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An Estimate from Above of the Number of Periodic Orbits for Semi-Dispersed Billiards

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Abstract. For a large class of semi-dispersed billiards an exponential estimate from above is found for the number of periodic points of the billiard ball map.

1. Introduction and Main Results

Let Q be a domain (bounded or unbounded) in $\mathbb{R}^d$, $d \geqq 2$, with the boundary

$$\partial Q = \Gamma = \Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_s \quad (s \geqq 3),$$

where each Γ_i is a compact convex C^2-smooth $(d-1)$-dimensional submanifold of $\mathbb{R}^d$ with piecewise smooth boundary $\partial\Gamma_i$, and

$$\Gamma_i \cap \Gamma_j \subset \partial\Gamma_i \cup \partial\Gamma_j$$

whenever $i \neq j$. Each $\partial\Gamma_i$ is the union of a finite number of compact $(d-2)$-dimensional submanifolds of $\mathbb{R}^d$. If $\partial\Gamma_i \neq \emptyset$, then clearly Γ_i is the boundary of a compact convex domain in $\mathbb{R}^d$.

Main Assumption. In the sequel we assume that each Γ_i is contained in the boundary of a convex domain in $\mathbb{R}^d$. Therefore if K_i is the convex hull of Γ_i, then $\Gamma_i \subset \partial K_i$.

The points of

$$\mathring{\Gamma} = (\Gamma_1 \backslash \partial\Gamma_1) \cup \cdots \cup (\Gamma_s \backslash \partial\Gamma_s)$$

will be called *regular points* of Γ. For $q \in \mathring{\Gamma}$ we denote by $N(q)$ the *normal unit* vector to Γ at q directed to the interior of Q. With respect to this framing the second fundamental form of Γ is non-negative definite at each $q \in \mathring{\Gamma}$.

We consider the billiard in Q, that is the dynamical system generated by the motion of material point in Q (see [4, 13]). The point is moving with constant velocity in the interior of Q with reflections at ∂Q according to the rule "the angle of incidence is equal to the angle of reflection."

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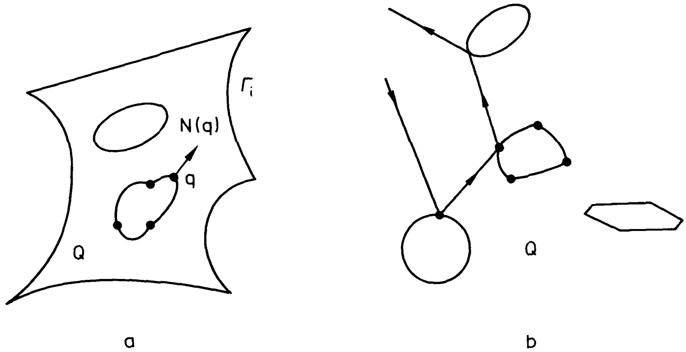


Fig. 1

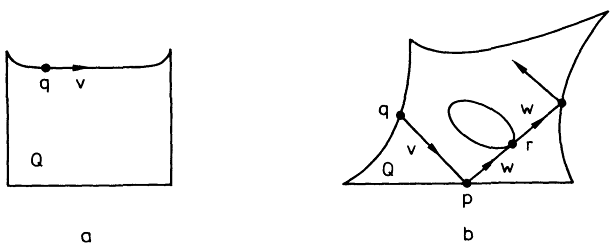


Fig. 2

Denote by $\langle .,. \rangle$ the *scalar product* in $\mathbb{R}^d$ and by $L_q\Gamma$ the *tangent hyperplane* to Γ at q. Then $L_q\Gamma = q + L'_q\Gamma$, where $L'_q\Gamma$ is a linear subspace of $\mathbb{R}^d$, and $T_q\Gamma = \{q\} \times L'_q\Gamma$ is the *tangent space* to Γ at q.

A point $x = (q, v) \in \Gamma \times S^{d-1}$ will be called *admissible* if it satisfies the following two conditions:

(i) q is regular and $\langle N(q), v \rangle \geqq 0$;
(ii) if $\langle N(q), v \rangle = 0$, then there exists in Γ a neighbourhood U of q such that $U \cap L_q\Gamma = \{q\}$.

Set

$$M' = \{(q, v) \in \mathring{\Gamma} \times S^{d-1} : \langle N(q), v \rangle \geqq 0\}.$$

Denote by M the set of $x = (q, v) \in M'$ such that if $\gamma(x)$ is the billiard semi-trajectory in Q starting at q in the direction v, then $\gamma(x) \cap \Gamma \subset \mathring{\Gamma}$, $\gamma(x)$ intersects Γ, and whenever $\gamma(x)$ is passing through a point $p \in \Gamma$ with reflected direction w, then (p, w) is an admissible point of $\Gamma \times S^{d-1}$. For $x \in M$ let p be the first point of reflection of $\gamma(x)$, that is $p \in \gamma(x) \cap \mathring{\Gamma}$ and the open segment (q, p) is contained in the interior of Q. Set

$$T(x) = T(q, v) = (p, w),$$

where $w = v - 2\langle N(p), v \rangle N(p)$. Thus we obtain a map

$$T: M \to M'$$

which is called the *billiard ball map* related to Q. In fact, it is more natural to consider T as a map

$$T: M_0 \to M_0,$$

where $M_0 = \bigcap_{m=0}^{\infty} T^{-m}(M)$. Note that if Q is bounded, then $M' \backslash M$ has a Lebesgue measure zero (cf. [4]).

