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# At the Other Side of a Saddle-Node

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**Abstract.** We describe phenomena occurring just before a saddle-node bifurcation for one-parameter families of interval maps. In particular, as a parameter approaches the bifurcation value, attracting periodic orbits of periods $k, k+1, k+2, k+3, \ldots$ can appear. We make a detailed study of a family of "cusp-shaped" maps, where this phenomenon occurs in a pure form.

## 1. Introduction

For a one parameter family $f_\mu$ of maps of an interval into itself, a saddle-node bifurcation occurs when the graph of $f_\mu$ (or $f_\mu^n$) touches the diagonal and then crosses it. A fixed point and immediately after it – a pair of fixed points (respectively a periodic point of period $n$ and then a pair of them) appears; one of these points is attracting and the other one repelling. However, here we will not be interested in these fixed (periodic) points. Instead, we shall look what happens at the other side of the saddle-node bifurcation, i.e. for these parameters for which a fixed point is not created yet.

This situation has been considered by Newhouse, Palis, and Takens in [NPT]. However, their aim was different than ours and hence [NPT] does not contain explicit statements of the results interesting to us. We shall restate (and reprove) these results in the form showing clearly what is going on.

The main phenomenon that may be observed in some families, is period adding. As the parameter approaches the bifurcation value, attracting periodic orbits of period $k$, $k+1$, $k+2$, $k+3, \ldots$ (or $k$, $k+n$, $k+2n$, $k+3n, \ldots$ if the bifurcation occurs for $f_\mu^n$) appear. However, in many cases there are many other periods of attracting periodic orbits which appear in the considered interval of parameters. This has to happen for instance if the maps $f_\mu$ are smooth and unimodal (see [MSS, CE]). This is why we turned to the investigation of unimodal maps which are smooth except at the critical point, where the derivative is discontinuous (and bounded away from zero).

Such "cusp-shaped" maps appear in many experimental or model systems, e.g. the Lorenz model [L], models of flip-flop process in visual perception [AB] or a bromate-chlorite-iodide oscillator [MAE]. We have also obtained this type of a map in a model of coupled enzymatic reactions with inhibition by an excess of substrates and products [KML]. The properties of such maps are much less investigated than the properties of smooth unimodal maps.

The main purpose of this paper is to present an example of a one-parameter family of such maps, in which the period adding phenomenon appears in a pure form. That is, every periodic attracting orbit is of the type predicted by the Newhouse-Palis-Takens theory. We prove this rigorously for parameter values close to the bifurcation value. However, the computer experiments suggest that this is so for all parameter values.

## 2. Local Theory

Let $J$ be a closed interval, $c \in \operatorname{int}(J)$, $\alpha > 0$. Let $f:[0,\alpha] \times J \to \mathbb{R}$ be a map of class $C^k$ $(k \geqq 3)$ such that if we denote $f_\mu = f(\mu, x)$ then

$$f_0(c) = c, \tag{2.1}$$

$$f_\mu(x) > x \quad \text{for each} \quad (\mu, x) \neq (0, c), \tag{2.2}$$

$$f'_\mu(x) > 0 \quad \text{for each} \quad \mu, x, \tag{2.3}$$

$$f''_\mu(x) \geqq 0 \quad \text{for each} \quad \mu, x, \tag{2.4}$$

$$f_\mu(x) > f_\nu(x) \quad \text{for each} \quad x \quad \text{and} \quad \mu > \nu. \tag{2.5}$$

We shall use the following theorem on embedding of our family of maps into a family of flows. It has been proved by Yoccoz [Y]; another proof has been given independently by Skrzypczak [S]. It is more convenient to use this theorem than the weaker result of Newhouse, Palis, and Takens [NPT], which gives only an approximate embedding.

**Theorem 2.1** ([Y, S]). *If $f$ satisfies* (2.1)–(2.5) *then there exists a map $X:[0,\alpha] \times J \to \mathbb{R}$ of class $C^1$ which is of class $C^{k-1}$ except at the point* $(0, c)$, *such that if we denote $X_\mu(x) = X(\mu, x)$ and $(\varphi^t_\mu)_{t \in \mathbb{R}}$ is the flow of the vectorfield $X_\mu$ then $\varphi^1_\mu = f_\mu$ for all $\mu$. Moreover, the vectorfield $X_0$ is uniquely determined by $f_0$.*

From this theorem we can derive the main technical tool for further proofs.

**Theorem 2.2.** *Assume that $f$ of class $C^k$ satisfies* (2.1)–(2.5). *Let $a, b \in J$, $a < c < b$. Then*

(a) *For any sufficiently large $n$ there exists a unique $\mu_n$ such that $f^n_{\mu_n}(a) = b$.*
(b) *For every $d \in (a, c)$ there exists $l \geqq 0$ such that for $n$ sufficiently large, the iterates $f^{n-l}_{\mu_n}$ are defined on $[a, d]$ and $f^{n-l}_{\mu_n}([a, d]) \subset J$. Then the sequence $(f^{n-l}_{\mu_n})$ is uniformly convergent with $k-1$ derivatives on the interval $[a, d]$. The number $l$ can be chosen arbitrarily large.*
(c) *If $a$ and $d$ are fixed and $b$ is sufficiently close to $c$ then the integer $l$ above can be chosen equal to* 0.

