

Area-Preserving Diffeomorphisms and Higher-Spin Algebras

E. Bergshoeff¹, M. P. Blencowe² and K. S. Stelle²

¹ Theory Division, CERN, CH-1211 Geneva 23, Switzerland

² The Blackett Laboratory, Imperial College, London SW7, England

Abstract. We show that there exists a one-parameter family of infinite-dimensional algebras that includes the bosonic $d = 3$ Fradkin–Vasiliev higher-spin algebra and the non-Euclidean version of the algebra of area-preserving diffeomorphisms of the two-sphere S^2 as two distinct members. The non-Euclidean version of the area preserving algebra corresponds to the algebra of area-preserving diffeomorphisms of the hyperbolic space $S^{1,1}$, and can be rewritten as $\lim_{N \rightarrow \infty} su(N, N)$. As an application of our results, we formulate a new $d = 2 + 1$ massless higher-spin field theory as the gauge theory of the area-preserving diffeomorphisms of $S^{1,1}$.

1. Introduction

Infinite-dimensional Lie algebras play an increasingly important role in the development of theoretical physics. One of the best-known examples is the Virasoro algebra, which underlies the physics of two-dimensional conformal field theories. As such, they are important for string theories and for critical phenomena in certain statistical-mechanical models.

Recently, two new types of infinite-dimensional Lie algebras have become relevant. One of them is the algebra of volume-preserving diffeomorphisms of a manifold \mathcal{M} , which we denote by $s\text{diff}(\mathcal{M})$. This algebra is a subalgebra of the general diffeomorphism algebra of \mathcal{M} and corresponds to the residual symmetry of an extended object in the light-cone gauge. A basic example of such an algebra is $s\text{diff}(S^2)$. This algebra occurs in the description of a spherical membrane, which can be viewed as a gauge theory of $s\text{diff}(S^2)$. An interesting feature of the algebra $s\text{diff}(S^2)$ is that it can be obtained by taking the limit as $N \rightarrow \infty$ of the finite-dimensional Lie algebra $su(N)$, i.e., $s\text{diff}(S^2) = \lim_{N \rightarrow \infty} su(N)$ [1]. Replacing the gauge theory of $s\text{diff}(S^2)$ by a gauge theory of $su(N)$ then provides a form of regularization. The original spherical membrane theory is reobtained in the limit $N \rightarrow \infty$.

The other type of infinite-dimensional Lie algebra that has recently emerged is the set of (super-) higher-spin algebras of Fradkin and Vasiliev [2, 3]. These algebras may play an important role in the construction of interacting massless higher-spin field theories in $3 + 1$ dimensions. In [2, 3], a perturbative construction of the higher-spin theories has been begun, in which the higher-spin algebra is the gauge algebra that guarantees masslessness. An essential ingredient in this approach is the introduction of an infinite number of massless higher-spin fields propagating in a curved anti-de Sitter background. The actions constructed in this approach are non-analytic in the cosmological constant. As a consequence, these higher-spin theories do not admit expansions over a flat background.

Recently, one of the authors (M.P.B.) has used the approach of [2, 3] to construct a consistent interacting supersymmetric higher-spin theory in $2 + 1$ dimensions, describing all integer and half-integer spins $\geq 3/2$ [4]. The algebra corresponding to the symmetry of the vacuum in this theory is one of the super-higher-spin algebras of [2, 3]. We will denote it by $\text{shs}(1|2) \oplus \text{shs}(1|2)$. The notation indicates the finite-dimensional subalgebra of $\text{shs}(1|2)$, which is $\text{osp}(1|2)$. The action is given by the integral of the Chern–Simons 3-form associated to the superalgebra $\text{shs}(1|2) \oplus \text{shs}(1|2)$. As in the four-dimensional case, the action contains an infinite number of massless higher-spin fields in a curved anti-de Sitter background. However, unlike the four-dimensional case, the action of [4] contains only positive powers of the cosmological constant and hence the limit to a flat background can be taken. A consistent truncation of the theory can be obtained by omitting all the half-integer spins. In that case, the symmetry of the vacuum is given by the bosonic subalgebra of $\text{shs}(1|2) \oplus \text{shs}(1|2)$, which we denote by $\text{hs}(1, 1) \oplus \text{hs}(1, 1)$. The notation again indicates the finite-dimensional subalgebra, which for $\text{hs}(1, 1)$ is $\text{su}(1, 1)$.

In Sect. 2 of this paper we show that the bosonic higher spin algebra $\text{hs}(1, 1)$ and the non-Euclidean version $s\text{diff}(S^{1,1})$ of the spherical membrane algebra $s\text{diff}(S^2)$ are two distinct members of a one-parameter family \mathcal{B}_λ of infinite-dimensional algebras. Each member of this family could be used to construct an interacting higher spin theory in $2 + 1$ dimensions. In Sect. 3 we show that the non-Euclidean non-compact algebra $s\text{diff}(S^{1,1})$ can be rewritten as $\lim_{N \rightarrow \infty} \text{su}(N, N)$.

The latter relation shows that the algebra $s\text{diff}(S^{1,1})$ considered as an alternative bosonic infinite-dimensional higher-spin algebra in $2 + 1$ dimensions admits truncation to an arbitrary but finite number of higher spins $s = 2, 3, \dots, 2N$. A theory involving a finite number of spins in this way can be considered as a gauge theory of the group $\text{su}(N, N)$. In the limit $N \rightarrow \infty$, one reobtains the infinite higher-spin theory. A geometric construction of the $s\text{diff}(S^{1,1})$ algebra as a subalgebra of the area-preserving diffeomorphisms of the hyperbolic space $S^{1,1}$ can also be made. In such a formulation the algebra $s\text{diff}(S^{1,1})$ is defined on a basis of real analytic functions on $S^{1,1}$.

In Sect. 4 of this paper, we apply our results in formulating a geometric bosonic higher-spin theory in $2 + 1$ dimensions as the gauge theory of the algebra $s\text{diff}(S^{1,1}) \oplus s\text{diff}(S^{1,1})$. More precisely, we formulate the higher-spin theory in terms of gauge fields taking their values in the algebra of functions on $S^{1,1} \times S^{1,1}$.

The advantage of such a geometrically formulated higher-spin theory lies in the fact that certain calculations which are very difficult in a non-geometric infinite-component formulation become almost trivial for a geometric theory that is formulated in terms of a finite set of fields defined on an extended manifold. For example, the Jacobi identities for the higher-spin algebras $s\text{diff}(S^{1,1})$ are manifest in this geometric formulation, in contrast to the lengthy algebraic calculations of [2, 3].

In the conclusion, we briefly consider superextensions of the higher-spin algebras. These superalgebras might be relevant to the description of spinning membranes, i.e., membranes with world-volume supersymmetry. In the Appendix we give some technical details concerning the proof of the Jacobi identities for the Fradkin–Vasiliev higher-spin algebra $hs(1, 1)$.

