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Analytic Torsion and Holomorphic Determinant Bundles

III. Quillen Metrics on Holomorphic Determinants

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Abstract. In this paper, we prove that in the case of holomorphic locally Kähler fibrations, the analytic and algebraic geometry constructions of determinant bundles for direct images coincide. We calculate the curvature of the holomorphic Hermitian connection for the Quillen metric on the determinant bundle. We study the behavior of the Quillen metric under change of metrics in the fibers, and also on the twisting vector bundles. We thus generalize the conformal anomaly formula to Kähler manifolds of arbitrary dimension. We also study the Quillen metrics on determinants associated with exact sequences of vector bundles. We prove that the Quillen metric is smooth on the Grothendieck-Knudsen-Mumford determinant for arbitrary holomorphic fibrations.

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This is the third of a series of three papers devoted to the study of holomorphic determinant bundles and direct images. Parts I and II of this work will be referred to as [BGS 1, BGS 2]. Also the Introduction of [BGS 1] contains a general description of our results.

Let $\pi: M \rightarrow B$ be a proper holomorphic map of complex manifolds, and let ξ be a complex holomorphic vector bundle on M. For $y \in B$, let $Z_y=\pi^{-1}(y)$ be the fiber over y. Let g^{Z_y} be a Kähler metric on Z_y which depends smoothly on y, and let h^{ξ} be a smooth metric on ξ. If $\ell=\dim Z_y$, let

$$0 \rightarrow E_y^0 \xrightarrow[\bar{\partial}]{} E_y^1 \ldots \xrightarrow[\bar{\partial}]{} E_y^{\ell} \rightarrow 0 \tag{0.1}$$

denote the $\bar{\partial}$ complex associated with the restriction of ξ to Z_y.

In [BGS 2], when the metrics g^{Z_y} are the restriction to the fibers Z_y of a Kähler metric on M, we constructed analytic torsion forms of any degree associated with the direct image of the vector bundle ξ by the map π.

In this paper, we study in detail the determinant bundle of the direct image.

In Sect. 1, when the fibration π is locally Kähler, we construct a holomorphic structure on the C^{∞} line bundle λ of Bismut and Freed [BF 1, 2] and the corresponding Quillen metric. We calculate the curvature of the associated holomorphic Hermitian connection, which is given by a differential form version of the Riemann-Roch-Grothendieck Theorem. We use two facts:

- The truncation procedure of [Q 2, BF 1, 2], which approximates the line bundle λ by the determinant bundle of finite dimensional eigenspaces of $(\bar{\partial}+\bar{\partial}^*)^2$, is compatible with the holomorphic structure on the infinite dimensional Hermitian vector bundles $E^0, \ldots, E^{\ell}$, at least when the metric g^Z is the restriction to Z of a Kähler metric on M.
- We establish in Theorem 1.23 a generalization of the conformal anomaly formula for Kähler manifolds of arbitrary dimension, i.e. we prove the result stated in [BGS 1, Theorem 0.2].

Observe that the construction of Sect. 1 is purely analytical, and that in particular the holomorphic structure on the line bundle λ has been constructed analytically. One of the purposes of the next two sections is to compare the holomorphic line bundle λ with the holomorphic line bundle λ^{KM} of Knudsen and Mumford [KM].

In Sect. 2, we first establish [BGS 1, Theorem 0.3]. Namely we calculate the Quillen norm of the canonical section σ of an alternate product of determinant bundles associated with an exact sequence of holomorphic Hermitian vector bundles on M. To obtain this result, we use in an essential way the results of [BGS 2]. When π is projective, we prove in Theorem 2.12 that λ and λ^{KM} are canonically isomorphic as smooth holomorphic line bundles.

In Sect. 3, we prove that in general, the canonical isomorphism of fibers $\lambda_y \simeq \lambda_y^{\mathrm{KM}}$ defines a smooth isomorphism of line bundles, even when π is not locally Kähler. We also prove that when π is locally Kähler, this isomorphism preserves the holomorphic structures. We thus prove [BGS 1, Theorem 0.1] in full generality.

We use in this paper many of the techniques we developed in [BGS 1, BGS 2]. In particular we are able to calculate the term of order 0 in a singular asymptotic expansion as $t\downarrow\downarrow 0$ in order to establish the generalization of the conformal anomaly formula of [BGS 1, Theorem 0.2]. Also we constantly use the formalism of the Bott-Chern classes of [BGS 1], and the superconnections of Quillen [Q 1].

We refer to [BGS 1, BGS 2] for notations and terminology. In particular if K is a Z_2 graded algebra, and if $A, B \in K$, $[A, B]$ denotes the supercommutator of A and B. Also the notations Tr and Tr_s are used for traces and supertraces.

On a complex manifold B, P denotes the set of smooth differential forms which are sums of forms of type (p, p). P' is the subspace of P which consists of the forms $\omega \in P$ such that $\omega = \partial^B \eta + \bar{\partial}^B \eta'$.

The results contained in this paper were announced in [BGS 3].

