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Quantum R Matrices Related to the Spin Representations of B_n and D_n

Masato Okado

Department of Mathematical Science, Faculty of Engineering Science, Osaka University, Toyonaka, Osaka 560, Japan

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Abstract. We present the explicit form of the trigonometric R matrices related to the spin representations of the simple Lie algebras $X_n = B_n, D_n$. We conjecture that one dimensional configuration sums of the corresponding vertex models in the face formulation are the string functions of $X_n^{(1)}$ modules.

1. Introduction

The importance of quantum R matrices has been recognized widely these days because of its deep relationship with quantum groups, q-analysis, operator algebras, like invariants, conformal field theories, statistical mechanical models, etc. In constructing trigonometric R matrices, the quantized universal enveloping algebra $U_q\mathfrak{g}$ plays a significant role. In [1] V. G. Drinfeld constructed a "universal R matrix" $\mathscr{R} \in U_q\mathfrak{g} \otimes U_q\mathfrak{g}$. This, in principle, enables us to write down the form of the R matrix corresponding to an arbitrary pair of a nontwisted affine Lie algebra $\hat{\mathfrak{g}}$ and an irreducible representation π of $\mathfrak{g}$. From the statistical mechanical point of view, each R matrix defines a solvable vertex model on the two dimensional square lattice. In order to carry out its analysis, we have to deal with the explicit form of the R matrix. So far, such explicit expressions have been obtained in the case of $\hat{\mathfrak{g}} = A_1^{(1)}, \pi =$ an arbitrary representation [2, 3] and in the case of $\hat{\mathfrak{g}} = A_n^{(1)}$, $B_n^{(1)}, C_n^{(1)}, D_n^{(1)}, \pi =$ the vector representation [4, 5]. Very recently, an exceptional case $G_2^{(1)}$ is also treated in [6].

In [3] a method was initiated to construct the R matrices related to higher representations from the one related to a basic representation. This method is called the "fusion procedure." In the case of $A_n^{(1)}$, the key R matrix is the one corresponding to the vector representation. I. V. Cherednik worked out the fusion procedure in the elliptic parametrization [7]. When we consider the cases of $B_n^{(1)}$ and $D_n^{(1)}$, the R matrices corresponding to the spin representations are necessary for the fusion procedure. The purpose of this article is to give a concise form of the trigonometric R matrices related to the spin representations of B_n and D_n.

Let us explain our method in the case $\hat{\mathfrak{g}} = B_n^{(1)}$. Let V_{sp} be the spin representation of $U_q = U_q(B_n)$. Let us recall the characterization of the quantum R matrix $R(x)$. $R(x) \in \mathrm{End}_{\mathbf{C}}(V_{\mathrm{sp}} \otimes V_{\mathrm{sp}})$ is uniquely determined up to a scalar multiple by the following relations [5]:

$$(1)\ [R(x), \Delta(x)] = 0 \quad \text{for all} \quad X \in U_q, \tag{1.1}$$

$$(2)\ R(x)(xq^{H_0} \otimes X_0^+ + X_0^+ \otimes q^{-H_0}) = (q^{H_0} \otimes X_0^+ + xX_0^+ \otimes q^{-H_0})R(x). \tag{1.2}$$

Here Δ is the comultiplication defined in (2.1), and X_0^+, H_0 are as in (3.1). Since the tensor product decomposition of $V_{\mathrm{sp}} \otimes V_{\mathrm{sp}}$ is multiplicity free, (1.1) implies that $R(x)$ must be written as

$$R(x) = \sum_k \rho_k(x) P_k,$$

where P_k is the projector onto the irreducible component V_k and $\rho_k(x)$ is the eigenvalue of $R(x)$ on V_k. Therefore, out first task is to obtain a set of orthonormal vectors in V_k. This is done inductively on the rank n of B_n (Proposition 3.5, 3.6). Next we determine $\rho_k(x)$ by solving (1.2). Let W_λ denote the weight space of $V_{\mathrm{sp}} \otimes V_{\mathrm{sp}}$ of weight λ. Thanks to the invariance (1.1), we have $R(x) W_\lambda \subset W_\lambda$. Put $k = n - (\lambda, \lambda)$, where the inner product is so normalized that the short roots have length 1. Exhibiting the rank n by the upper index (n), we find the following commutative diagram:

$$\begin{array}{ccccl} & W_\lambda^{(n)} & \cong & W_0^{(k)} & \\ R^{(n)}(x) & \downarrow & & \downarrow & (\text{scalar}) \times R^{(k)}(x). \\ & W_\lambda^{(n)} & \cong & W_0^{(k)} & \end{array} \tag{1.3}$$

Thus the calculation of $R^{(n)}(x)$ is reduced to that of the restriction $R^{(k)}(x)|_{W_0^{(k)}}$. The explicit form of $R(x)$ is given in (5.9).

In the case $\hat{\mathfrak{g}} = D_n^{(1)}$, there are two spin representation spaces, $V_{\mathrm{sp}}^{\varepsilon}$ $(\varepsilon = \pm)$. Accordingly, there are four R matrices corresponding to $V_{\mathrm{sp}}^{\varepsilon} \otimes V_{\mathrm{sp}}^{\varepsilon}$ and $V_{\mathrm{sp}}^{\varepsilon} \otimes V_{\mathrm{sp}}^{-\varepsilon}$ $(\varepsilon = \pm)$. Because of the symmetry of the Dynkin diagram, it suffices to consider two cases. Both R matrices have a property similar to (1.3). The form of $R(x)$ is in (5.14).

In our recent analysis on restricted face models, there emerged an intimate relation between the computation of the local state probabilities (LSPs) and the representation theory of affine Lie algebras [8, 9]. The situation is quite similar when we consider vertex models in the face formulation. By Baxter's corner transfer matrix method [10], the calculation of the LSP reduces to the evaluation of a quantity called "one dimensional (1D) configuration sum." As investigated in [11], 1D configuration sums of the vertex models related to the vector representation of classical simple Lie algebras turn out to be the string functions in the sense of [12]. Computer experiments suggest a similar conjectural result in the case of the spin representations. Let us explain the conjecture in the case of $D_{n+1}^{(1)}$ $(n \geqq 2)$. Let Λ_j $(j = 0, \dots, n+1)$ be the fundamental weights of $D_{n+1}^{(1)}$, let I be the set of level 1 integral weights in $\mathbf{Z}\Lambda_0 \oplus \cdots \oplus \mathbf{Z}\Lambda_{n+1}$, and let Λ be a level 1 dominant integral weight. With each Λ we associated a particular sequence $p_\Lambda = (p_\Lambda^{(j)})_{j \geqq 1}$ of elements of I (see (6.2)). This corresponds to a "ground state" in the statistical mechanical language. We call a sequence $p = (p^{(j)})_{j \geqq 1} (p^{(j)} \in I)$ Λ-path, if it satisfies the following conditions:

(1) $p^{(j+1)} - p^{(j)}$ is a weight of the spin representation $(\pi^+, V_{\mathrm{sp}}^+)$ of $U_q(D_{n+1})$ for all j,
(2) $p^{(j)} = p_\Lambda^{(j)}$ if $j \gg 1$.

For the definition of $(\pi^+, V_{\mathrm{sp}}^+)$, see (4.1). By $\mathscr{P}(\Lambda)$ we denote the set of Λ-paths. Let $L(\Lambda)$ be the irreducible highest weight module of highest weight Λ, and let a be an element in I. In our case, the 1D configuration sum $f(a,\Lambda;q)$ reads as follows:

$$f(a,\Lambda;q) = \sum_{\substack{p\in\mathscr{P}(\Lambda)\\ p^{(1)}=a}} q^{\omega(p)},$$

$$\omega(p) = \sum_{j=1}^{\infty} j(H(\eta^{(j)}(p), \eta^{(j+1)}(p)) - H(\eta^{(j)}(p_\Lambda), \eta^{(j+1)}(p_\Lambda))),$$

where $\eta^{(j)}(p) = p^{(j+1)} - p^{(j)}$. For the definition of the function $H = H^{D_{n+1}^{(1)}}$, see (6.1). Now our conjecture is

$$f(a,\Lambda;q) = \sum_i \dim L(\Lambda)_{a-i\delta} q^i.$$

Here δ is the null root. The right-hand side is the string function studied in [12]. If we admit this conjecture, we can show that the 1D configuration sums in the case of $B_n^{(1)}$ coincide with the string functions of a highest weight $D_{n+1}^{(1)}$ module viewed as $B_n^{(1)}$ module.

After having accomplished this work, the author came to know the work by N. Yu. Reshetikhin [13], in which he obtained a recursive formula for the rational R matrices related to the spin representations. The author thanks E. Date for informing him of this work.

The text is organized as follows. In Sect. 2, we recall the definition of $U_q(B_n)$ and $U_q(D_n)$. Their spin representations are described and the Clebsch–Gordan coefficients for the tensor product of two representations are calculated in Sects. 3 and 4. The explicit form of the quantum R matrices is given in Sect. 5. In Sect. 6, we define the corresponding vertex model in the face formulation, and give a conjecture on the 1D configuration sums. In the Appendix, we give a trigonometric version of Reshetikhin's recursive formula in the case of $B_n^{(1)}$.

2. The Algebras $U_q(B_n)$ and $U_q(D_n)$

Let us review the definition of the quantized universal enveloping algebra $U_q\mathfrak{g}(\mathfrak{g} = B_n, D_n)$. We follow the conventions in [14]. Let $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{g}$, let $(,)$ be the invariant bilinear form on $\mathfrak{h}^*$ so normalized that the short roots are of length 1, and let $\varepsilon_j (j = 1, \dots, n)$ be the standard orthonormal basis with respect to $(,)$. The simple roots of $\mathfrak{g}$ are given as follows:

$$\begin{aligned} \alpha_j &= \varepsilon_j - \varepsilon_{j+1} \quad (1 \leqq j < n) && \text{for} \quad \mathfrak{g} = B_n, D_n, \\ &= \varepsilon_n \quad (j = n) && \text{for} \quad \mathfrak{g} = B_n, \\ &= \varepsilon_{n-1} + \varepsilon_n \quad (j = n) && \text{for} \quad \mathfrak{g} = D_n. \end{aligned}$$

Let q be a nonzero complex number. We assume that q is not a root of unity. We define $U_q\mathfrak{g}$ to be the associative $\mathbf{C}$-algebra with generators $H_j, X_j^{\pm}$ $(j = 1, \dots, n)$ and relations

$$[H_i, H_j] = 0, \quad [H_i, X_j^{\pm}] = \pm(\alpha_i, \alpha_j) X_j^{\pm},$$

$$[X_i^+, X_j^-] = \delta_{ij} \frac{q^{2H_i} - q^{-2H_i}}{q^2 - q^{-2}},$$

$$\sum_{k=0}^{1-a_{ij}} (-)^k \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix}_{q_i} (X_i^{\pm})^{1-a_{ij}-k} X_j^{\pm} (X_i^{\pm})^k = 0 \quad (i \neq j).$$