2. The Algebras $s\text{diff} S^{1,1}$ and $hs(1, 1)$

We will first describe and compare the constructions of the algebras $s\text{diff} S^{1,1}$ and $hs(1, 1)$ as well as their Euclidean versions, $s\text{diff} S^2$ and $hs(2)$ respectively. We shall specify an algebra by giving the commutators $[\xi_1, \xi_2]^A = f_{BC}^A \xi_1^B \xi_2^C$, where the ξ^A are the generators and the f_{BC}^A are the structure constants of the algebra. The $hs(1, 1)$ algebra is then given by¹

$$[\xi_1, \xi_2]^{\alpha(n)} = \sum_{p,q,s=1}^{\infty} (-1)^{s/2-1/2} \frac{n!}{p!q!s!} \delta(n-p-q) \xi_{1\beta(s)}^{\alpha(p)} \xi_2^{\beta(s)\alpha(q)} \quad \begin{array}{l} n \geq 0, \text{ even} \\ p, q, s \geq 1, \text{ odd} \end{array} \quad (2.1)$$

where $\delta(\cdot)$ is the usual Kronecker delta. The spinorial index α ($\alpha = 1, 2$) is an $su(1, 1)$ index, corresponding to the finite-dimensional subalgebra of $hs(1, 1)$ generated by the $n = 2$ generators $\xi^{\alpha(2)}$:

$$[\xi_1, \xi_2]^{\alpha(2)} = 2\xi_{1\beta}^{\alpha} \xi_2^{\beta\alpha}. \quad (2.2)$$

The spinorial indices are raised and lowered with the aid of the symplectic form

$$\Omega = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad (2.3)$$

e.g. $A^\alpha = \Omega^{\alpha\beta} A_\beta$, $A_\alpha = A^\beta \Omega_{\beta\alpha}$. Covariant conjugation of the fundamental $su(1, 1)$ representation $A_\alpha = (A_1, A_2)$ is given by $A_c^\alpha = (A_1^*, -A_2^*)$. For $su(1, 1)$, it is possible to define real spinors satisfying $A_c^\alpha = A^\alpha$. Note that the $n = 0$ generator ξ commutes with itself and the rest of the algebra. Henceforth, we consider only the $n \geq 2$ generators.

The $su(1, 1)$ spin content of a generator $\xi^{\alpha(n)}$ is given by $l = n/2$. One can easily

¹ Our conventions are those of [4]. In particular, upper or lower indices denoted by one symbol are symmetrized with strength 1, i.e. $\frac{1}{2}(\alpha_1 \alpha_2 + \alpha_2 \alpha_1)$. For a given superscript or subscript $\alpha(n)$, the n denotes the number of symmetrized α indices. Symmetrization is performed before contraction between upper and lower indices

check that the structure constants $f_{\alpha(2l_1)\alpha(2l_2)}^{\alpha(2l_3)}$ differ from zero only if

$$|l_1 - l_2| + 1 \leq l_3 \leq l_1 + l_2 - 1, \quad \text{with } l_1 + l_2 + l_3 \text{ odd.} \tag{2.4}$$

The Euclidean version $hs(2)$ of $hs(1,1)$ is given by the same equation (2.1). However, the spinor indices in this case refer to $su(2)$, which is now the finite-dimensional subalgebra generated by the $\zeta^{\alpha(2)}$. Convariant conjugation is now given by $A_c^\alpha = (A_1^*, A_2^*)$. It is not possible to define an $su(2)$ -covariant reality condition for a single spinor. However, the bosonic algebra (2.1) contains only generators carrying even numbers of spinor indices. For these representations, reality conditions can be defined, e.g. $A_c^{\alpha\beta} = A^{\alpha\beta}$.

The explicit forms of the structure constants of $s \text{ diff}(S^2)$, which involve products of $3 - j$ symbols, have been given in [1, 5]. From them one can immediately derive the structure constants of $s \text{ diff}S^{1,1}$. Since the expressions are rather involved we will not give them here. General relations between the structure constants of the area-preserving algebra in a convenient basis have been given in [6] and we will make use of them below (see (2.22–24)).

In order to compare the algebras $hs(2)$ and $s \text{ diff}(S^2)$ (as well as their non-Euclidean versions), it is instructive to first consider the ways in which these algebras are constructed. Consider first the $s \text{ diff}(S^2)$ algebra. Let S^2 be a two-sphere of radius r and let x_i ($i = 1, 2, 3$) be three Cartesian coordinates,

$$x_i x_i = x_1^2 + x_2^2 + x_3^2 = r^2. \tag{2.5}$$

Then consider the set of all symmetric traceless homogeneous polynomials of the form

$$A(x) = \sum_{n=0}^{\infty} \frac{1}{n!} a^{i_1 \dots i_n} x_{i_1} \dots x_{i_n}, \tag{2.6}$$

where the coefficients $a^{i_1 \dots i_n}$ are symmetric and traceless. A polynomial $A(x)$ is said to be of degree l if the only nonzero coefficient is $a^{i_1 \dots i_l}$. One can introduce the following Lie bracket of x_i and x_j :

$$\{x_i, x_j\} = \varepsilon_{ijk} x_k. \tag{2.7}$$

This induces the Lie bracket of a pair of polynomials:

$$\{A(x), B(x)\} = \varepsilon_{ijk} x_i \partial_j A \partial_k B, \tag{2.8}$$

which is the $s \text{ diff}(S^2)$ algebra. Given two polynomials $A(x)$ and $B(x)$, of degrees l_1 and l_2 respectively, the right-hand side of (2.8) is equal to the product of three polynomials, x_i , $\partial_j A$ and $\partial_k B$, which are of degree 1, $(l_1 - 1)$ and $(l_2 - 1)$ respectively. This can be rewritten as a finite sum of polynomials of degree l_3 . One can verify that the Lie bracket (2.8) differs from zero only if the degrees l_1 , l_2 and l_3 satisfy (2.4) [7]. For $l_1 = l_2 = l_3 = 1$, we recover the finite dimensional $so(3) \simeq su(2)$ subalgebra (2.7) of $s \text{ diff}(S^2)$. The constant polynomial $A(x) = 1$ commutes with the rest of the algebra and will henceforth be omitted.

Although the two-sphere used in the construction of $s \text{ diff}(S^2)$ has been taken to be of radius r , taking different values of r does not change the resulting algebra.

To see this, one can make a rescaling of the basis polynomials $A^{(l)}(x) = a^{i_1 \dots i_l} x_{i_1} \dots x_{i_l}$,

$$A^{(l)} \rightarrow (r^2)^{l/2 - 1/2} A^{(l)}. \tag{2.9}$$

After this rescaling, the structure constants of the algebra in the new basis become identical to those in the original basis, but now for a sphere with radius $r = 1$.

We next consider the construction of the $hs(2)$ algebra. Following [3], we introduce operators \hat{q}_α ($\alpha = 1, 2$) where α is a spinor index of $su(2)$. We then take the set of all symmetric traceless homogeneous polynomials $F(\hat{q})$ of \hat{q}_α of the form

$$F(\hat{q}) = \sum_{n=0}^{\infty} \frac{1}{n!} b^{\alpha(n)} \hat{q}_{\alpha_1} \dots \hat{q}_{\alpha_n}, \tag{2.10}$$

where the c -number coefficients $b^{\alpha(n)}$ are totally symmetric multispinors. A polynomial $F(\hat{q})$ is said to be of degree l if the only non-zero coefficient is $b^{\alpha(2l)}$. The operators \hat{q}_α satisfy the following commutation relations:

$$[\hat{q}_\alpha, \hat{q}_\beta] = \Omega_{\alpha\beta}. \tag{2.11}$$

From the commutation relation (2.11), one can derive the commutators of pairs of basis polynomials. These commutators define the supersymmetric extension $shs(2, \mathbb{C})$ of the complexification $hs(2; \mathbb{C})$ of $hs(2)$.