I. The Analysis of Holomorphic Determinant Bundles

In this section, we construct a holomorphic structure on the C^∞ line bundle of Bismut and Freed [BF 1] associated with the family of operators $\bar{\partial}_y + \bar{\partial}_y^*$, and we derive the essential properties of this line bundle.

In a), we give the main assumptions and notations. In b), we describe the line bundle λ. Although our construction imitates [BF 1], it is somewhat different. In fact, we use the same approximating line bundles λ^a as in [BF 1]. However, our transition maps are not the same.

In c), we prove that the approximating line bundles λ^a are naturally holomorphic, and that λ inherits the corresponding holomorphic structure.

In d), we construct the Quillen metric on the bundle λ. This metric differs from the metric in [BF 1].

In e) we calculate the curvature of the holomorphic Hermitian line bundle λ in the special case where the metric on Z is the restriction of a Kähler metric on M.

In f) we briefly identify our line bundle λ with the bundle λ' of [BF 1, BF 2] as smooth bundles with metric and connection.

In g) we prove that the holomorphic structure on λ does not depend on the special Kähler metric on M which was used to construct it. This is done by purely analytic methods. Of course in the light of our final result $\lambda = \lambda^{\mathrm{KM}}$, it is very natural that such a direct proof can be given.

In h), we prove the result stated in [BGS 1, Theorem 0.2] which describes how the Quillen metric varies with the metric g^Z in one given fiber. This is done by using again non-trivial identities on traces, anticommuting variables, the heat equation and Brownian motion.

Finally in i), we prove that if the fibers Z are endowed with any family of Kähler metrics g^Z, the curvature of the holomorphic line bundle λ endowed with the Quillen metric is given by a differential form version of the Riemann-Roch-Grothendieck Theorem.

a) Assumptions and Notations

We now do the same assumptions as in [BGS 2, Sect. 2], and we use the same notations. In particular $(\pi, g^Z, T^H M)$ is a Kähler fibration and $\omega = \omega^Z + \omega^H$ is an associated $(1,1)$ form.

Let ξ be a holomorphic Hermitian vector bundle on M and ∇^ξ the corresponding Hermitian holomorphic connection, $L^\xi = (\nabla^\xi)^2$ its curvature. We use the notations of [BGS 2, Sect. 2] with $m=0$. In other words, we consider the case of one single twisting bundle ξ. The associated trivial chain complex ξ is simply $0 \to \xi \to 0$. In particular $v=0$, $v^*=0$.

Recall that for $0 \leqq p \leqq \ell$, $y \in B$, E_y^p is the set of smooth sections of $\Lambda^p T^{*(0,1)} Z \otimes \xi$ on Z_y. Also:

$$E^+ = \bigoplus_{p\,\text{even}} E^p, \qquad E^- = \bigoplus_{p\,\text{odd}} E^p, \qquad E = E^+ \oplus E^- .$$

$D_y = \bar{\partial}_y + \bar{\partial}_y^*$ acts on the fiber E_y. Let $D_\pm$ be the restriction of D to $E^\pm$. Then we write:

$$\mathrm{D} = \begin{bmatrix} 0 & D_- \\ D_+ & 0 \end{bmatrix}.$$

b) Description of the Determinant Bundle

We now describe the determinant bundle associated with D. Our description is inspired from Quillen [Q 2] and from Bismut and Freed [BF 1]. However, since we are especially interested in the explicit construction of an holomorphic structure on this line bundle, we must proceed differently from [BF 1].

For every $y \in B$, the spectrum of D_y^2 is discrete. For $b > 0$, $0 \leqq p \leqq \ell$, let $K_y^{b,p}$ be the sum of the eigenspaces of the operator D_y^2 acting on E_y^p for eigenvalues $< b$. Since D_y^2 is elliptic on Z_y, $K_y^{b,p} \subset E_y^p$.

Let U^b be the open set:

$$U^b = \{ y \in B; b \notin \operatorname{Spec} D_y^2 \} .$$

On the open set U^b, $K^{b,p}$ is a smooth finite dimensional vector bundle. Set

$$K^{b,+} = \bigoplus_{p\,\text{even}} K^{b,p}, \qquad K^{b,-} = \bigoplus_{p\,\text{odd}} K^{b,p}, \qquad K^b = K^{b,+} \oplus K^{b,-} . \tag{1.1}$$

We now define the line bundles λ^b and λ'^b on U^b,

$$\lambda^b = (\det K^{b,0})^{-1} \otimes (\det K^{b,1}) \otimes (\det K^{b,2})^{-1} \otimes \dots,$$

$$\lambda'^b = (\det K^{b,+})^{-1} \otimes (\det K^{b,-}) .$$

By [BGS 1, Sect. 1a) 3.], there is a canonical isomorphism i^b from λ^b into λ'^b.

