First Order Phase Transitions in Unbounded Spin Systems. II. Completeness of the Phase Diagram

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Abstract. We continue our analysis of unbounded spin systems with nearest neighbor interaction W and a single spin potential V which has N deep and widely separated minima. In this second part we show that all translation invariant phases obeying a certain regularity condition are convex combinations of the stable phases determined in the first part of this paper. For periodic boundary conditions each stable phase contributes with the same weight in the infinite volume limit.

1. Introduction

In the first part [1] of this work we have shown that for an unbounded spin model with Hamiltonian

$$H = \sum_x V(R_x) + \sum_{\langle xy \rangle} W(R_x, R_y), \qquad (1.1)$$

where V has N deep and widely separated minima and W is a kinetic energy type interaction (see Sect. 1 of [1] for the precise assumptions), the stable phases[1] are characterized by the condition that the free energy, h'_q, of a certain truncated model is minimal. As we have seen this is enough to construct translation invariant states, $\langle \cdot \rangle_q$, which are small perturbations of the corresponding Gaussian approximations. Furthermore these states show exponential clustering and hence are, in the usual langauge of statistical mechanics, pure states of the sytem (1.1).

The goal of this paper is to show that these are in fact *all* pure states, i.e. that any translation invariant equilibrium state, $\langle \cdot \rangle$, is a convex combination of the stable states, $\langle \cdot \rangle_q$, constructed in [1]. To be more precise assume that V and W obey assumption A.0 through A.3 of [1] (guaranteeing the convergence of the cluster expansions developed in [1] and hence the existence of $\langle \cdot \rangle_q$). Assume in addition the following

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[1] Throughout this paper we will use letters q, q', etc. to denote stable phases, while the letters m, m', etc. denote arbitrary (stable or unstable) phases

Assumption R (Regularity of W). *Let*

$$\alpha(R) = V(R) - \min_{R' \in \mathbf{R}} V(R').$$ (1.2)

Then there is a constant $k_1 < \infty$, possibility depending on β, such that

$$W(R, R') \leqq k_1 (1 + \alpha(R) + \alpha(R')).$$

For a configuration $\underline{R} : \mathbf{Z}^\nu \to \mathbf{R}$ we set

$$\| \underline{R} \| = \sup_{j \geqq 0} \frac{1}{|\Lambda(j)|} \sum_{x \in \Lambda(j)} \alpha(R_x),$$ (1.3)

where $\Lambda(j)$ is the set of all points which have distance $\leqq j$ from the origin[2]. Generalising a little bit the notion of temperedness from [2,4], we define

Definition 1.1. *An equilibrium state $\langle \cdot \rangle$ is called tempered[3], if $\| \underline{R} \| < \infty$ with probability one, that is, if $\left\langle \chi \left(\bigcup_{k \in \mathbf{N}} \{ \underline{R} \mid \| \underline{R} \| < k \} \right) \right\rangle = 1$, where $\chi(E)$ denotes the characteristic function of the event E.*

The main result of this paper is the following theorem. We recall that V depends on $N - 1$ parameters $\mu = (\mu_i)_{i = 1, \ldots, N - 1}$, and that all the results of [1] were derived under the assumption that μ lies in a certain neighborhood $\mathscr{V}_1$ of $0 \in \mathbf{Z}^{N-1}$.

Theorem A. *There is a constant $\beta^* < \infty$ such that all translation invariant tempered equilibrium states $\langle \cdot \rangle$ are convex combinations of the stable states $\langle \cdot \rangle_q$, provided $\beta \geqq \beta^*$ and $\mu \in \mathscr{V}_1$.*

The proof of Theorem A is based on an idea of [5], where the corresponding theorem is shown for the Ising model at low temperatures. To explain the underlying idea consider an observable A which depends only on one spin variable. Using the DLR-equations [6,7] together with the translation invariance of the state $\langle \cdot \rangle$ we may rewrite the expectation value of A as

$$\langle A \rangle = \int dP(s_{\partial \Lambda^c}) \langle \bar{A} \rangle_{\Lambda, s_{\partial \Lambda^c}},$$ (1.4)

where Λ is an arbitrary finite subvolume of $\mathbf{Z}^\nu$; $\bar{A}$ denotes the average of A over all translations in Λ and $\langle \cdot \rangle_{\Lambda, s_{\partial \Lambda^c}}$ denotes the finite volume expectations in Λ with boundary conditions $s_{\partial \Lambda^c}$. Choosing $0 < \delta < 1/\nu$ and fixing all contours in the low temperature expansion of $\langle \bar{A} \rangle_{\Lambda, s_{\partial \Lambda^c}}$ which touch the boundary of Λ (we call their union the outer contour network), one then shows that the size of this outer contour network is smaller than $|\partial \Lambda^c|^{1 + \delta}$ with probability $1 - \varepsilon(\Lambda)$, where $\varepsilon(\Lambda) \to 0$ as $\Lambda \to \mathbf{Z}^\nu$.

On the other hand, for a given outer contour network, Out, with total size smaller than $|\partial \Lambda^c|^{1 + \delta}$, and a single term in the average $\bar{A}$, the translated observable

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[2] We recall that we usually use the l_∞ distance $\mathrm{dist}(x, y) = |x - y|_\infty$, except for the notion of nearest neighbors, where $|x - y| = 1$ denotes the l_1 distance

[3] The definition of "tempered" in the sense of [2,4] is obtained by substituting the norm $\| \underline{R} \|^{(2)} = \sup_{j \geqq 0} (1/|\Lambda(j)|) \sum_{x \in \Lambda(j)} R_x^2$ for $\| \underline{R} \|$

$t_x(A)$ is typically far away from the outer contour network. Therefore its conditional expectation value $\langle t_x(A)|\text{Out}\rangle$ is well approximated by either $\langle t_x(A)\rangle_+ = \langle A\rangle_+$ or $\langle t_x(A)\rangle_- = \langle A\rangle_-$, depending on whether $t_x(A)$ lies in a $+$ or in a $-$ region of Out. Summing over Out and over the translations in Λ one obtains the theorem by a 3ε argument.

In our context two main difficulties arise when one tries to generalize this argument. The first one, the existence of unstable phases, already had to be faced in [8], where general finite range spin models with bounded, discrete spins were considered. The second is due to the fact that we are dealing with unbounded spins.

This difficulty arises at two different stages. First, the boundary conditions $R_{\partial\Lambda^c}$ arising in the analog of (1.4) may be arbitrarily large. This spoils the probability estimates on the existence of outer contour networks, Out, of size larger than $|\partial\Lambda^c|^{1+\delta}$. Secondly the conditional expectation values, $\langle\cdot|\text{Out}\rangle$, for a fixed outer contour network, Out, still involve boundary conditions in the whole neighborhoods $\mathcal{U}_q$ of the minimas R_q of V; while these boundary conditions are bounded for any fixed β, they are still too large[4] to guarantee the convergence of the cluster expansion for $\langle\cdot\rangle_{\Lambda,R_{\partial\Lambda^c}}$ which we would like to use to control the difference between $\langle t_x(A)|\text{Out}\rangle$ and $\langle A\rangle_q$.

We will solve these problems as follows: First we use the arguments of [2–4] to show that with probability $1 - \varepsilon_1(\Lambda)$, where $\varepsilon_1(\Lambda)\to 0$ as $\Lambda\to\mathbf{Z}^\nu$, the boundary conditions appearing in the analog of (1.4) do not grow faster than $\log j$ with j if $\Lambda = \Lambda(j)$. For such boundary conditions the desired probability estimates on the size of the outer contour network, Out, can be proven by a combination of the methods developed in [1] and [8]. Concerning the second problem we will in fact avoid boundary conditions growing like $\tilde{\tau}(\beta)^{1/2}$ in the conditional expectation values $\langle\cdot|\text{Out}\rangle$ by adding a partition into small and large fields to the expansion into contours before defining the notion of an outer contour network, see Sect. 2 for details.

The organization of this paper is as follows: In Sect. 2 we recall the definition of a general equilibrium state and define the notion of an outer contour network. Assuming the validity of the probability estimates for large boundary conditions (Lemma 2.3) and those for large outer contour networks (Lemma 2.4), we well as a lemma (Lemma 2.5) which controls the difference between $\langle A\rangle_{\Lambda,R_{\partial\Lambda^c}}$ and $\langle A\rangle_q$ for boundary conditions $R_{\partial\Lambda^c}$ in a small neighborhood of R_q we then prove Theorem A. Lemma 2.3 is proven in the appendix, using the methods of [2–4], and Lemma 2.4 is proven in Sect. 3. In Sect. 4 we prove Lemma 2.5 using the cluster expansion developed in [1], Sect. 7. Finally Sect. 5 is devoted to the study of periodic boundary conditions. The main result of this last section will be the following:

Theorem B. *Assume that V and W obey the conditions A.0 through A.3 of [1]. Then, for $\mu\in\mathcal{V}_1$ and for β large enough, the limit*

$$\langle\cdot\rangle_{\text{per}} = \lim_{\Lambda\to\mathbf{Z}^\nu}\langle\cdot\rangle_\Lambda^{\text{per}},$$

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[4] They grow with β like $\tilde{\tau}(\beta)^{1/2}$; see Sect. 1 of [1] for the definition of $\tilde{\tau}(\beta)$

where $\langle \cdot \rangle_\Lambda^{\mathrm{per}}$ denotes expectations in the volume Λ with periodic boundary conditions, is well defined and is a convex combination of the stable states, $\langle \cdot \rangle_q$, with equal weight for each of them.

2. The Translation Invariant Equilibrium States

In this section we present the main steps of the proof of Theorem A. The technical details of some auxiliary estimates are deferred to Sect. 3, Sect. 4 and Appendix A. For the convenience of the reader we recall some basic notations.

We denote by Ω the set of all functions $\underline{R} : \mathbf{Z}^\nu \to \mathbf{R}$, and by R_Λ the restriction of $\underline{R}$ to a bounded set $\Lambda \subset \mathbf{Z}^\nu$. A local observable is a complex valued, Borel measurable function on Ω which, for a suitable bounded set $\Lambda \subset \mathbf{Z}^\nu$, depend only on R_Λ. One says that A is an observable in Λ, or also "A has support in Λ," in this case. Finite volume expectation values with boundary conditions $R_{\partial \Lambda^c}$ are defined as

$$\langle A \rangle_{\Lambda, R_{\partial \Lambda^c}} = \frac{1}{Z(\Lambda | R_{\partial \Lambda^c})} \int A e^{-H(R_\Lambda | R_{\partial \Lambda^c})} dR_\Lambda \qquad (2.1)$$

with

$$H(R_\Lambda | R_{\partial \Lambda^c}) = \sum_{x \in \Lambda} V(R_x) + \sum_{\langle xy \rangle \cap \Lambda \neq \varnothing} W(R_x, R_y) \qquad (2.2)$$

and the obvious definition of $Z_q(\Lambda | R_{\partial \Lambda^c})$. One says, a measure dP on Ω obeys the DLR-equations [6, 7] if, for any bounded set Λ and all bounded observables A in Λ[5]

$$\int dP A = \int dP(\underline{R}) \langle A \rangle_{\Lambda, R_{\partial \Lambda^c}}. \qquad (2.3)$$

A Borel measure on Ω obeying the DLR-equations is called an equilibrium measure, and the corresponding linear functional $\langle \cdot \rangle = \int \cdot dP$ is called an equilibrium state. If an equilibrium state $\langle \cdot \rangle$ fulfils the condition that $\| \underline{R} \| < \infty$ with probability 1 (see Def. 1.1 of Sect. 1) it is called tempered.