A useful alternative way of describing the $shs(2; \mathbb{C})$ algebra is to use commuting variables q_α instead of the operators \hat{q}_α . One then takes the set of all symmetric traceless homogeneous polynomials $F(q)$ of q_α which, except for the substitution $\hat{q}_\alpha \rightarrow q_\alpha$, are identical in form to the polynomials $F(\hat{q})$ given in (2.10). One can then define the following composition law for two polynomials $F(q)$ & $G(q)$:

$$(F * G)(q) = \exp\left(\Omega_{\alpha\beta} \frac{\partial}{\partial q_{1\alpha}} \frac{\partial}{\partial q_{2\beta}}\right) F(q_1) F(q_2) \Big|_{q_1 = q_2 = q}. \tag{2.12}$$

The $shs(2; \mathbb{C})$ algebra is then given by the Lie bracket

$$[F(q), G(q)] = (F * G)(q) - (G * F)(q). \tag{2.13}$$

In order to construct the bosonic $hs(2)$ algebra, it is sufficient to consider only polynomials of even degree, i.e., $b^{\alpha(n)} = 0$ for n odd. In this case, one can introduce a real form of the algebra. Equivalently, instead of the operators \hat{q}_α , one can use the vectorial operators $\hat{S}_i \sim \hat{q}_{(\alpha} \hat{q}_{\beta)}$ ($i = 1, 2, 3$). From (2.11), one finds that the operators \hat{S}_i satisfy the commutation relations

$$[\hat{S}_i, \hat{S}_j] = \varepsilon_{ijk} \hat{S}_k \tag{2.14}$$

and the constraint²

$$\hat{S}_i \hat{S}_i = \frac{3}{16}. \tag{2.15}$$

Using the \hat{S}_i , instead of (2.10) we consider the set of all symmetric traceless

² Note that the value of Casimir operator $\hat{S}_i \hat{S}_i$ in (2.15) does not correspond to any finite-dimensional unitary representation of $su(2)$. This is also obvious from the construction of the \hat{S}_i from the Heisenberg algebra (2.11)

polynomials of the form

$$F(\hat{S}) = \sum_{n=0}^{\infty} \frac{1}{n!} b^{i_1 \dots i_n} \hat{S}_{i_1} \dots \hat{S}_{i_n}, \tag{2.16}$$

where the coefficients $b^{i_1 \dots i_n}$ are symmetric and traceless. The polynomial $F(\hat{S})$ is of degree l if the only nonzero coefficient is $b^{i_1 \dots i_l}$.

From the commutation relation (2.14) we can derive the commutator of a pair of polynomials. These commutators define the $hs(2)$ algebra. Given the polynomials $F(\hat{S})$ & $G(\hat{S})$, of degrees l_1 and l_2 respectively, their commutator can be written as a finite series of terms, each containing a single commutator $[\hat{S}_i, \hat{S}_j]$. Each of these terms consists of the product of three polynomials of degree 1, $l_1 - 1$ and $l_2 - 1$. These products can then be decomposed by repeated use of (2.14) and (2.15) into a finite number of irreducible (i.e., homogeneous symmetric traceless) polynomials of degrees l_3 . As in the case of the $s\text{diff}(S^2)$ algebra, the resulting structure constants differ from zero only if the degrees l_1, l_2 and l_3 satisfy (2.4). For $l_1 = l_2 = l_3 = 1$, we recover the $so(3)$ algebra (2.14). The constant polynomial commutes with every polynomial and can be omitted from the algebra.

Although the above constructions of $s\text{diff}(S^2)$ and $hs(2)$ are similar, they are not identical. The difference lies in the fact that the x_i are commuting while the \hat{S}_i are non-commuting operators. This affects the values of the non-zero structure constants, even though the pattern of zero and non-zero structure constants agrees for the two cases. As we have seen above, when calculating the Lie bracket or commutator of two polynomials of degrees l_1 and l_2 , in either algebra one ends up with a product of three polynomials $\mathcal{P}_1, \mathcal{P}_2$ and \mathcal{P}_3 of degrees 1, $l_1 - 1$ and $l_2 - 1$. This product has to be decomposed into a finite number of irreducible polynomials of degrees l_3 ranging from $|l_1 - l_2| + 1$ to $l_1 + l_2 - 1$. The polynomial of highest degree is obtained by completely symmetrizing the indices of the polynomials $\mathcal{P}_1, \mathcal{P}_2$ and \mathcal{P}_3 and removing all the traces. In the $s\text{diff}(S^2)$ construction, the polynomials of lower degree are given by the various repeated traces excluded from the highest degree polynomial. In taking these traces, $x_i x_i$ is replaced by 1 using (2.5) (where r has been set to 1 here without loss of generality, as we have discussed). In the case of the $hs(2)$ construction, exactly the same manipulations are performed using $\hat{S}_i \hat{S}_i = 1$. However, in the $hs(2)$ case there are additional contributions to the non-leading structure constants due to the fact that, in the process of decomposition into irreducible polynomials of the operators \hat{S}_i , the commutation relations (2.14) must be used. The commutators $[\hat{S}_i, \hat{S}_j]$ give additional contributions to the non-leading structure constants that have no analogues in the $s\text{diff}(S^2)$ construction, where the x_i are commuting. Thus, the above constructions of the two algebras $s\text{diff}(S^2)$ and $hs(2)$ do not give the same values for the non-leading non-zero structure constants.

Despite the fact that the structure constants for $s\text{diff}(S^2)$ and $hs(2)$ are not exactly the same, one can not a priori exclude the possibility that the two algebras are isomorphic. This is possible because the basis polynomials for the $hs(2)$ algebra can be rescaled in a similar fashion to that we used in discussing the $s\text{diff}(S^2)$ algebra itself, where different values of r in the equation for the sphere (2.5) nonetheless yield the same algebra. Of course, *a priori*, there are many more

non-zero non-leading structure constants than there are basis polynomials, so it is not obvious that such rescalings of the basis for the $hs(2)$ algebra will be sufficient. In fact, we will now show that despite the possibility to make rescalings of the basis polynomials the structure constants of $s\text{diff}(S^2)$ and $hs(2)$ can not be made the same and therefore the two algebras are not isomorphic. In the proof we will apply some of the results of [6].

In order to make use of the results of [6], it is convenient to represent the structure constants f_k^{mn} of $hs(2)$ in a particular basis as follows:

$$[\zeta^{\alpha(2m)}, \zeta^{\alpha(2n)}] = \sum_k f_k^{mn} \underbrace{\varepsilon^{\alpha\beta} \varepsilon^{\alpha\beta} \dots \varepsilon^{\alpha\beta}}_{m+n-k} \zeta^{\alpha(m-n+k)\beta(n-m+k)}. \tag{2.17}$$

In general we can obtain different values for the structure constants by making a rescaling of the basis of the algebra,

$$\zeta^{\alpha(n)} \rightarrow \zeta'^{\alpha(n)} = \Psi(n)\zeta^{\alpha(n)} \quad n \geq 0, \text{ even}, \tag{2.18}$$

where $\Psi(n)$ is an arbitrary function of the non-negative integers n satisfying the conditions

$$\begin{aligned} \Psi(n) &\neq 0 \quad \forall n \geq 0, \text{ even} \\ \Psi(n) &= 0 \quad \forall n \geq 1, \text{ odd}. \end{aligned} \tag{2.19}$$