Here $q_i = q^{(\alpha_i, \alpha_i)}$, $a_{ij} = 2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i)$ and

$$\begin{bmatrix} m \\ n \end{bmatrix}_t = \prod_{j=1}^{n} \frac{t^{m-n+j} - t^{-(m-n+j)}}{t^j - t^{-j}}.$$

The usual universal enveloping algebra $U\mathfrak{g}$ is obtained by letting q tend to 1. Our $U_q\mathfrak{g}$ is a Hopf algebra with the following comultiplication:

$$\begin{aligned} &\Delta: U_q\mathfrak{g} \to U_q\mathfrak{g} \otimes U_q\mathfrak{g}, \\ &\Delta(H_i) = H_i \otimes 1 + 1 \otimes H_i, \\ &\Delta(X_i^{\pm}) = q^{H_i} \otimes X_i^{\pm} + X_i^{\pm} \otimes q^{-H_i}. \end{aligned} \tag{2.1}$$

Note that only here the notation is different from that in [14]. In Sects. 3 and 4, we consider the tensor product decomposition with respect to the above comultiplication. Let us prepare several notations for the subsequent sections. Set $\mathbf{C}^2 = \mathbf{C}e_{1/2} \oplus \mathbf{C}e_{-1/2}$. We define 2×2 matrices $X^{\pm}, H$ acting on $\mathbf{C}^2$ as follows:

$$X^{\pm} e_\varepsilon = e_{\varepsilon \pm 1}, \quad He_\varepsilon = \varepsilon e_\varepsilon \quad \text{for} \quad \varepsilon = \pm \tfrac{1}{2}. \tag{2.2}$$

e_ε should be understood as 0 if $\varepsilon \neq \pm\frac{1}{2}$. Consider $\mu = \mu_1\varepsilon_1 + \cdots + \mu_n\varepsilon_n \in \mathfrak{h}^*$ such that $\mu_j = \pm\frac{1}{2}$ for all j. With such μ, we associate an orthonormal vector in $(\mathbf{C}^2)^{\otimes n}$ as follows:

$$e_\mu \stackrel{\text{def}}{=} e_{\mu_1} \otimes \cdots \otimes e_{\mu_n}. \tag{2.3}$$

3. Spin Representation of $U_q(B_n)$

Following [14], we recall the spin representation (π, V_{sp}) of $U_q(B_n)$. The representation space V_{sp} is identified with $(\mathbf{C}^2)^{\otimes n}$. It is spanned by the weight vector e_μ (2.3) of weight μ. Hence, we have $\dim V_{\text{sp}} = 2^n$. The actions of the generators $X_i^{\pm}, H_i$ $(i = 1, \ldots, n)$ of $U_q(B_n)$ are as follows:

$$X_i^+ = -1 \otimes \cdots \otimes \overset{i}{X^+} \otimes \overset{i+1}{X^-} \otimes \cdots \otimes 1 \qquad (1 \leqq i < n),$$

$$X_n^+ = \frac{1}{\sqrt{q + q^{-1}}} 1 \otimes \cdots \otimes \overset{n}{X^+},$$

$$H_i = 1 \otimes \cdots \otimes (\overset{i}{H} \otimes \overset{i+1}{1} - \overset{i}{1} \otimes \overset{i+1}{H}) \otimes \cdots \otimes 1 \quad (1 \leqq i < n),$$

$$H_n = 1 \otimes \cdots \otimes \overset{n}{H},$$

$$X_i^- = {}^t(X_i^+) \qquad (1 \leqq i < n).$$

Here ${}^tA$ denotes the transpose of A. In Sect. 5, we will also need the operators X_0^+ and H_0 defined as follows:

$$X_0^+ = -X^- \otimes X^- \otimes 1 \otimes \cdots \otimes 1,$$
$$H_0 = -(H \otimes 1 + 1 \otimes H) \otimes 1 \otimes \cdots \otimes 1. \tag{3.1}$$

These operators correspond to generators of the affine quantized universal enveloping algebra $U_q(B_n^{(1)})$. The tensor product $V_{sp} \otimes V_{sp}$ decomposes as follows:

$$V_{sp} \otimes V_{sp} = V_0 \oplus V_1 \oplus \cdots \oplus V_n.$$

Here V_k $(0 \leqq k < n)$ denotes the irreducible highest weight module with highest weight $\varepsilon_1 + \cdots + \varepsilon_{n-k}$, and V_n denotes the trivial module.

We construct the whole orthonormal vectors in each space V_k. We identify $V_{sp} \otimes V_{sp}$ with $(\mathbf{C}^4)^{\otimes n}$ via the following map:

$$\begin{array}{ccc} V_{sp} \otimes V_{sp} = (\mathbf{C}^2)^{\otimes n} \otimes (\mathbf{C}^2)^{\otimes n} & \to & (\mathbf{C}^4)^{\otimes n} \\ e_\mu \otimes e_\nu & \mapsto & e_{\mu_1 \nu_1} \otimes \cdots \otimes e_{\mu_n \nu_n}. \end{array}$$

Namely $\mathbf{C}^4$ is spanned by the four orthonormal vectors e_{++}, e_{+-}, e_{-+} and e_{--}. Here $e_{\pm\pm}$ is an abbreviation of $e_{\pm 1/2 \pm 1/2}$. The operators $\Delta(X_i^\pm)$ $(1 \leqq i \leqq n)$ act on $(\mathbf{C}^4)^{\otimes n}$ as follows:

$$\begin{aligned} \Delta(X_i^\pm)(\cdots \otimes \overset{i}{e_{\varepsilon\varepsilon'}} \otimes \overset{i+1}{e_{\eta\eta'}} \otimes \cdots) &= -q^{\varepsilon-\eta}(\cdots \otimes \overset{i}{e_{\varepsilon\, \varepsilon' \pm 1}} \otimes \overset{i+1}{e_{\eta\, \eta' \mp 1}} \otimes \cdots) \\ &\quad - q^{-(\varepsilon'-\eta')}(\cdots \otimes \overset{i}{e_{\varepsilon \pm 1\, \varepsilon'}} \otimes \overset{i+1}{e_{\eta \mp 1\, \eta'}} \otimes \cdots) \quad (1 \leqq i < n), \\ \Delta(X_n^\pm)(\cdots \otimes \overset{n}{e_{\varepsilon\varepsilon'}}) &= \frac{q^{\varepsilon}}{\sqrt{q+q^{-1}}}(\cdots \otimes \overset{n}{e_{\varepsilon\, \varepsilon' \pm 1}}) + \frac{q^{-\varepsilon'}}{\sqrt{q+q^{-1}}}(\cdots \otimes \overset{n}{e_{\varepsilon \pm 1\, \varepsilon'}}), \end{aligned}$$

where $\varepsilon, \varepsilon', \eta, \eta' = \pm\frac{1}{2}$. When we want to stress the rank n of B_n, we write $V_k^{(n)}$ instead of V_k. $(V_k^{(n)})_\lambda$ denotes the weight space in $V_k^{(n)}$ of weight λ. Note that $(V_k^{(n)})_\lambda \neq \{0\}$ only if $\lambda = \lambda_1\varepsilon_1 + \cdots + \lambda_n\varepsilon_n$ with $\lambda_i \in \{0, \pm 1\}$ for all i and $(\lambda, \lambda) \leqq n - k$. The next lemma is easy to show.

Lemma 3.1. *A normalized highest weight vector $v_k^{(n)} \in V_k^{(n)}$ of unit length is given as follows*:

$$v_k^{(n)} = N^{-1/2} \overbrace{e_{++} \otimes \cdots \otimes e_{++}}^{n-k} \otimes \bigotimes_{j=1}^{k} (q^{-(k-j+1/2)} e_{+-} + (-)^{k-j+1} q^{k-j+1/2} e_{-+}),$$

where $N = \prod_{j=1}^{k} (q^{-(2k-2j+1)} + q^{2k-2j+1})$ and $\bigotimes_{j=1}^{k} u(j)$ signifies $u(1) \otimes \cdots \otimes u(k)$.

Hereafter, we use the convention $V_{-1}^{(n)} = V_0^{(n)}$, $V_k^{(n)} = \{0\}$ $(k > n)$.

Lemma 3.2. *Let $\lambda = \lambda_1\varepsilon_1 + \cdots + \lambda_n\varepsilon_n$ be a weight of $V_k^{(n)}$, then we have*

$$\dim (V_k^{(n)})_\lambda = \begin{pmatrix} n - (\lambda, \lambda) \\ \left[\dfrac{n - (\lambda, \lambda) - k}{2}\right] \end{pmatrix},$$

where $[m]$ signifies the largest integer that does not exceed m. In particular, we can *state the following*:

(1) *If there exists an integer* j $(1 \leqq j \leqq n)$ *such that* $\lambda_j = \pm 1$, *then*

$$\dim (V_k^{(n)})_\lambda = \dim (V_k^{(n-1)})_{\lambda'},$$

where

$$\lambda' = \lambda_1 \varepsilon_1 + \cdots + \lambda_{j-1}\varepsilon_{j-1} + \lambda_{j+1}\varepsilon_j + \cdots + \lambda_n \varepsilon_{n-1}. \tag{3.2}$$

(2) $\dim (V_k^{(n)})_0 = \dim (V_{k-1}^{(n-1)})_0 + \dim (V_{k+1}^{(n-1)})_0$.

Proof. In [15], it is proved that if q is generic the dimensionality of the weight space is equal to the one when $q = 1$. This reduces the lemma to the calculation of the dimensionality of weight spaces of the fundamental modules over the Lie algebra B_n. q.e.d.

