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$SU(2)$ Chern–Simons Theory at Genus Zero

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Abstract. We present a detailed study of the Schrödinger picture space of states in the $SU(2)$ Chern–Simons topological gauge theory in the simplest geometry. The space coincides with that of the solutions of the chiral Ward identities for the WZW model. We prove that its dimension is given by E. Verlinde's formulae.

1. Introduction

The most characteristic feature of two-dimensional conformal field theories is the separation of left-moving and right-moving degrees of freedom. In the euclidean world, this translates to the factorization of Green functions into sums of products of holomorphic and antiholomorphic expressions: the conformal blocks. In the simplest case of rational conformal theories the sums are finite. Conformal blocks are multivalued and may be naturally viewed as sections of finite-dimensional holomorphic vector bundles over the moduli spaces of punctured Riemann surfaces. The way they are put together to form Green functions is determined by a hermitian metric on the bundles. This point of view was advocated by Friedan and Shenker [3] who formulated conformal field theories in terms of modular analytic geometry. A lot of effort has been invested in analysis of the general structure of Friedan–Shenker bundles, especially in translating the information they encode into an algebraic language [17, 16, 13, 14]. In particular, E. Verlinde's work, based on the expected factorization properties of the bundles and their modular properties, has allowed to come up with a formula for their dimensions, see also [14].

Among rational conformal field theories, a special role is played by Wess–Zumino–Witten (WZW) sigma model with fields taking values in a compact group G [18, 10, 6]. They generate through the so-called "coset construction" [7], a rich, possibly exhaustive, family of rational theories. For the WZW models, the Friedan–Shenker bundles are composed of solutions of current-algebra Ward identities. In [19], Witten has observed that they may also be viewed as bundles of Schrödinger-picture quantum states for a three-dimensional non-abelian gauge theory with action given by an integral of the Chern–Simons form. The insertion points of the

Green function correspond in the three-dimensional theory to Wilson lines. Both pictures lead to a mathematical description of fibers of the Friedan–Shenker bundles as composed of holomorphic sections of a power of the determinant bundle over the moduli spaces of stable holomorphic $G^{\mathbf{C}}$-bundles [1] with parabolic structure at the insertion points [12].

In the present paper, we give a detailed rigorous analysis of spaces of Chern–Simons theory quantum states in the genus zero case for $G = SU(2)$. Instead however of employing algebraic geometry of moduli spaces of parabolic bundles, we shall start in an infinite-dimensional setup which will be reduced to finite-dimensional analysis differently. The reduction realizes the Friedan–Shenker bundles as subbundles of a trivial bundle with a space of $SU(2)$-invariant tensors as the fibre. We show that they are invariant under a holomorphic connection in the trivial bundle, introduced by Knizhnik and Zamolodchikov [10]. We study the behavior of the subbundles when two punctures in the Riemann sphere coincide or, equivalently, when the sphere is pinched leaving two insertion points in one component. Using the Knizhnik–Zamolodchikov connection, we prove factorization of the subbundles in the special case. This allows us to show by a simple inductive argument that the dimensions of the subbundles are given by Verlinde's formulae.

The present paper is a preparation to a detailed study of the hermitian metric on the Friedan–Shenker bundles which, in the Chern–Simons picture, corresponds to the scalar product of the gauge-theory states. As argued in [4], this scalar product is given by a finite-dimensional (Coulomb-gas) integral representation. In the simplest cases, the integral converges producing exact solutions for the WZW model Green functions at genus zero. We have conjectured in [4] that this is generally the case, i.e. that every Chern–Simons theory state is normalizable. The dimensional count proven in the present paper provides a strong argument in support of that conjecture: this is for the normalizable states that we would expect Verlinde's formulae to apply.

An alternative way to construct the genus zero Friedan–Shenker bundles for the WZW model is from invariant tensors of the quantum deformation G_q of group G for a suitable root of unity q [11]. The hermitian metric on the bundle plays an important role also in this construction. We plan to discuss the relation of the two approaches in a future publication.

2. Chern–Simons States on the Riemann Sphere

The basic object of our study will be the space of quantum states of the Chern–Simons topological gauge theory with the action

$$k/(4\pi) \int_{\mathscr{M}} \operatorname{tr}(AdA + \tfrac{2}{3}A^3),$$

where $A = -A^*$ is a 1-form with values in the Lie algebra $sl(2, \mathbf{C})$ on a three-dimensional manifold $\mathscr{M}$ (a connection on the trivial bundle over $\mathscr{M}$). We shall take $\mathscr{M}$ to be $\Sigma \times \mathbf{R}$, where Σ is a compact Riemann surface without boundary. Coupling constant k, called the level, is a positive integer. The coherent state quantization of the theory in the presence of Wilson lines $\{\xi_n\} \times \mathbf{R}$, $n = 1, \ldots, N$, in representations of spin j_n, leads to the following description of the Schrödinger picture states ψ of the theory [19, 5, 2]:

ψ is a holomorphic functional of $sl(2,\mathbf{C})$-valued (0, 1)-forms $A^{01} = A_{\bar{z}}d\bar{z}$ (the $d\bar{z}$ part of the two-dimensional gauge fields) on Σ. $\psi(A^{01}) \in \bigotimes_{n=1}^{N} V_{j_n}$, where V_j denotes the space of spin j representation of $SU(2)$ (and $SL(2,\mathbf{C})$). Besides ψ has to satisfy the following differential constraint:

$$(F^a(\xi) - \sum_{n=1}^{N} 2\pi/k\, \delta(\xi - \xi_n)t^a_{j_n})\,\psi(A^{01}) = 0, \tag{1}$$

where $t^a = 1/2\,\sigma^a$ are the standard generators of $su(2)$, the subscript j_n indicating that they act on the n-th factor in $\bigotimes_n V_{j_n}$;

$$\sum_a t^a F^a \equiv F = \partial_z A_{\bar{z}} + \partial_{\bar{z}}(\pi/k)\delta/(\delta A_{\bar{z}}) - [(\pi/k)\delta/(\delta A_{\bar{z}}), A_{\bar{z}}]$$

stands for the quantized version of the curvature with A_z replaced by $(-\pi/k)\delta/(\delta A_{\bar{z}})$ and the convention

$$\delta\psi(A^{01}) = \int \mathrm{tr}\,(\delta/(\delta A_{\bar{z}})\psi(A^{01}))\delta A_{\bar{z}} d^2 z,$$

where $d^2z = i/2 dz d\bar{z}$. To be completely precise, by a holomorphic function of A^{01} we mean a C^∞-map (in the Frechét sense [8]) with complex-linear derivatives on the space $\mathscr{A}^{01}$ of smooth forms A^{01} with the C^∞ topology.