This freedom leads to the following arbitrariness for the structure constants F_k^{mn} in an arbitrary basis:

$$F_k^{mn} = f_k^{mn} \frac{\Psi(2k)}{\Psi(2m)\Psi(2n)}. \tag{2.20}$$

From (2.1) we deduce that the structure constants f_k^{mn} of $hs(2)$ in a particular basis are given by

$$f_k^{mn} = \frac{(2k)!}{(m-n+k)!(n-m+k)!(m+n-k)!}. \tag{2.21}$$

A similar closed expression for the structure constants of the $s\text{diff}(S^2)$ algebra is not known. For our purposes however, it is sufficient to use the following relations between the structure constants of $s\text{diff}(S^2)$ [6]:

$$g_k^{mn} = g_k^{nm}, \tag{2.22}$$

$$g_{m+n-1}^{mn} = -8\pi mn, \tag{2.23}$$

$$g_{m-1}^{m2} = -32\pi \frac{m(m-1)}{2m+1}. \tag{2.24}$$

Here we have indicated the structure constants of $s\text{diff}(S^2)$ by g_k^{mn} in order to distinguish them for those of $hs(2)$ given in (2.21). The question now is whether there exists a function $\Psi(m)$ such that

$$g_k^{mn} = f_k^{mn} \frac{\Psi(2k)}{\Psi(2m)\Psi(2n)} \tag{2.25}$$

for some choice of $\Psi(2m)$. We will now show that Eq. (2.25) is inconsistent with the relations (2.22–2.24). Clearly, substituting (2.25) into (2.22) does not lead to any restriction on $\Psi(2m)$. The second relation (2.20) restricts $\Psi(2m)$ to be

$$\Psi(2m) = \frac{g(2m)}{2\pi(2m)!} \tag{2.26}$$

with

$$g(2(m+n-1)) = g(2m)g(2n). \tag{2.27}$$

The third relation leads to the following additional restriction on $g(2m)$:

$$g(2(m+1)) = (2m+1)(2m-1)g(2(m-1)). \tag{2.28}$$

One can easily convince oneself that there is no function $g(2m)$ that satisfies both (2.27) and (2.28). For instance, from (2.27) taken at $n=3$ and for $m \rightarrow m-1$ it follows that $g(2(m+1)) = g(6)g(2(m-1))$. It would then follow from (2.28) that $g(6) = (2m+1)(2m-1) \forall m$ which is inconsistent. Our conclusion therefore is that the algebras $hs(2)$ and $s\,diff\,S^2$ are not isomorphic. From the above proof it is immediately clear that also the non-Euclidean algebras $hs(1, 1)$ and $s\,diff(S^{1,1})$ are not isomorphic.

We will now show that in fact the algebras $hs(2)$ and $s\,diff(S^2)$ are two distinct members of a one-parameter family \mathcal{A}_λ of infinite-dimensional algebras with different values of λ leading to inequivalent algebras³. To keep the discussion simple we will only consider positive values of λ , i.e. $\lambda \geq 0$. To describe the algebras \mathcal{A}_λ we introduce operators $\hat{S}_i (i = 1, 2, 3)$ which satisfy the commutation relations

$$[\hat{S}_i, \hat{S}_j] = \lambda \varepsilon_{ijk} \hat{S}_k \tag{2.29}$$

and the constraint

$$\hat{S}_i \hat{S}_i = 1. \tag{2.30}$$

We next consider the set of all symmetric traceless polynomials $F(\hat{S})$ like in (2.16). The algebra of commutators between these polynomials define the algebra \mathcal{A}_λ . Clearly, by construction $\mathcal{A}_{4/\sqrt{3}} \simeq hs(2)$, as can be seen by comparing with (2.15). We observe that we also can consider the limit of the algebra \mathcal{A}_λ as $\lambda \rightarrow 0$. In this limit the operators \hat{S}_i effectively behave like commuting variables and the algebra is given by $s\,diff(S^2)$. This particular limiting procedure has been described in more detail in [8]. Hence we have

$$\lim_{\lambda \rightarrow 0} \mathcal{A}_\lambda \simeq s\,diff(S^2) \quad \text{and} \quad \mathcal{A}_{4/\sqrt{3}} \simeq hs(2). \tag{2.31}$$

In order to show that different values of λ always lead to inequivalent algebras it is sufficient to consider the commutation relations between the polynomials of degree 1, 2, 3 and 4:

³ The discussion that now follows is due to a conversation with *M. Vasiliev* who pointed out to us the existence of the one-parameter family of algebras \mathcal{A}_λ

$$\begin{aligned}
 F_i &= \psi(2)\widehat{S}_i, \\
 F_{ij} &= \psi(4)\{\widehat{S}_{(i}\widehat{S}_{j)} - \frac{1}{3}\delta_{ij}\}, \\
 F_{ijk} &= \psi(6)\{\widehat{S}_{(i}\widehat{S}_j\widehat{S}_{k)} - \frac{1}{5}(\lambda^2 + 3)\delta_{(ij}\widehat{S}_{k)}\}, \\
 F_{ijkl} &= \psi(8)\{\widehat{S}_{(i}\widehat{S}_j\widehat{S}_k\widehat{S}_{l)} - \frac{1}{7}(5\lambda^2 + 6)\delta_{(ij}\widehat{S}_k\widehat{S}_{l)} + \frac{3}{35}(2\lambda^2 + 1)\delta_{(ij}\delta_{kl)}\}. \tag{2.32}
 \end{aligned}$$

Hence $\psi(2)$, $\psi(4)$, $\psi(6)$ and $\psi(8)$ are arbitrary functions of λ which reflect the freedom we have in the choice of normalization of the basis polynomials (see the discussion around (2.19)). Using the basic operator relations (2.29) and (2.30) one can now calculate commutation relations between the F 's. In particular, we find

$$\begin{aligned}
 [F_i, F_j] &= a_1 \varepsilon_{ijk} F^k, \\
 [F_{ij}, F^{kl}] &= a_2 \varepsilon_{(i}^{(k} F_{j)}^{l)s} + a_3 \varepsilon_{(i}^{(k} \delta_{j)}^l F^s, \\
 [F_{ijk}, F^{pq}] &= a_4 \varepsilon_{(i}^{(p} F_{j)}^{q)s} + a_5 \varepsilon_{(i}^{(p} \delta_{j)}^q F_k^s - \frac{1}{5} \delta_{jk} F^q\}, \tag{2.33}
 \end{aligned}$$

with a_1, \dots, a_5 given by

$$\begin{aligned}
 a_1 &= \lambda\psi(2), & a_2 &= 4\lambda \frac{(\psi(4))^2}{\psi(6)}, \\
 a_3 &= \frac{1}{5}\lambda(3\lambda^2 + 4) \frac{(\psi(4))^2}{\psi(2)}, & a_4 &= 6\lambda \frac{\psi(4)\psi(6)}{\psi(8)}, \\
 a_5 &= \frac{12}{7}\lambda(2\lambda^2 + 1)\psi(6). \tag{2.34}
 \end{aligned}$$

The question we would like to address now is the following. Is it possible that for different values of the parameter λ we can obtain the same value for all structure constants a_1, \dots, a_5 by means of a suitable choice of the arbitrary functions $\psi(2)$, $\psi(4)$, $\psi(6)$ and $\psi(8)$? To answer this question we first set $a_1 = a_2 = a_3 = a_4 = 1$ to fix the values of $\psi(2)$, $\psi(4)$, $\psi(6)$ and $\psi(8)$. In particular we find for $\psi(6)$:

$$\psi(6) = \frac{20}{\lambda(3\lambda^2 + 4)}. \tag{2.35}$$

We next substitute this value of $\psi(6)$ into the expression for the structure constant a_5 . We thus find that the condition for two algebras \mathcal{A}_{λ_1} and \mathcal{A}_{λ_2} ($\lambda_1 \neq \lambda_2$; $\lambda_1, \lambda_2 \geq 0$) to yield identical structure constants requires that the expression

$$a_5(\lambda) = \frac{240\lambda^2 + 1}{7\lambda(3\lambda^2 + 4)} \tag{2.36}$$

take the same value for λ_1 and λ_2 , i.e. $a_5(\lambda_1) = a_5(\lambda_2)$. Clearly this is not the case. We therefore conclude that different values of λ always lead to inequivalent algebras \mathcal{A}_λ .