For $0 < b < c$, if $y \in U^b \cap U^c$, let $K_y^{(b,c),p}$ be the sum of the eigenspaces of D_y^2 in E_y^p for eigenvalues μ such that $b < \mu < c$. Set:

$$K_y^{(b,c),+} = \bigoplus_{p\,\text{even}} K_y^{(b,c),p}, \qquad K_y^{(b,c),-} = \bigoplus_{p\,\text{odd}} K_y^{(b,c),p},$$

$$K_y^{(b,c)} = K_y^{(b,c),+} \oplus K_y^{(b,c),-} . \tag{1.2}$$

We also define $\lambda^{(b,c)}$, $\lambda'^{(b,c)}$ as before, and denote by $i^{(b,c)}$ the canonical isomorphism from $\lambda^{(b,c)}$ into $\lambda'^{(b,c)}$.

Let $\bar{\partial}^{(b,c)}$ and $D^{(b,c)}$ be the restrictions of $\bar{\partial}$ and D to $K^{(b,c)}$. $D_{\pm}^{(b,c)}$ is the restriction of D to $K^{(b,c),\pm}$.

The chain complex

$$0 \to K^{(b,c),0} \xrightarrow[\bar{\partial}^{(b,c)}]{} K^{(b,c),1} \xrightarrow[\bar{\partial}^{(b,c)}]{} \dots \xrightarrow[\bar{\partial}^{(b,c)}]{} K^{(b,c),\ell} \to 0 \tag{1.3}$$

is acyclic. By [BGS 1, Definition 1.1], $\lambda^{(b,c)}$ has a canonical non-zero section $T(\bar{\partial}^{(b,c)})$ which is smooth on $U^b \cap U^c$. Also $\det D_{+}^{(b,c)}$ is a smooth non-zero section of $\lambda'^{(b,c)}$ over $U^b \cap U^c$.

For $0<b<c$, over $U^b \cap U^c$, we have the C^∞ identifications

$$\begin{aligned} \lambda^c &= \lambda^b \otimes \lambda^{(b,c)}, \\ \lambda'^c &= \lambda'^b \otimes \lambda'^{(b,c)}. \end{aligned} \tag{1.4}$$

We identify λ^b and λ^c over $U^b \cap U^c$ by the C^∞ maps

$$\begin{aligned} s \in \lambda^b &\to s \otimes T(\bar{\partial}^{(b,c)}) \in \lambda^c, \\ s' \in \lambda'^b &\to s' \otimes \det D_{+}^{(b,c)} \in \lambda'^c. \end{aligned} \tag{1.5}$$

Note that the identifications (1.5) are *not* compatible with the isomorphisms i^b and i^c.

Definition 1.1. λ (respectively λ') is the C^∞ line bundle over B which coincides with λ^b (respectively λ'^b) on U^b with the transition functions (1.5) on $U^b \cap U^c$.

Only the line bundle λ' was considered in [BF 1] (under the name of λ).

c) A Holomorphic Structure on λ

As a smooth subbundle of E over U^b, K^b inherits the Hermitian product defined in [BGS 2, Eq. (1.38)]. It follows that over U^b, λ^b, and λ'^b are endowed with smooth Hermitian metrics $|\ |^b$ and $|\ |'^b$. The map i^b is an isometry from λ^b into λ'^b.

For $y \in U^b$, let P_y^b be the orthogonal projection operator from E_y on K_y^b. Since $K_y^b \subset E_y$, P_y^b is a smooth family of regularizing operators. Similarly for $0<b<c<+\infty$, over $U^b \cap U^c$, $P_y^{(b,c)}$ is the orthogonal projection operator from E_y into $K_y^{(b,c)}$.

Definition 1.2. Let ∇^b denote the connection on K^b over U^b such that if h is a C^∞ section of K^b over U^b, then:

$$\nabla^b h = P^b \tilde{\nabla} h. \tag{1.6}$$

The connection ∇^b preserves the metric of K^b, and induces connections ${}^0\nabla^b$ and ${}^0\nabla'^b$ on λ^b and λ'^b, which preserve the metrics $|\ |^b$ and $|\ |'^b$.

In the same way, for $0<b<c<+\infty$, over $U^b \cap U^c$, $\lambda^{(b,c)}$, and $\lambda'^{(b,c)}$ are endowed with smooth metrics $|\ |^{(b,c)}$ and $|\ |'^{(b,c)}$ and unitary connections ${}^0\nabla^{(b,c)}$ and ${}^0\nabla'^{(b,c)}$, which are induced by the connection $\nabla^{(b,c)}$ on $K^{(b,c)}$.

We now prove an essential result:

Theorem 1.3. *Over U^b, there is a uniquely defined holomorphic structure on the smooth Hermitian line bundle $(\lambda^b, |\ |^b)$ such that ${}^0\nabla^b$ is the corresponding holomorphic Hermitian connection.*

Similarly for $0<b<c<+\infty$, over $U^b \cap U^c$, there exists a uniquely defined holomorphic structure on the smooth Hermitian line bundle $(\lambda^{(b,c)}, |\ |^{(b,c)})$ such that ${}^0\nabla^{(b,c)}$ is the corresponding holomorphic Hermitian connection. $T(\bar\partial^{(b,c)})$ is a non-zero holomorphic section of $\lambda^{(b,c)}$ over $U^b \cap U^c$.

The holomorphic structures on (λ^b, U^b) patch into a uniquely defined holomorphic structure on the line bundle λ on B.