If Q is a bounded and Γ is strictly convex (convex) at each $q \in \mathring{\Gamma}$, then the billiard in Q is called *dispersed* (respectively *semi-dispersed*). Dispersed billiards were introduced by Sinai [15]. Various properties of dispersed and semi-dispersed billiards were studied by many authors in connection with some problems in statistical mechanics and mathematical physics (cf. [4, 2, 3, 5, 6, 9–18] and the references given there).

For each integer $k \geqq 2$ denote by $\mathscr{A}_k$ the set of those k-tuples $\alpha = (i_1, \ldots . i_k)$ such that $i_j = 1, 2, \ldots, s$ for all $j, i_j \neq i_{j+1}$ for $j = 1, \ldots, k-1$ and $i_k \neq i_1$. Let

$$\pi: \Gamma \times S^{d-1} \to \Gamma$$

be the natural projection. A point $x = (q, v) \in M_0$ is called a *periodic point of type* α for T if $T^k(x) = x$ and

$$q_j = \pi \circ T^{j-1}(x) \in \Gamma_{i_j}$$

for any $j = 1, 2, \ldots, k$. If the segment $[q_j, q_{j+1}]$ is tangent to Γ at q_j, then q_j will be called a *tangent reflection point* of $\gamma(x)$, otherwise it will be called a *proper reflection point* of $\gamma(x)$.

The main result in this paper is the following

Theorem 1.1. *Let Q satisfy the above assumptions and let $\alpha \in \mathscr{A}_k$. Let there exist two different periodic points (q, v) and (p, w) of type α for T and let $q_j = \pi \circ T^{j-1}(q, v)$, $p_j = \pi \circ T^{j-1}(p, w), j = 1, 2, \ldots$. Then $v = w$, and for every $j \geqq 1$ the segments $[q_j, q_{j+1}]$ and $[p_j, p_{j+1}]$ are parallel. If q_j is a proper reflection point, then $tq_j + (1-t)p_j \in \Gamma_{i_j}$ for all $t \in (0, 1)$ sufficiently close to 1. If all q_j are proper reflection points, then for every $t \in (0, 1)$ sufficiently close to 1 the points $(tq + (1-t)p, v)$ are periodic points of type α for T generating periodic billiard trajectories in Q of the same length, and these trajectories have parallel corresponding segments.*

In other words, for every $\alpha \in \mathscr{A}_k$ there are three possibilities: (a) there are no periodic points of type α; (b) there exists exactly one periodic point of type α; (c) the periodic points of type α generate a family (which might be discrete, see Fig. 3 (a)) of parallel periodic billiard trajectories in Q of the same period (length). The assumption that q_j is a proper reflection point is essential for the second part of the theorem (cf. again Fig. 3 (a)).

Since every periodic billiard trajectory has at least two proper reflection points, the following is an immediate consequence of the above theorem.

Corollary 1.2. *If $\alpha = (i_1, \ldots, i_k) \in \mathscr{A}_k$ and Γ_{i_j} is strictly convex for some $j = 1, \ldots, k$, then there exists at most one periodic point of type α for T.*

We should mention that Theorem 1.1 and Corollary 1.2 fail if we drop our main assumption (cf. Fig. 3 (b)). They fail also if one considers domains Q in an

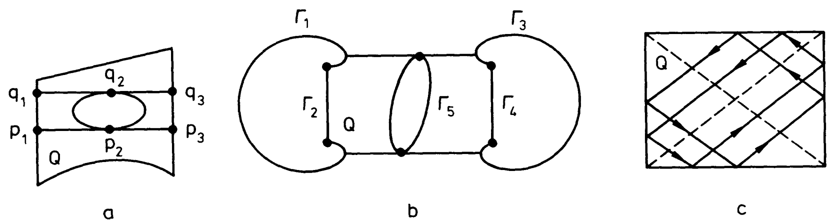


Fig. 3

arbitrary Riemannian manifold. It is easy to construct counterexamples with $Q \subset \mathrm{Tor}^2$ or $Q \subset S^2$.

If (q, v) and (p, w) are periodic points of period k for T, we will say that (q, v) and (p, w) are *equivalent* if they are of the same type and generate parallel periodic billiard trajectories of equal lengths. Denote by $P_k = P_k(Q)$ the *number of equivalent classes* of periodic points of period k for T.

Counting the cardinality of $\mathscr{A}_k$ and applying Theorem 1.1 one gets immediately the following.