We shall denote the space of holomorphic maps

$$\psi: \mathscr{A}^{01} \to \bigotimes_n V_{j_n}$$

satisfying condition (1) by $\mathscr{W}_{j_1 \cdots j_N}(\Sigma, (\xi_n))$. It will be more convenient to rewrite Eq. (1) in a global rather than in the infinitesimal form. Let us denote by $\mathscr{G}^{\mathbf{C}}$ the space of smooth maps $h: \Sigma \to SL(2,\mathbf{C})$ (chiral gauge transformations) acting on $\mathscr{A}^{01}$ by

$$A^{01} \mapsto {}^h A^{01} \equiv h A^{01} h^{-1} + h\bar{\partial}h^{-1}.$$

Equation (1) is equivalent to the following condition describing the behavior of ψ along the orbits of $\mathscr{G}^{\mathbf{C}}$

$$\psi({}^h A^{01}) = \exp[kS(h^{-1}, A^{01})] \bigotimes_n h(\xi_n)_{j_n} \psi(A^{01}), \tag{2}$$

where on the right-hand side $h(\xi_n)_{j_n}$ act on the n-th factor in the space $\bigotimes_n V_{j_n}$, where $\psi(A^{01})$ takes values. $S(h, A^{01})$ denotes the action functional of the two-dimensional Wess–Zumino–Witten (WZW) field theory model coupled to the right-handed gauge field, given up to the multiples of $2\pi i$ by

$$S(h, A^{01}) = -i/(4\pi) \int_\Sigma \mathrm{tr}\,(h^{-1}\partial h)(h^{-1}\bar{\partial}h) - i/(12\pi) \int_{\mathscr{B}} \mathrm{tr}\,(\tilde{h}^{-1} d\tilde{h})^3$$
$$+ i/(2\pi) \int_\Sigma \mathrm{tr}\,(h\partial h^{-1}) A^{01}, \tag{3}$$

where $\tilde{h}: \mathscr{B} \to SL(2,\mathbf{C})$ is an extension of h to a three-dimensional manifold $\mathscr{B}$ with $\partial\mathscr{B} = \Sigma$.

Action (3) may be shown to satisfy

$$S(hh', {}^hA^{01}) = S(h', A^{01}) - S(h^{-1}, A^{01}). \tag{4}$$

If one defines formally the Green functions $\Gamma_{j_1\cdots j_N}(\Sigma, (\xi_n), A^{01})$ of the WZW model by the functional integral

$$\int \bigotimes_n g(\xi_n)_{j_n} \exp\left[-kS(g, A^{01})\right] \prod_{\xi\in\Sigma} dg(\xi)$$

over $SU(2)$ valued fields g then Eq. (4) implies formally the chiral Ward identity

$$\Gamma_{j_1\cdots j_N}(\Sigma, (\xi_n), {}^hA^{01}) = \exp\left[kS(h^{-1}, A^{01})\right] \bigotimes_n h(\xi_n)_{j_n} \Gamma_{j_1\cdots j_N}(\Sigma, (\xi_n), A^{01}).$$

Consequently, the space of states satisfying (2) may be viewed as the space of solutions of the chiral Ward identity in the WZW model.

Equation (2) may be regarded as a defining relation for sections of a vector bundle with fiber $\bigotimes_n V_{j_n}$ over the space of orbits $\mathscr{A}^{01}/\mathscr{G}^{\mathbf{C}}$ which is effectively finite-dimensional. In the simplest case when $\Sigma = \mathbf{C}P^1$, the orbit of $A^{01} = 0$ forms an open dense subset of $\mathscr{A}^{01}$ [1] and as a consequence, $\psi(0)$ determines ψ completely. Notice that taking in (2) $A^{01} = 0$ and $h = \text{const.}$, we infer that

$$\psi(0) \in \mathrm{Inv}\left(\bigotimes_n V_{j_n}\right),$$

i.e. the subspace of tensors invariant under the diagonal action of $SL(2, \mathbf{C})$. Putting it differently, the map

$$\mathscr{W}_{j_1\cdots j_N}(\mathbf{C}P^1, (\xi_n)) \ni \psi \mapsto \psi(0) \in \mathrm{Inv}\left(\bigotimes_n V_{j_n}\right) \tag{5}$$

is 1 to 1. We would like to describe its image $W_{j_1\cdots j_N}(\mathbf{C}P^1, (\xi_n)) \equiv W((\xi_n))$.

Given $\psi_0 \in \mathrm{Inv}\left(\bigotimes_n V_{j_n}\right)$, it determines via (2) a functional ψ on the $\mathscr{G}^{\mathbf{C}}$-orbit of $A^{01} = 0$:

$$\psi(h^{-1}\bar{\partial}h) = \exp\left[kS(h, 0)\right] \bigotimes_n h(\xi_n)_{j_n}^{-1} \psi_0. \tag{6}$$

$\psi(0) = \psi_0$ and ψ is holomorphic on its domain since the map

$$\mathscr{G}^{\mathbf{C}}_{\xi_0} = \{h \in \mathscr{G}^{\mathbf{C}} \,|\, h(\xi_0) = 1\} \ni h \mapsto h^{-1}\bar{\partial}h \in \mathscr{A}^{01}$$

is a holomorphic diffeomorphism onto the open dense orbit $\mathscr{G}^{\mathbf{C}}\cdot 0$. Functional ψ does not have, however, to extend smoothly to the whole $\mathscr{A}^{01}$. To understand this in detail, we need some information, which may be extracted from [1], about the "geography" of the $\mathscr{G}^{\mathbf{C}}$-orbits in $\mathscr{A}^{01}$.

Given $A^{01} \in \mathscr{A}^{01}$, we may write locally over a sufficiently small covering $(\mathscr{U}_\alpha)$ of Σ

$$A^{01}|_{\mathscr{U}_\alpha} = h_\alpha^{-1}\bar{\partial}h_\alpha$$

for $h_\alpha : \mathscr{U}_\alpha \to SL(2, \mathbf{C})$. If $\Sigma = \mathbf{C}\mathbf{P}^1 \equiv \mathbf{C} \cup \{\infty\}$, $\mathscr{U}_\alpha$'s may be taken as two discs having an annulus centered around the equator $\{z \,|\, |z| = 1\}$ as their intersection. $(g_{\alpha\beta} = h_\alpha h_\beta^{-1})$ form a 1-cocycle of holomorphic transition functions of an $SL(2, \mathbf{C})$ holomorphic bundle over Σ. The space or orbits $\mathscr{A}/\mathscr{G}^{\mathbf{C}}$ coincides with the space of isomorphism classes of $SL(2,\mathbf{C})$ bundles. In particular, the orbit of $A^{01}=0$ corresponds to the tirivial bundle. By the Birkhoff theorem [15], on CP^1 each $SL(2,\mathbf{C})$ bundle is a direct sum $L\oplus L^{-1}$, where L is a holomorphic line bundle characterized by its Chern number n (L^{-1} is the dual bundle). In other words, the transition functions of an $SL(2,\mathbf{C})$ holomorphic bundle on $\mathbf{C}P^1$ around the equator can always be written in the form

$$\begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}.$$

Orbits in $\mathscr{A}^{01}/\mathscr{G}^{\mathbf{C}}$ are just enumerated by $|n|=0,1,\dots$. Let us denote them by $\mathcal{O}_{|n|}$. In particular $\mathcal{O}_0$ is the open dense orbit of $A^{01}=0$. Orbits $\mathcal{O}_n$ stratify $\mathscr{A}^{01}$ in the following sense:

$\bigcup_{|n|\leqq n_0} \mathcal{O}_{|n|}$ is an open subset of $\mathscr{A}^{01}$ and $\mathcal{O}_{n_0}$ is its closed complex submanifold of codimension $2n_0-1$.