Summarizing, we have found that $hs(2)$ and $s\text{diff}(S^2)$ are two distinct members of a one-parameter family \mathcal{A}_λ of infinite-dimensional algebras, with different values of λ always leading to inequivalent algebras. It is clear also that the non-Euclidean algebras $hs(1,1)$ and $s\text{diff}(S^{1,1})$ belong to a one-parameter family \mathcal{B}_λ of infinite-dimensional algebras where every \mathcal{B}_λ is the obvious non-compact version of \mathcal{A}_λ . At first sight one might be surprised by the existence of the algebras \mathcal{B}_λ in

view of the uniqueness theorem of [2]. In [2], the Jacobi identity of the higher spin algebra was used to prove, subject to certain assumptions, a uniqueness theorem for the infinite-dimensional extension of $so(3)$ whose generators decompose under $so(3)$ into a sequence of representations where each symmetric traceless $so(3)$ representation occurs just once. Clearly, both $s\text{diff}(S^{1,1})$ and $hs(1, 1)$ share this property. We would like to point out that our result does not contradict this uniqueness theorem. In particular one of the assumptions on which the derivation of the uniqueness theorem in [2] was based is that no Fierz identity need be used in the verification of the Jacobi identities. This is indeed the case for the higher spin algebra $hs(1, 1)$. However, one can easily convince oneself that for all other algebras $\mathcal{B}_\lambda(\lambda \neq 1)$ the verification of the Jacobi identities does require the use of Fierz identities. In the Appendix we will rederive the uniqueness theorem of [2] for the case of $2 + 1$ dimensions and will point out at which point one makes the assumption that no Fierz identities need be used.

In principle, each \mathcal{B}_λ could be used to construct an interacting higher spin theory in $2 + 1$ dimensions. This has already been done in the case of the algebra $\mathcal{B}_{4/\sqrt{3}} = hs(1, 1)$ [4]. In the remaining part of this paper we will consider $\lim_{\lambda \rightarrow 0} \mathcal{B}_\lambda = s\text{diff}(S^{1,1})$ as an alternative candidate for a consistent higher spin theory in $2 + 1$ dimensions.

3. Geometric Formulation of $d = 2 + 1$ Higher-Spin Theory

First we will show that the infinite-dimensional algebra $s\text{diff}(S^{1,1})$ can be rewritten as $\lim_{M \rightarrow \infty} SU(M, M)$. This is a non-compact version of the theorem in ref. [1] that the infinite dimensional algebra $s\text{diff}(S^2)$ can be rewritten as $\lim_{N \rightarrow \infty} su(N)$. To be more precise, it has been shown recently that $s\text{diff}(S^2) \simeq su_+(\infty)$ [9].

Let \mathcal{T}_i ($i = 1, 2, 3$) be the three generators of $su(1, 1)$. We take the compact generator \mathcal{T}_3 to be hermitian and the two non-compact ones \mathcal{T}_1 and \mathcal{T}_2 to be antihermitian. The matrix representations of these generators are related to those of the three hermitian generators \mathcal{U}_i of $su(2)$ by $\mathcal{T}_1 = i\mathcal{U}_1$, $\mathcal{T}_2 = i\mathcal{U}_2$ and $\mathcal{T}_3 = \mathcal{U}_3$. Using these relations, one can verify that when the value of the $su(2)$ Casimir operator $\mathcal{U}_1^2 + \mathcal{U}_2^2 + \mathcal{U}_3^2$ is $N^2 - 1/4$, then the value of the $su(1, 1)$ Casimir operator $\mathcal{T}_1^2 + \mathcal{T}_2^2 - \mathcal{T}_3^2$ is given by $-(N^2 - 1/4)$. Thus, the \mathcal{T}_i satisfy the commutation relations

$$[\mathcal{T}_i, \mathcal{T}_j] = \varepsilon_{ijk} \mathcal{T}^k \tag{3.1}$$

and the constraint

$$\mathcal{T}_i \mathcal{T}^i \equiv \mathcal{T}_1^2 + \mathcal{T}_2^2 - \mathcal{T}_3^2 = -\left(\frac{N^2 - 1}{4}\right) \mathbb{1}. \tag{3.2}$$

The $s\text{diff}(S^2)$ algebra was constructed in [1] by taking commutators between all irreducible, i.e., homogeneous symmetric traceless, polynomials in the \mathcal{U}_i of degrees $1 \leq l \leq N - 1$, decomposing the results into irreducible polynomials and then taking the limit as $N \rightarrow \infty$. For any finite value of N , the set of irreducible

polynomials of degrees $1 \leq l \leq N - 1$ in the \mathcal{U}_i forms the set of hermitian generators of $su(N)$ in the adjoint representation. We now perform the same construction with the \mathcal{T}_i instead of the \mathcal{U}_i . In this case, the set of irreducible polynomials of degrees $1 \leq l \leq N - 1$ form the set of (hermitian and antihermitian) generators of a non-compact version of $su(N)$. To identify the particular non-compact version, recall that the non-compact $su(P, Q)$ algebras has $P^2 + Q^2 - 1$ compact hermitian generators and $2PQ$ non-compact antihermitian generators. Now, a homogeneous symmetric traceless polynomial of degree l gives l hermitian and $l + 1$ antihermitian matrices if l is odd; if l is even, it gives $l + 1$ hermitian and l antihermitian matrices. Thus, the set of all irreducible polynomials of degrees $1 \leq l \leq N - 1$ contains the following numbers of hermitian and antihermitian matrices:

	hermitian matrices	antihermitian matrices	
N even	$\frac{1}{2}N^2 - 1$	$\frac{1}{2}N^2$	(3.3)
N odd	$\frac{1}{2}N^2 - \frac{3}{2}$	$\frac{1}{2}N^2 + \frac{1}{2}$.	

For even N , these matrices form the generators of the adjoint representation of the non-compact algebra $su(M, M)$ with $N = 2M$; for odd N , they form the adjoint representation generators of $su(M + 1, M)$ with $N = 2M + 1$. Thus, the non-compact version of $s \text{ diff}(S^2)$ can be obtained as the limit of the finite-dimensional Lie algebra $su(M, M)$ as $M \rightarrow \infty$. Note that we have not been specific about the particular real form that one gets and our counting of hermitian and antihermitian generators corresponds either to $SL(N, R)$ or $SU(N, N)$.

The algebra $s \text{ diff}(S^{1,1})$ can be given a more geometrical interpretation by considering its relation to the algebra of area-preserving diffeomorphisms of the hyperbolic space $S^{1,1}$. First, note that the Lie bracket (2.7, 2.8) may equally well be defined in a $d = 2 + 1$ “internal” Minkowski space \mathbb{M}^3 . It then turns the space of all functions on \mathbb{M}^3 into a Lie algebra. A proper subalgebra of this is the algebra of *polynomials* in the coordinates of \mathbb{M}^3 , which is isomorphic to $s \text{ diff}(S^{1,1})$.