Corollary 1.3. *Let Q satisfy the assumptions at the beginning of this section. Then for every integer $k \geqq 3$ we have*

$$P_k \leqq s(s-1)^{k-2}(s-2) < (s-1)^k.$$

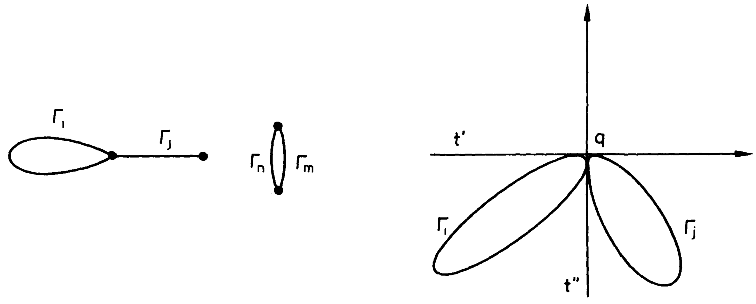
In particular, $\limsup_{k \to \infty} (\log P_k/k) \leqq s - 1$.

There is a large class of unbounded domains Q for which $P_k = s(s-1)^{k-2}(s-2)$ for all $k \geqq 3$. One may take for example all domains Q which are exteriors of several disjoint strictly convex compact domains in $\mathbb{R}^d$ and satisfy the condition (H) below (cf. [5]). Note that if Γ_i is strictly convex for every i, then P_k is exactly the number of all periodic points of period k for T.

The growth rate of the number $P(t)$ of closed geodesics of length $\leqq t$ on Riemannian manifolds, as well as that of the number $P_k(f)$ of periodic points of period k for diffeomorphisms f on compact manifolds, have been studied by many authors and in different contexts (cf. Katok [7, 8] for more details and some historical remarks). For example, for manifolds of negative curvature $\lim_{t \to \infty} P(t)/t$ exists and equals the topological entropy of the geodesic flow (Margulis [12]). If f is an Axiom A diffeomorphisms, then $\limsup_{k \to \infty} (\log P_k(f)/k)$ equals the topological entropy $h(f)$ of f(Bowen [1]). Katok [7] proved that if f is a $C^{1+\varepsilon}$ $(\varepsilon > 0)$ diffeomorphism of a compact manifold and μ is a Borel probability f-invariant measure with non-zero Lyapunov exponents, then $\limsup_{k \to \infty} (\log P_k(f)/k$ is not less than the metric entropy $h_\mu(f)$. Concerning the billiard ball map T we do not know any estimates of $P_k(T)$ by means of the (metric) entropy of T.

As N. Chernov pointed out, Theorem 1.1 has some consequences in the case when Γ_i are cylinders, which may have some applications to the study of systems of elastic hard spheres (cf. [17, 11]).

Let $Q \subset \mathbb{R}^2$ and $\partial Q = \Gamma_1 \cup \dots \cup \Gamma_s$. Every Γ_i is a smooth curve in $\mathbb{R}^2$ which may have one or two endpoints. If $i \neq j$ and $\Gamma_i \cap \Gamma_j \neq \varnothing$, then $\Gamma_i \cap \Gamma_j$ consists of



a b

Fig. 4

one or two points. Let $q \in \Gamma_i \cap \Gamma_j$. We will say that the pair (Γ_i, Γ_j) is *singular* at q if there is a common tangent line t to Γ_i and Γ_j at q such that Γ_i and Γ_j lie in different halfplanes with respect to t. Note that there could be two different common tangents to Γ_i and Γ_j at q (cf. Fig. 4).

Corollary 1.4. *Let Q be bounded, $Q \subset \mathbb{R}^2$, and let Γ_i be strictly convex for every $i = 1, \ldots, s$. Suppose moreover that for all $i \neq j$ with $\Gamma_i \cap \Gamma_j \neq \emptyset$ the pair (Γ_i, Γ_j) is non-singular at any point $q \in \Gamma_i \cap \Gamma_j$. Then there exists constants $c > 0, b > 0$ such that*

$$\tilde{P}_t \leqq (s-1)^{ct+b} \quad (t > 0),$$

where $\tilde{P}_t$ denotes the number of those $(q, v) \in M_0$ which generate periodic billiard trajectories in Q with lengths $\leqq t$.

An exponential estimate from below of P_k for semi-dispersed billiards in $\mathbb{R}^2$ is found by Bunimovich et al. [3]. It is also shown in [3] that the periodic points of the billiard map T are dense in the phase space M_0. These results are obtained as consequences of the existence of Markov partitions for such billiards established in [3].

Note that Theorem 1.1 works also in the case when Q is a polyhedron in $\mathbb{R}^d$, however in this case much better estimates for P_k and $\tilde{P}_t$ were found by Katok [9].