By a direct calculation, we have

Lemma 3.3. *If* $1 \leqq i < n$ *and* $\varepsilon = \pm$, *then*

$$\Delta(X_i^\pm)(\cdots \otimes \overset{i}{e_{\mp\mp}} \otimes \overset{i+1}{e_{\varepsilon-\varepsilon}} \otimes \cdots) = (\cdots \otimes \overset{i}{e_{\varepsilon-\varepsilon}} \otimes \overset{i+1}{e_{\mp\mp}} \otimes \cdots).$$

Let us define the **C**-linear map ι_j^ε $(1 \leqq j \leqq n, \varepsilon = \pm)$ as follows:

$$\begin{array}{ccc} V_{\mathrm{sp}}^{(n-1)} \otimes V_{\mathrm{sp}}^{(n-1)} & \xrightarrow{\iota_j^\varepsilon} & V_{\mathrm{sp}}^{(n)} \otimes V_{\mathrm{sp}}^{(n)} \\ e_{\mu_1 \nu_1} \otimes \cdots \otimes e_{\mu_{n-1}\nu_{n-1}} & \mapsto & e_{\mu_1\nu_1} \otimes \cdots \otimes \overset{j}{e_{\varepsilon\varepsilon}} \otimes \cdots \otimes e_{\mu_{n-1}\nu_{n-1}}. \end{array}$$

Fix an integer j. For the operators $\Delta(X_i^\pm)$ in $\Delta(U_q(B_{n-1}))$, we set

$$\begin{aligned} Z_i^{\pm(j)} &= \Delta(X_i^\pm) && (1 \leqq i < j-1), \\ &= -\Delta(X_{j-1}^\pm)\Delta(X_j^\pm) && (i = j-1), \\ &= \Delta(X_{i+1}^\pm) && (j \leqq i < n), \end{aligned} \tag{3.3}$$

which must be regarded as operators in $\Delta(U_q(B_n))$. Then we have

Lemma 3.4. *The following diagram is commutative.*

$$\begin{tikzcd} V_{\mathrm{sp}}^{(n-1)} \otimes V_{\mathrm{sp}}^{(n-1)} \arrow[r, "\iota_j^\mp"] \arrow[d, "\Delta(X_i^\pm)"'] & V_{\mathrm{sp}}^{(n)} \otimes V_{\mathrm{sp}}^{(n)} \arrow[d, "Z_i^{\pm(j)}"] \\ V_{\mathrm{sp}}^{(n-1)} \otimes V_{\mathrm{sp}}^{(n-1)} \arrow[r, "\iota_j^\mp"] & V_{\mathrm{sp}}^{(n)} \otimes V_{\mathrm{sp}}^{(n)} \end{tikzcd}.$$

Now we can state our propositions on the construction of vectors in $(V_k^{(n)})_\lambda$.

Proposition 3.5. *Let* λ *be a weight of* $V_{\mathrm{sp}}^{(n)} \otimes V_{\mathrm{sp}}^{(n)}$ *such that* $\lambda_j = \pm 1$. *We have*

$$(V_k^{(n)})_\lambda = \iota_j^\mp (V_k^{(n-1)})_{\lambda'}.$$

Here λ' *is given in (3.2).*

Proof. Let us consider the case $\lambda_j = 1$. From Lemmas 3.1 and 3.3, we have

$$\begin{aligned} \iota_j^+(v_k^{(n-1)}) &= v_k^{(n)} && \text{if} \quad j \leqq n-k, \\ &= (-)^{j-n+k} \Delta(X_{j-1}^-)\Delta(X_{j-2}^-)\cdots\Delta(X_{n-k}^-)v_k^{(n)} && \text{if} \quad j > n-k. \end{aligned}$$

Together with Lemma 3.4, we see

$$\iota_j^+(V_k^{(n-1)}) \subset V_k^{(n)}.$$

Lemma 3.2 completes the proof.

The following is an isomorphism of algebras.

$$\begin{array}{ccc} U_q(B_n) & \xrightarrow{\theta} & U_{q^{-1}}(B_n), \\ X_i^{\pm} & \mapsto & X_i^{\mp}, \\ H_i & \mapsto & -H_i. \end{array}$$

We also consider the map defined by

$$\begin{array}{ccc} V_{\mathrm{sp}}^{(n)} \otimes V_{\mathrm{sp}}^{(n)} & \xrightarrow{\varphi} & V_{\mathrm{sp}}^{(n)} \otimes V_{\mathrm{sp}}^{(n)}, \\ \sum r(q) e_\mu \otimes e_\nu & \mapsto & \sum r(q^{-1}) e_{-\mu} \otimes e_{-\nu}. \end{array}$$

Here $e_{-\mu}$ signifies $e_{-\mu_1} \otimes \cdots \otimes e_{-\mu_n}$ if $e_\mu = e_{\mu_1} \otimes \cdots \otimes e_{\mu_n}$. Then we find the following commutative diagram:

$$\begin{array}{cccc} & V_{\mathrm{sp}}^{(n)} \otimes V_{\mathrm{sp}}^{(n)} & \xrightarrow{\varphi} & V_{\mathrm{sp}}^{(n)} \otimes V_{\mathrm{sp}}^{(n)} & \\ \Delta(X_i^{\pm}) & \Big\downarrow & & \Big\downarrow & \Delta(\theta(X_i^{\pm})). \\ & V_{\mathrm{sp}}^{(n)} \otimes V_{\mathrm{sp}}^{(n)} & \xrightarrow{\varphi} & V_{\mathrm{sp}}^{(n)} \otimes V_{\mathrm{sp}}^{(n)} & \end{array}$$

It is easy to see that the proof in the case of $\lambda_j = -1$ is reduced to that in the case of $\lambda_j = 1$. q.e.d.

Proposition 3.6. *For* $0 \leqq k \leqq n$,

$$\begin{aligned} (V_k^{(n)})_0 = &(q^{-(2k-1)/2} e_{+-} + (-)^k q^{(2k-1)/2} e_{-+}) \otimes (V_{k-1}^{(n-1)})_0 \\ &\oplus (q^{(2k+3)/2} e_{+-} + (-)^{k+1} q^{-(2k+3)/2} e_{-+}) \otimes (V_{k+1}^{(n-1)})_0. \end{aligned}$$

Proof. We are going to show

(1) $(q^{-(2k-1)/2} e_{+-} + (-)^k q^{(2k-1)/2} e_{-+}) \otimes V_{k-1}^{(n-1)} \subset V_k^{(n)}$,

(2) $(q^{(2k+3)/2} e_{+-} + (-)^{k+1} q^{-(2k+3)/2} e_{-+}) \otimes V_{k+1}^{(n-1)} \subset V_k^{(n)}$.

Note that the two spaces in the left-hand side are mutually orthogonal. Comparing the dimension of the subspace of weight 0 (Lemma 3.2), we find that showing (1) and (2) proves this proposition.

Let us first consider (1). From Lemmas 3.1 and 3.3, we can show

$$\begin{aligned} &\Delta(X_1^-)\cdots\Delta(X_{n-k}^-)v_k^{(n)} \\ &= \frac{(-)^{n-k}}{(q^{-(2k-1)} + q^{2k-1})^{1/2}} (q^{-(2k-1)/2} e_{+-} + (-)^k q^{(2k-1)/2} e_{-+}) \otimes v_{k-1}^{(n-1)}. \end{aligned}$$

Applying the creation operators $\Delta(X_2^-),\ldots,\Delta(X_n^-)$, we get (1).

We prove (2) by the induction on n. From Proposition 3.5, we know

$$\overbrace{e_{++}\otimes\cdots\otimes e_{++}}^{n-k-1}\otimes e_{--}\otimes v_k^{(k)}\in V_k^{(n)}.$$

Applying $\Delta(X_{n-k-1}^-)$, we have

$$\overbrace{e_{++}\otimes\cdots\otimes e_{++}}^{n-k-2}\otimes(qe_{+-}\otimes e_{-+}+q^{-1}e_{-+}\otimes e_{+-})\otimes v_k^{(k)}\in V_k^{(n)}. \tag{3.4}$$

On the other hand, from $v_k^{(k)}\in V_k^{(k)}$ and the assumption of the induction, we have

$$(q^{(2k+1)/2}e_{+-}+(-)^k q^{-(2k+1)/2}e_{-+})\otimes v_k^{(k)}\in V_{k-1}^{(k+1)}.$$

Using (1) and Proposition 3.5, the following belongs to $V_k^{(n)}$:

$$\begin{aligned}\overbrace{e_{++}\otimes\cdots\otimes e_{++}}^{n-k-2}&\otimes(q^{-(2k-1)/2}e_{+-}+(-)^k q^{(2k-1)/2}e_{-+})\\ &\otimes(q^{(2k+1)/2}e_{+-}+(-)^k q^{-(2k+1)/2}e_{-+})\otimes v_k^{(k)}.\end{aligned} \tag{3.5}$$

The vector space spanned by (3.4) and (3.5) contains the following vector:

$$\overbrace{e_{++}\otimes\cdots\otimes e_{++}}^{n-k-2}\otimes(q^{(2k+3)/2}e_{+-}+(-)^{k+1}q^{-(2k+3)/2}e_{-+})\otimes v_{k+1}^{(k+1)}. \tag{3.6}$$

Applying $\Delta(X_1^-)\cdots\Delta(X_{n-k-2}^-)$ to (3.6), we see

$$(q^{(2k+3)/2}e_{+-}+(-)^{k+1}q^{-(2k+3)}e_{-+})\otimes v_{k+1}^{(n-1)}\in V_k^{(n)}.$$

The rest of the proof is similar to (1). q.e.d.

4. Spin Representations of $U_q(D_n)$

Our next task is to review the spin representation of $U_q(D_n)$. The difference from the case $U_q(B_n)$ lies in the existence of two spin representations $(\pi^\varepsilon, V_{sp}^\varepsilon)$ $(\varepsilon=\pm)$. Following [14] again, let us recall their features. Let us define the operator $\mathscr{H}$ acting on $(\mathbf{C}^2)^{\otimes n}$ as follows:

$$\mathscr{H}=2^n H\otimes\cdots\otimes H,$$

where H is as in (2.2). Then we have

$$\begin{gathered}V_{sp}^\varepsilon=\{e_\mu\in(\mathbf{C}^2)^{\otimes n}\,|\,\mathscr{H}e_\mu=\varepsilon e_\mu\}\quad(\varepsilon=\pm),\\ V_{sp}^+\oplus V_{sp}^-=(\mathbf{C}^2)^{\otimes n}.\end{gathered} \tag{4.1}$$

Each vector e_μ is again a weight vector of weight $\mu_1\varepsilon_1+\cdots+\mu_n\varepsilon_n$. We easily see $\dim V_{sp}^\varepsilon=2^{n-1}$. The actions of the generators $X_i^\pm, H_i$ $(i=1,\ldots,n)$ of $U_q(D_n)$ on $(\mathbf{C}^2)^{\otimes n}$ are as follows:

$$\begin{aligned}X_i^+&=-1\otimes\cdots\otimes\overset{i}{X^+}\otimes\overset{i+1}{X^-}\otimes\cdots\otimes 1\qquad(1\leqq i<n),\\ X_n^+&=-1\otimes\cdots\otimes\overset{n-1}{X^+}\otimes\overset{n}{X^+},\end{aligned}$$

$$H_i = 1 \otimes \cdots \otimes (\overset{i}{H} \otimes \overset{i+1}{1} - \overset{i}{1} \otimes \overset{i+1}{H}) \otimes \cdots \otimes 1 \quad (1 \leqq i < n),$$

$$H_n = 1 \otimes \cdots \otimes (\overset{n-1}{H} \otimes \overset{n}{1} + \overset{n-1}{1} \otimes \overset{n}{H}),$$

$$X_i^- = {}^t(X_i^+) \qquad (i = 1, \ldots, n).$$

Two representation spaces are preserved under these actions.