We now fix a translation invariant, tempered equilibrium state $\langle \cdot \rangle = \int \cdot dP$. Our goal is to show that it is a convex combination of the stable states $\langle \cdot \rangle_q$ constructed in [1]. We therefore have to show that for certain nonnegative numbers α_q, where $\sum_q \alpha_q = 1$, and all bounded local observables A,

$$\langle A \rangle = \sum_q \alpha_q \langle A \rangle_q. \qquad (2.4)$$

We recall that we introduced the convention that a sum or product over q always denotes a sum or product, respectively, over all stable values of q, whereas sums or products over m, m', etc. are sums or products, respectively, over all $m \in \{1, \ldots, N\}$.

Consider a volume Λ of the form $\Lambda = \Lambda(j) = \{x \in \mathbf{Z}^\nu | \mathrm{dist}(0, x) \leq j\}$ which is a union of L-blocks centered in $L\mathbf{Z}^\nu$, and a bounded observable A with support in

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[5] In a more probabilistic language this is often formulated as: The conditional probability $P(dR_\Lambda | R_{\Lambda^c})$ is given by the Gibbs measure $1/Z(\Lambda | R_{\partial \Lambda^c}) e^{-H(R_\Lambda | R_{\partial \Lambda^c})} dR_\Lambda$

Λ. We denote by $\bar{\Lambda}$ the set of all $x \in \mathbf{Z}^\nu$ such that the translate $t_x(A)$ has support in Λ and define

$$\bar{A} = \frac{1}{|\bar{\Lambda}|} \sum_{x \in \bar{\Lambda}} t_x(A). \tag{2.5}$$

Using the translation invariance of $\langle \cdot \rangle$ (which implies that $\langle A \rangle = \langle \bar{A} \rangle$) together with the DLR-equations we may rewrite $\langle A \rangle$ as

$$\langle A \rangle = \int dP(\underline{R}) \langle \bar{A} \rangle_{\Lambda, R_{\partial \Lambda^c}}. \tag{2.6}$$

In order to analyse the expectation values $\langle \bar{A} \rangle_{\Lambda, R_{\partial \Lambda^c}}$ we introduce the notation of an outer contour network. We first rewrite the partition function $Z(\Lambda | R_{\partial \Lambda^c})$ using the same partition of unity as in [1], Sect. 2:

$$Z(\Lambda | R_{\partial \Lambda^c}) = \sum_\omega \int dR_\Lambda \prod_{x \in \Lambda} \chi_{\omega(x)}(R_x) e^{-H(R_\Lambda | R_{\partial \Lambda^c})}, \tag{2.7}$$

where the sum goes over all $\omega : \Lambda \to \{0, 1, \dots, N\}$ and $\chi_\omega(R)$ is the characteristic function of the set $\mathcal{U}_\omega$ (we recall that $\mathcal{U}_1, \dots, \mathcal{U}_N$ are certain neighborhoods of the minima, $R_1, \dots, R_N$, of the potential V, with growing diameter $2\tau(\beta)^{1/2}$ as the inverse temperature $\beta \to \infty$, and that $\mathcal{U}_0 = \mathbf{R} \backslash \bigcup_{m=1}^N \mathcal{U}_m$).

Differing a little bit from the notation introduced in [1] we define: a block $\square(x')$ in Λ is *m-correct*, if $\square(x')$ does not touch $\partial \Lambda^c$ and if $\omega(x) = m \forall x$ with $\mathrm{dist}(x, \square(x')) \leq 1$. If $\square(x')$ is *q-correct* for some stable q we say that $\square(x')$ is a *stable book*. As before the union of all incorrect blocks is denoted $B(\omega)$.

Let $\chi_{\mathrm{small}}^{(m)}(R_{\square(x')})$ be the characteristic function of the event that $|R_x - R_m| \leq B \forall x \in \square(x')$, with some constant B to be chosen later. We define

$$\chi_{\mathrm{small}}(R_{\square(x')}) = \sum_{m=1}^N \chi_{\mathrm{small}}^{(m)}(R_{\square(x')}),$$

$$\chi_{\mathrm{large}}(R_{\square(x')}) = 1 - \chi_{\mathrm{small}}(R_{\square(x')}),$$

and indroduce the notation

$$\chi_{\mathrm{small}}(R_X) = \prod_{\square(x') \subset X} \chi_{\mathrm{small}}(R_{\square(x')})$$

for a union, X, of L-blocks in Λ, and similarly for $\chi_{\mathrm{large}}(R_X)$. Introducing the above partition of unity for all stable blocks we rewrite $Z(\Lambda | R_{\partial \Lambda^c})$ as

$$Z(\Lambda | R_{\partial \Lambda^c}) = \sum_{\omega, X} \int dR_\Lambda \chi_{\mathrm{large}}(R_X) \chi_{\mathrm{small}}(R_{X^c}) \prod_{x \in \Lambda} \chi_{\omega(x)}(R_x) e^{-H(R_\Lambda | R_{\partial \Lambda^c})}, \tag{2.8}$$

where the sum goes over pairs (ω, X) consisting of a function $\omega : \Lambda \to \{0, \dots, N\}$ and a union of stable blocks (stable with respect to ω). Given X and ω, X^c denotes the union of all stable blocks which are not in X.

We now define the outer contour network of a configuration (ω, X). We consider the union, $\mathrm{Irr} = X \cup B(\omega)$, and define: The connected component, E, of Irr which

touches $\partial\Lambda^c$ is called the *boundary component* of Irr. A component, C, of Irr is called an *m-component* of Irr if C does not touch $\partial\Lambda^c$ and if $\omega = m$ on $\partial_e C$. A *m-component*, C, of Irr is called large, if $a_m \operatorname{diam} C > \tau/3$ (with a_m as defined in [1], Sect. 3), and small otherwise. Finally, a component, C, of Irr is called a *canonical component* of Irr, if C is large and if there is no small component, C', of Irr for which C lies in the interior of C'. We define the *outer contour network* of the configuration (ω, X) as the triple $\underline{\text{Out}} = (\text{Out}, \omega_{\text{Out}}, X(\text{Out}))$, where Out is the union of the boundary component with all canonical components of Irr, ω_{Out} is the restriction of ω to Out and $X(\text{Out}) = X \cap \text{Out}$; for $m = 1, \ldots, N$; V_m denotes the union of all components, C, of $\Lambda\backslash\text{Out}$ for which $\omega = m$ on ∂C.

Resumming all (ω, X) which lead to the same outer contour network we rewrite

$$Z(\Lambda|R_{\partial\Lambda^c}) = \sum_{\underline{\text{Out}}} \int dR_{\text{Out}} \chi_{\text{large}}(R_{X(\text{Out})}) \prod_{x\in\text{Out}} \chi_{\omega(x)}(R_x) e^{-H(R_{\text{Out}}|R_{\partial\Lambda^c})}$$

$$\cdot \prod_m{}' Z_m^{\text{small}}(V_m|R_{\partial V_m^c}) \prod_q \tilde{Z}_q(V_q|R_{\partial V_q^c}), \qquad (2.9)$$

where the product $\prod'$ runs over all unstable m, and the product over q runs over all stable q. Z_m^{small} is the partition function obtained from Z_m^{dil} by restricting the sum over ω in the definition of Z_m^{dil} (see Sect. 2 of [1]) to a sum over all configurations containing only small external contours, and $\tilde{Z}_q(V_q|\cdot)$ is obtained from $Z_q^{\text{dil}}(V_q|\cdot)$ by inserting an additional characteristic function $\chi_{\text{small}}(R_{\square(x')})$ for each block touching ∂V_q.

Remark 2.1. We recall that $a_q = 0$ for stable q; therefore all canonical components of Irr must be m-components for some unstable m. It follows that the volumes V_q in (2.9) are always volumes without holes, while V_m may have holes if m is unstable.

We define approximations $\alpha_q(\Lambda)$ to α_q by

$$\alpha_q(\Lambda) = \int dP(\underline{R}) \sum_{\underline{\text{Out}}} P(\underline{\text{Out}}|R_{\partial\Lambda^c}) \alpha_q(\underline{\text{Out}}), \qquad (2.10)$$

with

$$\alpha_q(\underline{\text{Out}}) = \frac{1}{\sum_{q'}|V_{q'}|} \times |V_q| \qquad (2.11)$$

and

$$P(\underline{\text{Out}}|R_{\partial\Lambda^c}) = \frac{1}{Z(\Lambda|R_{\partial\Lambda^c})} \int dR_{\text{Out}} \chi_{\text{large}}(R_{X(\text{Out})}) \prod_{x\in\text{Out}} \chi_{\omega(x)}(R_x)$$

$$\cdot e^{-H(R_{\text{Out}}|R_{\partial\Lambda^c})} \prod_m{}' Z_m^{\text{small}}(V_m|R_{\partial V_m^c}) \prod_q \tilde{Z}_q^{\text{dil}}(V_q|R_{\partial V_m^c}). \qquad (2.12)$$

We want to show that the limits $\alpha_q = \lim_{\Lambda\to\mathbf{Z}^\nu} a_q(\Lambda)$ exist and satisfy Eq. (2.4) for all bounded local observables A. Both statements, and hence Theorem A, are an immediate corollary of the following Proposition 2.1. We denote by $\Lambda \nearrow \mathbf{Z}^\nu$ a sequence of volumes $\Lambda(j)$, $j \nearrow \infty$, which are unions of L-cubes. $\mathscr{V}_1 \subset \mathbf{R}^{N-1}$ is the region defined in Lemma 3.1 of [1].

Proposition 2.1. *There is a constant* $\beta^* < \infty$ *such that, for* $\mu \in \mathscr{V}_1$ *and* $\beta \geq \beta^*$, *and a suitable choice of* L *and* B,

$$\left| \langle A \rangle - \sum_q \alpha_q(\Lambda) \langle A \rangle_q \right| \to 0 \tag{2.13}$$

as $\Lambda \nearrow \mathbf{Z}^\nu$, *for all bounded, local observables,* A.

Corollary 2.2. *For* μ, β, B *and* L *as in Proposition 2.1, the limits*

$$\lim_{\Lambda \nearrow \mathbf{Z}^\nu} \alpha_q(\Lambda) = \alpha_q \tag{2.14}$$

exist, and fullfill Eq. (2.4) for all bounded local observables A.

Proof of Corollary 2.2. We only have to prove (2.14). Consider the observables $A_q = \chi_q(R_x)$, where x is some fixed point in $\mathbf{Z}^\nu$ (e.g. the origin) and q is stable. For β large,

$$\langle A_q \rangle_{q'} = M_{qq'} = \delta_{qq'} + \delta M_{qq'},$$

where $\delta M_{qq'} \to 0$ as $\beta \to \infty$. Therefore M is invertible for β large enough, and M^{-1} is bounded. Using (2.13) it follows that

$$\sum_{q'} (M^{-1})_{qq'} \langle \chi_{q'}(R_x) \rangle - \alpha_q(\Lambda)$$

goes to zero as $\Lambda \nearrow \mathbf{Z}^\nu$. This proves (2.14) and hence the corollary. ∎

We are now going to prove the proposition. The proof is based on three technical lemmas which will be proved in separate sections. For a volume Λ without holes (that is, a volume Λ such that Λ^c is connected) we denote

$$\langle A \rangle^q_{\Lambda, R_{\partial \Lambda^c}} = \frac{1}{Z_q(\Lambda | R_{\partial \Lambda^c})} \int A \prod_{x \in \partial \Lambda} \chi_q(R_x) e^{-H(R_\Lambda | R_{\partial \Lambda^c})} dR_\Lambda \tag{2.15}$$

with the obvious definition of $Z_q(\Lambda | R_{\partial \Lambda^c})$.