Proof. By [AHS, Theorem 5.1], to prove the first part of the theorem, we only need to prove that the curvature of the unitary connection ${}^0\nabla^b$ is of type $(1,1)$.

Since P^b is a smooth family of regularizing operators, for any $Y \in TB$, $\tilde\nabla_Y P^b$ is regularizing. If $K^{(b,+\infty)}$ is the orthogonal of K^b in E, by [BF 1, Proposition 1.13], $\tilde\nabla_Y P^b$ interchanges K^b and $K^{(b,+\infty)}$. Thus if $Y, Y' \in TB$, $[\tilde\nabla_Y P^b, \tilde\nabla_{Y'} P^b]$ map K^b and $K^{(b,+\infty)}$ into themselves.

By [BGS 2, Theorem 1.14], $\tilde\nabla^2(Y, Y')$ is a first order differential operator acting fiberwise. It follows that $P^b \tilde\nabla^2(Y, Y') P^b$ maps K^b into itself. An obvious computation shows that the curvature of the connection ${}^0\nabla^b$ on λ^b is given by

$$\mathrm{Tr}_s\{P^b \tilde\nabla^2 P^b\} + \tfrac{1}{2}\mathrm{Tr}_s\{P^b[\tilde\nabla P^b, \tilde\nabla P^b] P^b\}.$$

By [BGS 2, Theorem 1.14], $\tilde\nabla^2$ is of type $(1,1)$. So we should prove that $\mathrm{Tr}_s\{P^b[\tilde\nabla P^b, \tilde\nabla P^b] P^b\}$ is of type $(1,1)$.

Set $Q^b = I - P^b$. Since Tr_s vanishes on supercommutators, we find that

$$\mathrm{Tr}_s\{P^b[\tilde\nabla P^b, \tilde\nabla P^b] P^b\} = -\mathrm{Tr}_s\{Q^b[\tilde\nabla P^b, \tilde\nabla P^b] Q^b\}. \tag{1.7}$$

Since $[D^2, \bar\partial] = 0$ it is clear that:

$$P^b \bar\partial = \bar\partial P^b. \tag{1.8}$$

Also by [BGS 2, Theorem 1.14], $\tilde\nabla'' \bar\partial = 0$. Using (1.8), we find that

$$[(\tilde\nabla'' P^b), \bar\partial] = 0. \tag{1.9}$$

Take $Y, Y' \in T^{(0,1)}Z$. Set

$$S = \tilde\nabla_Y P^b \tilde\nabla_{Y'} P^b. \tag{1.10}$$

From (1.9), we find that

$$[S, \bar\partial] = 0. \tag{1.11}$$

We claim that

$$\mathrm{Tr}_s[Q^b S Q^b] = 0. \tag{1.12}$$

If μ is an eigenvalue of D^2, let T_μ be the orthogonal projection operator on the corresponding eigenspace. Since S is trace class, we know that

$$\mathrm{Tr}_s[Q^b S Q^b] = \sum_{\mu > b} \mathrm{Tr}_s[T_\mu S T_\mu]. \tag{1.13}$$

If $K_p^{\{\mu\}}$ is the eigenspace in E^p corresponding to the eigenvalue μ, we have the exact sequence

$$0 \to K_0^{\{\mu\}} \xrightarrow[\bar\partial]{} K_1^{\{\mu\}} \to \dots \xrightarrow[\bar\partial]{} K_\ell^{\{\mu\}} \to 0. \tag{1.14}$$

By [BGS 1, Proposition 1.3], we find that:

$$\mathrm{Tr}_s[T_\mu S T_\mu] = 0. \tag{1.15}$$

Equation (1.12) follows from (1.13) and (1.15).

Using (1.12), we find that $\mathrm{Tr}_s\{P^b[\tilde\nabla P^b, \tilde\nabla P^b]P^b\}$ is of type $(1,1)$. Therefore we have proved that $({}^0\nabla^b)^2$ is of type $(1,1)$. Similarly since $\nabla''\bar\partial = 0$, if $Y, Y' \in T^{(0,1)}Z$, $(\nabla^{(b,c)})^2(Y, Y')$ commutes with $\bar\partial^{(b,c)}$. Since the chain complex (1.3) is acyclic, by [BGS 1, Proposition 1.3], we find that:

$$\mathrm{Tr}_s[(\nabla^{(b,c)})^2(Y, Y')] = 0, \tag{1.16}$$

and so $({}^0\nabla^{(b,c)})^2$ is of complex type $(1,1)$.