Next let us consider the following tensor product decompositions:

$$\begin{aligned} V_{\mathrm{sp}}^{\varepsilon} \otimes V_{\mathrm{sp}}^{\varepsilon} &= V_0^{\varepsilon} \oplus V_2^{\varepsilon} \oplus \cdots \oplus V_{2[n/2]}^{\varepsilon}, \\ V_{\mathrm{sp}}^{\varepsilon} \otimes V_{\mathrm{sp}}^{-\varepsilon} &= V_1^{\varepsilon} \oplus V_3^{\varepsilon} \oplus \cdots \oplus V_{2[(n-1)/2]+1}^{\varepsilon}. \end{aligned} \tag{4.2}$$

Here V_k^{ε} is the irreducible highest weight module of highest weight $\varepsilon_1 + \cdots + \varepsilon_{n-1} + \varepsilon\varepsilon_n$ $(k=0)$, $\varepsilon_1 + \cdots + \varepsilon_{n-k}$ $(0<k<n)$, 0 $(k=n)$. Note that $(V_k^{\varepsilon(n)})_\lambda = \{0\}$ only if $\lambda = \lambda_1\varepsilon_1 + \cdots + \lambda_n\varepsilon_n$ with $\lambda_i \in \{0, \pm 1\}$ for all i and $(\lambda, \lambda) \leqq n-k$. For later use, it is convenient to introduce the following module:

$$\tilde{V}_k = \text{the module generated from the vector } v_k^{(n)} \text{ over } \Delta(U_q(D_n)).$$

Here $v_k^{(n)}$ is defined as follows:

$$\begin{aligned} v_k^{(n)} &= \overbrace{e_{++} \otimes \cdots \otimes e_{++}}^{n-1} \otimes (e_{++} + e_{--}) && \text{if} \quad k=0, \\ &= N^{-1/2} \overbrace{e_{++} \otimes \cdots \otimes e_{++}}^{n-k} \otimes \bigotimes_{j=1}^{k} (q^{-(k-j)} e_{+-} + (-q)^{k-j} e_{-+}) && \text{if} \quad 0 < k \leqq n. \end{aligned}$$

Here $N = \prod_{j=1}^{k-1} (q^{-2(k-j)} + q^{2(k-j)})$. $\tilde{V}_0$ coincides with $V_0^+ \oplus V_0^-$. Let p^{ε} $(\varepsilon = \pm)$ be the following projection:

$$\begin{array}{ccc} (\mathbf{C}^2)^{\otimes n} \otimes (\mathbf{C}^2)^{\otimes n} & \xrightarrow{p^{\varepsilon}} & V_{\mathrm{sp}}^{\varepsilon} \otimes (\mathbf{C}^2)^{\otimes n} \\ e_\mu \otimes e_\nu & \mapsto & \delta_{\varepsilon\eta} e_\mu \otimes e_\nu, \end{array} \tag{4.3}$$

where η is defined by $\mathscr{H} e_\mu = \eta e_\mu$ $(\eta = \pm)$. p^{ε} commutes with the action of $\Delta(U_q(D_n))$.

We need several lemmas similar to those in Sect. 3.

Lemma 4.1. *A highest weight vector in $V_k^{\varepsilon(n)}$ of unit length is given by $p^{\varepsilon}(v_k^{(n)})$.*

Hereafter, we will use the convention $V_k^{\varepsilon(n)} = \{0\}$ $(k > n)$.

Lemma 4.2. *Let $\lambda = \lambda_1\varepsilon_1 + \cdots + \lambda_n\varepsilon_n$ be a weight of $V_k^{\varepsilon(n)}$, then we have*

$$\begin{aligned} \dim (V_k^{\varepsilon(n)})_\lambda &= \frac{1}{2} \binom{n - (\lambda,\lambda)}{\frac{n-(\lambda,\lambda)}{2}} && (k=0), \\ &= \binom{n - (\lambda,\lambda)}{\frac{n-(\lambda,\lambda)-k}{2}} && (0 < k \leqq n). \end{aligned}$$

In particular, we can state the following:

(1) *If there exists an integer* j $(1 \leqq j \leqq n)$ *such that* $\lambda_j = \pm 1$, *then*

$$\dim (V_k^{\varepsilon(n)})_\lambda = \dim (V_k^{\varepsilon'(n-1)})_{\lambda'},$$

where $\varepsilon' = \pm \varepsilon$ *and* λ' *is defined in (3.2).*

(2)

$$\begin{aligned} \dim (V_k^{\varepsilon(n)})_0 &= \dim (V_1^{\varepsilon(n-1)})_0 && (k=0), \\ &= 2\dim (V_0^{\varepsilon(n-1)})_0 + \dim (V_2^{\varepsilon(n-1)})_0 && (k=1), \\ &= \dim (V_{k-1}^{\varepsilon(n-1)})_0 + \dim (V_{k+1}^{\varepsilon(n-1)})_0 && (k>1). \end{aligned}$$

We define the operators $Z_i^{\pm(j)}$ from $\Delta(X_i^\pm)$ in $\Delta(U_q(D_{n-1}))$ as follows: If $1 \leqq j < n-1$, then our definition is the same as in (3.3). Otherwise we set

$$\begin{aligned} Z_i^{\pm(n-1)} &= \Delta(X_i^\pm) && (1 \leqq i < n-2), \\ &= -\Delta(X_{n-2}^\pm)\Delta(X_{n-1}^\pm) && (i = n-2), \\ &= -\Delta(X_{n-2}^\pm)\Delta(X_n^\pm) && (i = n-1), \\ Z_i^{\pm(n)} &= \Delta(X_i^\pm) && (1 \leqq i < n-1), \\ &= \Delta(X_{n-1}^\pm)\Delta(X_{n-2}^\pm)\Delta(X_n^\pm) && (i = n-1). \end{aligned}$$

As in the $U_q(B_n)$ case, $Z_i^{\pm(j)}$ must be considered as operators in $\Delta(U_q(D_n))$. By a direct calculation, we have

Lemma 4.3. *Let* $\varepsilon, \eta = \pm$. *Then the following diagram is commutative.*

$$\begin{array}{ccccc} & V_{\mathrm{sp}}^{\varepsilon(n-1)} \otimes (\mathbf{C}^2)^{\otimes(n-1)} & \xrightarrow{\iota_j^\eta} & V_{\mathrm{sp}}^{\varepsilon\eta(n)} \otimes (\mathbf{C}^2)^{\otimes n} & \\ \Delta(X_i^{-\eta}) & \big\downarrow & & \big\downarrow & Z_i^{-\eta(j)}. \\ & V_{\mathrm{sp}}^{\varepsilon(n-1)} \otimes (\mathbf{C}^2)^{\otimes(n-1)} & \xrightarrow{\iota_j^\eta} & V_{\mathrm{sp}}^{\varepsilon\eta(n)} \otimes (\mathbf{C}^2)^{\otimes n} & \end{array}$$

We state our propositions on the construction of vectors in $(V_k^{\varepsilon(n)})_\lambda$.

Proposition 4.4. *Let* λ *be a weight of* $V_{\mathrm{sp}}^{\varepsilon(n)} \otimes (\mathbf{C}^2)^{\otimes n}$ *such that* $\lambda_j = \pm 1$. *We have*

$$(V_k^{\varepsilon(n)})_\lambda = \iota_j^\pm (V_k^{\varepsilon'(n-1)})_{\lambda'}.$$

Here $\varepsilon' = \pm \varepsilon$ *and* λ' *is given in (3.2).*

Proof. The proof is similar to that of Proposition 3.5. The difference is that Lemmas 3.1, 3.4 and 3.2 are replaced by Lemma 4.1, 4.3 and 4.2. q.e.d.

Proposition 4.5. *For* $0 \leqq k \leqq n$, $(V_k^{\varepsilon(n)})_0 = p^\varepsilon((\tilde{V}_k^{(n)})_0)$, *where* $(\tilde{V}_k^{(n)})_0$ *is obtained inductively as follows*:

$$\begin{aligned} (\tilde{V}_k^{(n)})_0 &= \bigoplus_{\varepsilon=\pm} p^\varepsilon((qe_{+-} + q^{-1}e_{-+}) \otimes (\tilde{V}_1^{(n-1)})_0) && (k=0), \\ &= (e_{+-} + e_{-+}) \otimes (\tilde{V}_0^{(n-1)})_0 \oplus (q^2 e_{+-} - q^{-2} e_{-+}) \otimes (\tilde{V}_2^{(n-1)})_0 && (k=1), \\ &= (q^{-(k-1)} e_{+-} + (-q)^{k-1} e_{-+}) \otimes (\tilde{V}_{k-1}^{(n-1)})_0 \\ &\quad \oplus (q^{k+1} e_{+-} - (-q)^{-(k+1)} e_{-+}) \otimes (\tilde{V}_{k+1}^{(n-1)})_0 && (1 < k \leqq n). \end{aligned}$$

Proof. We prove by the induction on n. First we show the following:

(1) $\tilde{V}_k^{(n)} \subset (q^{-(k-1)} e_{+-} + (-q)^{k-1} e_{-+}) \otimes \tilde{V}_{k-1}^{(n-1)}$ $(1 < k \leqq n)$,

(2) $\tilde{V}_1^{(n)} \subset (e_{+-} + e_{-+}) \otimes \tilde{V}_0^{(n-1)}$,

(3) $\tilde{V}_k^{(n)} \subset (q^{k+1} e_{+-} - (-q)^{-(k+1)} e_{-+}) \otimes \tilde{V}_{k+1}^{(n-1)} \quad (0 < k \leqq n)$,

(4) $\tilde{V}_0^{(n)} \subset \bigoplus_{\varepsilon = \pm} p^{\varepsilon}((q e_{+-} + q^{-1} e_{-+}) \otimes \tilde{V}_1^{(n-1)})$.

The proofs of (1) and (3) are similar to those of (1) and (2) in Proposition 3.6. We only deal with (2) and (4) here.

Let us prove (2). From the definition of $v_k^{(n)}$, we have

$$(\Delta(X_{n-1}^-) + \Delta(X_n^-)) v_1^{(n)} = e_{++} \otimes \cdots \otimes (e_{+-} + e_{-+}) \otimes (e_{++} + e_{--}).$$

Applying $\Delta(X_1^-) \cdots \Delta(X_{n-2}^-)$, we have $(e_{+-} + e_{-+}) \otimes v_0^{(n-1)} \in \tilde{V}_1^{(n)}$. The creation operators $\Delta(X_2^-), \ldots, \Delta(X_n^-)$ produce $(e_{+-} + e_{-+}) \otimes \tilde{V}_0^{(n-1)}$ from $(e_{+-} + e_{-+}) \otimes v_0^{(n-1)}$.

Let us proceed to (4). We have

$$\Delta(X_n^-) v_0^{(n)} = \overbrace{e_{++} \otimes \cdots \otimes e_{++}}^{n-2} \otimes (q e_{+-} \otimes e_{+-} + q^{-1} e_{-+} \otimes e_{-+}),$$

$$\Delta(X_{n-1}^-) v_0^{(n)} = \overbrace{e_{++} \otimes \cdots \otimes e_{++}}^{n-2} \otimes (q e_{+-} \otimes e_{-+} + q^{-1} e_{-+} \otimes e_{+-}).$$

Therefore, $p^{\varepsilon}(e_{++} \otimes \cdots \otimes e_{++} \otimes (q e_{+-} + q^{-1} e_{-+}) \otimes (e_{+-} + e_{-+})) \in \tilde{V}_0^{(n)}$ for $\varepsilon = \pm$. Note that p^{ε} commutes with $\Delta(U_q(D_n))$. We immediately have (4).