*Proof.* The existence of $\mu_n$ follows from (2.1), (2.2), (2.5) and continuity. The uniqueness of $\mu_n$ follows from (2.5). This proves (a).

Now we use Theorem 2.1. We have $f_{\mu_n}=\varphi^1_{\mu_n}$, so $f^{n-l}_{\mu_n}(x)=\varphi^{n-l}_{\mu_n}(x)$ for all $l, x$ such that $f^{n-l}_{\mu_n}(x)$ is defined and belongs to $J$. For any $x\in[a,c]$ there exists a unique $t_n(x)$ such that $x=\varphi^{t_n(x)}_{\mu_n}(a)$. Then

$$f^{n-l}_{\mu_n}(x)=\varphi^{n-l+t_n(x)}_{\mu_n}(a)=\varphi^{t_n(x)-l}_{\mu_n}(b).$$

By the Implicit Function Theorem, $t_n(x)$ depends on $x$ in a $C^{k-1}$ way and the functions $t_n$ converge to $t_\infty$, defined by $x=\varphi_0^{t_\infty(x)}(a)$, uniformly with $k-1$ derivatives on the interval $[a,d]$. There exists $l\geqq 0$ such that $\varphi_0^{t_\infty(d)-l}(b)$ exists and belongs to $\operatorname{int}J$. Then for $n$ sufficiently large, $\varphi^{t_n(x)-l}_{\mu_n}(b)$ exists and belongs to $J$ for all $x\in[a,d]$. Moreover, $\varphi^{t_n(\cdot)-l}_{\mu_n}(b)$ converges uniformly with $k-1$ derivatives to $\varphi^{t_\infty(\cdot)-l}_{\mu_n}$ on $[a,d]$. It is clear that $l$ can be chosen arbitrarily large. This proves (b).

If $a$ and $d$ are fixed and $b$ is sufficiently close to $c$ then, since $X_0(c)=0$, it follows that $\varphi_0^{t_\infty(d)}(b)$ exists and belongs to $\operatorname{int}J$. This proves (c). □

*Remark 2.3.* Sometimes we shall use not only Theorem 2.2, but also the explicit form of the limit of the sequence $(f^{n-l}_{\mu_n})$. This limit is equal to $g_{l,a,b}(x)=\varphi_0^{t(x)-l}(b)$, where $\varphi_0^{t(x)}(a)=x$. Note that $g_{l,a,b}$ depends only on $f_0$, $l$, $a$, and $b$. If $f_0$ is fixed then we get in fact only one-parameter family of limit maps. This is due to the fact that if for some $s$ we have $a_1=\varphi_0^s(a)$ and $b_1=\varphi_0^{s-l+l_1}(b)$, then $g_{l,a,b}=g_{l_1,a_1,b_1}$.

## 3. Global Theory

In this section we assume that locally the situation is as in the previous section. We shall investigate the global behaviour of the iterates of $f_\mu$.

Let $I$ be a closed interval, $\alpha>0$ and $f:[0,\alpha]\times I\to I$ a map of class $C^k$ ($k\geqq 3$). We denote as before, $f_\mu(x)=f(\mu,x)$. We assume that there exists an interval $J\subset I$ and $c\in\operatorname{int}J$ such that (2.1)–(2.5) are satisfied. Under these assumptions we have the following theorem.

**Theorem 3.1.** *Let $a$, $b$, and $\mu_n$ be as in Theorem 2.2. If for some $x\in I$ and $m\geqq 0$ we have $f_0^m(x)\in(a,c)$ then the sequence $(f^n_{\mu_n})_{n=1}^\infty$ converges uniformly with $k-1$ derivatives in some neighbourhood of $x$.*

*Proof.* For some neighbourhood $U$ of $x$, some $d\in(a,c)$ and all $n$ sufficiently large we have $f^m_{\mu_n}(U)\subset(a,d)$. Then for $y\in U$ we have

$$f^n_{\mu_n}(y)=f^{l-m}_{\mu_n}\circ f^{n-l}_{\mu_n}(f^m_{\mu_n}(y)),$$

where $l\geqq m$ is chosen as in Theorem 2.2 (b). Since $f^m_{\mu_n}$ and $f^{l-m}_{\mu_n}$ converge uniformly with $k$ derivatives to $f_0^m$ and $f_0^{l-m}$ respectively as $n\to\infty$, and $f^{n-l}_{\mu_n}$ converges uniformly on $[a,d]$ with $k-1$ derivatives, the sequence $(f^n_{\mu_n})$ converges uniformly on $U$ with $k-1$ derivatives. □

Under the same assumptions we have also the following theorem.

**Theorem 3.2.** *Let $a$, $b$, and $\mu_n$ be as in Theorem 2.2 and let $X_0$ be as in Theorem 2.1. Assume that for some $p>0$ we have $f_0^p(b)=a$ and*

$$(f_0^p)'(b)\cdot\frac{X_0(b)}{X_0(a)}\neq 1.$$

*Then there exists a neighbourhood U of b such that if n is sufficiently large then there is a unique $x_n \in U$ with $f_{\mu_n}^{p+n}(x_n) = x_n$. Moreover, as $n \to \infty$ then $x_n \to b$ and*

$$(f_{\mu_n}^{p+n})'(x_n) \to (f_0^p)'(b) \cdot \frac{X_0(b)}{X_0(a)}.$$

Before proving this theorem, we shall recall the following simple lemma (see e.g. [S]).