We now prove that $T(\bar\partial^{(b,c)})$ is a holomorphic section of $\lambda^{(b,c)}$. Since the complex (1.14) is acyclic, each $K^{(b,c),p}$ splits into

$$K^{(b,c),p} = \bar\partial^{(b,c)}[K^{(b,c),p-1}] \oplus \bar\partial^{(b,c)*}[K^{(b,c),p+1}], \tag{1.17}$$

and the two vector spaces in the right-hand side of (1.17) are orthogonal. Also

$$\dim K^{(b,c),p} = \dim(\bar\partial^{(b,c)} K^{(b,c),p-1}) + \dim(\bar\partial^{(b,c)} K^{(b,c),p}). \tag{1.18}$$

Since $K^{(b,c)}$ is a C^∞ bundle on $U^b \cap U^c$, $\dim K^{(b,c),p}$ is locally constant on $U^b \cap U^c$. Equation (1.18) shows that $\dim(\bar\partial^{(b,c)} K^{(b,c),p})$ is locally constant on $U^b \cap U^c$. Therefore, $\bar\partial^{(b,c)} K^{(b,c),p}$ and $\bar\partial^{(b,c)*} K^{(b,c),p}$ are smooth bundles on $U^b \cap U^c$.

Let $s^0, \dots, s^{\ell-1}$ be locally defined C^∞ non-zero sections of

$$\det[\bar\partial^{(b,c)*} K^{(b,c),1}], \dots, \det[\bar\partial^{(b,c)*} K^{(b,c),\ell}].$$

Then by [BGS 1, Definition 1.1], we know that:

$$T(\bar\partial^{(b,c)}) = (s^0)^{-1} \otimes \bar\partial s^0 \wedge s^1 \otimes (\bar\partial s^1 \wedge s^2)^{-1} \otimes \dots. \tag{1.19}$$

Let $U^{(b,c),p}$ be the orthogonal projection operator from $K^{(b,c),p}$ on $\bar\partial^{(b,c)*} K^{(b,c),p+1}$. The connection $\nabla^{(b,c)}$ splits into $\nabla^{(b,c)} = \nabla^{(b,c)'} + \nabla^{(b,c)''}$, where $\nabla^{(b,c)'}$, $\nabla^{(b,c)''}$ are the holomorphic and antiholomorphic parts of $\nabla^{(b,c)}$. Since by [BGS 2, Theorem 1.14], $\tilde\nabla''\bar\partial = 0$, we know that:

$$\nabla^{(b,c)''} \bar\partial^{(b,c)} = 0.$$

Since $\bar\partial^{(b,c)}$ vanishes on $\bar\partial^{(b,c)}(K^{(b,c),p-1})$ and since in the right-hand side of (1.17), the two vector spaces are orthogonal, we find that:

$$\begin{aligned} \nabla^{(b,c)''}(\bar\partial^{(b,c)} s^{p-1} \wedge s^p) &= (\bar\partial^{(b,c)} U^{(b,c),p-1} \nabla^{(b,c)''} s^{p-1}) \wedge s^p \\ &\quad + \bar\partial^{(b,c)} s^{p-1} \wedge U^{(b,c),p} \nabla^{(b,c)''} s^p, \end{aligned} \tag{1.20}$$

and so

$$\frac{\nabla^{(b,c)''}(\bar\partial s^{p-1} \wedge s^p)}{\bar\partial s^{p-1} \wedge s^p} = \frac{U^{(b,c),p-1} \nabla^{(b,c)''} s^{p-1}}{s^{p-1}} + \frac{U^{(b,c),p} \nabla^{(b,c)''} s^p}{s^p}. \tag{1.21}$$

${}^0\nabla^{(b,c)}$ also splits into ${}^0\nabla^{(b,c)} = {}^0\nabla^{(b,c)'} + {}^0\nabla^{(b,c)''}$. Using (1.21), we find that:

$$ {}^0\nabla^{(b,c)''} T(\bar{\partial}^{(b,c)}) = 0 . \tag{1.22} $$

$T(\bar{\partial}^{(b,c)})$ is thus a holomorphic section of $\lambda^{(b,c)}$.

We now prove that for $0 < b < c < +\infty$, over $U^b \cap U^c$, the canonical map

$$ s \in \lambda^b \to s \otimes T(\bar{\partial}^{(b,c)}) \in \lambda^c \tag{1.23} $$

is a holomorphic map from the holomorphic bundle λ^b into the holomorphic bundle λ^c. In fact, since $K^{c,p} = K^{b,p} \oplus K^{(b,c),p}$, it is clear that if s is a smooth section of λ^b,

$$ {}^0\nabla^c (s \otimes T(\bar{\partial}^{(b,c)})) = {}^0\nabla^b s \otimes T(\bar{\partial}^{(b,c)}) + s \otimes {}^0\nabla^{(b,c)} T(\bar{\partial}^{(b,c)}) . \tag{1.24} $$

Using (1.22), we find that:

$$ {}^0\nabla^{c''} (s \otimes T(\bar{\partial}^{(b,c)})) = {}^0\nabla^{b''} s \otimes T(\bar{\partial}^{(b,c)}) . \tag{1.25} $$

(1.25) exactly means that the map (1.23) is holomorphic.