If ψ is a holomorphic functional on $\bigcup_{|n|\leqq n_0} \mathcal{O}_{|n|}$ for $n_0\geqq 1$ then, by the Hartogs theorem, it uniquely extends to a holomorphic functional on $\bigcup_{|n|\leqq n_0+1} \mathcal{O}_{|n|}$ differing from $\bigcup_{|n|\leqq n_0} \mathcal{O}_{|n|}$ by a complex submanifold of codimension >1. Consequently, ψ extends then to a holomorphic functional on the whole $\mathscr{A}^{01}$. We have only to verify whether ψ given by (6) may be extended to $\mathcal{O}_0\cup\mathcal{O}_1$.

Consider a 1-parameter analytic family of smooth forms (A_t^{01}) given by

$$A_t^{01}=\begin{cases} 0 & \text{for } |z|\leqq 1 \\ \begin{pmatrix} 1 & 0 \\ -tz^{-1} & 1 \end{pmatrix} g_0^{-1}\bar{\partial} g_0 \begin{pmatrix} 1 & 0 \\ tz^{-1} & 1 \end{pmatrix} & \text{for } |1/z|\leqq 1 \end{cases},$$

where g_0 is a smooth $SL(2,\mathbf{C})$-valued map,

$$g_0=\begin{pmatrix} z^{-1} & 0 \\ 0 & z \end{pmatrix}$$

around $|z|=1$. A_0^{01} is clearly in $\mathcal{O}_1$. On the other hand, if $t\neq 0$ then

$$A_t^{01}=h_t^{-1}\bar{\partial} ht,$$

where

$$h_t=\begin{cases} \begin{pmatrix} 1 & t^{-1}z \\ 0 & 1 \end{pmatrix} & \text{for } |z|\leqq 1 \\ \begin{pmatrix} 0 & t^{-1} \\ -t & z^{-1} \end{pmatrix} g_0 \begin{pmatrix} 1 & 0 \\ tz^{-1} & 1 \end{pmatrix} & \text{for } |1/z|\leqq 1 \end{cases}, \tag{7}$$

are smooth $SL(2,\mathbf{C})$-valued maps on $\mathbf{C}P^1$. Thus for $t\neq 0$ $A_t^{01}\in\mathcal{O}_0$. Complex curve $t\mapsto A_t^{01}$ crosses $\mathcal{O}_1$ transversally. Indeed. The subspace tangent to $\mathcal{O}_1$ at A_0^{01} is

$\{\bar{\partial}\Lambda + [A_0^{01}, \Lambda] | \Lambda : \mathbf{C}P^1 \to sl(2, \mathbf{C})\}$. Let

$$a^{01} = \begin{cases} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} d\bar{z} & \text{for} \quad |z| \leqq 1 \\ g_0^* \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} g_0^{*-1} d(1/\bar{z}) & \text{for} \quad |1/z| \leqq 1 \end{cases}$$

Since

$$\int \operatorname{tr}(a^{01})^* (\bar{\partial}\Lambda + [A_0^{01}, \Lambda]) \equiv 0,$$

a^{01} spans the subspace orthogonal to $\mathcal{O}_1$ at A_0^{01}. On the other hand,

$$\begin{aligned} \int \operatorname{tr}(a^{01})^* \partial_t A_0^{01} &= \int_{|1/z| \leqq 1} \operatorname{tr} g_0^{-1} \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} g_0 d(1/z) \left[g_0^{-1} \bar{\partial} g_0, \begin{pmatrix} 1 & 0 \\ z^{-1} & 1 \end{pmatrix} \right] \\ &= - \int_{|1/z| \leqq 1} d \operatorname{tr} \begin{pmatrix} 1 & 0 \\ z^{-1} & 1 \end{pmatrix} g_0^{-1} \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} g_0 d(1/z) \\ &= \oint_{|z|=1} \operatorname{tr} \begin{pmatrix} 1 & 0 \\ z^{-1} & 1 \end{pmatrix} g_0^{-1} \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} g_0 d(1/z) = 2\pi i \end{aligned}$$

which shows the transversality of the intersection. Locally around A_0^{01}, $\mathscr{A}^{01}$ may be parametrized by ${}^h A_t^{01}$ for h from some codimension four submanifold of $\mathscr{G}^{\mathrm{C}}$ and this parameterization may be carried to other points of $\mathcal{O}_1$ by chiral gauge transformations. Hence ψ extends to a holomorphic functional on $\mathcal{O}_0 \cup \mathcal{O}_1$ if and only if the map

$$t \mapsto \psi(A_t^{01})$$

is holomorphic at zero.

Let us examine this condition more closely. It will be convenient to describe spin j representation of $SL(2, \mathbf{C})$ as acting on polynomials P of degree $\leqq 2j$ in complex variable v by fractional transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}_j^{-1} P(v) = (cv + d)^{2j} P((av + b)/(cv + d)).$$

The scalar product rendering the action of $SU(2)$ unitary is

$$\| P \|^2 = \int |P(v)|^2 (1 + |z|^2)^{-2j-2} d^2 z.$$

In this realization, $\operatorname{Inv}\left(\bigotimes_n V_{j_n} \right)$ becomes the space of polynomials P in variables v_n, of degree $\leqq 2j_n$ in v_n, such that

$$P((v_n)) \equiv \prod_n (cv_n + d)^{2j_n} P(((av_n + b)/(cv_n + d))). \tag{8}$$

In particular, P is translation invariant and homogeneous of degree $\sum j_n$. Let us assume that all points ξ_n lie in the disc $|z| \leqq 1$. By Eqs. (6) (7), with polynomial P representing ψ_0,

$$\psi(A_t^{01}) = \exp[kS(h_t, 0)] P((v_n + t^{-1}\xi_n)) = t^{-\sum j_n} \exp[kS(h_t, 0)] P((\xi_n + tv_n)). \tag{9}$$

A direct computation using the formula

$$\partial_t S(h,0) = i/(2\pi) \int \mathrm{tr}\,(h^{-1}\partial_t h)\partial(h^{-1}\bar{\partial}h)$$

shows that $\exp[kS(h_t,0)] \propto \text{const.}\ t^k$. As a result, we infer that expression (8) is analytic at $t=0$ if and only if $P((\xi_n + tv_n))$ vanishes to order $\sum j_n - k - 1$ in t.

Recall that $W((\xi_n))$ denoted the space of the initial values $\psi(0)$ of states ψ. Clearly, $W((\xi_n))$ has to be constant along the orbit of the Möbius group (of holomorphic transformations of $\mathbf{C}P^1$) in the space X_N of sequences (ξ_n), $n=1,\dots,N$, of non-coincident points in $\mathbf{C}P^1$. It is easy to see that, due to invariance (8), the condition that $P((\xi_n + tv_n))$ vanishes to some order in t is invariant under the Möbius transformations of (ξ_n) inside the subspace X_N^0 composed of sequences with all ξ_n's different from ∞. Since any such sequence may be brought by a Möbius transformation to one with $|\xi_n| \leqq 1$, we obtain

Proposition 1. *For* $(\xi_n)\in X_N^0$, $W((\xi_n)) = \left\{ P\in \mathrm{Inv}\left(\bigotimes_n V_{j_n}\right) \middle| \partial_{v_1}^{l_1}\cdots\partial_{v_N}^{l_N} P((\xi_n)) = 0 \text{ for } 0 \leqq \sum l_n \leqq \sum j_n - k - 1 \right\}$.