One may then restrict attention to functions defined on the hyperboloid $S^{1,1}$ defined by $(x^0)^2 - (x^1)^2 - (x^2)^2 = a^{-2}$. It is convenient to choose coordinates σ, τ, ρ in \mathbb{M}^3 such that ∂_σ and ∂_ρ are tangent to $S^{1,1}$ and ∂_τ is normal to it. In terms of these coordinates, (2.8) reduces to

$$\{A, B\} \sim \partial_\sigma A \partial_\rho B - \partial_\rho A \partial_\sigma B. \tag{3.4}$$

Thus, $s \text{ diff}(S^{1,1})$ is isomorphic to a subalgebra of the area-preserving diffeomorphisms on $S^{1,1}$.

A given function on $S^{1,1}$ belongs to the subalgebra isomorphic to $s \text{ diff}(S^{1,1})$ if it can be continued to a polynomial defined on the rest of \mathbb{M}^3 . The simplest formulation of this condition is to consider only *real analytic* functions on $S^{1,1}$. These form a subalgebra of the algebra of general functions since (3.4) preserves analyticity. The analytic continuations of these to the rest of \mathbb{M}^3 are power series in the x^i , so they belong to $s \text{ diff}(S^{1,1})$. Moreover, the generators of $s \text{ diff}(S^{1,1})$ are the traceless homogeneous symmetric polynomials, and hence are analytic. So the algebra $s \text{ diff}(S^{1,1})$ is isomorphic to the algebra of real analytic functions on $S^{1,1}$.

It would be interesting to relate $s \text{ diff}(S^{1,1})$ to other spaces of functions defined on $S^{1,1}$, such as the bounded-energy scalar wavefunctions considered in [10]. These wavefunctions of the massive scalar wave equation on $S^{1,1}$ also form a discrete

basis of functions on \mathbb{M}^3 . However, it is not immediately clear what relation, if any, this has to $s \text{diff}(S^{1,1})$, for one may see by inspection that the polynomial basis functions that we have considered above do not satisfy the conserved positive-energy boundary conditions of [10].

We now proceed to formulate an alternative $d = 2 + 1$ higher-spin theory as a Chern–Simons gauge theory associated to $s \text{diff}(S^{1,1}) \oplus s \text{diff}(S^{1,1})$. In the Euclidean case, the $d = 3$ higher spin theory is a Chern–Simons theory associated to $s \text{diff}(S^2) \oplus s \text{diff}(S^2)$. One may call these “geometric” formulations to distinguish them from the usual “algebraic,” or component, formulation of a theory such as that of ref. [4].

Since we have two copies of the algebra $s \text{diff}(S^{1,1})$, we consider the manifold $S^{1,1} \times S^{1,1}$. We choose an *identical* coordinate system on each $S^{1,1}$ and denote the coordinates by $(\sigma^a, \tilde{\sigma}^b)$ $a, b = 1, 2$. $s \text{diff}(S^{1,1}) \oplus s \text{diff}(S^{1,1})$ is then given by the set of all real analytic functions that depend only on σ or on $\tilde{\sigma}$, i.e., $A = A(\sigma)$ or $\tilde{C} = \tilde{C}(\tilde{\sigma})$, with Lie brackets

$$\begin{aligned} \{A, B\} &= g^{-1} \varepsilon^{ab} \partial_a A \partial_b B, \\ \{\tilde{C}, \tilde{D}\} &= \tilde{g}^{-1} \varepsilon^{ab} \tilde{\partial}_a \tilde{C} \tilde{\partial}_b \tilde{D}, \\ \{A, \tilde{C}\} &= 0, \end{aligned} \quad (3.5)$$

where $\partial_a \equiv \partial/(\partial\sigma^a)$ and $g = \sqrt{-\det g_{ab}}$, with g_{ab} the usual metric on $S^{1,1}$.

We denote gauge fields taking their values in the above algebra by $\Gamma_\mu(x, \sigma)$, $\tilde{\Gamma}_\mu(x, \tilde{\sigma})$, where x^μ , $\mu = 1, 2$, are the coordinates on the (arbitrary) spacetime manifold \mathcal{M}^3 . Thus, we see that in geometrical formulation, all higher-spin fields are combined into a *single* field on the extended manifold $\mathcal{M}^3 \times S^{1,1} \times S^{1,1}$. This is to be contrasted with the component formulation, where we have an infinite number of fields on \mathcal{M}^3 .

Now define

$$\{\Gamma, \Gamma\} \equiv g^{-1} \varepsilon^{ab} \partial_a \Gamma \wedge \partial_b \Gamma, \quad (3.6)$$

where $\Gamma(x, \sigma) \equiv \Gamma_\mu(x, \sigma) dx^\mu$. The higher-spin gauge transformations and curvatures can then be rewritten in the compact form

$$\begin{aligned} \delta_\varepsilon \Gamma &= d\Gamma - \{\varepsilon, \Gamma\}, \\ R(\Gamma) &= d\Gamma - \frac{1}{2} \{\Gamma, \Gamma\}, \\ \delta_{\tilde{\varepsilon}} \tilde{\Gamma} &= d\tilde{\Gamma} - \{\tilde{\varepsilon}, \tilde{\Gamma}\}, \\ \tilde{R}(\tilde{\Gamma}) &= d\tilde{\Gamma} - \frac{1}{2} \{\tilde{\Gamma}, \tilde{\Gamma}\}. \end{aligned} \quad (3.7)$$

Note that the exterior derivative d acts on the spacetime coordinates only.

The $d = 2 + 1$ higher-spin equations of motion are [4]

$$R(\Gamma) = \tilde{R}(\tilde{\Gamma}) = 0. \quad (3.8)$$

These can be obtained from the $s \text{diff}(S^{1,1}) \wedge s \text{diff}(S^{1,1})$ Chern–Simons action

$$I = \Omega \int_{\mathcal{M}^3 \times S^{1,1}} (\Gamma \wedge d\Gamma + \frac{1}{3} \{\Gamma, \Gamma\} \wedge \Gamma) + \Xi \int_{\mathcal{M}^3 \times S^{1,1}} (\tilde{\Gamma} \wedge d\tilde{\Gamma} + \frac{1}{3} \{\tilde{\Gamma}, \tilde{\Gamma}\} \wedge \tilde{\Gamma}), \quad (3.9)$$

where

$$\int_{\mathbb{R}^3 \times S^{1,1}} \equiv \int_{\mathbb{R}^3 \times S^{1,1}} d^3x \left\{ \begin{array}{l} d^2\sigma g(\sigma) \\ \text{respectively } d^2\tilde{\sigma}\tilde{g}(\tilde{\sigma}) \end{array} \right\} \quad (3.10)$$

and Ω and Ξ are arbitrary (at the classical level) non-zero constants. In order for the spin-two sector of the action (3.9) to contain just the standard Einstein kinetic term, we must have $\Xi = -\Omega$ [11].

In order to express the higher-spin theory in terms of the more usual e and ω gauge fields, one may write the algebra $s\text{diff}(S^{1,1}) \oplus s\text{diff}(S^{1,1})$ in the following way:

$$\begin{aligned} \{A, B\}'(\sigma) &= g^{-1}(\sigma)\varepsilon^{ab}\partial_a A(\sigma)\partial_b B(\sigma), \\ \{A, \tilde{B}\}'(\tilde{\sigma}) &= \tilde{g}^{-1}(\tilde{\sigma})\varepsilon^{ab}\tilde{\partial}_a \tilde{A}(\tilde{\sigma})\tilde{\partial}_b \tilde{B}(\tilde{\sigma}), \\ \{\tilde{A}, \tilde{B}\}'(\tilde{\sigma}) &= \lambda^2 g^{-1}(\sigma)\varepsilon^{ab}\partial_a A(\sigma)\partial_b B(\sigma), \end{aligned} \quad (3.11)$$

with $\lambda \neq 0$, where we have used the fact that, given any $C(\sigma)$, there is a corresponding $\tilde{C}(\tilde{\sigma})$ defined by $\tilde{C}(\tilde{\sigma}) = C(\sigma)|_{\sigma=\tilde{\sigma}}$.