We have thus shown that the holomorphic structures on (λ^b, U^b) patch together into a uniquely defined holomorphic structure on the line bundle λ on B. The theorem is proved. □

d) A Metric and a Holomorphic Connection on λ

We now construct a natural metric on the holomorphic bundle λ. Our construction is inspired from Quillen [Q 2] and Bismut and Freed [BF 1]. Take $0 < b < c < +\infty$. Over $U^b \cap U^c$, consider the acyclic chain complex (1.3). The $K^{(b,c),p}$ are Hermitian bundles. We can thus define the analytic torsion $\tau(\bar{\partial}^{(b,c)})$ of the chain complex (1.3). By [BGS 1, Proposition 1.5], we know that:

$$ |T(\bar{\partial}^{(b,c)})|^{(b,c)} = \tau(\bar{\partial}^{(b,c)}) . \tag{1.26} $$

Also, since K^b and $K^{(b,c)}$ are orthogonal subspaces of K^c, we find that if $s \in \lambda^b$,

$$ |s \otimes T(\bar{\partial}^{(b,c)})|^c = |s|^b \tau(\bar{\partial}^{(b,c)}) . \tag{1.27} $$

The metrics $|\ |^b$ clearly do not patch into a metric on λ because of the discrepancy (1.27).

Recall that $N_V = -i\omega^{Z,c} + \frac{\ell}{2}$ is the number operator on E, i.e. if $\eta \in E^p$, $N_V \eta = p\eta$. Set $Q^b = I - P^b$.

Definition 1.4. For $y \in U^b$, $\operatorname{Re}(s) > \ell$, set

$$ \theta_y^b(s) = -\operatorname{Tr}_s[N_V [D^2]^{-s} Q^b] . \tag{1.28} $$

Similarly if $0 < b < c < +\infty$, for $y \in U^b \cap U^c$, $s \in C$, set

$$ \theta_y^{(b,c)}(s) = -\operatorname{Tr}_s[N_V [D^2]^{-s} P^{(b,c)}] . \tag{1.29} $$

Equivalently:

$$ \begin{aligned} \theta_y^b(s) &= \frac{-1}{\Gamma(s)} \int_0^{+\infty} u^{s-1} \operatorname{Tr}_s[N_V \exp(-uD^2) Q^b] du , \\ \theta_y^{(b,c)}(s) &= \frac{-1}{\Gamma(s)} \int_0^{+\infty} u^{s-1} \operatorname{Tr}_s[N_V \exp(-uD^2) P^{(b,c)}] du . \end{aligned} \tag{1.30} $$

θ_y^b extends into a meromorphic function which is holomorphic at $s=0$. The same is true for $\theta_y^{(b,c)}$. Also, for $0<b<c<+\infty$, on $U^b \cap U^c$,

$$\theta^b = \theta^{(b,c)} + \theta^c . \tag{1.31}$$

Finally, by [BGS 1, Eq. (1.64)], we know that

$$\operatorname{Log}[\tau^2(\bar{\partial}^{(b,c)})] = -\theta^{(b,c)\prime}(0) . \tag{1.32}$$

We now extend (1.32) to the case where $c=+\infty$.

Definition 1.5. For $y \in U^b$, let $\tau_y(\bar{\partial}^{(b,+\infty)})$ be the positive real number

$$\tau_y(\bar{\partial}^{(b,+\infty)}) = \exp\{-\tfrac{1}{2}\theta_y^{b\prime}(0)\} . \tag{1.33}$$

Let $\| \; \|^b$ denote the metric on the line bundle (λ^b, U^b),

$$\| \; \|^b = | \; |^b \tau_y(\bar{\partial}^{(b,+\infty)}) \tag{1.34}$$

and ${}^1\nabla^b$ the connection on (λ^b, U^b),

$${}^1\nabla^b = {}^0\nabla^b + \partial^B \operatorname{Log} \tau^2(\bar{\partial}^{(b,+\infty)}) . \tag{1.35}$$

We now prove the following key result:

Theorem 1.6. *The metrics $\| \; \|^b$ on (λ^b, U^b) patch into a smooth metric $\| \; \|$ on the line bundle λ. The connections ${}^1\nabla^b$ patch into a connection ${}^1\nabla$ on λ. The connection ${}^1\nabla$ is the unique holomorphic Hermitian connection on the Hermitian line bundle $(\lambda, \| \; \|)$.*

Proof. For $0<b<c<+\infty$, take s in λ^b.
Then

$$\begin{aligned} \|s\|^b &= |s|^b \tau(\bar{\partial}^{(b,+\infty)}) , \\ \|s \otimes T(\bar{\partial}^{(b,c)})\|^c &= |s|^b \tau(\bar{\partial}^{(b,c)}) \tau(\bar{\partial}^{(c,+\infty)}) . \end{aligned} \tag{1.36}$$

By (1.31), we know that

$$\tau(\bar{\partial}^{(b,c)}) \tau(\bar{\partial}^{(c,+\infty)}) = \tau(\bar{\partial}^{(b,+\infty)}) . \tag{1.37}$$