If $k = 0$, we have $\tilde{V}_k^{(n)} = V_0^{+(n)} \oplus V_0^{-(n)}$ and $\dim (V_0^{+(n)})_0 = \dim (V_0^{-(n)})_0$. If $0 < k \leqq n$, each vector $v \in (\tilde{V}_k^{(n)})_0$ constructed as above is of the form $v = p^+(v) + p^-(v)$. So we easily have

$$\begin{aligned} \dim (\tilde{V}_k^{(n)})_0 &= 2 \dim (V_0^{\varepsilon(n)})_0 \quad (k = 0), \\ &= \dim (V_k^{\varepsilon(n)})_0 \quad (0 < k \leqq n). \end{aligned}$$

(1) $\sim$ (4) together with Lemma 4.2 show the equality of the sets in the inductive construction of $(\tilde{V}_k^{(n)})_0$.

This completes the proof. q.e.d.

5. Quantum R Matrices

Let us proceed to construct the quantum R matrices corresponding to the spin representations. Following [5], we recall the characterization of $R(x)$. Let V_λ, V_μ be two irreducible representation spaces of $U_q (= U_q(B_n), U_q(D_n))$. The quantum R matrix $R(x)$ for $\hat{\mathfrak{g}} (= B_n^{(1)}, D_n^{(1)})$ is an element of $\mathrm{Hom}_{\mathbf{C}}(V_\lambda \otimes V_\mu, V_\mu \otimes V_\lambda)$, and is characterized uniquely (if it exists) by the following conditions:

(1) $[R(x), \Delta(X)] = 0 \quad \text{for all} \quad X \in U_q$, (5.1)

(2) $R(x)(x q^{H_0} \otimes X_0^+ + X_0^+ \otimes q^{-H_0}) = (q^{H_0} \otimes X_0^+ + x X_0^+ \otimes q^{-H_0}) R(x)$. (5.2)

The definitions of X_0^+ and H_0 are in (3.1) for $\hat{\mathfrak{g}} = B_n^{(1)}$ and $D_n^{(1)}$. Note that our R matrix here is usually written as $\check{R}(x)$, and is equal to $PR(x)$ in [5], where P denotes the transposition $P(a \otimes b) = b \otimes a$.

Now let us assume that every irreducible component V_ν in $V_\lambda \otimes V_\mu$ appears with multiplicity 1. The condition (5.1) means that $R(x)$ must have the form

$$R(x) = \sum_{\nu} \rho_\nu(x) P_\nu,$$

where $P_\nu \in \mathrm{Hom}_{U_q}(V_\lambda \otimes V_\mu, V_\mu \otimes V_\lambda)$ is such that $P_\nu|_{V_{\nu'}} \neq 0$ iff $\nu' = \nu$. By fixing the normalization of P_ν, the scalars $\rho_\nu(x)$ will be determined by the condition (5.2).

In this paper, we construct three R matrices. In the case of $B_n^{(1)}$, there is just one in $\mathrm{End}_{\mathbf{C}}(V_{\mathrm{sp}} \otimes V_{\mathrm{sp}})$. For the consideration in the case of $D_n^{(1)}$, we define the map τ as follows:

$$\begin{array}{ccc} (\mathbf{C}^4)^{\otimes n} & \xrightarrow{\tau} & (\mathbf{C}^4)^{\otimes n} \\ e_{\mu_1 \nu_1} \otimes \cdots \otimes e_{\mu_n \nu_n} & \mapsto & e_{\mu_1 \nu_1} \otimes \cdots \otimes e_{\mu_{n-1} \nu_{n-1}} \otimes e_{-\mu_n - \nu_n}. \end{array}$$

Set $\sigma(X_i^\pm) = X_i^\pm$ $(1 \leqq i < n-1)$, $X_n^\pm$ $(i = n-1)$, $X_{n-1}^\pm$ $(i = n)$. Then the following diagram is commutative for $\varepsilon = \pm$:

$$\begin{array}{ccccc} & V_{\mathrm{sp}}^+ \otimes V_{\mathrm{sp}}^\varepsilon & \xrightarrow{\tau} & V_{\mathrm{sp}}^- \otimes V_{\mathrm{sp}}^{-\varepsilon} & \\ \Delta(X_i^\pm) & \downarrow & & \downarrow & \Delta(\sigma(X_i^\pm)) \\ & V_{\mathrm{sp}}^+ \otimes V_{\mathrm{sp}}^\varepsilon & \xrightarrow{\tau} & V_{\mathrm{sp}}^- \otimes V_{\mathrm{sp}}^{-\varepsilon} & \end{array}$$

Thanks to this commutativity, we only need to consider two R matrices, which are elements of $\mathrm{End}_{\mathbf{C}}(V_{\mathrm{sp}}^+ \otimes V_{\mathrm{sp}}^+)$ and $\mathrm{Hom}_{\mathbf{C}}(V_{\mathrm{sp}}^+ \otimes V_{\mathrm{sp}}^-, V_{\mathrm{sp}}^- \otimes V_{\mathrm{sp}}^+)$, respectively.

The Case of $B_n^{(1)}$. From the above consideration, $R(x)$ is of the form

$$R(x) = \sum_{k=0}^{n} \rho_k^{(n)}(x) P_k^{(n)},$$

where $P_k^{(n)}$ is the projector onto $V_k^{(n)}$. Let $W_\lambda^{(n)}$ be the weight space in $V_{\mathrm{sp}}^{(n)} \otimes V_{\mathrm{sp}}^{(n)}$ of weight λ. $W_\lambda^{(n)} \neq \{0\}$ only if $\lambda = \lambda_1 \varepsilon_1 + \cdots + \lambda_n \varepsilon_n$ with $\lambda_i \in \{0, \pm 1\}$ for all i. From the condition (5.1), we easily see $R(x) W_\lambda^{(n)} \subset W_\lambda^{(n)}$. Putting $k = n - (\lambda, \lambda)$, we have

$$\begin{array}{ccc} W_\lambda^{(n)} & \cong & (\mathbf{C}^2)^{\otimes k} \\ e_\mu \otimes e_\nu & \mapsto & e_{\mu_{j_1}} \otimes \cdots \otimes e_{\mu_{j_k}}, \end{array} \tag{5.3}$$

where $j_1, \ldots, j_k$ are determined from the condition

$$\{j_1, \ldots, j_k\} = \{j \mid \lambda_j = 0\}, \quad j_1 < \cdots < j_k. \tag{5.4}$$

Let us define the operator $u(j)$ acting on $\mathbf{C}^2 = \mathbf{C}e_+ \oplus \mathbf{C}e_-$ as follows:

$$\begin{aligned} u(j)e_+ &= \frac{1}{q^j + q^{-j}}(q^j e_+ + (-)^{(j-1)/2} e_-), \\ u(j)e_- &= \frac{1}{q^j + q^{-j}}((-)^{(j-1)/2} e_+ + q^{-j} e_-). \end{aligned} \tag{5.5}$$

Next we define the operator $Q_j^{(k)}$ acting on $(\mathbf{C}^2)^{\otimes k}$ inductively,

$$\begin{aligned} & Q_j^{(k)} = u(-2j+1) \otimes Q_{j-1}^{(k-1)} + u(2j+3) \otimes Q_{j+1}^{(k-1)}, \\ & (Q_{-1}^{(k)} = Q_0^{(k)}, Q_j^{(k)} = 0 \, (j > k)). \end{aligned} \tag{5.6}$$

From Propositions 3.5 and 3.6, we immediately have

Proposition 5.1. *Put $k = n - (\lambda, \lambda)$. Then the following diagram is commutative.*

$$\begin{array}{ccccc} & W_\lambda^{(n)} & \cong & (\mathbf{C}^2)^{\otimes k} & \\ P_j^{(n)} & \downarrow & & \downarrow & Q_j^{(k)}. \\ & W_\lambda^{(n)} & \cong & (\mathbf{C}^2)^{\otimes k} & \end{array}$$

To determine $\rho_k^{(n)}(x)$, we need to show

Proposition 5.2.

(1) $P_j^{(n)}(q^{H_0}\otimes X_0^+)P_k^{(n)}=0,\quad P_j^{(n)}(X_0^+\otimes q^{-H_0})P_k^{(n)}=0$

$$(j\neq k, k\pm 2, (j,k)\neq(0,1),(1,0)),$$

(2) $P_{k-2}^{(n)}(q^{H_0}\otimes X_0^+)P_k^{(n)}=-q^{4k-2}P_{k-2}^{(n)}(X_0^+\otimes q^{-H_0})P_k^{(n)}\qquad (1\leqq k\leqq n)$,

$P_{k+2}^{(n)}(q^{H_0}\otimes X_0^+)P_k^{(n)}=-q^{-(4k+6)}P_{k+2}^{(n)}(X_0^+\otimes q^{-H_0})P_k^{(n)}\quad (-1\leqq k\leqq n-2)$,

where $P_{-1}^{(n)}=P_0^{(n)}$,

(3) $P_k^{(n)}(q^{H_0}\otimes X_0^+)P_k^{(n)}=P_k^{(n)}(X_0^+\otimes q^{-H_0})P_k^{(n)}$.

Proof. It suffices to check (1), (2) and (3) by applying these operators to the base vectors in $V_{\mathrm{sp}}^{(n)}\otimes V_{\mathrm{sp}}^{(n)}$ constructed in Sect. 3. Put $Z_1=q^{H_0}\otimes X_0^+, Z_2=X_0^+\otimes q^{-H_0}$. Since Z_1 and Z_2 change only the first and second components of $(\mathbf{C}^4)^{\otimes n}=V_{\mathrm{sp}}^{(n)}\otimes V_{\mathrm{sp}}^{(n)}$, it is convenient to represent the base vectors in $V_k^{(n)}=P_k^{(n)}(V_{\mathrm{sp}}^{(n)}\otimes V_{\mathrm{sp}}^{(n)})$ as $u_1\otimes u_2\otimes u_l^{(n-2)}$, where $u_1,u_2\in\mathbf{C}^4, u_l^{(n-2)}\in V_l^{(n-2)}$. From Propositions 3.5 and 3.6, each base vector has one of the following forms:

(A-1) $(q^{-(2k-1)/2}e_{+-}+(-)^k q^{(2k-1)/2}e_{-+})\otimes(q^{-(2k-3)/2}e_{+-}+(-)^{k-1}q^{(2k-3)/2}e_{-+})$
$\otimes u_{k-2}^{(n-2)}$,

(A-2) $(q^{-(2k-1)/2}e_{+-}+(-)^k q^{(2k-1)/2}e_{-+})\otimes(q^{(2k+1)/2}e_{+-}+(-)^k q^{-(2k+1)/2}e_{-+})$
$\otimes u_k^{(n-2)}$,

(A-3) $(q^{(2k+3)/2}e_{+-}+(-)^{k+1}q^{-(2k+3)/2}e_{-+})\otimes(q^{-(2k+1)/2}e_{+-}+(-)^{k+1}q^{(2k+1)/2}e_{-+})$
$\otimes u_k^{(n-2)}$,

(A-4) $(q^{(2k+3)/2}e_{+-}+(-)^{k+1}q^{-(2k+3)/2}e_{-+})\otimes(q^{(2k+5)/2}e_{+-}+(-)^{k+2}q^{-(2k+5)/2}e_{-+})$
$\otimes u_{k+2}^{(n-2)}$,

(B-1) $e_{++}\otimes(q^{-(2k-1)/2}e_{+-}+(-)^k q^{(2k-1)/2}e_{-+})\otimes u_{k-1}^{(n-2)}$,

(B-2) $e_{++}\otimes(q^{(2k+3)/2}e_{+-}+(-)^{k+1}q^{-(2k+3)/2}e_{-+})\otimes u_{k+1}^{(n-2)}$,

(C-1) $(q^{-(2k-1)/2}e_{+-}+(-)^k q^{(2k-1)/2}e_{-+})\otimes e_{++}\otimes u_{k-1}^{(n-2)}$,

(C-2) $(q^{(2k+3)/2}e_{+-}+(-)^{k+1}q^{-(2k+3)/2}e_{-+})\otimes e_{++}\otimes u_{k+1}^{(n-2)}$,

(D) $e_{++}\otimes e_{++}\otimes u_k^{(n-2)}$.