Finally, consider the case when $Q = \mathbb{R}^d \quad \bigcup_{i=1}^{s} K_i$, where K_i are disjoint strictly convex compact domains in $\mathbb{R}^d$ with C^2-smooth boundaries $\partial K_i = \Gamma_i$. In this case Ikawa [5] proved Theorem 1.1 under the following additional assumption:

$$(H) \quad \begin{cases} \text{For } i,j \in \{1, \ldots, s\},\ i \neq j, \text{ the convex hull of} \\ K_i \cup K_j \text{ contains no points of the set} \\ \cup \{K_m : m \neq i,j\}. \end{cases}$$

Using this fact and the technique of [5], Ikawa [6] proved that in the latter case there exists $\varepsilon > 0$ such that the domain $\{z \in \mathbb{C}: 0 < \operatorname{Im} z < \varepsilon\}$ contains infinitely many poles of the scattering matrix $S(z)$ related to the wave equation in Q with Neumann boundary conditions on ∂Q. On the other hand, it follows by [14] that for generic Q in $\mathbb{R}^d$ (see [14] for the precise definition of "generic") all periodic billiard trajectories in Q have only proper reflection points. It seems that using this fact, Theorem 1.1 and the technique of Ikawa [5, 6] one can derive that for generic Q

in $\mathbb{R}^d$ (but without assuming (H)) there always exists $\varepsilon > 0$ such that the scattering matrix $S(z)$ related to Q has infinitely many poles z with $0 < \operatorname{Im} z < \varepsilon$.

The proofs of Theorem 1.1 and Corollary 1.4 are given in Sect. 3 of this paper.

2. Periodic Points and Local Minima of Length Functions

In this section we assume that Q satisfies the assumptions at the beginning of Sect. 1. Denote by K_i the convex hull of Γ_i in $\mathbb{R}^d$. Then K_i is a compact convex subset of $\mathbb{R}^d$ and $\Gamma_i \subset \partial K_i$ by the main assumption (cf. Sect. 1). It may occur that K_i and K_j have common interior points for some $i \neq j$, but this will not interfere with our considerations.

Fix an $\alpha = (i_1, \ldots, i_k) \in \mathscr{A}_k$. For convenience we set $q_{k+1} = q_1$ and $q_0 = q_k$. Consider the *length function*

$$F = F_\alpha : K_\alpha = K_{i_1} \times \cdots \times K_{i_k} \to \mathbb{R} \tag{1}$$

defined by

$$F(q_1, \ldots, q_k) = \sum_{j=1}^{k} \| q_j - q_{j+1} \| . \tag{2}$$

Clearly, if (q, v) is a periodic point of type α for T, then for $q_j = \pi \circ T^{j-1}(q, v)$ we have that $F(q, \ldots, q_k)$ is the length of the corresponding periodic billiard trajectory.

Set $\Gamma_\alpha = \Gamma_{i_1} \times \cdots \times \Gamma_{i_k} \subset K_\alpha$ (this is not the boundary of K_α in $(\mathbb{R}^d)^k$). It is well-known that if the restriction $F_{|\Gamma_\alpha}$ of F to Γ_α has a local minimum at some point $\tilde{q} = (q_1, \ldots, q_k) \in \Gamma_\alpha$ and if for every $j = 1, \ldots, k$ the open segment (q_j, q_{j+1}) is contained in the interior of Q, then $q_1, \ldots, q_k$ are the consecutive reflection points of a periodic billiard trajectory in Q. Our aim in this section is to prove the converse.

Lemma 2.1. *Let (q, v) be a periodic point of type α for T and let $q_j = \pi \circ T^{j-1}(q, v)$, $j = 1, \ldots, k$. Then F has a local minimum at $\tilde{q} = (q_1, \ldots, q_k)$ as a function on K_α.*

Proof. Clearly, F is smooth in a neighbourhood of $\tilde{q}$. Since the case $k = 2$ is clear, we will assume $k \geqq 3$.

Every q_j is a regular point of Γ, therefore there is a C^2-smooth cart

$$\varphi_j : \mathbb{R}^{d-1} \to U_j \subset \Gamma_{i_j}$$

such that $\varphi_j(0) = q_j$. Then $\{\partial \varphi_j / \partial u_j^{(n)})(0)\}_{n=1}^{d-1}$ is a basis in the tangent space $T_{q_j}\Gamma$ to Γ at q_j. Hence $u_j = (u_j^{(1)}, \ldots, u_j^{(d-1)})$ belongs to $\mathbb{R}^{d-1}$. Consider the function

$$G : (\mathbb{R}^{d-1})^k \to \mathbb{R},$$

defined by

$$G(u_1, \ldots, u_k) = F(\varphi_1(u_1), \ldots, \varphi_k(u_k)).$$

First, we are going to prove that G has a local minimum at 0. This would imply that $F_{|\Gamma_\alpha}$ has a local minimum at $\tilde{q}$.

Let $\varphi_j(u_j) = (\varphi_j^{(1)}(u_j), \ldots, \varphi_j^{(d)}(u_j))$, and let $u = (u_1, \ldots, u_k) \in (\mathbb{R}^{d-1})^k$. In what follows we will use the following notation: $I_j = \{j-1, j+1\}$,

$$a_{ji} = 1/\| q_j - q_i \|, \; v_{ji} = (q_j - q_i)/\| q_j - q_i \| \quad (i \in I_j).$$

Clearly, $a_{ji}>0$ and $v_{ji}\in S^{d-1}$. Moreover, $a_{ij}=a_{ji}$ and $v_{ij}=-v_{ji}$.