**Lemma 3.3.** *Under the assumptions of Theorem 2.1, if $x, f_\mu^s(x) \in J$, then*

$$(f_\mu^s)'(x) = \frac{X_\mu(f_\mu^s(x))}{X_\mu(x)}.$$

*Proof*. By Theorem 2.1, we have $s = \int_x^{f_\mu^s(x)} \frac{dt}{X_\mu(t)}$. Taking the derivative of both sides of this equality, we obtain

$$0 = (f_\mu^s)'(x) \cdot \frac{1}{X_\mu(f_\mu^s(x))} - \frac{1}{X_\mu(x)}.$$

Hence, $(f_\mu^s)'(x) = \dfrac{X_\mu(f_\mu^s(x))}{X_\mu(x)}$. □

*Proof of Theorem 3.2.* By Theorem 3.1 applied to $m = p+1$ and $x = b$ we get that the sequence $(f_{\mu_n}^n)_{n=1}^\infty$ converges uniformly with $k-1$ derivatives in some neighbourhood of $b$. Hence the same is true for the sequence $(f_{\mu_n}^{n+p})_{n=1}^\infty$. The limit function of this sequence is $h = f^{l-1} \circ g_{l,a,b} \circ f_0^{p+1}$ (see Remark 2.3 and the proof of Theorem 3.1). We have $f_0^{p+1}(b) = f_0(a)$ and $g_{l,a,b}(f_0(a)) = (f_0|_J)^{-l+1}(b)$, so $h(b) = b$.

Since $g_{l,a,b}$ is the limit of the sequence $(f_{\mu_n}^{n-l})$, we get by Lemma 3.3 that $g'_{l,a,b}$ is the limit of the sequence $\left(\dfrac{X_{\mu_n} \circ f_{\mu_n}^{n-l}}{X_{\mu_n}}\right)$. This limit is equal to $\dfrac{X_0 \circ g_{l,a,b}}{X_0}$. Hence, we get

$$h'(b) = (f_0^{l-1})'((f_0|_J)^{-l+1}(b)) \cdot \frac{X_0((f_0|_J)^{-l+1}(b))}{X_0(f_0(a))} \cdot (f_0^{p+1})'(b).$$

Again by Lemma 3.3, we have

$$(f_0^{l-1})' = \frac{X_0 \circ f_0^{l-1}}{X_0}$$

and

$$(f_0^{p+1})'(b) = (f_0)'(a) \cdot (f_0^p)'(b) = \frac{X_0(f_0(a))}{X_0(b)} \cdot (f_0^p)'(b).$$

Therefore

$$h'(b) = \frac{X_0(b)}{X_0(a)} \cdot (f_0^p)'(b).$$

Now the assertion of the theorem follows immediately. □

*Remark 3.4.* If in Theorem 3.2 we have

$$\left|\frac{X_0(b)}{X_0(a)}\cdot(f_0^p)'(b)\right|<1$$

then for sufficiently large $n$ the periodic orbit of the point $x_n$ is attracting.

Theorem 3.2 together with Remark 3.4 give us a period-adding phenomenon, described in the introduction.

## 4. Scaling

Another well-known thing is the scaling law. If the tangency of the graph of $f_0$ to the diagonal is of $k^{\text{th}}$ order ($k \geqq 2$) then we can write $X_\mu(x) \approx \alpha x^k + \beta\mu$ $\left(\text{we set here } c=0 \text{ and assume that } \frac{\partial f}{\partial \mu} > 0\right)$. Let us make computations for $X_\mu(x) = x^k + \mu$:

$$n = \int_a^b \frac{dz}{z^k+\mu_n} = \frac{1}{\mu_n}\int_a^b \frac{dz}{\frac{1}{\mu_n}z^k+1} = \frac{1}{\mu_n}\int_{\mu_n^{-1/k}a}^{\mu_n^{-1/k}b} \frac{dt}{t^k+1}\cdot\mu_n^{1/k}$$

(we have used the substitution $t=\mu_n^{-1/k}z$). Since $a<0<b$ and $\mu_n^{-1/k}\to\infty$ as $k\to\infty$, the last integral tends to the finite limit $\int_{-\infty}^{\infty}\frac{dt}{t^k+1}$, and therefore

$$n \approx \text{const}\cdot\mu_n^{1/k-1}.$$

In the general case the result will be the same, only the constant will perhaps change. Hence, we get the following scaling law:

$$\mu_n \approx \text{const}\cdot n^{-\frac{k}{k-1}}.$$

In the generic case we have $k=2$ and then

$$\mu_n \approx \text{const}\cdot n^{-2}.$$

## 5. An Example

We want to give an example where the described phenomena appear in a "pure" form. Set

$$f_{\gamma,\varphi,\beta,\eta}(x) = \begin{cases} \dfrac{\varphi}{\beta}\left(1-\dfrac{\varphi-\beta}{\varphi-x}\right) & \text{if } x \leqq \beta, \\[2ex] \gamma^2\left(\dfrac{1}{\beta-\eta}-\dfrac{1}{x-\eta}\right) & \text{if } x \geqq \beta. \end{cases}$$

If $0<\beta<1$, $\eta<\beta<\varphi$ and $\gamma^2 \leqq \dfrac{(\beta-\eta)(1-\eta)}{1-\beta}$ then $f_{\gamma,\varphi,\beta,\eta}$ maps the interval $[0,1]$ onto itself.