Now notice that for $(\xi_n)\in X_N\backslash X_N^0$ with $\xi_m = \infty$ and $\xi_n \neq 0$ for $n\neq m$,

$$\begin{aligned}
&\partial_{v_1}^{l_1}\cdots\partial_{v_N}^{l_N} P((v_n))|_{v_n=\xi_n,\ n\neq m,\ v_m=0} \\
&= \partial_{v_1}^{l_1}\cdots\partial_{v_N}^{l_N} \prod_n v_n^{2j_n} P((v_n^{-1}))|_{v_n=\xi_n,\ n\neq m,\ v_m=0} \\
&= l_m!/(2j_m - l_m)!\, \partial_{v_1}^{l_1}\cdots\partial_{v_{m-1}}^{l_{m-1}}\partial_{v_m}^{2j_m-l_m}\partial_{v_{m+1}}^{l_{m+1}}\cdots\partial_{v_N}^{l_N} \\
&\quad\cdot \prod_{n\neq m} v_n^{2j_n} P((v_1^{-1},\dots,v_{m-1}^{-1},v_m,v_{m+1}^{-1},\dots,v_N^{-1}))|_{v_n=\xi_n,\ n\neq m,\ v_m=0}.
\end{aligned} \tag{10}$$

Equation (10) implies that vanishing of $\partial_{v_1}^{l_1}\cdots\partial_{v_N}^{l_N}P$ at $v_n=\xi_n$, $n\neq m$, $v_m=0$ for $\sum l_n + 2(j_m - l_m) \leqq \sum j_n - k - 1$ and at $v_n = \xi_n^{-1}$, $n\neq m$, $v_m=0$ for $\sum l_n \leqq \sum j_n - k - 1$ are equivalent conditions. Since $W((\xi_n)) = W((\xi_n^{-1}))$, we may use the first condition to describe $W((\xi_n))$. Notice that since

$$\partial_{v_1}^{l_1}\cdots\partial_{v_N}^{l_N}P|_{v_n=\xi_n,\ n\neq m,\ v_m=0} = e^{b\partial_{v_m}}\partial_{v_1}^{l_1}\cdots\partial_{v_N}^{l_N}P|_{v_n=\xi_n+b,\ n\neq m,\ v_m=0},$$

vanishing of the left-hand side for $\sum l_n + 2(j_m - l_m) \leqq \sum j_n - k - 1$ is a condition translationally invariant in $(\xi_n)_{n\neq m}$ and our assumption that $\xi_n \neq 0$ for $n\neq m$ is nonessential.

Proposition 1$^\infty$. *For* $(\xi_n)\in X_N\backslash X_N^0$ *with* $\xi_m=\infty$, $W((\xi_n)) = \left\{ P\in\mathrm{Inv}\left(\bigotimes_n V_{j_n}\right) \middle| \partial_{v_1}^{l_1}\cdots \partial_{v_N}^{l_N} P((\xi_1,\dots,\xi_{m-1},0,\xi_{m+1},\dots,\xi_N)) = 0 \text{ for } 0 \leqq \sum l_n + 2(j_m - l_m) \leqq \sum j_n - k - 1 \right\}$.

Suppose that one of the spins, say $j_m > k/2$. We may assume that $\xi_m = \infty$. Since, due to the homogenuity of invariant polynomials P, $\partial_{l_1}\cdots\partial_{l_N}P$ vanishes if $\sum_n l_n \geqq \sum_n j_n$, Proposition 1$^\infty$ implies that

$$\partial_{l_1}\cdots\partial_{l_N} P((\xi_1,\dots,\xi_{m-1},0,\xi_{m+1},\dots,\xi_N)) = 0 \tag{11}$$

for $l_m = 2j_m$ and any l_n, $n\neq m$. But, due to the translational invariance of P,

$$\sum_n \partial_{v_n} P((v_n)) = 0. \tag{12}$$

It follows then by induction on l_m that (11) holds for any l_n's and P has to vanish. This proves

Corollary 1. *Spaces $W((\xi_n))$ are non-zero only if all spins $j_n \leqq k/2$.*

Spins satisfying condition $j \leqq k/2$ are called integrable (they correspond to the highest weight representations of the affine Kac–Moody algebra which integrate to projective representations of the loop group). As we see, only such spins may appear on Wilson lines in the Chern–Simons theory or in Green functions of the WZW model [15, 6].

Let us consider some examples of spaces $W((\xi_n))$.

Example 1. Two-Point Case

$$\mathrm{Inv}(V_{j_1}\otimes V_{j_2}) = \begin{cases} \mathbf{C}(v_1-v_2)^{2j_1} & \text{if } j_1 = j_2, \\ \{0\} & \text{otherwise.}\end{cases}$$

$$W_{j_1 j_2}(\mathbf{C}P^1, (\xi_1,\xi_2)) = \begin{cases} \mathrm{Inv}(V_{j_1}\otimes V_{j_2}) & \text{if } j_1 = j_2 \leqq k/2, \\ \{0\} & \text{otherwise.}\end{cases}$$

Example 2. Three-Point Case

$$\mathrm{Inv}(V_{j_1}\otimes V_{j_2}\otimes V_{j_3}) = \mathbf{C}P_{CG}^{j_1 j_2 j_3}$$

if $j_{12} \equiv j_1 + j_2 - j_3$, $j_{13} \equiv j_1 + j_3 - j_2$ and $j_{23} \equiv j_2 + j_3 - j_1$ are non-negative integers and is equal to zero otherwise. The Clebsch–Gordan polynomial

$$P_{CG}^{j_1 j_2 j_3}(v_1, v_2, v_3) = (v_1 - v_2)^{j_{12}}(v_1 - v_3)^{j_{13}}(v_2 - v_3)^{j_{23}}. \tag{13}$$

$$W_{j_1 j_2 j_3}(\mathbf{C}P^1, (\xi_1,\xi_2,\xi_3)) = \begin{cases} \mathrm{Inv}(V_{j_1}\otimes V_{j_2}\otimes V_{j_3}) & \text{if } j_1 + j_2 + j_3 \leqq k \\ \{0\} & \text{otherwise.}\end{cases}$$

This reproduces the standard $SU(2)$ fusion rules [6]. From the point of view of the Chern–Simons theory, the fusion rules were also studied in [2].

For two or three insertion points, due to the Möbius invariance, spaces $W((\xi_n))$ do not depend on (ξ_n). This is no more the case for four or more insertion points forming a continuum of orbits under the Möbius group. For the sake of illustration, let us consider

Example 3. Four-Point Spin 1/2 Case

$$\mathrm{Inv}(V_{1/2}^{\otimes 4}) = \mathbf{C}(v_1 - v_2)(v_3 - v_4) + \mathbf{C}(v_1 - v_3)(v_2 - v_4).$$

Let us write any invariant tensor as

$$\alpha(v_1 - v_2)(v_3 - v_4) + \beta(2(v_1 - v_3)(v_2 - v_4) - (v_1 - v_2)(v_3 - v_4))$$

(which are spin 0 and spin 1 contributions in the intermediate channel of the tensor product decomposition). For level $k > 1$ there are no restrictions on the invariant tensors. For $k = 1$, $W((\xi_1, \ldots, \xi_4))$ is the subspace of $\mathrm{Inv}(V_{1/2}^{\otimes 4})$ given by the equation

$$\xi\alpha + (2 - \xi)\beta = 0,$$

where $\xi \equiv (\xi_1 - \xi_2)(\xi_3 - \xi_4)(\xi_1 - \xi_3)^{-1}(\xi_2 - \xi_4)^{-1}$ is the anharmonic ratio of four points.