For all $j=1,\dots,k$, $n=1,\dots,d-1$ and u sufficiently close to 0 we have

$$\frac{\partial G}{\partial u_j^{(n)}}(u)=\sum_{i\in I_j}\left\langle \frac{\varphi_j(u_j)-\varphi_i(u_i)}{\|\varphi_j(u_j)-\varphi_i(u_i)\|},\frac{\partial\varphi_j}{\partial u_j^{(n)}}(u_j)\right\rangle. \tag{3}$$

Since $v_{jj-1}+v_{jj+1}$ is collinear with $N(q_j)$, one gets

$$\frac{\partial G}{\partial u_j^{(n)}}(0)=\left\langle v_{jj-1}+v_{jj+1},\frac{\partial\varphi_j}{\partial u_j^{(n)}}(0)\right\rangle=0.$$

Therefore 0 is a critical point of G.

Next, we will show that the second fundamental form of G at 0 is non-negative definite. First, we have to compute $(\partial^2 G/\partial u_j^{(n)}\partial u_i^{(m)})(0)$ for all $j,i=1,\dots,k$ and $n,m=1,\dots,d-1$. Given j there are three possibilities for i.

Case 1. $i\notin I_j\cup\{j\}$. Then $(\partial^2 G/\partial u_j^{(n)}\partial u_i^{(m)})(0)=0$.

Case 2. $i\in I_j$. Now (3) implies

$$\begin{aligned}\frac{\partial^2 G}{\partial u_j^{(n)}\partial u_i^{(m)}}(0)=&-a_{ji}\left\langle\frac{\partial\varphi_j}{\partial u_j^{(n)}}(0),\frac{\partial\varphi_i}{\partial u_i^{(m)}}(0)\right\rangle\\&+a_{ji}\left\langle\frac{\partial\varphi_j}{\partial u_j^{(n)}}(0),v_{ji}\right\rangle\left\langle\frac{\partial\varphi_i}{\partial u_i^{(m)}}(0),v_{ji}\right\rangle.\end{aligned}$$

Case 3. $i=j$. Then

$$\begin{aligned}\frac{\partial^2 G}{\partial u_j^{(n)}\partial u_j^{(m)}}(0)=&\sum_{i\in I_j}\left\langle v_{ji},\frac{\partial^2\varphi_j}{\partial u_j^{(n)}\partial u_j^{(m)}}(0)\right\rangle+\sum_{i\in I_j}a_{ji}\left\langle\frac{\partial\varphi_j}{\partial u_j^{(n)}}(0),\frac{\partial\varphi_j}{\partial u_j^{(m)}}(0)\right\rangle\\&-\sum_{i\in I_j}a_{ji}\left\langle\frac{\partial\varphi_j}{\partial u_j^{(n)}}(0),v_{ji}\right\rangle\left\langle\frac{\partial\varphi_j}{\partial u_j^{(m)}}(0),v_{ji}\right\rangle.\end{aligned}$$

Fix an arbitrary vector $\xi=(\xi_j^{(n)})_{1\leqq j\leqq k,1\leqq n\leqq d-1}$ in $(\mathbb{R}^{d-1})^k$. We have to show that

$$\sigma=\sum_{j,i=1}^{k}\sum_{n,m=1}^{d-1}\frac{\partial^2 G}{\partial u_j^{(n)}\partial u_i^{(m)}}(0)\xi_j^{(n)}\xi_i^{(m)}\geqq 0.$$

Set $z_j=\sum_{n=1}^{d-1}\xi_j^{(n)}(\partial\varphi_j/\partial u_j^{(n)})(0)$, where $\xi_j=(\xi_j^{(1)},\dots,\xi_j^{(d-1)})$. Note that for $N_j=N(q_j)$ we have $v_{jj-1}+v_{jj+1}=-\lambda_j N_j$ for some $\lambda_j>0$.

Since $U_j=\varphi_j(\mathbb{R}^{d-1})\subset\Gamma$ is convex at q_j, the choice of the normal vector N_j shows that the second fundamental form B_j of U_j at q_j is non-positive definite. That is

$$B_j(\xi_j,\xi_j)=\sum_{n,m=1}^{d-1}\left\langle N_j,\frac{\partial^2\varphi_j}{\partial u_j^{(n)}\partial u_j^{(m)}}(0)\right\rangle\xi_j^{(n)}\xi_j^{(m)}\leqq 0$$

for every $\xi_j\in\mathbb{R}^{d-1}$.