We fix $\varphi$, $\beta$, and $\eta$ and consider the one parameter family obtained in such a way. There are many ways of doing this which suit our purposes. We choose one particular family and investigate it closer. Namely, we set $\varphi=0.27$, $\beta=0.25$, $\eta=0$ and denote $f_\gamma=f_{\gamma,\varphi,\beta,\eta}$. We obtain the following formulas.

$$f_\gamma(x)=\begin{cases} 1.08-\dfrac{0.0216}{0.27-x} & \text{if} \quad x\leqq 0.25, \\ \gamma^2\left(4-\dfrac{1}{x}\right) & \text{if} \quad x\geqq 0.25, \end{cases}$$

$0<\gamma\leqq\sqrt{1/3}$. We get the saddle-node bifurcation for $\gamma=0.5$ at the point $x=0.5$. We can use the previous results with $\mu=0.5-\gamma$ and replacing $x$ by $0.5-x$.

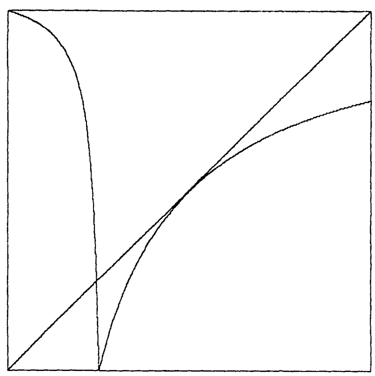

**Fig. 1.** The graph of $f_\gamma$ for $\gamma=0.5$

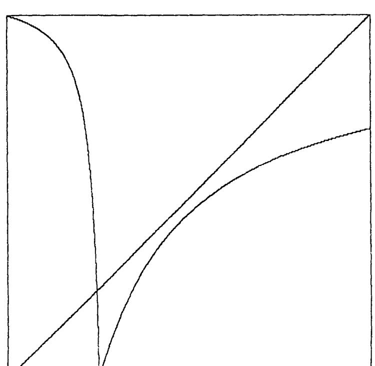

**Fig. 2.** The graph of $f_\gamma$ for $\gamma=0.4$

We need some more formulas. It is easy to compute that $X_0(x)=-2(x-\frac{1}{2})^2$. Denote $f=f_{0.5}$ and choose a point $b\in(0.25,0.5)$ such that $f(b)\in(0,0.25)$ and $a=f^2(b)\in(0.5,1)$. We have $p=2$ and according to Theorem 3.2 and Remark 3.4 we have to look at the value of the "limit derivative"

$$B=(f^2)'(b)\cdot\frac{X_0(b)}{X_0(a)}.$$

However, notice that $f'(b)=\dfrac{X_0(d)}{X_0(b)}$, where $d=f(b)$. Therefore $B=f'(d)\cdot\dfrac{X_0(d)}{X_0(a)}$ and as $d$ we can take any point of $(0, f_l^{-1}(0,5))$, where $f_l$ and $f_r$ denote the left and right branches of $f$ respectively $\left(\text{i.e. } f_l(x)=1.08-\dfrac{0.0216}{0.27-x},\ f_r(x)=1-\dfrac{1}{4x}\right)$. We have

$$B=f'(d)\cdot\frac{(d-\frac{1}{2})^2}{(f(d)-\frac{1}{2})^2}=-0.0216\left(\frac{d-0.5}{0.135-0.58d}\right)^2.$$

The inequality $|B|<1$, sufficient for applying Remark 3.4 (and Theorem 3.2) is

$$\left|\frac{0.135-0.58d}{d-0.5}\right|>\sqrt{0.0216}.$$

Since $d<0.5$ and $0.58d<0.135$, this is equivalent to

$$d<\frac{0.135-0.5\sqrt{0.0216}}{0.58-\sqrt{0.0216}}\approx 0.142.$$

The above computations show that indeed we can apply for our family all results of the previous sections. The following figures illustrate these applications.

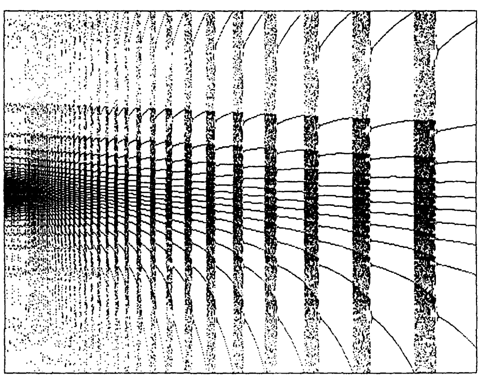

**Fig. 3.** The dependence of successive iterations on $\gamma$. 400 initial iterations were omitted and the next 400 ones are shown. The parameter $\gamma$ varies from 0.5 (left) to 0.48 (right)