Lemma 2.3. *There is a constant* $a < \infty$, *and a function* $\varepsilon_1(j) \to 0$ *as* $j \to \infty$, *such that*

$$\alpha(R_x) \leq a \log j \quad \forall x \in \partial \Lambda(j)^c$$

with probability $1 - \varepsilon_1(j)$, *provided the equilibrium measure* P *is tempered.*

Lemma 2.4. *Denote by* $\sum''$ *the sum over all outer contour networks such that*

$$\left| \Lambda \setminus \bigcup_q V_q \right| > |\partial \Lambda^c|^{1+\delta}. \tag{2.16}$$

Assume that $R_{\partial \Lambda^c}$ *is chosen in such a way that* $\alpha(R_x) \leq a \log j$ *for all* $x \in \partial \Lambda^c = \partial \Lambda(j)^c$, *and let* $L \geq L_0$, *where* L_0 *is the constant from Theorem 3.3 of [1]. Then there are constants* $\beta_1 = \beta_1(L)$ *and* $B_1 = B_1(L)$ *such that*

$$\sum_{\text{Out}}'' P(\underline{\text{Out}} | R_{\partial \Lambda^c}) \leq \varepsilon_2(\Lambda),$$

provided $\beta \geq \beta_1$, $\mu \in \mathscr{V}_1$ *and* $B \geq B_1$. $\varepsilon_2(\Lambda)$ *is a constant going to zero as* $\Lambda \nearrow \mathbf{Z}^\nu$.

Lemma 2.5. *There are constants $L_0 < \infty$ and $\beta_2 = \beta_2(L, B)$ such that, for $B < \infty$, $L \geq L_0, \mu \in \mathscr{V}_1, \beta \geq \beta_2(L, B)$ and all volumes Λ which are unions of L-blocks centered in $L\mathbb{Z}^v$*

$$|\langle A \rangle^q_{\Lambda, R_{\partial \Lambda^c}} - \langle A \rangle_q| \leq \|A\| K^{|\text{supp} A|} e^{-M \text{dist}(A, \partial \Lambda)}, \tag{2.17}$$

provided q is stable and $|R_x - R_q| \leq B \, \forall x \in \partial \Lambda^c$. $M > 0$ and $K < \infty$ are constants not depending on A or Λ.

Proof of Proposition 2.1. The proof relies on two facts:

i) outer contour networks for which $\left| \Lambda \backslash \bigcup_q V_q \right|$ is larger than $|\partial \Lambda^c|^{1+\delta}$ are suppressed as $\Lambda \nearrow \mathbb{Z}^v$ (Lemma 2.4);

ii) for a given outer contour network with $\left| \Lambda \backslash \bigcup_q V_q \right| \leq |\partial \Lambda^c|^{1+\delta}$ most terms $t_x(A)$ in the average $\bar{A}$ lie deep inside a region V_q; therefore their conditional expectation values $\langle t_x(A)|\underline{\text{Out}}\rangle_{\Lambda, R_{\partial \Lambda^c}}$ are well approximatized by $\langle t_x(A)\rangle_q = \langle A \rangle_q$ (Lemma 2.5).

Summing over $\underline{\text{Out}}$ and integrating over the boundary conditions $R_{\partial \Lambda^c}$ we will obtain the proposition by a 4ε-argument.

We fix $L \geq L_0$ according to Lemmas 2.4 and 2.5 and choose $B \geq B_1(L), \beta \geq \max\{\beta_1(L), \beta_2(L, B)\}$. We fix a according to Lemma 2.3 and $\delta < 1/v$. Given ε and a bounded local observable A, we will show that the left-hand side of (2.13) can be made smaller than 4ε by choosing Λ large enough.

Given A and ε, we choose d_0 so large that

$$\|A\| K^{|\text{supp} A|} e^{-Md_0} \leq \varepsilon.$$

For an outer contour network $\underline{\text{Out}}$ with $\left| \Lambda \backslash \bigcup_q V_q \right| \leq |\partial \Lambda^c|^{1+\delta}$, let D_q denote the union of all cubes, $\square(x')$, in V_q which touch the outer contour network, and set $\tilde{V}_q = V_q \backslash D_q$. By the definition of an outer contour network, all fields R_x with $x \in D_q$ obey the condition $|R_x - R_q| \leq B$. Therefore the conditional expectation value $\langle t_x(A)|\underline{\text{Out}}\rangle_{\Lambda, R_{\partial \Lambda^c}}$ can be rewritten as

$$\int d\tilde{P}(R_{\partial \tilde{V}_q^c}) \langle t_x(A)\rangle_{\tilde{V}_q, R_{\partial \tilde{V}_q^c}},$$

provided $t_x(A)$ has support in $\tilde{V}_q$; $d\tilde{P}$ is a probability measure with support in the set of all fields $R_{\partial \tilde{V}_q^c}$ which obey the condition $|R_x - R_q| \leq B \, \forall x \in \partial \tilde{V}_q^c$. Applying Lemma 2.5 we obtain that

$$|\langle t_x(A)|\underline{\text{Out}}\rangle_{\Lambda, R_{\partial \Lambda^c}} - \langle A \rangle_q| \leq \varepsilon,$$

provided $x \in \tilde{\Lambda}_q$, where $\tilde{\Lambda}_q$ is defined as the set of all $x \in \mathbb{Z}^v$ such that $t_x(A)$ has support in $\tilde{V}_q$ and $\text{dist}(t_x(A), \partial \tilde{V}_q) \geq d_0$. On the other hand,

$$\left| \frac{|\tilde{\Lambda}_q|}{|\bar{\Lambda}|} - \alpha_q(\underline{\text{Out}}) \right| = \left| \frac{|\tilde{\Lambda}_q|}{|\bar{\Lambda}|} - \frac{|V_q|}{\sum_{q'} |V_{q'}|} \right| \leq O\left(\frac{(L + d_0)|\partial \Lambda^c|^{1+\delta}}{|\Lambda|} \right),$$

provided $|\underline{\text{Out}}| \leq |\partial \Lambda^c|^{1+\delta}$. Using the fact that $|\langle t_x(A)|\underline{\text{Out}}\rangle_{\Lambda, R_{\partial \Lambda^c}}| \leq \|A\|$ to bound

the terms for which $x \notin \bigcup_q \tilde{\Lambda}_q$ we obtain that

$$\left| \langle \bar{A} | \underline{\text{Out}} \rangle_{\Lambda, R_{\partial \Lambda^c}} - \sum_q \alpha_q(\underline{\text{Out}}) \langle A \rangle_q \right| \leq 2\varepsilon,$$

provided $\Lambda = \Lambda(j)$ is large enough and $\left| \Lambda \setminus \bigcup_q V_q \right| \leq |\partial \Lambda^c|^{1+\delta}$. We finally use the identity

$$\langle A \rangle = \int dP(\underline{R}) \langle \bar{A} \rangle_{\Lambda, R_{\partial \Lambda^c}} = \int dP(\underline{R}) \sum_{\underline{\text{Out}}} P(\underline{\text{Out}} | R_{\partial \Lambda^c}) \langle \bar{A} | \underline{\text{Out}} \rangle_{\Lambda, R_{\partial \Lambda^c}} \quad (2.18)$$

together with Lemma 2.4 and Lemma 2.5 to conclude that

$$\left| \langle A \rangle - \sum_q \alpha_q(\Lambda) \langle A \rangle_q \right| \leq 4\varepsilon, \quad (2.19)$$

provided $\Lambda = \Lambda(j)$ is chosen so large that $2\|A\|\varepsilon_2(\Lambda) \leq \varepsilon$ and $2\|A\|\varepsilon_1(j) \leq \varepsilon$. This completes the proof of Proposition 2.1. ∎

3. The Suppression of Unstable Phases

In this section we prove Lemma 2.4. We first bound $P(\underline{\text{Out}} | R_{\partial \Lambda^c})$ for a given outer contour network; this will reduce us to an ordinary contour model with hard core interaction. The sum $\sum''$ over $\underline{\text{Out}}$ can then be estimated by the methods of [8]. Throughout this section we will assume that $\mu \in \mathcal{V}_1$ with $\mathcal{V}_1$ as defined in Lemma 3.1 of [1].

We start with a lower bound on $Z(\Lambda | R_{\partial \Lambda^c})$. We choose a stable value of q and use the fact that $1 \geq \chi_q$ to bound, for $\Lambda = \Lambda(j)$,

$$Z(\Lambda | R_{\partial \Lambda^c}) \geq Z_q^{\text{dil}}(\Lambda) e^{-k_1(1 + a \log j + k_2)|\partial \Lambda^c|},$$

where k_1 is the constant from Assumption R (see Sect. 1) and $k_2 = \max_{R \in \mathcal{U}_q} \alpha(R)$. Choosing $L \geq L_0$ and $\beta \geq \beta_2(L)$ as in Theorem 3.3 of [1] we may bound $Z_q^{\text{dil}}(\Lambda)$ from below by $e^{-h'_q|\Lambda| - C_1|\partial \Lambda^c|} = e^{-h|\Lambda| - C_1|\partial \Lambda^c|}$.

On the other hand $Z_m^{\text{small}}(V_m | R_{\partial V_m^c})$ and $\tilde{Z}_q^{\text{dil}}(V_q | R_{\partial V_q^c})$ may be bounded by $Z'_m(V_m)$ and $Z_q^{\text{dil}}(V_q) = Z'_q(V_q)$, respectively[6]. We use Lemma 3.2 of [1] together with the bound $\sum'_m |\partial V_m^c| + \sum_q |\partial V_q^c| \leq |\underline{\text{Out}}|$ to estimate the product of these partition functions. Inserting the resulting bounds into the definition (2.12) of $P(\underline{\text{Out}} | R_{\partial \Lambda^c})$ we obtain

$$P(\underline{\text{Out}} | R_{\partial \Lambda^c}) \leq e^{(k_1 + ak_1 \log j + k_1 k_2 + C_1)|\partial \Lambda^c|} e^{(h + C_1)|\underline{\text{Out}}|}$$

$$\cdot e^{-\sum_m a_m |V_m|} \int dR_{\text{Out}} \chi_{\text{large}}(R_{X(\text{Out})}) \prod_{x \in \text{Out}} \chi_{\omega(x)}(R_x) e^{-H(R_{\text{Out}})}. \quad (3.1)$$

---

[6] We recall that the truncated partition functions $Z'_m(\Lambda)$ were obtained from $Z_m^{\text{dil}}(\Lambda)$ by truncating the activities of the unstable contours, see [1], Sect 3, Definition 3 1

