde{N}$ and

$$\tilde{N}(y_1\otimes y_2\otimes y_3)=vy_1\otimes vy_2\otimes ny_3+vy_1\otimes ny_2\otimes v^*y_3$$
$$+ny_1\otimes v^*y_2\otimes v^*y_3$$

for any $y_1,y_2,y_3\in D(n)$. It shows that $(\phi\otimes\mathrm{id})\phi(n)=(\mathrm{id}\otimes\phi)\phi(n)$. According to the Universality property, any morphism from A into a C^*-algebra is completely determined by its values at v and n. Therefore, $(\phi\otimes\mathrm{id})\phi=(\mathrm{id}\otimes\phi)\phi$. Q.E.D.

It is not our aim to develop here the complete theory of the group G_μ of motions of the Euclidean quantum plane. It will be presented in a separate paper. We wanted only to convince the reader that the affiliation relation plays a crucial role in the theory of non-compact quantum groups.

We end this section with a brief description of the algebra B of all continuous vanishing at infinity functions on the Euclidean quantum plane $\mathbb{C}_\mu$. By definition B is the universal (i.e. obeying the suitable Universality property) C^*-algebra having a distinguished element $\zeta\eta B$ such that

$$D(\zeta)=D(\zeta^*),\tag{3.40}$$

$$(\zeta x)^*(\zeta y)=\mu^2(\zeta^*x)^*(\zeta^*y)\tag{3.41}$$

for any $x,y\in D(\zeta)$ and

$$\mathrm{Sp}(\zeta^*\zeta)\subset\{\mu^{2k}\colon k\in\mathbb{Z}\}\cup\{0\}.\tag{3.42}$$

Following the last remark of part A we shall look for B inside A. Using the Universality property of the algebra A one can easily show that there exists one-parameter group $\{\sigma_t\}_{t\in\mathbb{R}}$ of automorphisms of A such that

$$\sigma_t(v)=e^{-it}v, \qquad \sigma_t(n)=e^{it}n$$

for any $t\in\mathbb{R}$. Let

$$B=\{a\in A\colon \sigma_t(a)=a \text{ for all } t\in\mathbb{R}\},$$
$$\zeta=vn.$$

Then $\zeta\eta B$ and relations (3.40)–(3.42) including the suitable Universality property can be easily verified. Moreover, $\psi=\phi|_B$ belongs to $\mathrm{Mor}(B, A\otimes B)$ and describes the action of G_μ on $\mathbb{C}_\mu$.

4. The Quantum $SU(1,1)$-Group

This group has been introduced and analysed in many papers (see e.g. [9]). The main aim of this section is to show that it does not exist on the C^*-algebra level. In our opinion this fact does not undermine that general philosophy of the topological quantum groups [18] saying that the C^*-algebra language has to be used. Conversely, it shows that there is something essentially wrong with $S_\mu U(1,1)$ for real values of the parameter μ. It is very likely that a non-compact form of $SU(2)$ exists only for $\mu\in S^1$ [in this case $S_\mu U(1,1)$ is a deformation of $SL(2,\mathbb{R})$].

A. Hopf-Algebra Level

Let μ be a fixed real number such that $\mu\neq 0$ and $|\mu|<1$. The $*$-algebra $\mathscr{A}$ of all polynomials on $S_\mu U(1,1)$ is generated by two elements α and γ satisfying the following commutation relations:

$$\left.\begin{aligned} \alpha\gamma&=\mu\gamma\alpha\\ \alpha\gamma^*&=\mu\gamma^*\alpha\\ \gamma\gamma^*&=\gamma^*\gamma\\ \alpha^*\alpha-\gamma^*\gamma&=I\\ \alpha\alpha^*-\mu^2\gamma^*\gamma&=I\,. \end{aligned}\right\} \tag{4.1}$$

We endow $\mathscr{A}$ with the Hopf-algebra structure introducing the comultiplication ϕ, counit e and coinverse κ in the following way:

$$\phi(\alpha)=\alpha\otimes\alpha+\mu\gamma^*\otimes\gamma,$$
$$\phi(\gamma)=\gamma\otimes\alpha+\alpha^*\otimes\gamma,$$
$$e(\alpha)=1, \qquad e(\gamma)=0,$$
$$\kappa(\alpha)=\alpha^*, \qquad \kappa(\alpha^*)=\alpha,$$
$$\kappa(\gamma)=-\mu\gamma, \qquad \kappa(\gamma^*)=-\frac{1}{\mu}\gamma^*.$$

We recall that ϕ and e are *-algebra homomorphisms whereas κ is linear and antimultiplicative. One can easily verify that all axioms of the *-Hopf-algebra theory are satisfied. The simplest way to memorize the above formulae defining ϕ, e, and κ is to remember that

$$u=\begin{pmatrix}\alpha, & \mu\gamma^* \\ \gamma, & \alpha^*\end{pmatrix}$$

is the fundamental two-dimensional representation of $S_\mu U(1,1)$. It means that $(\mathrm{id}\otimes\phi)u=u \oplus u$, $(\mathrm{id}\otimes e)u=I$, and $(\mathrm{id}\otimes\kappa)u=u^{-1}$.