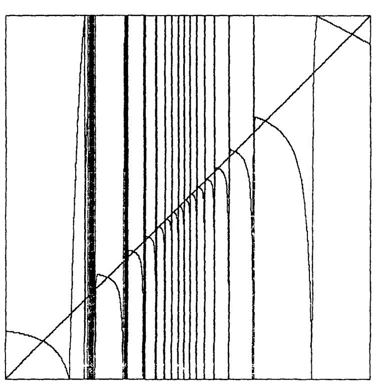

**Fig. 4.** The graph of the $16^{\text{th}}$ iterate of $f_\gamma$ for $\gamma\approx 0.4917797$. There is an attracting periodic orbit of period 16. In notations of Theorem 3.2, $p+n=16$ and $\mu_n\approx 0.0082203$. The situation described in Remark 3.4 occurs

The periodic attracting orbits predicted by Theorem 3.2 and Remark 3.4 pass once through the left lap (the interval $[0, 0.25]$) and the rest of times through the right lap (the interval $[0.25, 1]$). We shall call them good periodic orbits. Figure 3 suggests that all attracting periodic orbits are good, in particular that the period doubling phenomenon does not occur (what can be interpreted as one period doubling bifurcation, is obviously due to extremely slow attraction for these values of parameters). We are going to prove that indeed if $\gamma$ is sufficiently close to 0.5 (but smaller than 0.5) then either there is no attracting periodic orbit or there is only one and it is good.

Let us start with several simple observations.

(i) $f_\gamma$ is piecewise linear fractional and hence all its iterates are piecewise linear fractional.
(ii) $f_\gamma$ has Schwarzian derivative zero and therefore every attracting periodic orbit attracts either the critical point 0.25 or one of the endpoints of $[0, 1]$ (see e.g. [P]). However, $f_\gamma(0.25)=0$ and $f_\gamma(0)=1$, so $f_\gamma$ has at most one attracting periodic orbit.

**Lemma 5.1.** *Let $I$ be a closed interval and $g: I \to \mathbb{R}$ a linear fractional map with $g', g'' < 0$. Let $z < x$; $z, x \in I$ and $g(z) = z$. Then $(g^2)'(x) > 1$ (respectively $=1$, $<1$) if and only if $|g'(z)| > 1$ (respectively $=1$, $<1$).*

*Proof.* By a linear conjugacy we can reduce the problem to the case of the map $g(t) = \frac{1}{t} + c$ and $I$ to the left of 0 (so $x < 0$). We have

$$(g^2)'(y) = \frac{-1}{y^2} \cdot \frac{-1}{\left(\frac{1}{y} + c\right)^2} = \frac{1}{(1+cy)^2}.$$

Since $|g'|$ is increasing, $|g'(-1)| = 1$ and $g(-1) = -1 + c$, we have $|g'(z)| > 1$ (respectively $=1$, $<1$) if and only if $c > 0$ (respectively $=0$, $<0$). Therefore:

(a) If $|g'(z)| > 1$ then $c > 0$. Since $x > z$, then $g(x) < x$, so $\frac{1}{x} + c < x$. Since $x < 0$, then $1 + cx > x^2 > 0$, but $cx < 0$, so $1 > 1 + cx$. Therefore $|1 + cx| < 1$ and $(g^2)'(x) > 1$.
(b) If $|g'(z)| = 1$ then $c = 0$. Therefore $(g^2)'(x) = 1$.
(c) If $|g'(z)| < 1$ then $c < 0$. Therefore $1 + cx > 1$, so $(g^2)'(x) < 1$. □

Now let us look at the map $F_\gamma$ induced by $f_\gamma$ on $[0, 0.25]$ (the first return map). It has finitely many laps (i.e. maximal intervals on which it is continuous and monotone). On all of them $F_\gamma$ is decreasing and all of them, except perhaps the leftmost one, are mapped by $F_\gamma$ onto the whole $[0, 0.25]$. On all laps $F_\gamma$ is concave; this follows for each lap from the inductive use of the formula

$$(\varphi \circ \psi)'' = (\varphi'' \circ \psi) \cdot (\psi')^2 + (\varphi' \circ \psi) \cdot \psi''$$

when $\varphi'', \psi'' < 0$, $\varphi' > 0$.

Assume that $\gamma < 0.5$ but $\gamma$ is very close to 0.5. Then $f_\gamma|_{[0.25, 1]}$ is a time-one map of $X_{0.5-\gamma}$ which is very close to $X_0$. If $f_\gamma^n(a) = b$ and $a, f_\gamma(a), \ldots, f_\gamma^{n-1}(a) > 0.25$ then

$$(f_\gamma^n)'(a) = \frac{X_{0.5-\gamma}(b)}{X_{0.5-\gamma}(a)}.$$

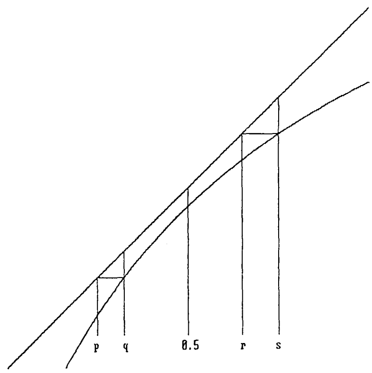


**Fig. 5.** The choice of $\delta_2$. Here $p=0.5-\xi$, $q=(f_\gamma|_{[0.25,1]})^{-1}(0.5-\xi)$, $r=f_\gamma(0.5+\xi)$, and $s=0.5+\xi$