3. Knizhnik–Zamolodchikov Connection

We would like to compare spaces $W((\xi_n))$ for different (ξ_n)'s. A convenient tool for this is the flat [11] Knizhnik–Zamolodchikov (KZ) connection in the trivial bundle $X_N^0 \times \mathrm{Inv}\left(\bigotimes_n V_{j_n}\right)$ over X_N^0 (recall that X_N^0 is the space of sequence (ξ_n) of non-coincident points in $\mathbf{C}$). It is given by [10]

$$\nabla_{\bar{\xi}_n} = \partial_{\bar{\xi}_n}, \tag{14}$$

$$\nabla_{\xi_n} = \partial_{\xi_n} - 2/(k+2) \sum_{m \neq n} (\xi_n - \xi_m)^{-1} \sum_{a=1}^{3} t^a_{j_n} t^a_{j_m}, \tag{15}$$

where $t^a_{j_n}$ are the $SU(2)$ generators acting on the n-th component of $\bigotimes V_{j_n}$. In the polynomial representation,

$$\begin{aligned} t^1_j &= 1/2\,(v^2-1)\partial_v - jv, \\ t^2_j &= i/2\,(v^2+1)\partial_v - ijv, \\ t^3_j &= -v\partial_v + j: \end{aligned} \tag{16}$$

Proposition 2. *Parallel transport in the KZ connection preserves spaces $W((\xi_n))$.*

Proof. Let for $L = (l_1, \ldots, l_N)$, $0 \leqq l_n \leqq 2j_n$, $\sum l_n \leqq \sum j_n - k - 1$, $D^L = \partial^{l_1}_{v_1} \cdots \partial^{l_N}_{v_N}$ and $P \in \mathrm{Inv}(\otimes V_{j_n})$,

$$R^L = D^L P((\xi_n)). \tag{17}$$

The map $((\xi_n), P) \mapsto ((\xi_n), (R^L))$ is a homomorphism J of trivial vector bundles $X_N^0 \times \mathrm{Inv}(\otimes V_{j_n})$ and $X_N^0 \times \mathbf{C}^{\# L}$. We shall find a connection (which is not unique) on the second bundle related by J to the KZ connection, i.e. such that

$$\nabla J P = J \nabla P. \tag{18}$$

We may put

$$\nabla_{\bar{\xi}_n} R^L = \partial_{\bar{\xi}_n} R^L. \tag{19}$$

In order to find the (1,0)-part of the connection, let us compute

$$\begin{aligned} J\nabla_{\xi_n} P &= D^L\Big(\partial_{\xi_n} P - 2/(k+2) \sum_{m \neq n} (\xi_n - \xi_m)^{-1} \sum_a t^a_{j_n} t^a_{j_m} P\Big)((v_r))|_{v_r = \xi_r} \\ &= \partial_{\xi_n}(D^L P((\xi_r))) - \partial_{v_n} D^L P((\xi_r)) - 2/(k+2) \sum_{m \neq n} (\xi_n - \xi_m)^{-1} D^L \\ &\quad \cdot (1/2\,(v_n - v_m)^2 \partial_{v_n}\partial_{v_m} + j_m(v_m - v_n)\partial_{v_n} + j_n(v_n - v_m)\partial_{v_m} + j_n j_m) P((v_r))|_{v_r \equiv \xi_r}, \end{aligned} \tag{20}$$

where we have used Eqs. (16). On the right-hand side of Eq. (20), we shall commute D^L with $\sum_a t^a_{j_n} t^a_{j_m}$ and shall show that the terms with more than $\sum j_n - k - 1$ v-derivatives cancel. Let us denote by L^r (L_r) multi-index L with l_r increased (lowered) by one (if $l_r = 0$ then terms with L_r should be omitted). We have then

$$\begin{aligned} J\nabla_{\xi_n} P = \partial_{\xi_n}(D^L P((\xi_r))) - \Big\{ D^{L^n} + 2/(k+2) \sum_{m \neq n} \Big[1/2\,(\xi_m - \xi_n) D^{L^{mn}} \\ + (l_m - j_m) D^{L^n} - (l_n - j_n) D^{L^m} + (\xi_m - \xi_n)^{-1} (1/2\, l_m(l_m - 1) D^{L^n_m} - l_m l_n D^L \\ + 1/2\, l_n(l_n - 1) D^{L^m_n} + j_m(l_n D^L - l_m D^{L^n_m}) + j_n(l_m D^L - l_n D^{L^m_n}) - j_m j_n D^L) \Big] \Big\} P((\xi_r)). \end{aligned} \tag{21}$$

By virtue of the translation invariance (12) of P,

$$\sum_{m\neq n} 1/2\,(\xi_m-\xi_n)D^{L^{mn}}P((\xi_r)) = \sum_{m\neq n} 1/2\,\xi_m D^{L^{mn}}P((\xi_n)) + 1/2\,\xi_n D^{L^{nn}}P((\xi_r))$$

$$= \sum_{m\neq n} 1/2\,D^{L^n} v_m \partial_{v_m} P((v_r))_{v_r\equiv\xi_r} - \sum_{m\neq n} 1/2\,l_m D^{L^n}P((\xi_r)) + 1/2\,\xi_n D^{L^{nn}}P((\xi_r)).$$

Invariance (8) implies also that

$$\sum_m \xi_m \partial_{v_m} P((\xi_r)) = \sum_m j_m P((\xi_r)).$$

Consequently,

$$\sum_{m\neq n} 1/2\,(\xi_m-\xi_n)D^{L^{mn}}P((\xi_r)) = -1/2\,D^{L^n} v_n \partial_{v_n} P((v_r))|_{v_r\equiv\xi_r} + 1/2\sum_m j_m D^{L^n}P((\xi_r))$$

$$- \sum_{m\neq n} 1/2\,l_m D^{L^n}P((\xi_r)) + 1/2\,\xi_n D^{L^{nn}}P((\xi_r))$$

$$= -1/2\left(1+\sum_m l_m - \sum_m j_m\right)D^{L^n}P((\xi_r)). \tag{22}$$

Similarly,

$$\sum_{m\neq n} (l_m D^{L^n} - l_n D^{L^m})P((\xi_r)) = \sum_m l_m D^{L^n}P((\xi_r)) \tag{23}$$

and

$$\sum_{m\neq n} (j_n D^{L^m} - j_m D^{L^n})P((\xi_r)) = -\sum_m j_m D^{L^n}P((\xi_r)). \tag{24}$$

Substituting Eqs. (22)–(24) to (21), we obtain

$$J\nabla_{\xi_n}P = \partial_{\zeta_n}(D^L P((\xi_r))) - (k+2)^{-1}\Bigg\{\left(\sum_m l_m - \sum_m j_m + k + 1\right)D^{L^n}$$

$$+ \sum_{m\neq n} (\xi_m-\xi_n)^{-1}[l_m(l_m-1)D^{L^n_m} - 2l_m l_n D^L + l_n(l_n-1)D^{L^m_n}$$

$$+ 2j_m(l_n D^L - l_m D^{L^n_m}) + 2j_n(l_m D^L - l_n D^{L^m_n}) - 2j_m j_n D^L]\Bigg\}P((\xi_r))$$

$$\equiv \partial_{\xi_n}(D^L P((\xi_r))) + \sum_{L'} A_{nLL'}((\xi_r))D^{L'}P((\xi_r)), \tag{25}$$

where on the right-hand side only $L'=(l'_1,\ldots,l'_N)$ with $\sum_n l'_n \leqq \sum_n j_n - k - 1$ contribute non-zero terms.