To bound the integral in (3.1) we use the fact that all blocks $\square(x')$ in Out which do not touch $\partial\Lambda^c$ are either incorrect or contain a large field. If $\square(x')$ contains a large field, $\omega(x)$ takes a constant value q on $\square(x')$ which is stable. We therefore may bound $H(R_{\square(x')})$ by

$$\sum_{x\in\square(x')}\left[e_q+\frac{\eta}{2}(R-R_q)^2\right].$$

by our assumption A.1, [1]. Using the definition of $\chi_{\text{large}}(R_{\square(x')})$ and bounding the integrals

$$\int_{|R|\geq B} e^{-(\eta/2)R^2}dR \leq e^{-(\eta/2)B^2}\int_{-\infty}^{+\infty}e^{-(\eta/2)R^2}dR,$$

one easily finds the estimate

$$\int dR_{\square(x')}\chi_{\omega(x)}(R_x)\chi_{\text{large}}(R_{\square(x')})e^{-H(R_{\square(x')})}\leq e^{-(\eta/2)B^2}(2(2\pi/\eta)^{1/2}e^{-e_0})^{|\square(x')|}$$

with $e_0=\min\limits_{q} e_q$. Continuing as in the proof of Lemma 3.1 of [1] to bound the remaining integrals we obtain the bound

$$\int dR_{\text{Out}}\chi_{\text{large}}(R_{X(\text{Out})})\prod_{x\in\text{Out}}\chi_{\omega(x)}(R_x)e^{-H(R_{\text{Out}})}\leq e^{k_3|\partial\Lambda^c|-(\tau'+e_0)|\text{Out}|}$$

with

$$e^{-\tau'}=\max\{e^{-\tau},2(2\pi/\eta)^{1/2}e^{-(\eta/2)B^2L^{-d}}\} \tag{3.2}$$

and $k_3=L(\tau'+\log C_0)$. τ and C_0 are the constants from Lemma 3.1 of [1]; the factor $k_3|\partial\Lambda^c|$ arises because we defined the outer contour network in such a way that it contains the blocks $\square(x')$ touching $\partial\Lambda^c$, even if none of these blocks contributes a small factor. Using Lemma 3.2 of [1] a second time to bound $h\leq e_0+C_1$ we finally obtain the estimate

$$P(\underline{\text{Out}}|R_{\partial\Lambda^c})\leq e^{(K+ak_1\log j)|\partial\Lambda^c|}e^{-(\tau'-2C_1)|\text{Out}|}e^{-\sum_m a_m|V_m|}, \tag{3.3}$$

with $K=C_1+k_1(1+k_2)+k_3$.

We are now ready to bound the sum $\sum''$ over $\underline{\text{Out}}$. We extract a factor

$$e^{-(\kappa/2)|\partial\Lambda^c|^{1+\rho}}\geq e^{-\sum_m(a_m/2)|V_m|}e^{-(\tau'/2)|\text{Out}|},$$

with $\kappa=\min\{\tau',\min\{a_m:a_m\neq 0\}\}$, resum all outer contour networks which differ only by $X(\text{Out})$, and finally relax the condition (2.16) in the sum $\sum''$. We obtain the bound

$$\sum_{\text{Out}}'' P(\underline{\text{Out}}|R_{\partial\Lambda^c})\leq e^{(K+ak_1\log j-(\kappa/2)|\partial\Lambda^c|^\rho)|\partial\Lambda^c|}$$

$$\cdot\sum_{(\text{Out},\omega_{\text{Out}})}' e^{-\sum_m(a_m/2)|V_m|}e^{-(\tau'/2-2C_1-\log 2)|\text{Out}|}, \tag{3.4}$$

where the sum $\sum'$ goes over all sets $\text{Out}\subset\Lambda$ which are unions of L-blocks and over all functions $\omega_{\text{Out}}:\text{Out}\rightarrow\{0,1,\ldots,N\}$ such that

i) Out contains all blocks $\square(x')$ in Λ which touch $\partial\Lambda^c$,

ii) ω_{Out} takes a constant value $m \neq 0$ on ∂C^c for each component C of $\Lambda \backslash \text{Out}$ and

iii) all components C of Out which do not touch $\partial\Lambda^c$ obey the condition diam $C > \tau/3a_m$ if $\omega = m$ on $\partial_e C$.

The sets V_m, $m = 1, \ldots, N$, are defined as before, see Sect. 2.

Lemma 2.4 now follows immediately from (3.4), the observation that τ' can be made arbitrarily large by choosing β and B large, and the following

Lemma 3.1. *There is a constant τ_0, depending only on N and v, such that*

$$\sum_{(\text{Out}, \omega_{\text{Out}})}^{'} e^{-\sum_m (a_m/2)|V_m|} e^{-\tilde{\tau}|\text{Out}|} \leqq e^{|\partial\Lambda^c|}, \qquad (3.5)$$

provided $\tau \geqq \tilde{\tau} \geqq \tau_0$.

Remark 3.1. Lemma 3.1 can actually be proven by the methods of [8], Sect. 2.3 to Sect. 3.3. We give an alternative proof below.

Proof. We introduce an auxiliary contour model as follows: A pair $\gamma = (\text{supp}\,\gamma, \omega_\gamma)$ where supp γ is a connected union of L-blocks centered in $L\mathbf{Z}^v$ and ω_γ is a function from supp γ into $\{0, 1, \ldots, N\}$ is called a contour, if ω_γ takes a constant value $m \neq 0$ on ∂C^c for each connected component of $\mathbf{Z}^v \backslash \text{supp}\,\gamma$. The sets Ext γ and $\text{Int}_m \gamma$, $m = 1, \ldots, N$, are defined as usual as the infinite component of $\mathbf{Z}^v \backslash \text{supp}\,\gamma$ and as the union of those components C for which $\omega_\gamma = m$ on ∂C^c, respectively. γ is called an m-contour if $\omega_\gamma = m$ on $\partial(\text{Ext}\,\gamma)^c$.

The activity of an m-contour γ is defined as

$$z(\gamma) = \begin{cases} 0 & \text{if } \text{diam}\,\gamma \leqq \tau/3a_m \\ e^{-\tilde{\tau}|\text{supp}\,\gamma|} & \text{if } \text{diam}\,\gamma > \tau/3a_m \end{cases}$$

and the partition function, $\mathscr{Z}_m^{\text{dil}}$, is defined as

$$\mathscr{Z}_m^{\text{dil}}(\Lambda) = \sum_{\{\gamma_1, \ldots, \gamma_n\}} \prod_{m'=1}^{N} e^{-(a_{m'}/2)|V_{m'}|} \prod_{i=1}^{n} z(\gamma_i). \qquad (3.6)$$

As usual the sum goes over sets of pairwise compatible contours $\gamma_1 = (\text{supp}\,\gamma_1, \omega_1), \ldots, \gamma_n = (\text{supp}\,\gamma_n, \omega_n)$ such that ω can be extended to a function on Λ which is constant on $C \cup \partial C^c$ for all connected components C of $\Lambda \backslash \cup \text{supp}\,\gamma_i$, and which takes the value m on $\partial\Lambda$. $V_{m'}$ denotes the union of all those components C for which $\omega = m'$ on ∂C. We only allow "dilute configurations," $\gamma_1, \ldots, \gamma_n$, i.e. we require that $\text{dist}(\Lambda^c, \text{supp}\,\gamma_i \cup \text{Int}\,\gamma_i) > L$ for all $i = 1, \ldots, n$. Note that $z(\gamma) = 0$ for all stable contours γ; therefore $\mathscr{Z}_q^{\text{dil}}(\Lambda) = 1$ if $a_q = 0$.

With the help of these definitions we rewrite the left-hand side of (3.5) as

$$\sum_{(E, \omega_E)} e^{-\tilde{\tau}|E|} \prod_m \mathscr{Z}_m^{\text{dil}}(\text{Int}_m E), \qquad (3.7)$$

where the sum goes over all boundary contours (E, ω_E), that is over all connected unions, E, of L-blocks in Λ and all functions, ω_E, from E into $\{0, 1, \ldots, N\}$ such that E contains all blocks touching $\partial\Lambda^c$ and such that ω_E is constant on ∂C^c for

each component C of $\Lambda \backslash E$. We will bound the partition functions, $\mathscr{L}_m^{\text{dil}}(\text{Int}_m E)$, in (3.7) by the analog of Theorem 3.3 ([1], Sect. 3) for $\mathscr{L}_m^{\text{dil}}$.

As usual (see, for example, [8], Sect. 1 or [9], Sect. 2) one may rewrite $\mathscr{L}_m^{\text{dil}}(\Lambda)$ as a sum over pairwise compatible m-contours $\gamma_1^m, \ldots, \gamma_n^m$:

$$\mathscr{L}_m^{\text{dil}}(\Lambda) = e^{-(a_m/2)|\Lambda|} \sum_{\{\gamma_1^m, \ldots, \gamma_n^m\}} \prod_{i=1}^{n} \rho(\gamma_i^m)$$

with

$$\rho(\gamma^m) = z(\gamma^m) e^{-(a_m/2)|\text{supp}\,\gamma^m|} \prod_{m'} \frac{\mathscr{L}_{m'}^{\text{dil}}(\text{Int}_{m'}\,\gamma^m)}{\mathscr{L}_m^{\text{dil}}(\text{Int}_{m'}\,\gamma^m)}.$$

We also introduce a truncated partition function, $\mathscr{L}_m'(\Lambda)$, by replacing $\rho(\gamma)$ by

$$\rho'(\gamma) = \min\{\rho(\gamma), e^{-(\tilde{\tau}/6)|\text{supp}\,\gamma|}\}.$$

For $\tilde{\tau}$ large enough, depending only on v and N, $\mathscr{L}_m'(\Lambda)$ can be analysed by a convergent cluster expansion. One obtains that

$$|\mathscr{L}_m'(\Lambda) + f_m'|\Lambda|| \leq O(e^{-(\tilde{\tau}/6)})|\partial\Lambda^c|,$$

where f_m' is the free energy of the truncated model. Since all contours contributing to f_m' are larger than $\tau/3a_m$, we may bound

$$\left| f_m' - \frac{a_m}{2} \right| \leq e^{-(\tilde{\tau}/12) \times (\tau/3a_m)} \leq \frac{a_m}{36\tilde{\tau}^2} \leq \frac{a_m}{2},$$

provided $\tilde{\tau}$ is large enough and $\tau \geq \tilde{\tau}$. We conclude that $f' = \min f_m' \geq 0$. Using the analog of Theorem 3.3 of [1] (see also Theorem 1.7 of [8]) this shows that

$$\mathscr{L}_m^{\text{dil}}(\text{Int}_m E) \leq e^{|\partial(\text{Int}_m E)^c|} \leq e^{|E|},$$

provided $\tau \geq \tilde{\tau} \geq \tau_0$, where τ_0 is a constant depending only on N and v. Inserting this bound into (3.7) and summing over (E, ω_E) one immediately obtains the lemma. ∎

4. Proof of Lemma 2.5

In this section we prove Lemma 2.5 by a suitable generalization of the cluster expansion for $\langle A \rangle_{q,\Lambda}^{\text{dil}}$ derived in Sect. 7 of [1]. We first derive a cluster expansion for

$$Z_q(\Lambda | R_{\partial\Lambda^c}) = \int dR_\Lambda \prod_{x \in \partial\Lambda} \chi_q(R_x) e^{-H(R_\Lambda | R_{\partial\Lambda^c})}. \tag{4.1}$$

Starting from the identity

$$\prod_{x \in \partial\Lambda} \chi_q(R_x) = \sum_{\omega:\, \Lambda \to \{0,\ldots,N\}}' \prod_{x \in \Lambda} \chi_{\omega(x)}(R_x), \tag{4.2}$$

where the sum $\sum'$ goes over all ω such that $\omega(x) = q$ for $x \in \partial\Lambda$, we proceed as in [1] to obtain a contour expansion for $Z_q(\Lambda | R_{\partial\Lambda^c})$,

$$Z_q(\Lambda | R_{\partial\Lambda^c}) = \sum_{\mathscr{C}} \int \chi_q(R_{\Lambda\backslash\mathscr{C}}) e^{-H(R_{\Lambda\backslash\mathscr{C}} | R_{\partial\Lambda^c})} \prod_{\gamma \in \mathscr{C}} e^{-\psi_q(\gamma | R_{\partial_e\gamma})} dR_{\Lambda\backslash\mathscr{C}}. \tag{4.3}$$

The sum $\sum_{\mathscr{C}}$ goes over sets of pairwise compatible q-contours γ with $\operatorname{supp}\gamma \subset \Lambda$.