If we fix some $M>0$ then there exist $\xi, \delta_1>0$ such that if $0.5-\delta_1<\gamma<0.5$ and $a\in[0.5-\xi,\ 0.5+\xi]$, $b\in[0,0.25]$, $f_\gamma^n(a)=b$, then $(f_\gamma^n)'(a)>M$. Now, there exists $\delta_2>0$ such that if $0.5-\delta_2<\gamma<0.5$ then $(f_\gamma|_{[0.25,1]})^{-1}(0.5-\xi)<0.5$, $f_\gamma(0.5+\xi)>0.5$, $f_\gamma'((f_\gamma|_{[0.25,1]})^{-1}(0.5-\xi))>1$ and $f_\gamma'(f_\gamma(0.5+\xi))<1$ (see Fig. 5).

Therefore if $\delta_3=\min(\delta_1,\delta_2)$ and $0.5-\delta_3<\gamma<0.5$ then we can divide the set of laps of $F_\gamma$ into 3 sets:

*1. Left Laps.* They are those laps whose image under $f_\gamma$ contains some point to the right of $0.5+\xi$. Therefore $f_\gamma'$ at this image is smaller than 1. If there are points $x<y$ belonging to (distinct) left laps and such that $F_\gamma(x)=F_\gamma(y)$ then $|f_\gamma'(x)|<|f_\gamma'(y)|$ and $f_\gamma(y)=f_\gamma^{1+k}(x)$ for some $k>0$. Since all the points $f_\gamma^{1+i}(x)$, $i=0,1,\dots,k-1$, are to the right of the whole image (under $f_\gamma$) of the lap to which $y$ belongs, we have $f_\gamma'(f_\gamma^{1+i}(x))<1$. Consequently, we get $|F_\gamma'(x)|<|F_\gamma'(y)|$.

*2. Central Laps.* They are those laps whose image under $f_\gamma$ is contained in the interval $[0.5-\xi,\ 0.5+\xi]$. On such a lap we have

$$|F_\gamma'|>M\cdot \inf_{y\in[0,0.25]}|f_\gamma'(y)|=M\cdot|f_\gamma'(0)|.$$

*3. Right Laps.* They are those laps whose image under $f_\gamma$ contains some point to the left of $0.5-\xi$. Therefore $f_\gamma'$ at this image is larger than 1. The derivatives on next images of this lap (until we came back to $[0,0.25]$) are also larger than 1. Therefore on this lap $|F_\gamma'|\geqq|f_\gamma'(y)|$, where $y$ is the left endpoint of this lap. If $\gamma$ is sufficiently close to 0.5 (which we can assume) then $f_\gamma(y)<0.5$. Therefore $y>v$, where $v\in[0,0.25]$ is the point at which $f_\gamma(v)=0.5$ (since $f_\gamma|_{[0,0.25]}$ does not depend on $\gamma$; $v$ and $|f_\gamma'(v)|$ also do not depend on $\gamma$). Hence, $|F_\gamma'|\geqq|f_\gamma'(v)|$ on any right lap.

Now we specify the value of $M$ as

$$M=\frac{|f'(v)|}{|f'(0)|}.$$

Notice that if $\gamma$ is sufficiently close to 0.5 then no right lap is a left lap. We take $\delta_4>0$ such that if $0.5-\delta_4<\gamma<0.5$ then this holds, the condition from the discussion of the behaviour of right laps holds and $\delta_4\leqq\delta_3$. Let $z_\gamma$ be the fixed point of $F_\gamma$ on the leftmost lap. Then we have the following result.

**Proposition 5.2.** *There exists $\delta>0$ such that if $0.5-\delta<\gamma<0.5$ and $|f_\gamma'(z_\gamma)|>1$ then the map $\Phi_\gamma$ induced on $[z_\gamma, 0.25]$ is piecewise expanding.*

*Proof.* We assume that $\delta \leqq \delta_4$. Notice that whether we think about $\Phi_\gamma$ as induced by $f_\gamma$ or by $F_\gamma$, we get the same map. We shall rather use $F_\gamma$.

Suppose that the following two conditions are satisfied.

$$|F_\gamma'|>1 \text{ on all laps except perhaps the leftmost one}, \tag{5.1}$$

$$|f_\gamma'(v)| \cdot |F_\gamma'(0)|>1. \tag{5.2}$$

If $x \geqq z_\gamma$ belongs to the leftmost lap of $F_\gamma$ then $\Phi_\gamma(x)=F_\gamma^2(x)$ and $|\Phi_\gamma'(x)|>1$ by Lemma 5.1. Suppose now that $x \in [z_\gamma, 0.25]$ does not belong to the leftmost lap. If $f_\gamma(x) \geqq z_\gamma$ then by (5.1), $|\Phi_\gamma'(x)|=|F_\gamma'(x)|>1$. Assume that $F_\gamma(x)<z_\gamma$. Then $\Phi_\gamma(x)=F_\gamma^2(x)$ and we have several possibilities.