According to the above formulas for the second derivatives of G at 0 we find:

$$\sigma = \sum_{j=1}^{k} \sum_{n,m=1}^{d-1} \frac{\partial^2 G}{\partial u_j^{(n)} \partial u_j^{(m)}}(0)\, \xi_j^{(n)} \xi_j^{(m)}$$

$$+ \sum_{j=1}^{k} \sum_{i \in I_j} \sum_{n,m=1}^{d-1} \frac{\partial^2 G}{\partial u_j^{(n)} \partial u_i^{(m)}}(0)\, \xi_j^{(n)} \xi_i^{(m)}$$

$$= \Bigg[- \sum_{j=1}^{k} \lambda_j \sum_{n,m=1}^{d-1} \left\langle N_j, \frac{\partial^2 \varphi_j}{\partial u_j^{(n)} \partial u_j^{(m)}}(0) \right\rangle \xi_j^{(n)} \xi_j^{(m)}$$

$$+ \sum_{j=1}^{k} \sum_{i \in I_j} \sum_{n,m=1}^{d-1} a_{ji} \left\langle \frac{\partial \varphi_j}{\partial u_j^{(n)}}(0), \frac{\partial \varphi_j}{\partial u_j^{(m)}}(0) \right\rangle \xi_j^{(n)} \xi_j^{(m)}$$

$$- \sum_{j=1}^{k} \sum_{i \in I_j} \sum_{n,m=1}^{d-1} a_{ji} \left\langle \frac{\partial \varphi_j}{\partial u_j^{(n)}}(0), v_{ji} \right\rangle \left\langle \frac{\partial \varphi_j}{\partial u_j^{(m)}}(0), v_{ji} \right\rangle \xi_j^{(n)} \xi_j^{(m)} \Bigg]$$

$$+ \Bigg[- \sum_{j=1}^{k} \sum_{i \in I_j} \sum_{n,m=1}^{d-1} a_{ji} \left\langle \frac{\partial \varphi_j}{\partial u_j^{(n)}}(0), \frac{\partial \varphi_i}{\partial u_i^{(m)}}(0) \right\rangle \xi_j^{(n)} \xi_i^{(m)}$$

$$+ \sum_{j=1}^{k} \sum_{i \in I_j} \sum_{n,m=1}^{d-1} a_{ji} \left\langle \frac{\partial \varphi_j}{\partial u_j^{(n)}}(0), v_{ji} \right\rangle \left\langle \frac{\partial \varphi_i}{\partial u_i^{(m)}}(0), v_{ji} \right\rangle \xi_j^{(n)} \xi_i^{(m)} \Bigg]$$

$$= - \sum_{j=1}^{k} \lambda_j B_j(\xi_j, \xi_j) + \sum_{j=1}^{k} \sum_{i \in I_j} a_{ji} \langle z_j, z_j \rangle - \sum_{j=1}^{k} \sum_{i \in I_j} a_{ji} \langle z_j, v_{ji} \rangle^2$$

$$- \sum_{j=1}^{k} \sum_{i \in I_j} a_{ji} \langle z_j, z_i \rangle + \sum_{j=1}^{k} \sum_{i \in I_j} a_{ji} \langle z_j, v_{ji} \rangle \langle z_i, v_{ji} \rangle .$$

Since $i \in I_j$ is equivalent to $j \in I_i$, according to $a_{ji} = a_{ij}$ and $v_{ji} = -v_{ij}$, one can rewrite the last expression for σ as follows:

$$\sigma = - \sum_{j=1}^{k} \lambda_j B_j(\xi_j, \xi_j) + \sum_{j=1}^{k} a_{jj+1} [\, \| z_j \|^2 - \langle z_j, v_{jj+1} \rangle^2$$

$$- \langle z_j, z_{j+1} \rangle + \langle z_j, v_{jj+1} \rangle \langle z_{j+1}, v_{jj+1} \rangle + \| z_{j+1} \|^2$$

$$- \langle z_{j+1}, v_{j+1j} \rangle^2 - \langle z_{j+1}, z_j \rangle + \langle z_{j+1}, v_{j+1j} \rangle \langle z_j, v_{j+1j} \rangle]$$

$$= - \sum_{j=1}^{k} \lambda_j B_j(\xi_j, \xi_j) + \sum_{j=1}^{k} a_{jj+1} [\, \| z_j - z_{j+1} \|^2 - \langle z_j - z_{j+1}, v_{jj+1} \rangle^2].$$

By definition $\| v_{jj+1} \| = 1$, therefore $\langle z_j - z_{j+1}, v_{jj+1} \rangle^2 \leqq \| z_j - z_{j+1} \|^2$, which yields $\sigma \geqq 0$.

In this way we have shown that G has a local minimum at 0, thus the restriction of F to Γ_α has a local minimum at $\tilde{q}$. Then there exist neighbourhoods V_j of q_j in K_{i_j} such that $F(\tilde{q}) \leqq F(\tilde{p})$ for every $\tilde{p} \in V \cap \Gamma_\alpha$, where $V = V_1 \times \cdots \times V_k$. Since $T^{j-1}(q, v)$ are admissible points for all $j \geqq 1$, we may choose the neighbourhoods V_j in such a way that for every $\tilde{p} \in V$ and every $j = 1, \ldots, k$ the segment $[p_j, p_{j+1}]$ intersects Γ_{i_j} and $\Gamma_{i_{j+1}}$ at points belonging to V_j and V_{j+1}, respectively. Indeed, if q_j is a tangent reflection point, we may define V_j by

$$V_j = \{ p_j \in K_{i_j} : \langle p_j - q_j, N(q_j) \rangle > -\varepsilon_j \}$$

for some $\varepsilon_j > 0$. If q_j is a proper reflection point, we take an open ball D_j with center q_j and a sufficiently small radius $\varepsilon_j > 0$ and set $V_j = K_{i_j} \cap D_j$.