ψ_q is defined as before (Eq. (2.11) of [1]) and obeys a bound[7]

$$e^{-\psi_q(\gamma|R_{\partial_e\gamma})} \leq e^{-(e_q + (\tau/3))|\operatorname{supp}\gamma|}. \tag{4.4}$$

Recall that $\Lambda\backslash\mathscr{C}$ stands for $\Lambda\backslash\bigcup_{\gamma\in\mathscr{C}}\operatorname{supp}\gamma$ and that $\partial_e\gamma$ was defined as $\partial(V(\gamma))^c$ (Eq. (2.7) of [1]). Note that (4.3) differs from the expansion for $Z_q^{\mathrm{dil}}(\Lambda)$ in [1] by the dependence of e^{-H} on $R_{\partial\Lambda^c}$ and the fact that the set $\mathscr{C}$ in (4.3) may contain contours γ which touch $\partial\Lambda^c$.

We continue as in [1] with an expansion of the Gibbs factor $e^{-H(R_{\Lambda\backslash\mathscr{C}}|R_{\partial\Lambda^c})}$ around the corresponding Gaussian approximation, with the difference that we absorb the perturbing interaction, $W_q(\phi_x, \phi_y)$, into the definition of the perturbing potential if $\langle xy\rangle$ intersects $\partial\Lambda^c$. More precisely we define (with, as before, $\phi_x = R_x - R_0$)

$$\widetilde{W}_q(\phi, \phi') = W(R_q + \phi, R_q + \phi') - \frac{\kappa_q}{2}(\phi - \phi')^2, \tag{4.5a}$$

$$\widetilde{V}_q(\phi_x) = V(R_q + \phi_x) - e_q - \frac{m_q^2}{2}\phi_x^2 + \sum_{\substack{y\in\partial\Lambda^c \\ |x-y|=1}} W_q(\phi_x, \phi_y), \tag{4.5b}$$

and[8], for $X \subset \Lambda\backslash\mathscr{C}$,

$$\widetilde{F}_X(\phi_x) = \sum_{\substack{X_1, B \\ X = X_1\cup X(B)}} \prod_{x\in X_1}\{\chi_q(\phi_x + R_q)e^{-\widetilde{V}_q(\phi_x)} - 1\} \prod_{\langle xy\rangle\in B}\{e^{-\widetilde{W}_q(\phi_x, \phi_y)} - 1\}, \tag{4.5c}$$

where B denotes a set of nearest neighbor pairs and $X(B)$ denotes the corresponding set of points. We obtain the expansion

$$Z_q(\Lambda|R_{\partial\Lambda^c}) = e^{-e_q|\Lambda|}Z_q^{(0)}(\Lambda|R_{\partial\Lambda^c})\sum_{\mathscr{C}}\sum_{X\subset\Lambda\backslash\mathscr{C}}\int d\phi_{X(\mathscr{C})}e^{-H_q^{(0)}(R_{X(\mathscr{C})}|R_{\partial\Lambda^c})}$$

$$\cdot\widetilde{F}_X(\phi_x)\frac{Z_q^{(0)}(\Lambda_1|\phi_{\partial X(\mathscr{C})\cup\partial\Lambda^c})}{Z_q^{(0)}(\Lambda|\phi_{\partial\Lambda^c})}\prod_{\gamma\in\mathscr{C}}e^{e_q|\operatorname{supp}\gamma| - \psi_q(\gamma|R_{\partial_e\gamma})}, \tag{4.6}$$

where $X(\mathscr{C}) = X\cup(\Lambda\cap\partial_e\mathscr{C})$ and $\Lambda_1 = \Lambda\backslash(X(\mathscr{C})\cup\operatorname{supp}\mathscr{C})$. $H_q^{(0)}$ and $Z_q^{(0)}$ are the quadratic and Gaussian approximations to H and Z, respectively, in $\mathscr{U}_q$ (see Eq. (4.3) and (4.6) of [1] for the definitions).

The above expansion is an expansion into "excitations" $(\mathscr{C}, X)$, coupled through a ratio of Gaussian partition functions. As in [1] we decouple the connected components of these excitations with an expansion for this ratio (see [1], Sect. 4, Eq. (4.8)). We obtain the polymer representation

$$Z_q(\Lambda|R_{\partial\Lambda^c}) = e^{-e_q|\Lambda|}Z_q^{(0)}(\Lambda|R_{\partial\Lambda^c})\sum_{\mathbb{P}}\prod_{P\in\mathbb{P}}\rho(P|R_{\partial\Lambda^c}), \tag{4.7}$$

---

[7] Throughout this section we assume that q is stable
[8] With a slight abuse of notation we suppress the dependence of $\widetilde{V}_q(\phi_x)$ and $\widetilde{F}_X(\phi_x)$ on $R_{\partial\Lambda^c}$

where the sum goes over sets $\mathbb{P} = \{P_1, P_2, \ldots\}$ of polymers in Λ (i.e. connected unions of L-blocks in Λ) such that P_i and P_j have no face in common for $i \neq j$. $\rho(P | R_{\partial\Lambda^c})$ is defined as

$$
\rho(P | R_{\partial\Lambda^c}) = \sum_{\mathscr{C}, X, \Gamma} \frac{1}{|\Gamma|!} \frac{1}{Z_q^{(0)}(\bar{X} \cup \operatorname{supp} \mathscr{C} | \phi_{\partial\Lambda^c})} \int d\phi_{\bar{X}} e^{-H_q^{(0)}(\phi_{\bar{X}} | \phi_{\partial\Lambda^c})}
$$

$$
\cdot \tilde{F}_X(\phi_X) \prod_{\gamma \in \mathscr{C}} e^{e_q |\operatorname{supp} \gamma| - \psi_q(\gamma | R_{\partial_e\gamma})} \prod_{\Gamma \in \Gamma} \tilde{W}(\Gamma | \phi_{\partial X(\mathscr{C}) \cup \partial\Lambda^c}), \tag{4.8a}
$$

where the sum goes over all triples $(\mathscr{C}, X, \Gamma)$ such that $\operatorname{supp} \mathscr{C} \cup \bar{X} \cup X(\Gamma) = P$, and

$$
\tilde{W}(\Gamma | \phi_{\partial X(\mathscr{C}) \cup \partial\Lambda^c}) = \int ds_\Gamma \partial^\Gamma \{\log Z_q^{(0)}(\Lambda_1, s_\Gamma | \phi_{\partial X(\mathscr{C}) \cup \partial\Lambda^c}) - \log Z_q^{(0)}(\Lambda, s_\Gamma | \phi_{\partial\Lambda^c})\}. \tag{4.8b}
$$

As in [1] Γ denotes a set of faces separating L-blocks in Λ, Γ denotes a sequence $(\Gamma_1, \ldots, \Gamma_{|\Gamma|})$, $X(\Gamma)$ is the set of L-blocks which have at least one face in common with Γ and $X(\Gamma) = \bigcup_{\Gamma \in \Gamma} X(\Gamma)$. The s-dependent partition functions $Z_q^{(0)}(\cdot, s_\Gamma | \cdot)$ are defined by interpolation in the Hamiltonian $H_q^{(0)}$, and $\bar{X}$ is the smallest union of L-blocks in $\Lambda \backslash \operatorname{supp} \mathscr{C}$ such that $X(\mathscr{C}) \subset \bar{X}$ and $\operatorname{dist}(\Lambda \backslash (\bar{X} \cup \operatorname{supp} \mathscr{C}), X(\mathscr{C})) < r_0$, where r_0 is a constant to be chosen below.

Note that for polymers P not touching $\partial\Lambda$ the activities $\rho(P | R_{\partial\Lambda^c})$ do not depend[9] on $R_{\partial\Lambda^c}$ and are equal to the activities $\rho(P)$ obtained in [1] for the partition function $Z_q^{\mathrm{dil}}(\Lambda)$. The following proposition immediately gives convergence of the Mayer expansion for $\log Z_q(\Lambda | \phi_{\partial\Lambda^c})$.

Proposition 4.1. *Given $b < \infty$ there is a choice of $L_0 < \infty$, such that, for all $L \geq L_0$ and all $B < \infty$ one can choose $r_0 = r_0(L)$, $\beta_1 = \beta_1(L, B)$, $\tau_1 = \tau_1(L, B)$ in such a way that*

$$
|\rho(P | R_{\partial\Lambda^c})| \leq e^{-b|P|},
$$

provided $|\phi_x| \leq B$ for all $x \in \partial\Lambda^c$ and $\beta \geq \beta_1$, $\tau = \tau(\beta, L) \geq \tau_1$.

We first prove the following lemma, which is the analog of Lemma 5.1 of [1]; η and η'' are the constants from Assumption A.1.

Lemma 4.2. *For all $c < \infty$ there is a constant $L_0 = L_0(c, \eta, \eta'', \nu)$ such that, for $L \geq L_0$, $\varepsilon > 0$ and a suitable choice of r_0 (depending on ε, L, η, η'' and ν)*

$$
\sum_{\substack{\Gamma \text{ s.t.} \\ \operatorname{supp} \mathscr{C} \cup \bar{X} \cup X(\Gamma) = P}} \frac{1}{|\Gamma|!} \prod_{\Gamma \in \Gamma} |\tilde{W}(\Gamma | \phi_{\partial X(\mathscr{C}) \cup \partial\Lambda^c})| \leq \exp\left\{ \frac{\varepsilon}{2} \sum_{x \in \partial X(\mathscr{C})} \phi_x^2 \right\}
$$

$$
\cdot \exp\{ f_1(1 + B^2) |\bar{X} \cup \operatorname{supp} \mathscr{C}| - cL^{-\nu} |P \backslash (\bar{X} \cup \operatorname{supp} \mathscr{C})| \},
$$

provided $|\phi_x| \leq B$ for all $x \in \partial\Lambda^c$. f_1 is a constant depending only on L, ν, η and η''.