If $x$ belongs to a left lap then by the properties of the left laps we have $|F_\gamma'(x)|>|F_\gamma'(y)|$, where $y$ is the point of the leftmost lap for which $F_\gamma(y)=F_\gamma(x)$. Then $|\Phi_\gamma'(x)|>|F_\gamma'(y)|$, which, as we already know, is larger than 1.

If $x$ belongs to a central lap then $|F_\gamma'(x)|>M \cdot |f_\gamma'(0)|=|f_\gamma'(v)|$. If $x$ belongs to a right lap then also $|F_\gamma'(x)|>|f_\gamma'(v)|$. In both cases, since on the leftmost lap the absolute value of the derivative of $F_\gamma$ is smallest at 0, we get by (5.2), $|\Phi_\gamma'(x)|>1$.

Therefore it remains to prove (5.1) and (5.2). As we have seen already, on the central and right laps we have $|F_\gamma'|>|f_\gamma'(v)|$, so on these laps (5.1) holds if only

$$|f_\gamma'(v)|>1. \tag{5.3}$$

By the properties of the left laps, the smallest value of $|F_\gamma'|$ on them (except the leftmost lap) is attained at the left endpoint of the second leftmost lap. We call this endpoint $c$ and consider at it a one-sided derivative (from the right). Hence, in all cases, (5.1) follows from (5.3) and the following inequality:

$$|F_\gamma'(c)|>1. \tag{5.4}$$

Now it remains to prove (5.2)–(5.4). We have the following formula for $F_\gamma'$:

$$F_\gamma'(t)=f_\gamma'(t) \cdot \frac{X_{0.5-\gamma}(F_\gamma(t))}{X_{0.5-\gamma}(f_\gamma(t))}. \tag{5.5}$$

Since we are interested only in $\gamma$'s sufficiently close to 0.5, we can replace $X_{0.5-\gamma}$ by $X_0$ in (5.5) and use it to get new versions of (5.2) and (5.4) (equivalent to them for $\gamma$'s sufficiently close to 0.5):

$$|f_\gamma'(v)| \cdot |f_\gamma'(0)| \cdot \frac{X_0(F_\gamma(0))}{X_0(f_\gamma(0))}>1, \tag{5.2a}$$

$$|f_\gamma'(c)| \cdot \frac{X_0(F_\gamma(c))}{X_0(f_\gamma(c))}>1. \tag{5.4a}$$

Denote $f(c)=b$. Clearly, $F_\gamma(c)=0.25$, and hence (after substituting the formula for $X_0$) (5.4a) is equivalent to

$$|f'(f_l^{-1}(b))| \cdot \frac{1}{4(2b-1)^2}>1. \tag{5.4b}$$

Clearly, $f_\gamma(0)=1$. Since $X_0(t)=X_0(1-t)$, we have

$$\int_0^{1-b}\frac{dt}{X_0(t)}=\int_b^1\frac{dt}{X_0(t)},$$

so if $\gamma$ is sufficiently close to 0.5 then $F_\gamma(0)$ is as close to $1-b$ as we want. Therefore, instead of (5.2a), it is enough to prove (again we substitute the formula for $X_0$)

$$|f'(v)|\cdot|f'(0)|\cdot(2b-1)^2>1. \tag{5.2b}$$

Our assumptions are that $|F'_\gamma(z_\gamma)|>1$. By Lemma 5.1, this is equivalent to $|F'_\gamma(c)|\cdot|F'_\gamma(0)|>1$, where this time the point $c$ is considered as belonging to the leftmost lap, so the derivative is taken from the left. By the same considerations as before we get that this inequality implies

$$|f'(f_l^{-1}(b))|\cdot\frac{X_0(0)}{X_0(b)}\cdot|f'(0)|\cdot\frac{X_0(1-b)}{X_0(1)}>1-\varepsilon, \tag{5.6}$$

where $\varepsilon>0$ is as small as we want, but $\delta$ depends on $\varepsilon$. Since $X_0(1-t)=X_0(t)$, (5.6) is equivalent to

$$|f'(f_l^{-1}(b))|\cdot|f'(0)|>1-\varepsilon. \tag{5.6a}$$

Now we are going to use the following form of the formula for $f$:

$$f_l(x)=4\varphi-\frac{4\varphi^2-\varphi}{\varphi-x},\qquad f_r(x)=1-\frac{1}{4x},\quad \text{where}\quad \varphi=0.27.$$

Substituting this formula, we see that (5.6a) is equivalent to $b<\varphi(4-\sqrt{1-\varepsilon})$. For a suitable $\varepsilon_1$ (as small as we want) this takes the form

$$b<3\varphi+\varepsilon_1. \tag{5.7}$$

We have $v=f_l^{-1}(0.5)$, so

$$|f'(v)|=\frac{(4\varphi-\frac{1}{2})^2}{4\varphi^2-\varphi}.$$

Therefore (5.3) is equivalent to $(4\varphi-\frac{1}{2})^2>4\varphi^2-\varphi$. After substituting the value of $\varphi$ we get the inequality $0.3364>0.27\cdot 0.08$, which is true. This proves (5.3). Therefore it remains to prove (5.2b) and (5.4b), which are equivalent respectively to:

$$\frac{(4\varphi-\frac{1}{2})^2}{4\varphi^2-\varphi}\cdot\frac{4\varphi^2-\varphi}{\varphi^2}\cdot(2b-1)^2>1 \tag{5.8}$$

and

$$\frac{(4\varphi-b)^2}{4\varphi^2-\varphi}\cdot\frac{1}{4(2b-1)^2}>1. \tag{5.9}$$

From the definition of $b$ we have $0.75\leq b$. Therefore to show (5.8) it is enough to prove that $(2\cdot 0.75-1)\cdot(4\varphi-0.5)>\varphi$. After substituting the value of $\varphi$ we get the inequality $0.5\cdot 0.58>0.27$, which is true. This proves (5.8).