Equation (25) shows that the relation

$$\nabla_{\xi_n}R^L = \partial_{\xi_n}R^L + \sum_{L'} A_{nLL'}R^{L'}, \tag{26}$$

together with Eq. (19), define a connection on $X_N^0 \times \mathbf{C}^{\#L}$ which is intertwined (see Eq. (18)) with the KZ connection by the homomorphism J. In particular, if $t \mapsto ((\xi_r^t), P_t)$ is a horizontal curve with respect to the KZ connection then $R_t^L \equiv D^L P_t((\xi_r^t))$ satisfy the system of linear differential equations

$$d/dt R_t^L = -\sum_{n,L'} d\xi_n/dt\, A_{nLL'}((\xi_r^t))R_t^{L'}. \tag{27}$$

As a result, if R_t^L vanish at $t=0$, they vanish for all t. This proves Proposition 2.

The following

Corollary 2. *The dimension of $W((\xi_n))$ does not depend on (ξ_n).*

is an immediate consequence of Proposition 2. Let us denote by $W_{j_1\cdots j_N}$ the kernel of homomorphism J. In the words, $W_{j_1\cdots J_N} = \bigcup_{(\xi_n)\in X_N^0} \{(\xi_n)\} \times W_{j_1\cdots j_N}((\xi_n))$. Corollary 2 implies

Corollary 3. *$W_{j_1\cdots j_N}$ is a holomorphic vector subbundle of $X_N^0 \times \mathrm{Inv}\left(\bigotimes_n V_{j_n}\right)$.*

4. Factorization

$W_{j_1\cdots j_N}$ may be viewed as a concrete realization of Friedan–Shenker bundles [3] over moduli spaces of punctured Riemann surfaces for genus zero WZW model with group $SU(2)$. In order to find the dimension of these bundles, we shall study, much in the spirit of [3], the behavior of $W_{j_1\cdots j_N}$ at the boundary of X_N^0 in a compactification of the moduli space of a punctured sphere. The boundary points correspond to pinched surfaces with punctures distributed between the smooth components of the surface. In the language of X_N^0, one approaches the boundary points of the moduli space by letting points ξ_n converge to each other in groups. More concretely, we shall study the situation when $\xi_2 \to \xi_1 = 0$, i.e. when we pinch off two punctures. We shall also assume that spin $j_2 = 1/2$. Values of spins at other points will be taken between $1/2$ and $k/2$ which is an inessential restriction since insertions with spin zero may be dropped altogether and presence of spins $> k/2$ reduces $W_{j_1\cdots j_N}$ to zero.

Let us restrict bundle $W_{j_1 1/2 j_3\cdots j_N}$ to points

$$(0, \zeta^{2(k+2)}, \xi_3, \ldots, \xi_N) \in X_N^0 \tag{28}$$

with $\zeta \neq 0$ from an ε-disc D_ε and $\xi_3, \ldots, \xi_N$ fixed. $\zeta = 0$ is the compactification point. We would like to extend the trivial bundle $(D_\varepsilon \backslash \{0\}) \times \mathrm{Inv}\left(\bigotimes_n V_{j_n}\right)$ to a buundle U over D_ε so that the (restricted) KZ connection becomes smooth at $\zeta = 0$. This will allow to extend subbundle $W_{j_1\cdots j_N}$ to $\zeta = 0$.

Let us decompose

$$\mathrm{Inv}\left(\bigotimes_n V_{j_n}\right) \cong \bigoplus_{\substack{j = j_1 - 1/2 \\ \text{or } j_1 + 1/2}} \mathrm{Inv}(V_{j_1} \otimes V_{1/2} \otimes V_j) \otimes \mathrm{Inv}\left(V_j \otimes \left(\bigotimes_{n=3}^{N} V_{j_n}\right)\right). \tag{29}$$

In the polynomial realization of V_j's, this becomes

$$\begin{aligned} P((v_n)) &= \sum_j \sum_{l=0}^{2j} (-1)^l \partial_v^{2j-l}|_{v=0} P_{CG}^{j_1 1/2 j}(v_1, v_2, v) \partial_{v'}^l|_{v'=0} P^{j j_3 \cdots j_N}(v', v_3, \ldots, v_N) \\ &\equiv P((P^{j j_3 \cdots j_N}))((v_n)), \end{aligned} \tag{30}$$

where $P_{CG}^{j_1 j_2 j_3}$ are given by Eq. (13). The terms of the KZ connection (15) which develop singularity when $\xi_2 \to \xi_1 = 0$,

$$\pm 2/(k+2)\,\xi_2^{-1} \sum_a t^a_{j_1} t^a_{1/2}$$

are diagonalized by decomposition (29) since

$$\begin{aligned} 2\sum_a t^a_{j_1} t^a_{1/2} &= \sum_a (t^a_{j_1} + t^a_{1/2})^2 - \sum_a (t^a_{j_1})^2 - \sum_a (t^a_{1/2})^2 \\ &= j(j+1) - j_1(j_1+1) - 3/4 = \begin{cases} -j_1 - 1 & \text{for} \quad j = j_1 - 1/2, \\ j_1 & \text{for} \quad j = j_1 + 1/2. \end{cases} \end{aligned}$$

Consequently, the restriction of the KZ connection to points (28) becomes in terms of representation (30)

$$\nabla_{\bar\zeta} P^{jj_3\cdots j_N} = \partial_{\bar\zeta} P^{jj_3\cdots j_N}, \tag{31}$$

$$\nabla_\zeta P^{jj_1\cdots j_N} = [\partial_\zeta - (2j(j+1) - 2j_1(j_1+1) - 3/2)\zeta^{-1}] P^{jj_3\cdots j_N} + \mathcal{O}(\zeta^{2k+3}), \tag{32}$$

where $\mathcal{O}(\zeta^{2k+3})$ is linear in $P^{jj_3\cdots j_N}$'s. It is easy to see that the gauge transformation

$$(P^{jj_3\cdots j_N}) \mapsto (\tilde P^{jj_3\cdots j_N}) \equiv (\zeta^{-2j(j+1)+2j_1(j_1+1)+3/2} P^{jj_3\cdots j_N}) \tag{33}$$

renders the connection given by (31) and (32) smooth at $\zeta = 0$. Indeed, transformation (33) removes $\mathcal{O}(\zeta^{-1})$ terms on the right-hand side of Eq. (32). Moreover, since $\zeta^{2k+3-2(2j_1+1)}$ is regular at zero, $\mathcal{O}(\zeta^{2k+3})$-terms in (32) stay regular. The resulting connection in the bundle

$$U \equiv D_\varepsilon \times \left[\bigoplus_j \operatorname{Inv}\left(V_j \otimes \left(\bigotimes_{n=3}^N V_{j_n} \right)\right)\right]$$

is clearly flat everywhere. We shall call it the $\widetilde{\text{KZ}}$ connection.