B. Hilbert Space Level

Let α and γ be closed operators acting on a Hilbert space H. We say that the pair (α,γ) is a representation of commutation relations (4.1) if $\mathscr{D}(\alpha)=\mathscr{D}(\alpha^*)=\mathscr{D}(\gamma)=\mathscr{D}(\gamma^*)$ and

$$(\alpha^*\varphi\mid\gamma\psi)=\mu(\gamma^*\varphi\mid\alpha\psi), \tag{4.2}$$

$$(\alpha^*\varphi\mid\gamma^*\psi)=\mu(\gamma\varphi\mid\alpha\psi),$$

$$(\gamma^*\varphi\mid\gamma^*\psi)=(\gamma\varphi\mid\gamma\psi), \tag{4.3}$$

$$(\alpha\varphi\mid\alpha\psi)-(\gamma\varphi\mid\gamma\psi)=(\varphi\mid\psi), \tag{4.4}$$

$$(\alpha^*\varphi\mid\alpha^*\psi)-\mu^2(\gamma\varphi\mid\gamma\psi)=(\varphi\mid\psi), \tag{4.5}$$

for any $\varphi,\psi\in\mathscr{D}(\gamma)$. Relation (4.3) shows that γ is normal. By virtue of (4.4) and (4.5) $\ker\alpha=\ker\alpha^*=\{0\}$. Therefore, in the polar decomposition $\alpha=v|\alpha|$, the first factor is unitary. Clearly, $|\alpha|=(I+\gamma^*\gamma)^{1/2}$. So we have

$$\alpha=v(I+\gamma^*\gamma)^{1/2}, \tag{4.6}$$

where v is a unitary operator acting on H. Next using (4.2) we get

$$v\gamma v^*=\mu\gamma.$$

Using the above two formulae one can easily describe all irreducible representations of (4.1). They are either one- or infinite-dimensional. Any one-dimensional representation is of the form $(cI,0)$, where $c\in\mathbb{C}$, $|c|=1$.

If (α,γ) is an irreducible infinite-dimensional representation of (4.1) then $\mathrm{Sp}\,\gamma=\{0\}\cup\Lambda$, where Λ is of the form

$$\Lambda=\{t_0\mu^k\colon k\in\mathbb{Z}\},$$

and $t_0\in\mathbb{C}-\{0\}$. Moreover, one can find an orthonormal basis $\{e(t)\colon t\in\Lambda\}$ such that

$$\alpha e(t)=\sqrt{1+|t|^2}\,e(\mu^{-1}t),$$

$$\gamma e(t)=te(t)$$

for any $t\in\Lambda$. It means that (α,γ) is uniquely determined (up to a unitary equivalence) by $\mathrm{Sp}\,\gamma$.

We shall prove the following "no go" theorem.

Theorem 4.1. *Let (α_1, γ_2) and (α_2, γ_2) be infinite-dimensional irreducible representations of* (4.1) *acting on Hilbert spaces H_1 and H_2, respectively. Then there exists no representation (α, γ) of* (4.1) *acting on $H_1 \otimes H_2$ such that*

$$\alpha \supset \alpha_1 \otimes \alpha_2 + \mu \gamma_1^* \otimes \gamma_2 , \tag{4.7}$$

$$\gamma \supset \gamma_1 \otimes \alpha_2 + \alpha_1^* \otimes \gamma_2 , \tag{4.8}$$

$$\alpha^* \supset \alpha_1^* \otimes \alpha_2^* + \mu \gamma_1 \otimes \gamma_2^* , \tag{4.9}$$

$$\gamma^* \supset \gamma_1^* \otimes \alpha_2^* + \alpha_1 \otimes \gamma_2^* , \tag{4.10}$$

where by definition the operators on the right-hand side are defined on $\mathscr{D}(\gamma_1) \otimes_{\mathrm{alg}} \mathscr{D}(\gamma_2)$.

We shall use the following two lemmae.