Proof. We closely follow the proof of Proposition 5.3 and Lemma 5.1 in [1]. For a

---

[9] There is *no* dependence of $\rho(P | R_{\partial\Lambda^c})$ on $R_{\partial\Lambda^c}$ via $\tilde{W}(\Gamma | \cdot)$ for these polymers, because the contributions from $\Lambda \backslash P$ in (4.8b) cancel if P does not touch $\partial\Lambda$

given Γ let $\tilde{C}^{(1)}$ and $\tilde{C}^{(2)}$ be the covariances corresponding to $Z_q^{(0)}(Y_1, s_\Gamma | \phi_{\partial X(\mathscr{C}) \cup \partial \Lambda^c})$ and $Z_q^{(0)}(Y_2, s_\Gamma | \phi_{\partial \Lambda^c})$, respectively, with $Y_2 = \bar{X} \cup \operatorname{supp} \mathscr{C} \cup X(\Gamma)$ and $Y_1 = Y_2 \setminus (X(\mathscr{C}) \cup \operatorname{supp} \mathscr{C})$. Using the fact that the contributions from $\Lambda \setminus Y_2$ cancel in (4.8b) we rewrite

$$\tilde{W}(\Gamma | \phi_{\partial X(\mathscr{C}) \cup \partial \Lambda^c}) = \tilde{W}(\Gamma) + \frac{1}{2} \sum_{x, y \in \partial X(\mathscr{C}) \cup \partial \Lambda^c} \phi_x B_{xy}^{(1)} \phi_y - \frac{1}{2} \sum_{x, y \in \partial \Lambda^c} \phi_x B_{xy}^{(2)} \phi_y,$$

where $\tilde{W}(\Gamma)$ is obtained from $W(\Gamma)$ as defined in [1] by substituting $\tilde{C}^{(i)}$ for $C^{(i)}$; $B_{xy}^{(i)}$ is defined as

$$B_{xy}^{(i)} = B_{xy}^{(i)}(\Gamma) = \sum_{\substack{u, v \in Y_i \\ |u - x| = |v - y| = 1}} \int ds_\Gamma \partial^\Gamma \tilde{C}_{uv}^{(i)}.$$

Using the inequality $\phi_x^2 + \phi_y^2 \geq 2 |\phi_x \phi_y|$ and the symmetry $B_{xy}^{(i)} = B_{yx}^{(i)}$ we bound

$$\sum_{\Gamma : |\Gamma| = n} |\tilde{W}(\Gamma | \phi_{\partial X(\mathscr{C}) \cup \partial \Lambda^c})|$$

$$\leq \sum_{\Gamma : |\Gamma| = n} |\tilde{W}(\Gamma)| + \frac{1}{2} \sum_{x \in \partial X(\mathscr{C})} \phi_x^2 \sum_{\substack{y \in \mathbf{Z}^\nu \\ \Gamma : |\Gamma| = n}} |B_{xy}^{(1)}(\Gamma)|$$

$$+ \frac{1}{2} \sum_{x \in \partial \Lambda^c} \phi_x^2 \sum_{\substack{y \in \mathbf{Z}^\nu \\ \Gamma : |\Gamma| = n}} |B_{xy}^{(1)}(\Gamma)| + \frac{1}{2} \sum_{x \in \partial \Lambda^c} \phi_x^2 \sum_{\substack{y \in \mathbf{Z}^\nu \\ \Gamma : |\Gamma| = n}} |B_{xy}^{(2)}(\Gamma)|.$$

We now extract a factor $e^{-m_0 r_0/2}$ from the sum over y in the second term on the right-hand side and use the fact that only those x contribute to the last two terms for which $x \in \partial \Lambda^c \cap (Y_2 \cup \partial Y_2^c) \subset \partial Y_2^c$, because $B_{xy}^{(i)} = 0$ if $\operatorname{dist}(x, Y_2) \geq 2$. Continuing as in the proof of Proposition 5.3 of [1] we obtain the estimate

$$\sum_{\Gamma : |\Gamma| = n} |\tilde{W}(\Gamma | \phi_{\partial X(\mathscr{C}) \cup \partial \Lambda^c})| \leq d_1^{n+1} e^{-n(m_0 L/4\nu)} \left\{ \frac{e^{m_0 L/2}}{L} |\bar{X} \cup \operatorname{supp} \mathscr{C}| \right.$$

$$\left. + \frac{e^{-(m_0/2)(r_0 - L)}}{2} \sum_{x \in \partial X(\mathscr{C})} \phi_x^2 + e^{m_0 L/2} B^2 |\partial Y_2^c| \right\},$$

where d_1 is the constant from Proposition 5.3 of [1] (it depends only on ν, η and η'). We now bound $|\partial Y_2^c| \leq (3^\nu/L) |Y_2|$ and use the fact that $|X(\Gamma)| \leq 2L^d |\Gamma| \leq 2 |\Gamma| |\bar{X} \cup \operatorname{supp} \mathscr{C}|$ for all terms which contribute to the above sum (if $\bar{X} \cup \operatorname{supp} \mathscr{C} = \varnothing$, $Y_2 = Y_1$ and $\tilde{W}(\Gamma | \phi_{\partial X(\mathscr{C}) \cup \partial \Lambda^c}) = 0$; if $\bar{X} \cup \operatorname{supp} \mathscr{C} \neq \varnothing$, the inequality is trivial); we therefore may bound

$$|\partial Y_2^c| \leq \frac{3^\nu}{L} (1 + 2 |\Gamma|) |\bar{X} \cup \operatorname{supp} \mathscr{C}| \leq \frac{3^{\nu+n}}{L} |\bar{X} \cup \operatorname{supp} \mathscr{C}|,$$

which shows that

$$|\tilde{W}(\Gamma | \phi_{\partial X(\mathscr{C}) \cup \partial \Lambda^c})| \leq (3 d_1 e^{-(m_0 L/4\nu)})^n \left\{ \frac{d_1 e^{m_0 L/2}}{L} 3^\nu (1 + B^2) |\bar{X} \cup \operatorname{supp} \mathscr{C}| \right.$$

$$\left. + \frac{d_1 e^{-(m_0/2)(r_0 - L)}}{2} \sum_{x \in \partial X(\mathscr{C})} \phi_x^2 \right\}. \tag{4.9}$$

Given $c < \infty$ we choose L_0 so large that

$$3d_1 e^{-(m_0 L/4v)} \leq \tfrac{1}{2} e^{-2c}$$

and $r_0 = r_0(\varepsilon, L, \eta, \eta'', v)$ so large that

$$d_1 e^{-(m_0/2)(r_0 - L)} \leq \varepsilon.$$

Lemma 4.2 now follows from the bound (4.9) in the same way as Lemma 5.1 follows from Proposition 5.3 in [1]. ■

Proof of Proposition 4.1. Using Lemma 4.2 together with the bound (4.4) we are left with a bound on a sum over terms of the form

$$\frac{1}{Z_q^{(0)}(\bar{X} \cup \operatorname{supp} \mathscr{C} | \phi_{\partial \Lambda^c})} e^{cL^{-v}|P|} e^{-(\tau/3 - cL^{-v} - f_1(1 + B^2))|\operatorname{supp} \mathscr{C}|}$$

$$\cdot e^{(cL^{-v} + f_1(1 + B^2))|\bar{X}|} \int d\phi_{\bar{X}} \tilde{F}_X(\phi_X) e^{-H_q^{(0)}(\phi_{\bar{X}} | \phi_{\partial \Lambda^c})} e^{(\varepsilon/2) \sum\limits_{x \in \partial X(\mathscr{C})} \phi_x^2}.$$

Bounding the integral by

$$Z_q^{(0)}(\bar{X} \setminus X(\mathscr{C}) | \phi_{\partial \Lambda^c}) \int d\phi_X \tilde{F}_X(\phi_X) e^{-H_q^{(0)}(\phi_{X(\mathscr{C})} | \phi_{\partial \Lambda^c})} e^{(\varepsilon/2) \sum\limits_{x \in \partial X(\mathscr{C})} \phi_x^2},$$

and the ratio

$$\frac{Z_q^{(0)}(\bar{X} \setminus X(\mathscr{C}) | \phi_{\partial \Lambda^c})}{Z_q^{(0)}(\bar{X} \cup \operatorname{supp} \mathscr{C} | \phi_{\partial \Lambda^c})} \leq e^{\tilde{f}_2(1 + B^2)|X(\mathscr{C}) \cup \operatorname{supp} \mathscr{C}|}$$

with a constant $\tilde{f}_2$ depending only on v, η, η' and η'', we continue as in [1] to complete the proof of Proposition 4.1. ■

At this point the proof of Lemma 2.5 is a standard exercise: one repeats the procedure leading to the polymer representation (4.7) for the unnormalized expectation value $[A]_{\Lambda, R_{\partial \Lambda^c}}^q = \langle A \rangle_{\Lambda, R_{\partial \Lambda^c}}^q Z_q(\Lambda | \phi_{\partial \Lambda^c})$ and uses the usual algebraic procedure to divide out the partition function $Z_q(\Lambda | \phi_{\partial \Lambda^c})$. Since we have assumed that the observable A is bounded, the bound proven in Proposition 4.1 immediately gives the bounds necessary to prove the convergence of the resulting cluster expansion for $\langle A \rangle_{z, R_{\partial \Lambda^c}}^q$. Lemma 2.5 is an immediate consequence.

5. Periodic Boundary Conditions

In this section we analyse periodic boundary conditions. We first consider the partition function,

$$Z_{\text{per}}(T) = \int dR_T e^{-H(R_T)}, \tag{5.1}$$

where T is the v-dimensional torus with length l in each direction. As usual we assume that T is a union of L-blocks, i.e. l is an integer multiple of L, where L is chosen according to Lemma 3.1 in [1]. Denoting by h'_q the free energy of the truncated contour model (see Sect. 3 of [1] for the precise definition), by h the minimum of the h'_q's and by a_q the difference $h'_q - h$, we recall that the phase q is

stable iff $a_q = 0$ and introduce the constant

$$\tilde{a} = \min_{m:a_m \neq 0} a_m.$$

Our first goal is the proof of the following theorem, which is the appropriate reformulation of the corresponding theorem in [9]. $\mathcal{V}_1$ is the neighborhood of $0 \in \mathbf{R}^{N-1}$ defined in Lemma 3.1 of [1].

Theorem 5.1. *Assume that V and W obey the conditions A.0 through A.3 of* [1]. *Then there are constants $\beta^* < \infty, B > 0$ and $l_0 = l_0(\tilde{a}) < \infty$, such that for $\mu \in \mathcal{V}_1$, $\beta \geq \beta^*$ and $l \geq l_0$,*

$$|Z_{\text{per}}(T)e^{hl^\nu} - N_0| \leq e^{-Bl}. \tag{5.3}$$

N_0 is the number of stable phases.

The proof of Theorem 5.1 is essentially the same as that of the corresponding theorem in [9]. For the convenience of the reader and for later reference, however, we give the complete proof. We start with the partition of unity,

$$1 = \sum_{\omega: T \to \{0,1,\dots,N\}} \prod_{x \in T} \chi_{\omega(x)}(R_x),$$

with $\chi_\omega(\cdot)$ as before and define a L-block, $\square(x')$, to be m-correct iff $\omega(x) = m$ for all x with dist$(x, \square(x')) \leq 1$. As before the connected components of the set, $B(\omega)$, of incorrect points are the supports, supp $\gamma_1, \dots,$ supp γ_n, of the contours, $\gamma_1 = (\text{supp } \gamma_1, \omega_{\gamma_1}), \dots, \gamma_n = (\text{supp } \gamma_n, \omega_{\gamma_n})$, of the configuration ω.