Consider an arbitrary $\tilde{p} = (p_1, \ldots, p_k) \in V$. Denote by p'_1 the intersection point of Γ_{i_1} and the segment $[p_1, p_2]$. Then $p'_1 \in V_1$, and it follows by the triangle inequality that

$$F(p_1, p_2, \ldots, p_k) \geqq F(p'_1, p_2, \ldots, p_k).$$

Next, denoting by p'_2 the intersection point of Γ_{i_2} and the segment $[p_1, p_2]$ we obtain

$$F(p'_1, p_2, p_3, \ldots, p_k) \geqq F(p'_1, p'_2, p_3, \ldots, p_k),$$

and so on. Thus we find for each $j, p'_j \in \Gamma_{i_j} \cap V_j$ such that $F(\tilde{p}) \geqq F(\tilde{p}')$, where $\tilde{p}' = (p'_1, \ldots, p'_k) \in \Gamma_\alpha \cap V$. It follows from above that $F(\tilde{p}') \geqq F(\tilde{q})$, therefore $F(\tilde{p}) \geqq F(\tilde{q})$. This proves the assertion.

Remark. If Γ_{i_j} is strictly convex at q_j for every j, then clearly F has a strict local minimum at $\tilde{q}$.

3. Proofs of the Main Results

Let Q be as at the beginning of Sect. 1 and let $\alpha \in \mathscr{A}_k$ be given. In what follows we will use the function (1) defined by (2). Note that F is convex, that is

$$F(t\tilde{q} + (1-t)\tilde{p}) \leqq tF(\tilde{q}) + (1-t)F(\tilde{p})$$

for all $\tilde{q}, \tilde{p} \in K_\alpha$ and $t \in [0, 1]$.

Proof of Theorem 1.1. Assume there exist two different periodic points (q, v) and (p, w) of type α for T. Set $\tilde{q} = (q_1, \ldots, q_k)$ and $\tilde{p} = (p_1, \ldots, p_k)$. Then $\tilde{q}, \tilde{p} \in K_\alpha$ and by Lemma 2.1 F has local minima at $\tilde{q}$ and $\tilde{p}$. For $t \in [0, 1]$ set $q_j^{(t)} = tq_j + (1-t)p_j$ and $\tilde{q}^{(t)} = (q_1^{(t)}, \ldots, q_k^{(t)})$. Clearly, $\tilde{q}^{(t)} = t\tilde{q} + (1-t)\tilde{p} \in K_\alpha$.

We will show that $F(\tilde{q}) = F(\tilde{p})$. Assume $F(\tilde{q}) > F(\tilde{p})$. Then for every $t \in (0, 1)$ we have

$$F(\tilde{q}^{(t)}) = F(t\tilde{q} + (1-t)\tilde{p}) \leqq tF(\tilde{q}) + (1-t)F(\tilde{p}) < F(\tilde{q}).$$

Since $\tilde{q}^{(t)} \to \tilde{q}$ as $t \to 1$, we get a contradiction with the fact that F has a local minimum at $\tilde{q}$. Thus $F(\tilde{q}) \leqq F(\tilde{p})$. Similarly one gets $F(\tilde{p}) \leqq F(\tilde{q})$, therefore $F(\tilde{q}) = F(\tilde{p})$. Moreover, by $F(\tilde{q}^{(t)}) = F(t\tilde{q} + (1-t)\tilde{p}) \leqq F(\tilde{q}) = F(\tilde{p})$ we find that $F(\tilde{q}^{(t)}) = F(\tilde{q}) = F(\tilde{p})$ for all $t \in (0, 1)$ sufficiently close to 0 or 1. It then follows that $F(\tilde{q}^{(t)}) = F(\tilde{q}) = F(\tilde{p})$ for all $t \in [0, 1]$. Note that for $t \in (0, 1)$ the equality

$$\| [tq + (1-t)p] - [tq' + (1-t)p'] \| = t \| q - q' \| + (1-t) \| p - p' \|$$

holds if and only if the segments $[q, q']$ and $[p, p']$ are parallel (we assume $q \neq q'$ and $p \neq p'$). Then it follows from above that the segments $[q_j, q_{j+1}]$ and $[p_j, p_{j+1}]$ are parallel for each $j = 1, 2, \ldots$. In particular, $v = w$.