The algebras (3.5) and (3.11) are easily shown to be isomorphic by considering the $1 \leftrightarrow 1$ map given by

$$\begin{aligned} T(A) &= \frac{1}{2}(A + \lambda^{-1}\tilde{A}), \\ T(\tilde{A}) &= \frac{1}{2}(A - \lambda^{-1}\tilde{A}), \end{aligned} \quad (3.12)$$

which is an isomorphism because

$$T(\{\cdot, \cdot\}) = \{T(\cdot), t(\cdot)\}'. \quad (3.13)$$

We denote gauge fields taking their values in the algebra (3.11) by $\omega_\mu(x, \sigma)$ and $e_\mu(x, \tilde{\sigma})$. In order to have the canonical normalization of the Einstein kinetic term when the action is written out in terms of e_μ and ω_μ , we must pick $\lambda = 1/(4\Omega)$ [11]. The $d = 2 + 1$ equations of motion for e_μ and ω_μ following from (3.9) are then

$$R(e) \equiv de + \{\omega, e\}' = 0, \quad (3.14)$$

$$R(\omega) \equiv d\omega + \frac{1}{2}\{\omega, \omega\}' + \frac{1}{2}\{e, e\}' = 0. \quad (3.15)$$

The last term in (3.15) is a cosmological term with cosmological constant $-\lambda^2$, as can be seen from (3.11). Equation (3.14) is the torsion constraint. One way to solve this constraint is first to “regularize” the theory by considering the finite-dimensional $su(N, N) \oplus su(N, N)$ higher-spin theory. Since there are the same number of ω component fields as there are torsion-constraint equations, there is an algebraic solution to the $su(N, N) \oplus su(N, N)$ torsion constraint. The solution to (3.14) can then be given as the $N \rightarrow \infty$ limit of the finite-component solution.

Finally, we may take the contraction limit $\lambda \rightarrow 0$ in (3.11). Note that in this limit the last term in (3.5) vanishes. The resulting contracted algebra and equations of motion describe a geometrical $d = 2 + 1$ Poincaré higher-spin theory alternative to the one given in [4]. Recently, also a $d = 2 + 1$ Chern–Simons gauge theory associated to the $d = 3 + 1$ higher spin algebra has been constructed [12]. It turns out that this theory describes conformal higher spins.

4. Conclusion

We have found that the three-dimensional higher-spin algebras and the area-preserving diffeomorphisms are distinct members of one-parameter families of algebras. This relation may have applications in several different areas. In the context of the higher-spin theories, we have applied our results in formulating an alternative higher-spin theory which has a more geometrical interpretation than the original algebraic one. In particular, the geometrical basis may be useful in solving some of the outstanding problems of the $d = 3 + 1$ higher-spin theory, such as finding and solving constraints on the necessary non-dynamical “extra fields” [2]. It also would be of interest to investigate whether alternative higher-spin algebras analogous to the ones we have found in $2 + 1$ dimensions exist in $3 + 1$ dimensions as well. In the context of membrane theories, the relations found may be of importance for the study of representations of the area preserving diffeomorphisms. This could be relevant for an investigation of the quantum spectra. Although the area-preserving diffeomorphisms appear as a constraint algebra, under which one might expect all the physical states to be singlets, quantum anomalies could change this picture in a way similar to what happens to the Virasoro algebra in string theory, where ghost and “physical” modes separately must carry non-trivial representations, with the BRS-invariant states being constructed as invariant products of ghost and physical mode creation operators. Already a considerable amount is known about the representations of the higher-spin algebras [13], and some of these results may be useful for the construction of representations of the area preserving algebra.

Another area in which the connection between higher-spin and area-preserving diffeomorphisms is suggestive is supersymmetry. Serious difficulties lie in the way of constructing membrane theories with local world-volume supersymmetry [14]⁴, although rigidly spacetime supersymmetric models certainly exist. On the other hand, there are known supersymmetric extensions of the higher-spin algebras [2–4]. Also, infinite-dimensional superalgebras corresponding to symplectic superdiffeomorphisms have been investigated recently [16]. It would be interesting to see whether these algebras are again members of a one-parameter family of infinite-dimensional superalgebras as in the bosonic case. We recall that both $hs(1, 1)$ and its complexification $hs(2; \mathbb{C})$ admit supersymmetric extensions [2–4]. The complex superalgebra is given by

$$\begin{aligned}
 [\xi_1, \xi_2]^{\alpha(n)} &= \sum_{p,q,r=1}^{\infty} (-1)^{r/2-1/2} \frac{n!}{p!q!r!} \delta(n-p-q) \xi_1^{\alpha(p)}{}_{\beta(r)} \xi_2^{\beta(r)\alpha(q)} \begin{cases} n \geq 0, & \text{even} \\ p, q, r \geq 1, & \text{odd} \end{cases} \\
 [\xi, Q]^{\alpha(n)} &= \sum_{\substack{p=0 \\ q,r=1}}^{\infty} (-1)^{r/2-1/2} \frac{n!}{p!q!r!} \delta(n-p-q) Q^{\alpha(p)}{}_{\beta(r)} \xi^{\beta(r)\alpha(q)} \begin{cases} p \geq 0, & \text{even} \\ n, q, r \geq 1 & \text{odd} \end{cases} \\
 [Q_1, Q_2]^{\alpha(n)} &= - \sum_{\substack{r=0 \\ p,q=1}}^{\infty} (-1)^{r/2} \frac{n!}{p!q!r!} \delta(n-p-q) Q_1^{\alpha(p)}{}_{\beta(r)} Q_2^{\beta(r)\alpha(q)} \begin{cases} n, r \geq 0, & \text{even} \\ p, q \geq 1, & \text{odd,} \end{cases}
 \end{aligned}
 \tag{4.1}$$

⁴ For a recent proposal of a spinning supermembrane action however, see [15]

where the spinorial index $\alpha = 1, 2$ is an $su(2)$ index. The algebra (4.1) contains a finite-dimensional graded subalgebra $su(2|1; \mathbb{C})$, which is generated by the supercharges $Q^{\alpha(1)}$ and by the $su(2)$ generators $\xi^{\alpha(2)}$. The bosonic subalgebra $s\text{diff}(S^2; \mathbb{C})$ is obtained from (4.1) by the truncation $Q^{\alpha(n)} = 0$. Since this bosonic subalgebra contains only generators with even numbers of spinorial indices, an $su(2)$ -covariant reality condition can be defined for them, e.g. $\xi_c^{\alpha\beta} = \xi^{\alpha\beta}_c$. In this way, we obtain the real form $s\text{diff}(S^2)$ of $s\text{diff}(S^2; \mathbb{C})$.

Another important issue is the nature of the area-preserving diffeomorphism algebras for spaces with non-spherical topology. Interesting work on the structure of the algebra on tori has been done in [17]. Furthermore, in [18], an interesting relation has been found between the area-preserving diffeomorphisms of the two-plane $s\text{diff}(R^2)$ and the W algebras [19]. There also exists a general classification of the area-preserving infinite-dimensional algebras [20]. Whether choices of the sets of basis functions on spaces with different topologies do in fact correspond to distinct algebras deserves more careful study.