Lemma 4.2. *Let α_0, γ_0, α_0^+, γ_0^+ be four operators acting on a Hilbert space H and having the same dense domain $\mathscr{D}$. Assume that there exists a representation (α, γ) of* (4.1) *acting on H such that $\alpha \supset \alpha_0$, $\gamma \supset \gamma_0$, $\alpha^* \supset \alpha_0^+$, and $\gamma^* \supset \gamma_0^+$. Then*

$$\dim[\ker(\alpha_0^*) \cap \mathscr{D}((\gamma_0^+)^*)] = \dim[\ker((\alpha_0^+)^*) \cap \mathscr{D}(\gamma_0^*)] . \tag{4.11}$$

Proof. Let (α, γ) be the representation of (4.1). Then α^{-1}, $\gamma\alpha^{-1}$, and $\gamma^*\alpha^{-1}$ are bounded operators and one can verify that the matrix

$$v = \begin{pmatrix} \alpha^{-1}, & -\gamma^*\alpha^{-1} \\ \gamma\alpha^{-1}, & \alpha^{-1} \end{pmatrix}$$

is unitary. Moreover, for any $\varphi, \psi \in \mathscr{D}(\gamma)$ we have

$$v \begin{pmatrix} \alpha\varphi + \mu\gamma^*\psi \\ \psi \end{pmatrix} = \begin{pmatrix} \varphi \\ \gamma\varphi + \alpha^*\psi \end{pmatrix} . \tag{4.12}$$

Let

$$\mathscr{D}_- = \left\{ \begin{pmatrix} \alpha_0\varphi + \mu\gamma_0^+\psi \\ \psi \end{pmatrix} : \varphi, \psi \in \mathscr{D} \right\},$$

$$\mathscr{D}_+ = \left\{ \begin{pmatrix} \varphi \\ \gamma_0\varphi + \alpha_0^+\psi \end{pmatrix} : \varphi, \psi \in \mathscr{D} \right\}.$$

According to (4.12) $v\mathscr{D}_- = \mathscr{D}_+$ and remembering that v is unitary we get $v\mathscr{D}_-^\perp = \mathscr{D}_+^\perp$. Therefore,

$$\dim \mathscr{D}_-^\perp = \dim \mathscr{D}_+^\perp .$$

On the other hand, one can easily check that

$$\mathscr{D}_-^\perp = \left\{ \begin{pmatrix} \chi \\ -\mu(\gamma_0^+)^*\chi \end{pmatrix} : \chi \in \ker(\alpha_0^*) \cap \mathscr{D}((\gamma_0^+)^*) \right\},$$

$$\mathscr{D}_+^\perp = \left\{ \begin{pmatrix} -\gamma_0^*\chi \\ \chi \end{pmatrix} : \chi \in \ker((\alpha_0^+)^*) \cap \mathscr{D}(\gamma_0^*) \right\},$$

and (4.11) follows. Q.E.D.

Lemma 4.3. *Let α_0 (γ_0, α_0^+, and γ_0^+, respectively) be the operator standing on the right-hand side of the relation* (4.7) *((4.8), (4.9), and (4.10), respectively). Then*

$$1^\circ \qquad \ker(\alpha_0^*)=\{0\}, \tag{4.13}$$

$$2^\circ \qquad \dim[\ker((\alpha_0^+)^*)\cap\mathscr{D}(\gamma_0^*)]>0.$$

Proof. Ad 1°. Let us notice that

$$\alpha_0=(I+\mu\gamma_1^*\alpha_1^{-1}\otimes\gamma_2\alpha_2^{-1})(\alpha_1\otimes\alpha_2). \tag{4.14}$$

We know [cf. (4.6)] that α_1 and α_2 are invertible. Therefore, $\alpha_1\mathscr{D}(\gamma_1)=H_1$, $\alpha_2\mathscr{D}(\gamma_2)=H_2$, and $(\alpha_1\otimes\alpha_2)(\mathscr{D}(\gamma_1)\otimes_{\text{alg}}\mathscr{D}(\gamma_2))=H_1\otimes_{\text{alg}}H_2$ is dense in $H_1\otimes H_2$. On the other hand, the first factor in (4.14) is a bounded invertible operator ($\|\mu\gamma_1^*\alpha_1^{-1}\otimes\gamma_2\alpha_2^{-1}\|=\mu<1$ and the Neumann series converges!). Therefore, range of α_0 is dense and (4.13) follows.