As we have seen above, the KZ connection preserves subbundle $W_{j_1 1/2 j_3\cdots j_N}$ which after restriction to points (28) and gauge transformation (33) becomes subbundle $\tilde W_{j_1 1/2 j_3\cdots j_N}$ of $U|_{D_\varepsilon \setminus \{0\}}$. We shall denote its fibers by $\tilde W_{j_1 1/2 j_3\cdots j_N}(\zeta)$. Obviously, subbundle $\tilde W_{j_1 1/2 j_3\cdots j_N}$ is preserved by the $\widetilde{\text{KZ}}$ connection which moreover allows to extend it to whole D_ε by addition of fiber $\tilde W_{j_1 1/2 j_3\cdots j_N}(0)$ related to fibers at $\zeta \neq 0$ by the parallel transport. We have the following factorization result:

Proposition 3. $\tilde W_{j_1 1/2 j_3\cdots j_N}(0) = \bigoplus_{\substack{j = j_1 - 1/2 \\ \text{or } j_1 + 1/2}} W_{jj_3\cdots j_N}((0, \xi_3, \ldots, \xi_N))$.

Proof. Let for $\tilde L = (l_2, l_3, \ldots, l_N)$, $l_2 = 0, 1$, $l_n \leqq 2j_n$ for $n \geqq 3$, $\sum_{n=2}^N l_n \leqq j_1 + 1/2 + \sum_{n=3}^N j_n - k - 1$, $D^{\tilde L} = \partial_{v_2}^{l_2} \cdots \partial_{v_N}^{l_N}$, $(\tilde P^{jj_3\cdots j_N}) \in \bigoplus_{\substack{j = j_1 - 1/2 \\ \text{or } j_1 + 1/2}} \operatorname{Inv}\left(V_j \otimes \left(\bigotimes_{n=3}^n V_{j_n} \right)\right)$ and $\zeta \neq 0$,

$$R^{\tilde L} = D^{\tilde L} P((\zeta^{2j(j+1) - 2j_1(j_1+1) - 3/2} \tilde P^{jj_3\cdots j_N}))|_{\substack{v_1 = 0,\ v_2 = \zeta^{2(k+2)}, \\ v_n = \xi_n,\ n \geqq 3}}, \tag{34}$$

(compare Eq. (17)). The vector-bundle homomorphism $\tilde J: U|_{D_\varepsilon \setminus \{0\}} \to (D_\varepsilon \setminus \{0\}) \times \mathbf{C}^{\# \tilde L}$ sending $(\zeta, (\tilde P^{jj_3\cdots j_N}))$ to $(\zeta, (R^{\tilde L}))$ intertwines the $\widetilde{\text{KZ}}$ connection with the one given by

$$\nabla_{\bar{\zeta}} R^{\tilde{L}} = \partial_{\bar{\zeta}} R^{\tilde{L}}, \tag{35}$$

$$\begin{aligned}\nabla_{\zeta} R^{\tilde{L}} = {} & \partial_{\zeta} R^{\tilde{L}} - 2\left[(j_1(1-2l_2) - l_2(2-l_2))R^{\tilde{L}} - \sum_{n=3}^{N} l_2(2-l_2) R^{\tilde{L}_2^n} \right]\zeta^{-1} \\ & - 2\zeta^{2k+3}\left(\sum_{n=2}^{N} l_n - \sum_{n=3}^{N} j_n + k + 1/2 \right) R^{\tilde{L}^2} + 2\zeta^{2k+3} \sum_{n=3}^{N} (\xi_n - \zeta^{2(k+2)})^{-1} \\ & \cdot [(1-2l_2)(j_n - l_n) R^{\tilde{L}} + l_2(2-l_2) R^{\tilde{L}_2^n} + l_n(2j_n - l_n + 1) R^{\tilde{L}_n^2}]. \end{aligned} \tag{36}$$

This follows immediately from relation (18) together with Eqs. (19), (25) and (26) defining the right-hand side connection of Eq. (18), provided that we observe that, due to the translational invariance of polynomials P,

$$R^{L_2^1} = - R^{L} - \sum_{n=3}^{N} R^{L_2^N}.$$

Equation (36) may be put into the form

$$\nabla_{\zeta} R^{\tilde{L}} = \partial_{\zeta} R^{\tilde{L}} - 2j_1\zeta^{-1} R^{\tilde{L}} + \sum_{\tilde{L}'} A_{\tilde{L}\tilde{L}'} R^{\tilde{L}'} \quad \text{for} \quad l_2 = 0,$$

$$\begin{aligned}\nabla_{\zeta}\left(R^{\tilde{L}} + (2j_1+1)^{-1} \sum_{n=3}^{N} R^{\tilde{L}_2^n} \right) = {} & \partial_{\zeta}\left(R^{\tilde{L}} + (2j_1+1)^{-1} \sum_{n=3}^{N} R^{\tilde{L}_2^n} \right) \\ & + 2(j_1+1)\zeta^{-1}\left(R^{\tilde{L}} + (2j_1+1)^{-1} \sum_{n=3}^{N} R^{\tilde{L}_2^n} \right) \\ & + \sum_{\tilde{L}'} A_{\tilde{L}\tilde{L}'} R^{\tilde{L}'} \quad \text{for} \quad l_2 = 1, \end{aligned}$$

where $A_{\tilde{L}\tilde{L}'} = \mathcal{O}(\zeta^{2k+3})$. As a result, the gauge transformation

$$(R^{\tilde{L}}) \mapsto (\tilde{R}^{\tilde{L}}),$$

where

$$\tilde{R}^{\tilde{L}} = \zeta^{-2j_1} R^{\tilde{L}} \quad \text{for} \quad l_2 = 0,$$

$$\tilde{R}^{\tilde{L}} = \zeta^{2(j_1+1)}\left(R^{\tilde{L}} + (2j_1+1)^{-1} \sum_{n=3}^{N} R^{\tilde{L}_2^n} \right) \quad \text{for} \quad l_2 = 1,$$

removes the singularity at $\zeta = 0$ from the covariant derivative (36). Consequently, if $t \mapsto (\zeta_t, (\tilde{P}_t^{jj_3\cdots j_N}))$ is a curve horizontal with respect to the KZ connection and $\zeta_0 = 0$ then

$$d/dt\, \tilde{R}^{\tilde{L}} = - d\zeta_t/dt \sum_{\tilde{L}'} \tilde{A}_{\tilde{L}\tilde{L}'}(\zeta_t) \tilde{R}^{\tilde{L}'} \tag{37}$$

with $\tilde{A}_{\tilde{L}\tilde{L}'}$ analytic at $\zeta = 0$. From Eqs. (30) and (34), it follows that

$$\begin{aligned}\tilde{R}^{\tilde{L}} = {} & (2j_1+1)!\, \partial_{v_3}^{l_3} \cdots \partial_{v_N}^{l_N} \tilde{P}^{(j_1+1/2)j_3\cdots j_N}(0, \xi_3, \ldots, \xi_N) \\ & + (2j_1)!\, \zeta^{2(k+2)} \partial_{v'} \partial_{v_3}^{l_3} \cdots \partial_{v_N}^{l_N} \tilde{P}^{(j_1+1/2)j_3\cdots j_N}(0, \xi_3, \ldots, \xi_N) \\ & + (2j_1-1)!\, \zeta^{2(k+1-2j_1)} \partial_{v_3}^{l_3} \cdots \partial_{v_N}^{l_N} \tilde{P}^{(j_1-1/2)j_3\cdots j_N}(0, \xi_3, \ldots, \xi_N) \quad \text{for} \quad l_2 = 0 \end{aligned}$$

and

$$\tilde{R}^{\tilde{L}} = (2j_1)!\,\zeta^{2(2j_1+1)}\partial_{v'}\partial_{v_3}^{l_3}\cdots\partial_{v_N}^{l_N}\tilde{P}^{(j_1+1/2)j_3\cdots j_N}(0,\xi_3,\ldots,\xi_N)$$
$$+(2j_1-1)!\,\partial_{v_3}^{l_3}\cdots\partial_{v_N}^{l_N}\tilde{P}^{(j_1-1/2)j_3\cdots j_N}(0,\xi_3,\ldots,\xi_N)$$
$$+(2j_1+1)^{-1}\zeta^{2(2j_1+1)}\sum_{n=3}^{N}\tilde{R}^{\tilde{L}_2^n}\quad \text{for}\quad l_2=1.$$