From (1.37), we get

$$\|s \otimes T(\bar{\partial}^{(b,c)}\|^c = \|s\|^b . \tag{1.38}$$

Equation (1.38) exactly means that the metrics $\| \; \|^b$ patch into a C^∞ metric $\| \; \|$ on λ. Clearly ${}^1\nabla^b$ is the unique holomorphic connection which is Hermitian with respect to $\| \; \|^b$. Since the holomorphic structures on (λ^b, U^b) and the metrics $\| \; \|^b$ patch together, the connections ${}^1\nabla^b$ also patch together into a connection ${}^1\nabla$ on λ, which is holomorphic and preserves the metric $\| \; \|$. □

e) Evaluation of the Curvature of ${}^1\nabla$

We now calculate the curvature of ${}^1\nabla$. By [BGS 2, Theorem 2.11], as $u \downarrow\downarrow 0$,

$$\operatorname{Tr}_s\left[\left(\sqrt{u}D + \frac{c(T)}{4\sqrt{u}}\right)\exp(-A_u^2)\right] = O(u) . \tag{1.39}$$

Since $c(T)$ is of degree 2 in the Grassmann variables of $\Lambda(T^*B)$,

$$\left[\operatorname{Tr}_s\left(\left(\sqrt{u}D+\frac{c(T)}{4\sqrt{u}}\right)\exp(-A_u^2)\right)\right]^{(1)}=[\operatorname{Tr}_s(\sqrt{u}D\exp-(\tilde{\nabla}+\sqrt{u}D)^2)]^{(1)}. \tag{1.40}$$

By [BF 2, Theorem 3.3], we know that

$$[\operatorname{Tr}_s(\sqrt{u}D\exp-(\tilde{\nabla}+\sqrt{u}D)^2)]^{(1)}=-u\operatorname{Tr}_s[\exp(-uD^2)\tilde{\nabla}DD]. \tag{1.41}$$

Using (1.40), (1.41), we thus reobtain the result of [BF 2, Theorem 3.4] which asserts that as $u\downarrow\downarrow 0$,

$$\operatorname{Tr}_s[\exp(-uD^2)\tilde{\nabla}DD]=O(1).$$

It follows that as $u\downarrow\downarrow 0$,

$$\operatorname{Tr}_s[\exp(-uD^2)\tilde{\nabla}DDQ^b]=O(1). \tag{1.42}$$

We now define the differential form δ_0^b as in [BF 1, Definition 1.14].

Definition 1.7. Let δ_0^b denote the smooth 1 form on U^b:

$$\delta_0^b=\int_0^{+\infty}\operatorname{Tr}_s[\exp(-uD^2)\tilde{\nabla}DDQ^b]du. \tag{1.43}$$

The integral in (1.43) is well defined by [BF 2, Theorem 3.4] or by [BGS 2, Theorem 2.11].

Let $\nabla^{(b,+\infty)}$ be the connection on $K^{(b,+\infty)}$ such that if h is a C^∞ section of $K^{(b,+\infty)}$ on U^b,

$$\nabla^{(b,+\infty)}h=Q^b\tilde{\nabla}h. \tag{1.44}$$

$\nabla^{(b,+\infty)}$ is the natural extension of $\nabla^{(b,c)}$ for $c=+\infty$. Similarly $D^{(b,+\infty)}$, $\bar{\partial}^{(b,+\infty)}$, $\bar{\partial}^{(b,+\infty)*}$ are the restrictions of D, $\bar{\partial}$, $\bar{\partial}^*$ to $K^{(b,+\infty)}$.

Using again [BF 2, Theorem 3.3], we find that

$$\begin{aligned}&\{\operatorname{Tr}_s[\sqrt{u}D^{(b,+\infty)}\exp-(\nabla^{(b,+\infty)}+\sqrt{u}D^{(b,+\infty)})^2]\}^{(1)}\\&\quad=-u\operatorname{Tr}_s[(\exp(-u(D^{(b,+\infty)})^2)(\nabla^{(b,+\infty)}D^{(b,+\infty)})D^{(b,+\infty)}].\end{aligned} \tag{1.45}$$

Since $\nabla^{(b,+\infty)}D^{(b,+\infty)}=Q^b\tilde{\nabla}DQ^b$, we get

$$\begin{aligned}&\{\operatorname{Tr}_s[\sqrt{u}D^{(b,+\infty)}\exp-(\nabla^{(b,+\infty)}+\sqrt{u}D^{(b,+\infty)})^2]\}^{(1)}\\&\quad=-u\operatorname{Tr}_s[\exp(-uD^2)\tilde{\nabla}DDQ^b].\end{aligned} \tag{1.46}$$

Using (1.42)–(1.46), we find

$$\delta_0^b=-\int_0^{+\infty}\frac{1}{\sqrt{u}}\{\operatorname{Tr}_s[D^{(b,+\infty)}\exp-(\nabla^{(b,+\infty)}+\sqrt{u}D^{(b,+\infty)})^2]\}^{(1)}du. \tag{1.47}$$

Theorem 1.8. *Over U^b, the following identity holds:*

$$\delta_0^b=-(\partial^B-\bar{\partial}^B)[\operatorname{Log}\tau^2(\bar{\partial}^{(b,+\infty)})]. \tag{1.48}$$

Proof. Clearly

$$-\operatorname{Log} \tau^2(\bar{\partial}^{(b,+\infty)}) = \theta^{b\prime}(0).$$