Ad 2°. Let t_1 and t_2 be non-zero elements of $\operatorname{Sp}\gamma_1$ and $\operatorname{Sp}\gamma_2$, respectively. For any integer s we set

$$\psi_s=(-\mu c)^s\prod_{k=s}^{\infty}\left(1+\frac{\mu^{2k}}{|t_1|^2}\right)^{-1/2}\left(1+\frac{\mu^{2k}}{|t_2|^2}\right)^{-1/2},$$

where

$$c=\frac{t_1|t_2|}{|t_1|t_2}\in S^1.$$

Let us notice that for $s\to+\infty$, ψ_s behaves like $(-\mu c)^s$, whereas for $s\to-\infty$ it tends to zero faster than any natural power of μ^{-s}. In particular, $\{\varphi_s\}_{s\in\mathbb{Z}}$ is square summable. Moreover, for any $m\in\mathbb{Z}$,

$$(1+\mu^{-2m+2}|t_1|^2)^{1/2}(1+\mu^{-2m+2}|t_2|^2)^{1/2}\psi_{m-1}=-\mu^{-2m+1}\bar{t}_1t_2\psi_m.$$

Let

$$\psi=\sum_{s=-\infty}^{+\infty}\psi_se(\mu^{-s}t_1)\otimes e(\mu^{-s}t_2).$$

We shall prove that $\psi\in\ker((\alpha_0^+)^*)$ and $\psi\in\mathscr{D}(\gamma_0^*)$. Indeed, for any integers m,n we have

$$\begin{aligned}
&(\psi\,|\,\alpha_1^*e(\mu^{-n}t_1)\otimes\alpha_2^*e(\mu^{-m}t_2))\\
&=(1+\mu^{-2n+2}|t_1|^2)^{1/2}(1+\mu^{-2m+2}|t_2|^2)^{1/2}\\
&\quad\times(\psi\,|\,e(\mu^{-n+1}t_1)\otimes e(\mu^{-m+1}t_2))\\
&=(1+\mu^{-2m+2}|t_1|^2)^{1/2}(1+\mu^{-2m+2}|t_2|^2)^{1/2}\delta_{mn}\overline{\psi_{m-1}}\\
&=-\mu^{-2m+1}t_1\bar{t}_2\delta_{mn}\bar{\psi}_m\\
&=-\mu^{1-n-m}t_1\bar{t}_2(\psi\,|\,e(\mu^{-n}t_1)\otimes e(\mu^{-m}t_2))\\
&=-\mu(\psi\,|\,\gamma_1e(\mu^{-n}t_1)\otimes\gamma_2^*e(\mu^{-m}t_2)).
\end{aligned}$$

It shows that $(\psi\,|\,\alpha_0^+(e(\mu^{-n}t_1)\otimes e(\mu^{-m}t_2)))=0$ for any $m,n\in\mathbb{Z}$ and $\psi\in\ker((\alpha_0^+)^*)$.

Let

$$\psi'=\sum_{s=-\infty}^{+\infty}\bar{t}_1(\mu^{2s}+\mu^2|t_2|^2)^{-1/2}\psi_se(\mu^{-s}t_1)\otimes e(\mu^{-s+1}t_2).$$

Using the same techniques as above one may check that

$$(\psi \mid \gamma_0(e(\mu^{-n}t_1)\otimes e(\mu^{-m}t_2))) = (\psi' \mid e(\mu^{-n}t_1)\otimes e(\mu^{-m}t_2))$$

for any $m, n \in \mathbb{Z}$. It shows that $\psi \in \mathscr{D}(\gamma_0^*)$ (and $\gamma_0^*\psi = \psi'$). Q.E.D.

Proof of Theorem 4.1. It follows immediately from Lemma 4.2 and Lemma 4.3. Q.E.D.

C. C^*-Algebra Level

Assume for the moment that we have a C^*-algebra A and two distinguished elements α and γ affiliated with it such that for any representation π of A acting on a Hilbert space H, $(\pi(\alpha), \pi(\gamma))$ is a representation of commutation relations (4.1). Using Theorem 4.1 we immediately conclude that there exists no $\phi \in \mathrm{Mor}(A, A\otimes A)$ such that

$$\begin{aligned}\phi(\alpha) &\supset \alpha\otimes\alpha + \mu\gamma^*\otimes\gamma,\\ \phi(\gamma) &\supset \gamma\otimes\alpha + \alpha^*\otimes\gamma,\\ \phi(\alpha^*) &\supset \alpha^*\otimes\alpha^* + \mu\gamma\otimes\gamma^*,\\ \phi(\gamma^*) &\supset \gamma^*\otimes\alpha^* + \alpha\otimes\gamma^*.\end{aligned}$$

Therefore, $S_\mu U(1,1)$ group does not exist on the C^*-algebra level.

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