Following [9] we first consider those configurations, ω, for which

$$\text{diam } \gamma \leq l/3 \quad \text{for all contours } \gamma. \tag{5.4}$$

For such configurations, the notation of exterior contours can be well defined: For a contour, γ, of ω let Int γ be the union of those components of $T\backslash\text{supp }\gamma$ which have diameter less or equal $l/3$. Ext γ is then the remaining component of $T\backslash\text{supp }\gamma$, and the set of exterior contours of the configuration ω is the set of contours, γ, for which there is no other contour, γ', of ω such that supp $\gamma \subset \text{Int } \gamma'$. Denoting the sum over all configurations, ω, which obey (5.4) and whose exterior contours are m-contours by $\sum^{(m)}$, we introduce the partition function

$$\tilde{Z}_m(T) = \sum_\omega^{(m)} \int \prod_{x \in T} \chi_{\omega(x)}(R_x) e^{-H(R_T)} dR_T. \tag{5.5a}$$

More generally, we define, for a subset $V \subset T$,

$$\tilde{Z}_m(V) = \sum_\omega^{(m)} \int \prod_{x \in V} \chi_{\omega(x)}(R_x) e^{-H(R_V)} dR_V, \tag{5.5b}$$

where we impose the additional constraint that all exterior contours, γ, obey the "diluteness condition" dist$(V(\gamma), T\backslash V) \geq L$. As before $V(\gamma)$ is the set Int $\gamma \cup \text{supp } \gamma$. We also introduce the partition function,

$$Z_{\text{Big}}(T) = \sum_\omega^{\text{Big}} \int \prod_{x \in T} \chi_{\omega(x)}(R_x) e^{-H(R_T)} dR_T, \tag{5.6}$$

where the sum goes over all configurations violating the condition (5.4).

With this notation we rewrite $Z_{per}(T)$ as

$$Z_{per}(T) = \sum_{m=1}^{N} \tilde{Z}_m(T) + Z_{Big}(T). \tag{5.7}$$

Theorem 5.1 now follows immediately from (5.7) and the following three lemmas, which are proven in the sequel. We assume from now on that V and W obey assumptions A.0 through A.3 of [1], and that $\mu \in \mathscr{V}_1$, $L \geq L_0$ and $\beta \geq \beta_2$, where L_0 and β_2 are the constants from Lemma 3.2 and Theorem 3.3 of [1], respectively.

Lemma 5.2. *Assume that q is stable. Then there are constants $K_2 < \infty$, $B_2 > 0$ such that*

$$|\log \tilde{Z}_q(T) - h|T|| \leq K_2 e^{-B_2 l}.$$

Lemma 5.3. *There is a constant $\beta_3 < \infty$, such that for $\beta \geq \beta_3$ the following statements are true:*

i) *For all $m \in \{1, \ldots, N\}$ and all $V \subset T$ which are unions of L-blocks*

$$\tilde{Z}_m(V)e^{h|V|} \leq \exp\{K_2 e^{-B_2 l} + (C_1 + 1)|\partial V^c|\}.$$

ii) *If m is unstable*

$$\tilde{Z}_m(T)e^{h|T|} \leq e^{K_2} \max\{e^{-(a_m/2)l^v}, e^{-B_3 \tau l^{v-1}}\}.$$

Here $\tau = \tau(\beta, L)$ is the constant from Lemma 3.1 [1] and $B_3 > 0$ is a constant depending only on the dimension.

Lemma 5.4. *There is a constant $\beta_4 < \infty$ such that*

$$Z_{Big}(T)e^{h|T|} \leq e^{-\tau l/6},$$

provided $\beta \geq \beta_4$.

Proof of Lemma 5.2. For the Gaussian approximation, $Z_q^{(0)}(T)$, to $\tilde{Z}_q(T)$, the statement is well known. It can be proven, for example, by comparing the random walk expansion for $\log Z_q^{(0)}(T)$ with the random walk expansion for $|T|h_q^{(0)}$. Since the difference between $\log Z_q^{(0)}(T)$ and $\log \tilde{Z}_q(T)$ can be controlled by a convergent cluster expansion if q is stable, the result immediately generalizes to $\log \tilde{Z}_q(T)$. ∎

Proof of Lemma 5.3. We proceed as in the proof of Theorem 3.3, [1] to bound

$$\tilde{Z}_m(V)e^{h|V|} \leq \exp\{C_1|\partial V^c| + K_2 e^{-B_2 l}\} \sum_{\mathscr{C}}' e^{-a_m|V \setminus \text{Int} \,\mathscr{C}|} \prod_{\gamma \in \mathscr{C}} e^{-(\tau - 3C_1 - 1)|\text{supp} \,\gamma|},$$

where the sum goes over sets $\mathscr{C}$ of mutally external m-contours which are all large (i.e. $a_m \operatorname{diam} \gamma > \tau/3$) and obey a bound $\operatorname{diam} \gamma < l/3$. Extracting a factor

$$M = \max_{\mathscr{C}} e^{-(a_m/2)|V \setminus \text{Int} \,\mathscr{C}|} \prod_{\gamma \in \mathscr{C}} e^{-(\tau/2)|\mathscr{C}|} \tag{5.8}$$

and continuing as in the proof of Theorem 3.3, [1], we obtain, for a suitable choice of β_3 and for $\beta \geq \beta_3$,

$$\tilde{Z}_m(V)e^{h|V|} \leq M \exp\{(C_1 + 1)|\partial V^c| + K_2 e^{-B_2 l}\}.$$

Bounding M by 1 we obtain i). To prove ii) we set $V = T$ and use the isoperimetric inequality to bound

$$|\mathscr{C}| \geq |\partial(\text{Int }\mathscr{C})^c| \geq K|\text{Int }\mathscr{C}|^{(v-1)/v},$$

where $K > 0$ is a constant depending only on the dimension v. We obtain that

$$M \leq \sup_{0 \leq x \leq |T|} e^{(a_m/2)(|T|-x)} e^{-(\tau/2)Kx^{(v-1)/v}}.$$

The supremum is obtained for either $x = 0$ or $x = |T|$, which shows that

$$M \leq \max\{e^{-(a_m/2)|T|}, e^{-(\tau/2)K|T|(v-1)/v}\}.$$

This proves ii), with, for example, $B_3 = K/2$. ∎

Proof of Lemma 5.4. For a given configuration ω contributing to (5.6) we let $\mathscr{C}_1$ be the union of those contours which violate (5.4) and V_m be the union of those components, K, of $T\backslash\text{supp }\mathscr{C}_1$ for which $\omega = m$ on ∂K. Resumming all configurations ω which give rise to the same set $\mathscr{C}_1$ we may bound

$$Z_{\text{Big}}(T) \leq \sum_{\mathscr{C}_1} \prod_m \tilde{Z}_m(V_m) \prod_{\gamma \in \mathscr{C}_1} e^{-(\tau+h-C_1)|\gamma|},$$

where we used Lemma 3.1 and Lemma 3.2, ii) of [1] to bound integrals of the form

$$\int dR_\gamma \chi_{\omega_\gamma}(R_\gamma) e^{-H(R_\gamma|R_{\partial(\text{supp }\gamma)^c})}, \quad \gamma \in \mathscr{C}_1.$$

Next we use Lemma 5.3, i), to conclude that

$$Z_{\text{Big}}(T)e^{h|T|} \leq e^{K_2} \sum_{\mathscr{C}_1} \prod_{\gamma \in \mathscr{C}_1} e^{-(\tau-2C_1-1)|\gamma|}.$$

Bounding the sum over all contours, γ, with fixed size $s = |\gamma|$ by $l^v e^{Ks}$, where K is a constant depending only on v and N, we obtain, for τ large enough,

$$Z_{\text{Big}}(T)e^{h|T|} \leq e^{K_2} \sum_{n=1}^{\infty} \frac{1}{n!}\left(l^v \sum_{s \geq l/3} e^{-(\tau-K-2C_1-1)s}\right)^n \leq e^{-\tau l/6}. \quad ∎$$

We are now ready to prove Theorem B of Sect. 1. We consider a bounded, local observable, A, and introduce the modified partition function,

$$\tilde{Z}_m(A, T) = \sum_{\omega:T\to\{0,1,\ldots,N\}}^{(m)} \int dR_T \prod_{x\in T} \chi_{\omega(x)}(R_x)Ae^{-H(R_T)}, \tag{5.9}$$

as well as

$$Z_{\text{Big}}(A, T) = \sum_{\omega:T\{0,1,\ldots,N\}}' \int dR_T \prod_{x\in T} \chi_{\omega(x)}(R_x)Ae^{-H(R_T)}, \tag{5.10}$$

where the sum $\sum'$ denotes a sum over all configurations, ω, which violate the condition (5.4). We also introduce the expectation values

$$\langle A \rangle_{m,T}' = \frac{\tilde{Z}_m(A, T)}{\tilde{Z}_m(T)} \tag{5.11}$$

and recall that $\langle A \rangle_q$ denotes the infinite volume limit of the finite volume expectation values with boundary condition q, see [1], Theorem 1. For stable

values of q, the difference, $\langle A \rangle_q - \langle A \rangle'_{m,T}$, can be controlled by a convergent cluster expansion and goes to zero as $T \to \mathbf{Z}^\nu$. Therefore Theorem B follows immediately from the following theorem.

Theorem 5.5. *Assume that V and W obey the assumptions A.0 through A.3 of* [1]. *Then there are constants $\beta^* < \infty$, $B > 0$ and $l_0 = l_0(\tilde{a})$, such that*

$$\left| \langle A \rangle_{\mathrm{per},T} - \frac{1}{N_0} \sum_q \langle A \rangle'_{q,T} \right| \leq \| A \| e^{-Bl},$$

provided $\mu \in \mathcal{V}_1$, $\beta \geq \beta^*$ and $l \geq l_0$. The sum goes over all stable values of q, and N_0 denotes the number of stable phases.

Proof. Theorem 5.5 immediately follows from the identity

$$\langle A \rangle_{\mathrm{per},T} = \frac{1}{Z_{\mathrm{per}}(T)} \left[\sum_{m=1}^N \tilde{Z}_m(A,T) + Z_{\mathrm{Big}}(A,T) \right],$$

Theorem 5.1, Lemma 5.2 through 5.4 and the bounds

$$\tilde{Z}_m(A,T) \leq \| A \| \tilde{Z}_m(T), \tag{5.13}$$

$$Z_{\mathrm{Big}}(A,T) \leq \| A \| Z_{\mathrm{Big}}(T). \tag{5.14}$$

∎

Remark. For an unbounded observable A, the bound (5.12) is useless because $\| A \| = \infty$. To prove Theorem B for the more general class of observables for which $\| A \|^{(\alpha)} < \infty$, see [1], Sect. 1, for the definition of $\| A \|^{(\alpha)}$, we show that

$$\tilde{Z}_m(A,V)e^{h|V|} \leq \| A \|^{(\alpha)} K^{|\mathrm{supp}\,A|} \exp\{K_2 e^{-B_2 l} + (C_1 + 1)|\partial V^c|\}, \tag{5.15}$$

$$\tilde{Z}_m(A,T)e^{h|T|} \leq \| A \|^{(\alpha)} K^{|\mathrm{supp}\,A|} e^{K_2} \max\{e^{-(a_m/2)l^\nu}, e^{-B_3 t l^{\nu-1}}\} \tag{5.16}$$

and

$$Z_{\mathrm{Big}}(A,T)e^{h|T|} \leq \| A \|^{(\alpha)} K^{|\mathrm{supp}\,A|} e^{-tl/6}, \tag{5.17}$$

with a constant K which does not depend on l. Given these bounds we immediately obtain Theorem 5.5 with $\| A \|$ replaced by $\| A \|^{(\alpha)} K^{|\mathrm{supp}\,A|}$, and hence Theorem B for all observables, A, with $\| A \|^{(\alpha)} < \infty$. Since the proof of (5.15) to (5.17) is a straightforward combination of the methods developed in Sect. 7 of [1] and the methods of this section, we don't present the details here.