Appendix

In this appendix we will give a simple rederivation of the uniqueness theorem of [2] for the case of $d = 2 + 1$ dimensions. The uniqueness theorem of [2] stated that there exists just one infinite-dimensional extension of $so(3)$ whose generators decompose under $so(3)$ into a sequence of representations where each symmetric traceless $so(3)$ representation occurs just once. We will point out at which point one makes the assumption that no Fierz identities need be used and also which further assumptions are needed in the proof of the theorem.

In order to make use of the results of [2], it is convenient to describe the algebras in terms of generators $\xi^{\alpha(n)}$ (with n even), where the α indices are spinorial indices of $su(2) \simeq so(3)$. We assume that the commutation relations for the $\xi^{\alpha(n)}$ take the general form

$$\begin{aligned}
 & [\xi_1, \xi_2]^{\alpha(n)} \\
 &= \sum_{p,q,s=1}^{\infty} (-1)^{s/2-1/2} \frac{n!}{p!q!s!} \delta(n-p-q) \alpha(p,q,s) \xi_1^{\alpha(p)} \xi_2^{\beta(s)\alpha(q)} \quad \begin{array}{ll} n \geq 0, & \text{even} \\ p, q, s \geq 1, & \text{odd.} \end{array}
 \end{aligned} \tag{A.1}$$

The factor $(-1)^{(s-1)/2} n! / (p!q!s!)$ has been introduced for convenience and $\alpha(p, q, s)$ is an arbitrary function of the non-negative odd integers p, q and s ; $\alpha(p, q, s) = 0$ if p, q or s is even. Antisymmetry of the commutator or Lie bracket $[\xi_1, \xi_2] = -[\xi_2, \xi_1]$ imposes the following symmetry property on $\alpha(p, q, s)$:

$$\alpha(p, q, s) = \alpha(q, p, s). \tag{A.2}$$

The Jacobi identities corresponding to the commutation relations (A.1) have been calculated in [2]. They lead to the following equation for $\alpha(p, q, s)$:

$$\begin{aligned}
 & \alpha(p+s, q+v, u) \alpha(p+q, r, s+v) \\
 &+ (-1)^{s+u+(p+s+u)(q+u+r+s)} \alpha(q+u, r+s, v) \alpha(q+r, p, u+s) \\
 &+ (-1)^{s+v+(r+s+v)(p+s+q+v)} \alpha(r+v, p+u, s) \alpha(r+p, q, u+v) = 0.
 \end{aligned} \tag{A.3}$$

It is in the derivation of (A.3) that one makes the assumption that it is never necessary to use Fierz identities of the form

$$\xi_{1\alpha}\xi_{2\gamma}\xi_3^\gamma + \xi_{1\gamma}\xi_2^\gamma\xi_{3\alpha} - \xi_{1\gamma}\xi_{2\alpha}\xi_3^\gamma = 0 \tag{A.4}$$

is verifying the Jacobi identities [2].

The identity (A.3) depends on the six integers p, q, s, u, v & r . Using the fact that the function $\alpha(p, q, s)$ vanishes if one of its entries is even, it follows that for any choice of these integers, one of the terms on the left-hand side of (A.3) is zero. The two terms are non-zero only if one takes two of the integers to be even and the remaining four to be odd. Different choices of the two even parameters lead to identical equations. We make the following choice:

$$\begin{aligned} p, u, v, r &\geq 1 \quad \text{odd} \\ q, s &\geq 0 \quad \text{even.} \end{aligned} \tag{A.5}$$

With this choice, the identity (A.3) reduces to

$$\alpha(p + s, q + v, u)\alpha(p + q, r, s + v) - \alpha(q + u, r + s, v)\alpha(q + r, p, u + s) = 0. \tag{A.6}$$

The identity (A.6) does not allow us to solve for $\alpha(p, q, s)$ uniquely. This is due to the freedom that one has to rescale the basis of the algebra, as has been explained in the discussion around (2.18). This freedom leads to the following invariance of the identity (A.3) under transformation of the basis:

$$\alpha(p, q, s) \rightarrow \alpha'(p, q, s) = \frac{\Psi(p + q)}{\Psi(p + s)\Psi(q + s)}\alpha(p, q, s). \tag{A.7}$$

Two solutions to the identity (A.3) that are related by the basis transformation (A.7) for some choice of $\Psi(n)$ correspond to isomorphic algebras.

In trying to solve the identity (A.6), it is convenient to choose a definite basis for the algebra, thus fixing the transformation freedom (A.7) completely. We now assume that the structure constant $\alpha(p, q, 1)$, which corresponds to the coefficient of the highest-degree polynomial in a given commutator, is always non-zero. This allows one to partially fix the freedom (A.7) by imposing the conditions

$$\alpha(p, q, 1) = 1. \tag{A.8}$$

These conditions restrict the function $\Psi(m)$ occurring in the transformation (A.7) so that it must satisfy

$$\Psi(p + q) = \Psi(p + 1)\Psi(q + 1). \tag{A.9}$$

From (A.8) and (A.9), it follows that $\Psi(2) = 1$. The values of $\Psi(4)$, $\Psi(6)$, etc. are not yet determined.

Next, consider the identity (A.6) for $q = s = 0$ and $u = 1$. Subject to the conditions (A.8), (A.6) reduces for these values to

$$\alpha(p, r, v) = \alpha(1, r, v). \tag{A.10}$$

Recalling the symmetry property (A.2), this implies

$$\alpha(p, q, s) = \alpha(s), \quad \alpha(1) = 1. \tag{A.11}$$

In terms of $\alpha(s)$, the remaining unfixed basis-transformation freedom is given by

$$\alpha(s) \rightarrow \alpha'(s) = \frac{\alpha(s)}{(\Psi(s+1))^2}, \quad (\text{A.12})$$

where $\Psi(m)$ satisfies (A.9). Substituting (A.11) back into the identity (A.6) leads to the relation

$$\alpha(u)\alpha(s+v) - \alpha(v)\alpha(u+s) = 0, \quad (\text{A.13})$$

which is invariant under the remaining basis freedom (A.12). For $u=1$, (A.13) reduces to

$$\alpha(s+v) = \alpha(v)\alpha(s+1). \quad (\text{A.14})$$

We now furthermore assume that the structure constant $\alpha(3)$, which corresponds to the coefficient of the next to highest degree polynomial in a given commutator, is always non-zero. This allows one to fix the remaining basis freedom by the condition

$$\alpha(3) = 1. \quad (\text{A.15})$$

From (A.12) it follows that this condition fixes $\Psi(4) = 1$, and it then follows from (A.9) that $\Psi(m) = 1$ for all m , so that the basis freedom is now fixed completely. Substituting the gauge condition (A.15) into Eq. (A.14), we similarly find that $\alpha(s) = 1$ for all odd s , and hence

$$\alpha(p, q, s) = 1 \quad \forall p, q, s \quad \text{odd}. \quad (\text{A.16})$$

This is the solution to the identity (A.6) in the basis fixed by (A.8, A.15). From it, we see that, up to rescaling of the basis polynomials of the algebra, there is a unique solution to (A.6). This concludes the proof of the uniqueness theorem.

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Note added. After submission of this paper, we received a preprint by Bordemann, Hoppe and Schaller [21] in which the existence of the one-parameter family of algebras discussed in this paper is also noted. These authors consider also the possibility of nonlinear changes on the bases of the family of algebras in demonstrating the inequivalence of the algebras for distinct values of the parameter λ .

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