Note that if a vector contains e_{--} in the first or second component, this vector vanishes when we apply Z_1 or Z_2.

Firstly, let us examine the case (A-1). Applying Z_1 and Z_2 to the vector (A-1), we get $q^{2k-1}e_{--}\otimes e_{--}\otimes u_{k-2}^{(n-2)}$ and $-q^{-2k+1}e_{--}\otimes e_{--}\otimes u_{k-2}^{(n-2)}$, respectively. From Proposition 2.5, the vector $e_{--}\otimes e_{--}\otimes u_{k-2}^{(n-2)}$ is contained in $V_{k-2}^{(n)}$. So only the action of $P_{k-2}^{(n)}$ is non-trivial. Since $P_{k-2}^{(n)}$ is the identity operator on $V_{k-2}^{(n)}$, we have the lemma. The proof in the case of (A-2) $\sim$ (A-4) are similar.

Next, we consider the case (B-1). After applying Z_1 and Z_2, we get

$$-(-)^k q^{(2k-1)/2}e_{+-}\otimes e_{--}\otimes u_{k-1}^{(n-2)} \tag{5.7}$$

and

$$-q^{-(2k-1)/2}e_{-+}\otimes e_{--}\otimes u_{k-1}^{(n-2)}, \tag{5.8}$$

respectively. It is easy to see that (5.7) and (5.8) are in $V_{k-2}^{(n)}\oplus V_k^{(n)}$. So the actions of $P_{k-2}^{(n)}$ and $P_k^{(n)}$ are non-trivial. Let us consider the case of $P_{k-2}^{(n)}$. From Proposition 5.1, this projector acts on $\mathbf{C}^2\otimes\mathbf{C}\otimes V_{k-1}^{(n-2)}$ as $u(2k-1)\otimes 1\otimes \mathrm{id}$. Here

$\mathbf{C}^2 = \mathbf{C}e_{+-} \oplus \mathbf{C}e_{-+}, \mathbf{C} = \mathbf{C}e_{--}$ and $u(j)$ is defined in (5.5). Applying $P_{k-2}^{(n)}$ to (5.7–8), we have

$$-(-)^k \frac{q^{(2k-1)/2}}{q^{2k-1}+q^{-2k+1}}(q^{2k-1}e_{+-}+(-)^{k-1}e_{-+})\otimes e_{--}\otimes u_{k-1}^{(n-2)},$$

and

$$-\frac{q^{-(2k-1)/2}}{q^{2k-1}+q^{-2k+1}}((-)^{k-1}e_{+-}+q^{-2k+1}e_{-+})\otimes e_{--}\otimes u_{k-1}^{(n-2)}.$$

This shows the lemma. The cases (B-2) and (C-1, 2) are similar.

We finish the proof by showing the last case (D). Applying Z_1 and Z_2, we get $q^{-1}e_{+-}\otimes e_{+-}\otimes u_k^{(n-2)}$ and $qe_{-+}\otimes e_{-+}\otimes u_k^{(n-2)}$. In this case, the actions of the three projectors $P_{k-2}^{(n)}, P_k^{(n)}$ and $P_{k+2}^{(n)}$ are non-trivial. The actions on the first two components $\mathbf{C}^2\otimes\mathbf{C}^2$ ($\mathbf{C}^2 = \mathbf{C}e_{+-}\oplus\mathbf{C}e_{-+}$) are $u(2k-1)\otimes u(2k+1)$, $u(-2k+1)\otimes u(2k+1)+u(2k+3)\otimes u(-2k-1)$ and $u(-2k-3)\otimes u(-2k-1)$, respectively. The rest of the proof is left to the reader. q.e.d.

From Proposition 5.2, (5.2) reduces to the following recursive formula for $\rho_j^{(n)}(x)$:

$$\frac{\rho_j^{(n)}(x)}{\rho_{j-2}^{(n)}(x)} = \frac{xq^{2j-1}-q^{-2j+1}}{q^{2j-1}-xq^{-2j+1}} \quad (0<j\leqq n, \rho_{-1}^{(n)}(x)=\rho_0^{(n)}(x)).$$

Setting

$$\rho_0^{(n)}(x) = \prod_{i=1}^{n}\frac{q^{2i-1}-xq^{-2i+1}}{q^{2i-1}-q^{-2i+1}},$$

we have

$$\rho_j^{(n)}(x) = \prod_{i=1}^{j/2}\frac{q^{4i-3}-xq^{-4i+3}}{q^{4i-3}-q^{-4i+3}}\frac{xq^{4i-1}-q^{-4i+1}}{q^{4i-1}-q^{-4i+1}}\prod_{i=j+1}^{n}\frac{q^{2i-1}-xq^{-2i+1}}{q^{2i-1}-q^{-2i+1}} \qquad (j\text{: even}),$$

$$= \prod_{i=1}^{(j+1)/2}\frac{xq^{4i-3}-q^{-4i+3}}{q^{4i-3}-q^{-4i+3}}\frac{q^{4i-1}-xq^{-4i+1}}{q^{4i-1}-q^{-4i+1}}\prod_{i=j+2}^{n}\frac{q^{2i-1}-xq^{-2i+1}}{q^{2i-1}-q^{-2i+1}} \qquad (j\text{: odd}).$$

Summing up, we obtain the form of $R(x)$ as follows:

$$R(x) = \bigoplus_{\lambda} R(x)|_{W_\lambda^{(n)}},$$

$$R(x)|_{W_\lambda^{(n)}} = \sum_{j=0}^{k}\rho_j^{(n)}(x)Q_j^{(k)} \quad (k = n-(\lambda,\lambda)). \tag{5.9}$$

Here $Q_j^{(k)}$ is defined in (5.6). We identify $W_\lambda^{(n)}$ with $(\mathbb{C}^2)^{\oplus k}$ via (5.3).

The Case of $D_n^{(1)}$. In this case, two R matrices are to be considered: (I) $R(x)\in\mathrm{End}_{\mathbf{C}}(V_{\mathrm{sp}}^+\otimes V_{\mathrm{sp}}^+)$ and (II) $R(x)\in\mathrm{Hom}_{\mathbf{C}}(V_{\mathrm{sp}}^+\otimes V_{\mathrm{sp}}^-, V_{\mathrm{sp}}^-\otimes V_{\mathrm{sp}}^+)$. From (5.1), $R(x)$ is of the form

$$R(x) = \sum_{\substack{k=0\\ k\equiv s(2)}}^{n}\rho_k^{(n)}(x)P_k^{(n)},$$

where

$$s = 0(I), 1(II). \tag{5.10}$$

For k even, $P_k^{(n)}$ is the projector onto $V_k^{+(n)}$. For k odd, it is the map which sends each vector $p^+(u)\in V_j^{+(n)}$ to $p^-(u)\in V_k^{-(n)}$ if $j=k$ and to 0 if $j\neq k$. For the definition of p^ε, see (4.3). As in the case of $U_q(B_n)$, these R matrices have a block structure. Let $W_\lambda^{(n)}$ be the weight space in $(\mathbf{C}^2)^{\otimes n}\otimes(\mathbf{C}^2)^{\otimes n}$ of weight λ. For $\varepsilon=\pm$, we set

$$W_\lambda^{\varepsilon(n)} \stackrel{\text{def}}{=} W_\lambda^{(n)} \cap (V_{\text{sp}}^{\varepsilon(n)}\otimes(\mathbf{C}^2)^{\otimes n}).$$

$W_\lambda^{\varepsilon(n)}\neq\{0\}$ only if $\lambda=\lambda_1\varepsilon_1+\cdots+\lambda_n\varepsilon_n$ with $\lambda_i\in\{0,\pm1\}$ for all i. Note that $W_\lambda^{\varepsilon(n)}\subset V_{\text{sp}}^{\varepsilon(n)}\otimes V_{\text{sp}}^{\varepsilon'(n)}$, where $\varepsilon'=(-)^k\varepsilon(k=n-(\lambda,\lambda))$. Putting $k=n-(\lambda,\lambda)$, we have

$$\begin{array}{ccc} W_\lambda^{(n)} & \cong & (\mathbf{C}^2)^{\otimes k} \\ e_\mu\otimes e_\nu & \mapsto & e_{\mu_{j_1}}\otimes\cdots\otimes e_{\mu_{j_k}}, \end{array} \tag{5.11}$$

The determination of $\{j_1,\ldots,j_k\}$ is the same as in (5.4). By abuse of notation, define the projection p^ε ($\varepsilon=\pm$) as follows:

$$\begin{array}{ccccc} (\mathbf{C}^2)^{\otimes k} & \xrightarrow{p^\varepsilon} & p^\varepsilon((\mathbf{C}^2)^{\otimes k}) & \subset & (\mathbf{C}^2)^{\otimes k} \\ e_{\mu_1}\otimes\cdots\otimes e_{\mu_k} & \mapsto & \delta_{\varepsilon\eta}e_{\mu_1}\otimes\cdots\otimes e_{\mu_k}, & & \end{array}$$

where η is the signature of the product $\mu_1\cdots\mu_k$. Then we have the following commutative diagram:

$$\begin{array}{ccccc} & W_\lambda^{(n)} & \cong & (\mathbf{C}^2)^{\otimes k} & \\ p^\varepsilon & \downarrow & & \downarrow & p^\eta \\ & W_\lambda^{\varepsilon(n)} & \cong & p^\eta((\mathbf{C}^2)^{\otimes k}), & \end{array} \tag{5.12}$$

where $\eta=\varepsilon(-)^{\#\{j|\lambda_j=-1\}}$.