Take neighbourhoods V_j of q_j in K_{i_j} as at the end of Sect. 2. There exists $t_0 \in (0, 1)$ such that $q_j^{(t)} \in V_j$ for all $t \in (t_0, 1]$. Set $V = V_1 \times \cdots \times V_k$. Clearly, F has a minimum at $\tilde{q}^{(t)}$ in V for every $t \in (t_0, 1]$. Let q_j be a proper reflection point for some $j \leqq k$, and suppose $q_j^{(t)} \notin \Gamma_{i_j}$ for some $t \in (t_0, 1)$. Set $\tilde{r} = (q_1^{(t)}, \ldots, q_{j-1}^{(t)}, q'_j, q_{j+1}^{(t)}, \ldots,$

$q_k^{(t)}$), where q_j' is the point of intersection of Γ_{i_j} and the segment $[q_j^{(t)}, q_{j+1}^{(t)}]$. Since q_j is a proper reflection point, if t_0 is sufficiently close to 1, then the segments $[q_{j-1}^{(t)}, q_j^{(t)}]$ and $[q_j^{(t)}, q_{j+1}^{(t)}]$ would be not collinear, so

$$\| q_{j-1}^{(t)} - q_j^{(t)} \| + \| q_j^{(t)} - q_{j+1}^{(t)} \| > \| q_{j-1}^{(t)} - q_j' \| + \| q_j' - q_{j+1}^{(t)} \|,$$

and therefore $F(\tilde{q}^{(t)}) > F(\tilde{r})$ in contradiction with the minimality of $F(\tilde{q}^{(t)})$. Hence $q_j^{(t)} \in \Gamma_{i_j}$ for all $t \in (t_0, 1]$ providing t_0 is sufficiently close to 1.

Finally, if all $q_1, \dots, q_k$ are proper reflection points, then it follows from above that for every $t \in (0,1)$ sufficiently close to 1 the points $(tq + (1-t)p, v)$ are periodic points of type α for T which generate periodic billiard trajectories in Q of length $F(\tilde{q}) = F(\tilde{p})$ and parallel corresponding segments.

Proof of Corollary 1.4. Let $i \neq j$ be such that $\Gamma_i \cap \Gamma_j \neq \emptyset$ and let $q \in \Gamma_i \cap \Gamma_j$. Denote by $\omega_{ij}(q)$ the minimal angle between two different tangents to Γ_i and Γ_j at q. Put

$$\omega = \min \{ \omega_{ij}(q) : i \neq j, q \in \Gamma_i \cap \Gamma_j \}$$

if the set on the right-hand side is non-empty, and $\omega = \pi$ otherwise. For $n = [\pi/2\omega] + 1$ a simple geometrical argument shows that if $\gamma(x)$, $x \in M_0$, is a billiard semi-trajectory in Q and if $\Gamma_i \cap \Gamma_j \neq \emptyset$, then there are no more than $2n$ consecutive reflection points of $\gamma(x)$ belonging to $\Gamma_i \cup \Gamma_j$.

Further, divide each Γ_i which has endpoints into two curves Γ_i' and Γ_i'' by an arbitrary point $q_i \in \Gamma_i$ ($\Gamma_i = \Gamma_i' \cup \Gamma_i''$ and $\Gamma_i' \cap \Gamma_i'' = \{q_i\}$ if Γ_i has two different endpoints, $\Gamma_i' \cap \Gamma_i'' = \{q_i\} \cup \partial \Gamma_i$ otherwise). If $\partial \Gamma_i = \emptyset$, i.e. Γ_i has no endpoints, set $\Gamma_i' = \Gamma_i'' = \Gamma_i$. Define the numbers

$$m_i' = \min \{ \operatorname{dist}(\Gamma_i', \Gamma_j) : \Gamma_j \cap \Gamma_i' = \emptyset \},$$

$$m_i'' = \min \{ \operatorname{dist}(\Gamma_i'', \Gamma_j) : \Gamma_j \cap \Gamma_i'' = \emptyset \},$$

$$m = \min \{ m_1', \dots, m_s', m_1'', \dots, m_s'' \}.$$

Clearly, $m > 0$. Moreover, it follows from above that if $p_k, p_{k+1}, \dots, p_{k+2n}$ are consecutive reflection points of a billiard semi-trajectory $\gamma(x)$ in Q, $x \in M_0$, then at least one of the segments $[p_j, p_{j+1}]$, $j = 1, \dots, k + 2n - 1$, has a length not less than m.

Take an arbitrary $t > 0$, and let $\gamma = \gamma(x)$, $x \in M_0$, be an arbitrary periodic billiard trajectory in Q with length $l_\gamma \leqq t$. If k is the number of reflections of γ, then

$$l_\gamma \geqq m[k/(2n+1)] \geqq m(k-2n)/(2n+1),$$

so $k \leqq (2n+1)t/m + 2n$. Therefore for $i = [(2n+1)t/m]$, according to Corollary 1.3, we find

$$\tilde{P}_t \leqq \sum_{j=2}^{i+2n} P_j < \sum_{j=2}^{i+2n} (s-1)^j < (s-1)^{i+2n} \leqq (s-1)^{ct+b},$$

where $c = (2n+1)/m$ and $b = 2n+1$. This proves the assertion.

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