It remains to prove (5.9). Set $\alpha = 0.02 = \varphi - 0.25$. We have $4\varphi - b = 1 + 4\alpha - b$ and $4(4\varphi^2 - \varphi) = 4\alpha + 16\alpha^2$, so (5.9) is equivalent to $P(b) > 0$, where

$$P(t) = (1 - 16\alpha - 64\alpha^2)t^2 + (-2 + 8\alpha + 64\alpha^2)t + (1 + 4\alpha).$$

Since $\alpha = 0.02$, we have $1 - 16\alpha - 64\alpha^2 > 0$ and $P(1) = -4\alpha < 0$. Therefore, if $P(3\varphi + \varepsilon_1) > 0$ then $P$ attains its minimum to the right of $3\varphi + \varepsilon_1$ and consequently $P(b) > 0$ for all $b$ satisfying (5.7). Hence, to complete the proof of (5.9), and thus the whole proposition, it remains to show that $P(3\varphi + \varepsilon_1) > 0$. However, since $\varepsilon_1$ is arbitrarily small, it is enough to show that $P(3\varphi) > 0$. This is equivalent to (5.9) with $b = 3\varphi$. However, we have

$$(4\varphi - 1) \cdot 4 \cdot (6\varphi - 1)^2 = 0.08 \cdot 4 \cdot 0.62^2 = 0.123008 < \varphi,$$

so (5.9) with $b = 3\varphi$ holds and this completes the proof. □

**Theorem 5.3.** *There exists $\delta > 0$ such that if $0.5 - \delta < \gamma < 0.5$ then exactly one of the following three possibilities occurs.*

1. $|F'_\gamma(z_\gamma)| > 1$ *and then the map $\Phi_\gamma$ induced on $[z_\gamma, 0.25]$ is piecewise expanding. There exists an ergodic probabilistic $f_\gamma$-invariant measure, absolutely continuous with respect to the Lebesgue measure. There is no periodic attracting orbit.*

2. $|F'_\gamma(z_\gamma)| = 1$ *and then $F^2_\gamma$ on the leftmost lap is the identity (it is equal there to some iterate of $f_\gamma$). There is no attracting periodic orbit.*

3. $|F'_\gamma(z_\gamma)| < 1$ *and then $f_\gamma$ has a unique attracting periodic orbit, namely the orbit of $z_\gamma$. This orbit is good. It attracts almost all points of* $[0, 1]$.

*Proof*. From Proposition 5.2, since $\Phi_\gamma$ has finitely many laps, it follows that $\Phi_\gamma$ has an invariant probabilistic ergodic measure $\nu$, absolutely continuous with respect to the Lebesgue measure (see [LY]). For each lap $\Delta_{\gamma,i}$ of $\Phi_\gamma$ there is $n(\gamma, i)$ such that $\Phi_\gamma = f_\gamma^{n(\gamma,i)}$ on $\Delta_{\gamma,i}$. The well known formula (see e.g. [R])

$$\tilde{\mu} = \sum_i \sum_{k=0}^{n(\gamma,i)-1} (f_\gamma^k)_* (\nu|_{\Delta_{\gamma,i}})$$

[where by $g_*(\kappa)$ we mean the image of $\kappa$ under $g: g_*(\kappa)(A) = \kappa(g^{-1}(A))$] defines a finite ergodic $f_\gamma$-invariant measure on $[0, 1]$, absolutely continuous with respect to the Lebesgue measure. We can normalize it by taking $\mu = \dfrac{1}{\tilde{\mu}([0,1])} \cdot \tilde{\mu}$. By Proposition 5.2, there cannot be any periodic attracting orbits.

2. By Lemma 5.1, $F^2_\gamma$ on the leftmost lap is the identity. The image under $f_\gamma$ of the critical point 0.25 is 0, which is in this case a periodic neutral point of $f_\gamma$. Therefore, by the observation (ii) before Lemma 5.1, there are no periodic attracting orbits.

3. From the same observation (ii) it follows that the attracting periodic orbit of $z_\gamma$ is the unique one. From the definition of a good orbit it follows that this orbit is good. The fact that in such a case almost all points of $[0, 1]$ are attracted by this orbit is well known (see e.g. [M]; in the proof given there the behaviour of the map close to the critical point is irrelevant). □

*Remark 5.4.* From the description of Case 2 it follows that at the moment when the periodic orbit becomes unstable, the whole interval of periodic neutral points appear. This behaviour is specific for piecewise linear fractional maps.

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