Define $\theta(j)$ to be 1 ($j>0$), 0 ($j<0$). For the case of $D_n^{(1)}$, let us define the operators $u(j)\in\operatorname{End}_{\mathbf{C}}(\mathbf{C}^2)$, $Q_j^{(k)}\in\operatorname{End}_{\mathbf{C}}((\mathbf{C}^2)^{\otimes k})$ as follows:

$$u(j)e_+=\frac{1}{q^j+q^{-j}}(q^je_++(-)^{(j/2)+\theta(j)}e_-),$$

$$u(j)e_-=\frac{1}{q^j+q^{-j}}((-)^{(j/2)+\theta(j)}e_++q^{-j}e_-)\quad\text{if}\quad j\neq0,$$

$$u(0)e_+=u(0)e_-=e_++e_-.$$

$$\begin{aligned} &Q_j^{(k)}=u(-2j+2)\otimes Q_{j-1}^{(k-1)}+u(2j+2)\otimes Q_{j+1}^{(k-1)}\quad(0<j\leqq k),\\ &Q_0^{(k)}=p^+\circ(u(2)\otimes Q_1^{(k-1)})\circ p^++p^-\circ(u(2)\otimes Q_1^{(k-1)})\circ p^-,\\ &(Q_j^{(k)}=0\ (j>k)). \end{aligned} \tag{5.13}$$

The next proposition is a direct consequence of Propositions 4.4 and 4.5.

Proposition 5.3. *Put $k=n-(\lambda,\lambda)$. Set $\varepsilon'=\varepsilon(-)^k$, $\eta=\varepsilon(-)^{\#\{j|\gamma_j=-1\}}$ and $\eta'=\eta(-)^k$. If $j\equiv k \bmod 2$, then the following diagram is commutative for $\varepsilon=\pm$:*

$$\begin{array}{ccccccc} & W_\lambda^{\varepsilon(n)} & \cong & p^\eta((\mathbf{C}^2)^{\otimes k}) & \xrightarrow{\text{inj.}} & (\mathbf{C}^2)^{\otimes k} & \\ P_j^{(n)} & \Big\downarrow & & & & \Big\downarrow & Q_j^{(k)}. \\ & W_\lambda^{\varepsilon'(n)} & \cong & p^{\eta'}((\mathbf{C}^2)^{\otimes k}) & \xleftarrow{\text{proj.}} & (\mathbf{C}^2)^{\otimes k} & \end{array}$$

Let $\tilde{P}_k^{(n)}$ be the projector onto $\tilde{V}_k^{(n)}$. Then we have

Proposition 5.4.

(1) $\tilde{P}_j^{(n)}(q^{H_0}\otimes X_0^+)\tilde{P}_k^{(n)}=0,\quad \tilde{P}_j^{(n)}(X_0^+\otimes q^{-H_0})\tilde{P}_k^{(n)}=0 \qquad (j\neq k, k\pm 2),$

(2) $\tilde{P}_{k-2}^{(n)}(q^{H_0}\otimes X_0^+)\tilde{P}_k^{(n)}=-q^{4k-4}\tilde{P}_{k-2}^{(n)}(X_0^+\otimes q^{-H_0})\tilde{P}_k^{(n)} \qquad (2\leqq k\leqq n),$

$\tilde{P}_{k+2}^{(n)}(q^{H_0}\otimes X_0^+)\tilde{P}_k^{(n)}=-q^{-(4k+4)}\tilde{P}_{k+2}^{(n)}(X_0^+\otimes q^{-H_0})\tilde{P}_k^{(n)} \qquad (0\leqq k\leqq n-2),$

(3) $\tilde{P}_k^{(n)}(q^{H_0}\otimes X_0^+)\tilde{P}_k^{(n)}=\tilde{P}_k^{(n)}(X_0^+\otimes q^{-H_0})\tilde{P}_k^{(n)}.$

The proof is similar to that of Proposition 5.2. Thanks to this proposition, we are led to the following formula:

$$\frac{\rho_j^{(n)}(x)}{\rho_{j-2}^{(n)}(x)}=\frac{xq^{2j-2}-q^{-2j+2}}{q^{2j-2}-xq^{-2j+2}}.$$

Setting

$$\rho_0^{(n)}(x)=\prod_{i=1}^{[n/2]}\frac{q^{4i-2}-xq^{-4i+2}}{q^{4i-2}-q^{-4i+2}},$$

$$\rho_1^{(n)}(x)=\prod_{i=1}^{[(n-1)/2]}\frac{q^{4i}-xq^{-4i}}{q^{4i}-q^{-4i}},$$

we have

$$\rho_j^{(n)}(x)=\prod_{i=1}^{j/2}\frac{xq^{4i-2}-q^{-4i+2}}{q^{4i-2}-q^{-4i+2}}\prod_{i=j/2+1}^{[n/2]}\frac{q^{4i-2}-xq^{-4i+2}}{q^{4i-2}-q^{-4i+2}}\qquad (j\text{: even}),$$

$$=\prod_{i=1}^{(j-1)/2}\frac{xq^{4i}-q^{-4i}}{q^{4i}-q^{-4i}}\prod_{i=(j+1)/2}^{[(n-1)/2]}\frac{q^{4i}-xq^{-4i}}{q^{4i}-q^{-4i}}\qquad (j\text{: odd}).$$

Fix s as in (5.10). Put $k=n-(\lambda,\lambda)$, $\eta=(-)^{\#\{j\mid\lambda_j=-1\}}$ and $\eta'=\eta(-)^s$. Then $R(x)$ is obtained as follows:

$$R(x)=\bigoplus_{\substack{\lambda\\(\lambda,\lambda)\equiv n-s(2)}} R(x)|_{W_\lambda^{+(n)}},$$

$$R(x)|_{W_\lambda^{+(n)}}=\sum_{\substack{j=0\\ j\equiv s(2)}}^{k}\rho_j^{(n)}(x)(p^{\eta'}\circ Q_j^{(k)}). \tag{5.14}$$

Here $Q_j^{(k)}$ is defined in (5.13). We identify $W_\lambda^{+(n)}$ with $p^\eta((\mathbf{C}^2)^{\otimes k})\subset(\mathbf{C}^2)^{\otimes k}$ via (5.12).

6. Conjecture on the 1D Configuration Sum

The quantum R matrix constructed in the previous section defines a vertex model on the two dimensional square lattice $\mathscr{L}$. Consider $R(x)$ in $\mathrm{End}_{\mathbf{C}}(V_{\mathrm{sp}}\otimes V_{\mathrm{sp}})$. We deal with (I) only and put $V_{\mathrm{sp}}=V_{\mathrm{sp}}^+$ in the case of $U_q(D_{n+1})$. Set

$$R(x)e_\alpha\otimes e_\beta=\sum R(x)_{\alpha\beta\mu\nu}e_\mu\otimes e_\nu.$$

The matrix element $R(x)_{\alpha\beta\mu\nu}$ is the Boltzmann weight of the configuration such that the fluctuation variables α,β,μ,ν are placed on the left, lower, upper and right bond around a vertex. This vertex model can also be formulated as a solvable face model as treated in [11]. We explain the formulation below. Let us consider the square lattice $\mathscr{L}^*$ dual to $\mathscr{L}$. On each site i of $\mathscr{L}^*$, associate a site variable σ_i, which takes values in the following set I:

$$I = \left\{ \sum_{j=0}^{n} a_j \Lambda_j \,|\, a_0 + a_1 + 2a_2 + \cdots + 2a_{n-1} + a_n = 1, a_j \in \mathbf{Z} \right\} \qquad \text{for } B_n^{(1)},$$

$$= \left\{ \sum_{j=0}^{n+1} a_j \Lambda_j \,|\, a_0 + a_1 + 2a_2 + \cdots + 2a_{n-1} + a_n + a_{n+1} = 1, a_j \in \mathbf{Z} \right\} \quad \text{for } D_{n+1}^{(1)}.$$

Here $\Lambda_0, \ldots, \Lambda_n (, \Lambda_{n+1})$ are the fundamental weights of the affine Lie algebra $\hat{\mathfrak{g}} = B_n^{(1)}, D_{n+1}^{(1)}$ [16]. I is the set of level 1 integral weights in $\mathbf{Z}\Lambda_0 \oplus \cdots \oplus \mathbf{Z}\Lambda_n (\oplus \mathbf{Z}\Lambda_{n+1})$. An element of I will be called a "local state." With each configuration of local states (a, b, c, d) around a face (ordered clockwise from the NW corner), we associate the Boltzmann weight $W(a, b, c, d)$ as follows:

$$W(a,b,c,d) = R(x)_{\alpha\beta\mu\nu} \quad \text{if} \quad b = a + \mu, \quad d = a + \alpha, \quad c = a + \mu + \nu = a + \alpha + \beta,$$
$$= 0 \qquad \text{otherwise.}$$

Now let us turn to the calculation of the local state probabilities (LSPs). By definition the LSP $P(a)$ is the probability of finding a at a particular site, say $i = 1$,

$$P(a) = \frac{1}{Z} \sum_{\text{config.}} \delta_{\sigma_1 = a} \prod_{\text{face}} W(\sigma_i, \sigma_j, \sigma_k, \sigma_l),$$

$$Z = \sum_{\text{config.}} \prod_{\text{face}} W(\sigma_i, \sigma_j, \sigma_k, \sigma_l).$$

In the calculation, we fix the σ_i on the boundary of the lattice to a "ground state," which will be specified below, and take the infinite volume limit.

Hereafter, we will assume $n \geqq 2$. Let π denote the projection from the weight lattice of $D_{n+1}^{(1)}$ to that of $B_n^{(1)}$ induced from the embedding $B_n^{(1)} \hookrightarrow D_{n+1}^{(1)}$, i.e.,

$$\pi(\Lambda_j) = \Lambda_j \quad (0 \leqq j \leqq n),$$
$$= \Lambda_n \quad (j = n + 1).$$

Let $\mathfrak{h}$ be the Cartan subalgebra of D_{n+1}. Note that the vectors ε_j $(j = 1, \ldots, n+1)$ in $\mathfrak{h}^*$ are contained in the weight lattice of $D_{n+1}^{(1)}$ and π sends ε_j to ε_j $(1 \leqq j \leqq n)$, 0 $(j = n+1)$. $\pi(\mathfrak{h}^*)$ is identified with the dual space of the Cartan subalgebra of B_n. Next let $\sigma = (\sigma_1, \ldots, \sigma_s)$ be a sequence such that $\sigma_j = \pm\frac{1}{2}$ for all j. With each σ we associate a "height" ht. The rule is given inductively as follows:

$$ht(\phi) = 0,$$
$$ht((\sigma_1, \ldots, \sigma_s)) = \max(0, ht((\sigma_2, \ldots, \sigma_s)) + 2\sigma_1).$$

Let $\mu = \mu_1\varepsilon_1 + \cdots + \mu_n\varepsilon_n$, $\nu = \nu_1\varepsilon_1 + \cdots + \nu_n\varepsilon_n$ be weights of the spin representation of $U_q(B_n)$. Set $\sigma(\mu, \nu) = (\mu_{j_1}, \ldots, \mu_{j_s})$, where $j = j_i$ iff $\mu_j + \nu_j = 0$ $(j_1 < \cdots < j_s)$.