Appendix: Superstability Estimates

In this appendix we prove Lemma 2.3, using the methods of [2–4]. In fact we prove a more general theorem on the probability that $\alpha(R_x) \geq \alpha_0$ for a given site, $s \in \mathbf{Z}^\nu$. Lemma 2.3 then follows by a three line argument. To make this appendix self-contained we restate those assumptions which are needed in the course of the proof.

We consider a spin model with spins $R_x \in \mathbf{R}$ and a Hamiltonian

$$H = \sum_x V(R_x) + \sum_{\langle xy \rangle} W(R_x, R_y), \tag{A.1}$$

where the sum $\sum\limits_{\langle xy\rangle}$ denotes a sum over nearest neighbor pairs, $\langle xy\rangle$. We assume that V and W are bounded from below and that W is normalized in such a way that $W(R_x, R_y)\geqq 0$. Putting

$$\alpha(R_x) = V(R_x) - \inf_{R\in\mathbf{R}} V(R)$$

we assume that W obeys the regularity condition

$$0 \leqq W(R_x, R_y) \leqq k_1(1 + \alpha(R_x) + \alpha(R_y)) \tag{A.2}$$

and that

$$\int e^{-(1-\lambda)\alpha(R)}dR < \infty \tag{A.3}$$

for some constants $k_1 < \infty$, $0 < \lambda < 1$. For the model considered in the main body of this paper assumption (A.2) is the assumption R of Sect. 1 and (A.3) immediately follows from assumptions A.1 i) and A.2 ii) of [1].

Let $\Lambda(j)$ be the set of all points in $\mathbf{Z}^\nu$ which have distance[10] less or equal to j from the origin and put, for a configuration $\underline{R}$ in $\mathbf{Z}^\nu$

$$\|\underline{R}\| = \sup_{j\geqq 0} \frac{1}{|\Lambda(j)|}\sum_{x\in\Lambda(j)} \alpha(R_x). \tag{A.4}$$

We define: an equilibrium state, $\langle\cdot\rangle = \int\cdot dP$, is *tempered* if $\|\underline{R}\| < \infty$ with probability one.

The goal of this appendix is to prove the following:

Theorem A.1. *Let $\langle\cdot\rangle = \int\cdot dP$ be a tempered equilibrium state. Then there is a constant $K < \infty$ such that the probability, $P(\alpha(R_x) > \alpha_0)$, that $\alpha(R_x)$ is larger than α_0 is bounded from above by $Ke^{-(\lambda/2)\alpha_0}$.*

Remarks

i) As the reader may easily verify from the following proof, the constant $\lambda/2$ in the above theorem may be replaced by any constant $\tilde{\lambda} < \lambda$, at the cost of making K larger if $\lambda/2 < \tilde{\lambda}$.

ii) Lemma 2.3 immediately follows from Theorem A.1 because, for $a > 2(\nu - 1)/\lambda$, the probability that $\alpha(R_x) > a\log j$ is bounded by

$$K\left(\frac{1}{j}\right)^{\lambda a/2} \leqq o\left(\left(\frac{1}{j}\right)^{\nu-1}\right).$$

Therefore the probability that $\alpha(R_x) > a\log j$ for at least one x with $\mathrm{dist}(x,0) = j$ is bounded by

$$O(j^{\nu-1})o\left(\left(\frac{1}{j}\right)^{\nu-1}\right),$$

which goes to zero as $j \to \infty$.

---

[10] We define the distance $\mathrm{dist}(x,y)$ between two points, $x, y\in\mathbf{Z}^\nu$, as $|x - y|_1$, where $|\cdot|_1$ denotes the l_1 distance in $\mathbf{Z}^\nu$

Proof of Theorem A.1. We introduce a growing sequence of positive numbers $\psi_k \to \infty$ as $k \to \infty$ and a sequence of volumes Λ_k, such that

i) Λ_0 contains all $y \in \mathbf{Z}^\nu$ with $\mathrm{dist}\,(x, y) \leq 1$, $\qquad\qquad\qquad\qquad\qquad$ (A.5)

ii) $\partial \Lambda_k^c \subset \Lambda_{k+1}$ and $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ (A.6)

iii) $\dfrac{|\Lambda_{k+1}|}{|\Lambda_k|}\dfrac{\psi_{k+1}}{\psi_k} \leq 1 + \varepsilon \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ (A.7)

for some small number ε to be chosen later (a possible choice would be, for example, $\Lambda_k = \Lambda(k + k_0)$, $\psi_k = k + k_0$, where k_0 has been chosen large enough, depending on ε). As in [2–4] we define sets, $\mathscr{C}_k$, of configurations $\underline{R}$,

$$\mathscr{C}_0 = \left\{ \underline{R} \,\bigg|\, \sum_{x \in \Lambda_k} \alpha(R_x) < \psi_k |\Lambda_k| \,\forall k \geq 0 \right\} \qquad\qquad\qquad\text{(A8.a)}$$

$$\mathscr{C}_{j+1} = \left\{ \underline{R} \,\bigg|\, \sum_{x \in \Lambda_j} \alpha(R_x) \geq \psi_j |\Lambda_j|, \sum_{x \in \Lambda_k} \alpha(R_x) < \psi_k |\Lambda_k| \,\forall k \geq j + 1 \right\}. \qquad\text{(A8.b)}$$

We also introduce the characteristic functions, $\chi_j = \chi_j(\underline{R})$, which are 1 if $\underline{R} \in \mathscr{C}_j$ and 0 otherwise. Since P is tempered, we may rewrite

$$P(\alpha(R_x) > \alpha_0) = \sum_{j=0}^{\infty} P_j, \qquad\qquad\qquad\qquad\qquad\qquad\text{(A.9)}$$

with

$$P_j = \int dP(\underline{R}) \chi_j(\underline{R}) \chi(\alpha(R_x) > \alpha_0).$$

Using the DLR equations, we rewrite

$$P_{j+1} = \int dP(R_{\Lambda_j^c}) \frac{1}{Z(\Lambda_j | R_{\partial \Lambda_j^c})} \int dR_{\Lambda_j} \chi(\alpha(R_x) > \alpha_0) \chi_{j+1}(\underline{R}) e^{-H(R_{\Lambda_j} | R_{\Lambda_j^c})}, \quad\text{(A.10)}$$

where we used the convention $\Lambda_{-1} = \{x\}$.

After these preparations we bound the probability $P(\alpha(R_x) > \alpha_0)$. We first estimate P_0. The necessary lower bound on $Z(\{x\} | R_{\partial\{x\}^c}) = \int d\tilde{R}_x e^{-H(\tilde{R}_x | R_{\partial\{x\}^c})}$ is obtained as follows. For $\underline{R} \in \mathscr{C}_0$ and $\tilde{R}_x \in \mathbf{R}$ we estimate

$$H(\tilde{R}_x | R_{\partial\{x\}^c}) \leq V(\tilde{R}_x) + k_1 \sum_{y:|y-x|=1} (1 + \alpha(\tilde{R}_x) + \alpha(R_y))$$

$$\leq V(\tilde{R}_x) + 2\nu k_1 (1 + \alpha(\tilde{R}_x)) + k_1 \sum_{y \in \Lambda_0} \alpha(R_y)$$

$$\leq V(\tilde{R}_x) + 2\nu k_1 (1 + \alpha(\tilde{R}_x)) + k_1 \psi_0 |\Lambda_0|,$$

which, for $\underline{R} \in \mathscr{C}_0$, gives the bound

$$\frac{1}{Z(\{x\} | R_{\partial\{x\}^c})} \leq e^{k_1 \psi_0 |\Lambda_0|} C,$$

with $C = e^{2\nu k_1} / \int d\tilde{R} e^{-V(\tilde{R}) - 2\nu k_1 \alpha(\tilde{R})}$. Inserting this bound into (A.10) we obtain, using

the lower bound $W(R, R') \geq 0$,

$$P_0 \leq e^{k_1\psi_0|\Lambda_0|} C \int_{\alpha(R_x) \geq \alpha_0} e^{-V(R_x)} dR_x \leq e^{k_1\psi_0|\Lambda_0|} \tilde{C} e^{-\lambda\alpha_0}, \qquad (A.11)$$

with

$$\tilde{C} = C \int e^{\lambda\alpha(R) - V(R)} dR = e^{2vk_1} \frac{\int e^{-V(R) + \lambda\alpha(R)} dR}{\int e^{-V(\tilde{R}) - 2vk_1\alpha(\tilde{R})} d\tilde{R}}. \qquad (A.12)$$

To bound P_{j+1} for $j \geq 0$ we estimate, for $\underline{R} \in \mathscr{C}_{j+1}$ and $\tilde{R}_{\Lambda_j}: \Lambda_j \to \mathbf{R}$,

$$H(\tilde{R}_{\Lambda_j} | R_{\partial\Lambda_j^c}) \leq \sum_{x \in \Lambda_j} V(\tilde{R}_x) + \sum_{\substack{x \in \Lambda_j, y \in \partial\Lambda_j^c \\ |x-y|=1}} (1 + \alpha(\tilde{R}_x) + \alpha(R_y))$$

$$\leq \sum_{x \in \Lambda_j} \{V(\tilde{R}_x) + 2vk_1(1 + \alpha(\tilde{R}_x))\} + 2vk_1 \sum_{y \in \Lambda_{j+1} \setminus \Lambda_j} \alpha(R_y)$$

$$\leq \sum_{x \in \Lambda_j} \{V(\tilde{R}_x) + 2vk_1(1 + \alpha(\tilde{R}_x))\} + 2vk_1 \left(\frac{|\Lambda_{j+1}|}{|\Lambda_j|} \frac{\psi_{j+1}}{\psi_j} - 1\right) \sum_{x \in \Lambda^j} \alpha(R_x),$$

where we have used (A.6) in the second step. Choosing the constant ε in (A.7) so small that $(2vk_1)\varepsilon \leq \lambda/4$ we obtain the bound

$$\frac{1}{Z(\Lambda_j | R_{\partial\Lambda_j^c})} \leq C^{|\Lambda_j|} e^{-(\lambda/4)\sum_{y \in \Lambda_j} \alpha(R_y)}$$

$$\leq e^{-(\lambda/2)\alpha_0} (Ce^{-(\lambda/4)\psi_j})^{|\Lambda_j|} e^{\lambda\sum_{y \in \Lambda_j} \alpha(R_y)},$$

provided $\underline{R} \in \mathscr{C}_{j+1}$ and $\alpha(R_x) \geq \alpha_0$ (we have used the fact that $x \in \Lambda_j$ for all $j \geq 0$ by our choice of Λ_j, see (A.5) and (A.6)). Inserting the above bound into (A.10) one gets the estimate

$$P_{j+1} \leq e^{-(\lambda/2)\alpha_0} (\tilde{C}e^{-(\lambda/4)\psi_j})^{|\Lambda_j|}.$$

Combined with (A.11) we finally obtain Theorem A.1 with

$$K = \tilde{C}e^{k_1\psi_0|\Lambda_0|} + \sum_{j \geq 0} (\tilde{C}e^{-(\lambda/4)\psi_j})^{|\Lambda_j|}. \qquad \blacksquare$$

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