But Eq. (37) implies that spaces $\tilde{W}_{j_1 1/2 j_3\cdots j_N}(\zeta)$ are selected by conditions

$$\tilde{R}^{\tilde{L}} = 0 \tag{38}$$

for all ζ. In particular, for $\zeta=0$, we obtain the equations

$$\partial_{v_3}^{l_3}\cdots\partial_{v_N}^{l_N}\tilde{P}^{(j_1+1/2)j_3\cdots j_N}(0,\xi_3,\ldots,\xi_N)=0 \quad\text{for}\quad \sum_{n=3}^{N} l_n \leqq j_1+1/2+\sum_{n=3}^{N} j_n-k-1$$

and

$$\partial_{v_3}^{l_3}\cdots\partial_{v_N}^{l_N}\tilde{P}^{(j_1-1/2)j_3\cdots j_N}(0,\xi_3,\ldots,\xi_N)=0 \quad\text{for}\quad \sum_{n=3}^{N} l_n \leqq j_1-1/2+\sum_{n=3}^{N} j_n-k-1$$

which, by virtue of Proposition 1 and of translational invariance of polynomials $P^{jj_3\cdots j_N}$, imply Proposition 3.

Let us denote $N_{j_1\cdots j_N}=\dim W_{j_1\cdots j_N}((\xi_n))$. Clearly, $N_{j_1\cdots j_N}$ is symmetric in $j_1,\ldots,j_N$. The following

Corollary 4. $N_{j_1 1/2 j_3\cdots j_N} = N_{(j_1-1/2)j_3\cdots j_N)} + N_{(j_1+1/2)j_3\cdots j_N}$.

Follows immediately from Proposition 2.

5. Count of Dimensions

E. Verlinde has observed [17] that the dimensions $N_{j_1j_2j_3}$ of Friedan–Shenker bundles of rational conformal field theories may be related to the matrix realizing the $\tau\mapsto -1/\tau$ modular transformation in the bundle corresponding to the toroidal geometry with no insertions. For the $SU(2)$ WZW model, the relation is

$$N_{j_1j_2j_3} = \sum_{j=0}^{k/2} S_{j_1j}S_{j_2j}S_{j_3j}/S_{0j}, \tag{39}$$

where

$$S_{jj'} = (2/(k+2))^{1/2}\sin[\pi(2j+1)(2j'+1)/(k+2)] \tag{40}$$

with $0\leqq j,j'\leqq k/2$. Formal factorization arguments [3, 17] suggest the relation

$$N_{j_1\cdots j_N} = \sum_{j=0}^{k/2} N_{j_1\cdots j_m j}N_{jj_{m+1}\cdots j_N} \tag{41}$$

for $m<N$. Since square of the symmetric matrix $(S_{jj'})$ is 1, Eqs. (39) and (41) imply that

$$N_{j_1\cdots j_N} = \sum_{j=0}^{k/2} S_{j_1j}\cdots S_{j_Nj}/(S_{0j})^{N-2}, \tag{42}$$

see [14].

Having, due to Proposition 3, a special case of factorization under control, we may now prove (42) rigorously. First, let us notice that since

$$N_{j_1 1/2 j} = \begin{cases} 1 & \text{if} \quad j = j_1 - 1/2 \geqq 0 \quad \text{or} \quad j = j_1 + 1/2 \leqq k/2, \\ 0 & \text{otherwise,} \end{cases}$$

Corollary 4 is a special case of (41). Its subsequent application allows to show formula (42) for $j_2, \ldots, j_N \equiv 1/2$ by induction on the number of insertion points. Next, let us order sequences $(j_1, \ldots, j_N)$, $1/2 \leqq j_n \leqq k/2$, so that the earlier ones have either smaller value of $\sum_n j_n$ or, if these are equal, the bigger number N of insertion points. Suppose that we have proven formula (42) up to some place in the set of sequences $(j_1 \cdots j_N)$. If the next sequence is composed only of spins 1/2, we may proceed further. If it has one spin (e.g. $j_1) > 1/2$ then, by Corollary 4,

$$N_{j_1 \cdots j_N} = N_{(j_1 - 1/2) 1/2 j_3 \cdots j_N} - N_{(j_1 - 1) j_3 \cdots j_N}, \tag{43}$$

where on the right-hand side only $N_{j'_1 \cdots j'_{N'}}$ with earlier sequences $(j'_1, \ldots, j'_{N'})$ appear. Representation (42) for the left-hand side follows then from that for the terms on the right-hand side. This establishes

Theorem. *Dimensions $N_{j_1 \cdots j_N}$ of spaces $W_{j_1 \cdots j_N}$ of $SU(2)$ Chern–Simons states on the Riemann sphere are given by formulae (40), (42).*

6. Conclusions

We have achieved a finite-dimensional realization of bundles of $SU(2)$ Chern–Simons quantum states in the presence of Wilson lines in representations j_n in the spherical geometry. The resulting holomorphic vector bundles $W_{j_1 \cdots j_N}$ over spaces X_N^0 of sequences (ξ_n) of non-coincident points in $\mathbf{C}$ carry (the restriction of) the flat Knizhnik–Zamolodchikov connection. Their dimensions $N_{j_1 \cdots j_N}$ are given by Verlinde's formulae.

The spaces of quantum states come usually with additional structure: the scalar product. For the Chern–Simons states this is formally given by

$$\| \psi \|^2 = \int_{\mathscr{A}^{01}} |\psi(A^{01})|^2 \exp[-ik/(2\pi) \int \operatorname{tr}(A^{01})^* A^{01}] DA^{01} D(A^{01})^*. \tag{44}$$

It was shown in [4] how to reduce functional integral (44) to a finite-dimensional integral which (if convergent, which remains to be proven in the general case) gives rise to a hermitian structure on the bundle $W_{j_1 \cdots j_N}$ preserved by the Knizhnik–Zamolodchikov connection. Existence of such a hermitian structure is equivalent to the unitarizability of the holonomy representation of $\pi_1(X_N^0)$ given by the connection. The latter is related to unitary representations of Jones algebra [9] with index $4\cos^2(\pi/(k+2))$. The hermitian metric on $W_{j_1 \cdots j_N}$ allows to express the Green functions of the WZW model in a simple way. For example, at zero gauge field,

$$\Gamma_{j_1 \cdots j_N}(\mathbf{C}P^1, (\xi_n)) = \sum_{a=1}^{N_{j_1 \cdots j_N}} P_a \otimes \bar{P}_a,$$

where (P_a) is an orthonormal basis of the fiber $W_{j_1 \cdots j_N}((\xi_n))$. We shall return to the study of metric properties of bundles $W_{j_1 \cdots j_N}$ in a future publication.

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Note added in proof. Bundles $W_{j_1 \cdots j_N}$ and their factorization were also studied in

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