We will need the following limiting value of $R(x)_{\alpha\beta\mu\nu}$.

Proposition 6.1.

$$\lim_{q \to 0,\, x:\text{fix}} R(x)_{\alpha\beta\mu\nu} = \delta_{\alpha\mu}\delta_{\beta\nu} x^{H(\mu,\nu)},$$

where

$$H^{B_n^{(1)}}(\mu, \nu) = n - \left[\frac{ht(\sigma(\mu, \nu)) + 1}{2}\right],$$

$$H^{D_{n+1}^{(1)}}(\mu, \nu) = H^{B_n^{(1)}}(\pi(\mu), \pi(\nu)) + \left[\frac{n+1}{2}\right] - n. \tag{6.1}$$

Now let us specify the ground states. The local states of our ground state are constant along NE–SW direction. Hence, a ground state can be specified by a one dimensional sequence of local states $(p^{(j)})_{j\in\mathbf{Z}}$. Hereafter, we deal exclusively with the case $0 < q < 1$, $|x| < 1$. For later use, it is convenient to label the ground states as follows. Let Λ be a level 1 dominant integral weight of $D_{n+1}^{(1)}$. Let i be either 0 or 1. We set for $j\in\mathbf{Z}$,

$$D_{n+1}^{(1)}(n\text{: odd}) \quad p_\Lambda^{(j)} = \begin{cases} \Lambda_i & (j\text{: odd}) \\ \Lambda_{n+1-i} & (j\text{: even}) \end{cases} \quad \text{if} \quad \Lambda = \Lambda_i,$$

$$= \begin{cases} \Lambda_{n+i} & (j\text{: odd}) \\ \Lambda_{1-i} & (j\text{: even}) \end{cases} \quad \text{if} \quad \Lambda = \Lambda_{n+i},$$

$$D_{n+1}^{(1)}(n\text{: even}) \quad p_\Lambda^{(j)} = \begin{cases} \Lambda_i & (j \equiv 1) \\ \Lambda_{n+1-i} & (j \equiv 2) \\ \Lambda_{1-i} & (j \equiv 3) \\ \Lambda_{n+i} & (j \equiv 0) \end{cases} \quad \text{if} \quad \Lambda = \Lambda_i,$$

$$= \begin{cases} \Lambda_{n+i} & (j \equiv 1) \\ \Lambda_i & (j \equiv 2) \\ \Lambda_{n+1-i} & (j \equiv 3) \\ \Lambda_{1-i} & (j \equiv 0) \end{cases} \quad \text{if} \quad \Lambda = \Lambda_{n+i}. \tag{6.2}$$

The symbol $\equiv$ signifies the congruence modulo 4. These are the ground states for $D_{n+1}^{(1)}$. In the case of $B_n^{(1)}$, $(\pi(p_\Lambda^{(j)}))_{j\in\mathbf{Z}}$ ($\Lambda = \Lambda_i, \Lambda_{n+i}, i = 0, 1$) are the ground states. Note that Λ is still a dominant integral weight of $D_{n+1}^{(1)}$.

For our purpose of calculating the LSPs, Baxter's corner transfer matrix [10] turns out be a powerful tool. In his argument, the essential part lies in the evaluation of the "one dimensional (1D) configuration sum." For the preparation, let us introduce a terminology. A sequence of local states $p = (p^{(j)})_{j\geqq 1}$ is called a "Λ-path" if it satisfies the following:

(1) $p^{(j+1)} - p^{(j)}$ is a weight in V_{sp} for all j,
(2) $p^{(j)} = p_\Lambda^{(j)}$ if $j \gg 1$.

Here $p_\Lambda^{(j)}$ should be replaced by $\pi(p_\Lambda^{(j)})$ in the case of $B_n^{(1)}$. Let $\mathscr{P}(\Lambda)$ denote the set of Λ-paths. Set $\eta^{(j)}(p) = p^{(j+1)} - p^{(j)}$. 1D configuration sum $f(a, \Lambda; q)$ ($a\in I$) is defined as follows:

$$f(a, \Lambda; q) = \sum_{\substack{p\in\mathscr{P}(\Lambda) \\ p^{(1)} = a}} q^{\omega(p)}$$

$$\omega(p) = \sum_{j=1}^{\infty} j(H(\eta^{(j)}(p), \eta^{(j+1)}(p)) - H(\eta^{(j)}(p_\Lambda), \eta^{(j+1)}(p_\Lambda))). \tag{6.3}$$

In the case of $B_n^{(1)}$, we replace $p_\Lambda^{(j)}$ with $\pi(p_\Lambda^{(j)})$ in (6.3). Let $L(\Lambda)$ be the irreducible highest weight module with highest weight Λ, and let $\hat{\mathfrak{h}}$ be the Cartan subalgebra of $\hat{\mathfrak{g}} = B_n^{(1)}, D_{n+1}^{(1)}$. If $\hat{\mathfrak{g}} = B_n^{(1)}$, we consider $L(\Lambda)$ as $B_n^{(1)}$ module. For $\mu \in \hat{\mathfrak{h}}^*$, we set

$$L(\Lambda)_\mu = \{v \in L(\Lambda) \mid hv = \mu(h)v \text{ for } h \in \hat{\mathfrak{h}}\}.$$

Let δ be the null root. The generating function $\sum_i \dim L(\Lambda)_{\mu - i\delta} q^i$ is the string function studied in [12].

Now we can state our conjecture on the 1D configuration sum.

Conjecture. *With these notations, we have*

$$f(a, \Lambda; q) = \sum_i \dim L(\Lambda)_{a - i\delta} q^i.$$

We remark here that if we admit the conjecture in the case of $D_{n+1}^{(1)}$, we can show the one in the case of $B_n^{(1)}$.

With a in I of $\hat{\mathfrak{g}}$ ($\mathfrak{g} = B_n, D_{n+1}$), we associate its classical part $\bar{a} \in \mathfrak{h}^*$, where $\mathfrak{h}$ is the Cartan subalgebra of $\mathfrak{g}$. If $a = \sum_j a_j \Lambda_j$, then $\bar{a} = \sum_j a_j \bar{\Lambda}_j$. Here

$$\begin{aligned}
\bar{\Lambda}_j &= 0 && (j = 0) && \text{for } B_n^{(1)}, D_{n+1}^{(1)}, \\
&= \varepsilon_1 + \cdots + \varepsilon_j && (1 \leqq j \leqq n-1) && \text{for } B_n^{(1)}, D_{n+1}^{(1)}, \\
&= \tfrac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_n) && (j = n) && \text{for } B_n^{(1)}, \\
&= \tfrac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_n - \varepsilon_{n+1}) && (j = n) && \text{for } D_{n+1}^{(1)}, \\
&= \tfrac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_n + \varepsilon_{n+1}) && (j = n+1) && \text{for } D_{n+1}^{(1)}.
\end{aligned}$$

Set $s(a, \Lambda) = \frac{1}{2}((\bar{a}, \bar{a}) - (\bar{\Lambda}, \bar{\Lambda}))$, $t(a, \Lambda) = (\bar{a} - \bar{\Lambda}, \bar{a} - \bar{\Lambda})$. In the case of $B_n^{(1)}, \bar{\Lambda}$ should be replaced by $\overline{\pi(\Lambda)}$.

We write down the explicit form of $f(a, \Lambda; q)$ below.

$$\begin{aligned}
&D_{n+1}^{(1)} && f(a, \Lambda; q) = q^{s(a, \Lambda)} / \varphi(q)^{n+1}, \\
&B_n^{(1)} && f(a, \Lambda; q) = q^{s(a, \Lambda)} E(-q^2, q^4) / \varphi(q)^{n+1} && \text{if } \Lambda = \Lambda_0, \Lambda_1 \text{ and } t(a, \Lambda) \text{ even}, \\
& && \qquad = q^{s(a, \Lambda) + 1/2} E(-1, q^4) / \varphi(q)^{n+1} && \text{if } \Lambda = \Lambda_0, \Lambda_1 \text{ and } t(a, \Lambda) \text{ odd}, \\
& && \qquad = q^{s(a, \Lambda)} E(-q, q^4) / \varphi(q)^{n+1} && \text{if } \Lambda = \Lambda_n, \Lambda_{n+1}.
\end{aligned}$$

Here $\varphi(q) = \prod_{k \geqq 1} (1 - q^k)$, $E(z, x) = \prod_{k \geqq 1} (1 - zx^{k-1})(1 - z^{-1}x^k)(1 - x^k)$.

Appendix

In this Appendix, we give a trigonometric version of Reshetikhin's recursive formula for our $R(x)$. We deal with the $B_n^{(1)}$ case only. Hereafter, the upper (n) indicates the rank of B_n. Let us consider the following isomorphism:

$$\begin{aligned}
V_{\mathrm{sp}}^{(n)} \otimes V_{\mathrm{sp}}^{(n)} &\cong \mathbf{C}^4 \otimes (V_{\mathrm{sp}}^{(n-1)} \otimes V_{\mathrm{sp}}^{(n-1)}), \\
e_\mu \otimes e_\nu &\mapsto e_{\mu_1 \nu_1} \otimes (e_{\mu'} \otimes e_{\nu'}).
\end{aligned}$$

Here $e_{\mu'}$ signifies $e_{\mu_2} \otimes \cdots \otimes e_{\mu_n}$ if $e_\mu = e_{\mu_1} \otimes \cdots \otimes e_{\mu_n}$. We identify these two vector spaces. Let v be a vector in $V_{\mathrm{sp}}^{(n-1)} \otimes V_{\mathrm{sp}}^{(n-1)}$. The Reshetikhin's formula reads as follows:

$$R^{(n)}(x) e_{\pm\pm} \otimes v = \frac{q^{2n-1} - xq^{-2n+1}}{q^{2n-1} - q^{-2n+1}} e_{\pm\pm} \otimes R^{(n-1)}(x) v,$$

$$R^{(n)}(x)e_{+-}\otimes v = e_{+-}\otimes R^{(n-1)}(xq^{-4})(R^{(n-1)}(q^{-4}))^{-1}v + \frac{q^{2n-2}(1-x)}{q^{2n-1}-q^{-2n+1}}e_{-+}\otimes R^{(n-1)}(xq^{-4})v,$$

$$R^{(n)}(x)e_{-+}\otimes v = \frac{q^{2n-2}(1-x)}{q^{2n-1}-q^{-2n+1}}e_{+-}\otimes R^{(n-1)}(xq^{-4})v + xe_{-+}\otimes R^{(n-1)}(xq^{-4})(R^{(n-1)}(q^{-4}))^{-1}v.$$

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