Also

$$\theta^b(s) = \frac{-1}{\Gamma(s)} \int_0^{+\infty} u^{s-1} \operatorname{Tr}_s[N_V \exp -(\nabla^{(b,+\infty)} + \sqrt{u} D^{(b,+\infty)})^2]^{(0)} du. \tag{1.49}$$

Then, by proceeding formally as in the proof of [BGS 1, Theorem 1.9], but using instead the C^∞ kernels of the relevant operators as in [B 1, Sect. 2], we find that:

$$\begin{aligned}
&d^B \operatorname{Tr}_s(N_V \exp -(\nabla^{(b,+\infty)} + \sqrt{u} D^{(b,+\infty)})^2) \\
&= \operatorname{Tr}_s([\nabla^{(b,+\infty)} + \sqrt{u} D^{(b,+\infty)}, N_V \exp -(\nabla^{(b,+\infty)} + \sqrt{u} D^{(b,+\infty)})^2]) \\
&= \sqrt{u} \operatorname{Tr}_s([D^{(b,+\infty)}, N_V] \exp -(\nabla^{(b,+\infty)} + \sqrt{u} D^{(b,+\infty)})^2) \\
&= \sqrt{u} \operatorname{Tr}_s((-\bar{\partial}^{(b,+\infty)} + \bar{\partial}^{(b,+\infty)*}) \exp -(\nabla^{(b,+\infty)} + \sqrt{u} D^{(b,+\infty)})^2).
\end{aligned} \tag{1.50}$$

Identifying the term of degree 1 in (1.50), we get

$$\begin{aligned}
d^B \operatorname{Tr}_s(N_V \exp -u(D^{(b,+\infty)})^2) &= \sqrt{u} \operatorname{Tr}_s((-\bar{\partial}^{(b,+\infty)} + \bar{\partial}^{(b,+\infty)*}) \\
&\times \exp -(\sqrt{u} \nabla^{(b,+\infty)} D^{(b,+\infty)} + u(D^{(b,+\infty)})^2))^{(1)}.
\end{aligned} \tag{1.51}$$

Since $\tilde{\nabla}'' \bar{\partial} = 0$, $\tilde{\nabla}' \bar{\partial}^* = 0$, we find that

$$\nabla^{(b,+\infty)\prime\prime} \bar{\partial}^{(b,+\infty)} = 0, \qquad \nabla^{(b,+\infty)\prime} \bar{\partial}^{(b,+\infty)*} = 0. \tag{1.52}$$

We can now use the degree counting argument of the proof of [BGS 1, Theorem 1.9] to obtain

$$\begin{aligned}
&\partial^B \operatorname{Tr}_s[N_V \exp -u(D^{(b,+\infty)})^2] \\
&\quad = \sqrt{u} [\operatorname{Tr}_s(\bar{\partial}^{(b,+\infty)*} \exp -(\nabla^{(b,+\infty)} + \sqrt{u} D^{(b,+\infty)})^2)]^{(1)}, \\
&\bar{\partial}^B \operatorname{Tr}_s[N_V \exp -u(D^{(b,+\infty)})^2] \\
&\quad = -\sqrt{u} [\operatorname{Tr}_s(\bar{\partial}^{(b,+\infty)} \exp -(\nabla^{(b,+\infty)} + \sqrt{u} D^{(b,+\infty)})^2)]^{(1)},
\end{aligned} \tag{1.53}$$

and so

$$\begin{aligned}
&(\partial^B - \bar{\partial}^B) \operatorname{Tr}_s[N_V \exp -u(D^{(b,+\infty)})^2] \\
&\quad = \sqrt{u} \operatorname{Tr}_s[D^{(b,+\infty)} \exp -(\nabla^{(b,+\infty)} + \sqrt{u} D^{(b,+\infty)})^2]^{(1)}.
\end{aligned} \tag{1.54}$$

Using (1.49), (1.54), we find easily that

$$(\partial^B - \bar{\partial}^B) \theta^b(s) = \frac{-1}{\Gamma(s)} \int_0^{+\infty} \frac{u^s}{\sqrt{u}} \operatorname{Tr}_s[D^{(b,+\infty)} \exp -(\nabla^{(b,+\infty)} + \sqrt{u} D^{(b,+\infty)})^2]^{(1)} du. \tag{1.55}$$

Now by (1.42), (1.46), we know that as $u \downarrow\downarrow 0$,

$$\frac{1}{\sqrt{u}} \operatorname{Tr}_s[D^{(b,+\infty)} \exp -(\nabla^{(b,+\infty)} + \sqrt{u} D^{(b,+\infty)})^2]^{(1)} = O(1).$$

We thus find that

$$(\partial^B-\bar{\partial}^B)\theta^{b\prime}(0)=-\int_0^{+\infty}\frac{1}{\sqrt{u}}[\mathrm{Tr}_s(D^{(b,+\infty)}\exp-(\nabla^{(b,+\infty)}+\sqrt{u}D^{(b,+\infty)})^2)]^{(1)}du. \tag{1.56}$$

and by (1.47), we get:

$$(\partial^B-\bar{\partial}^B)\theta^{b\prime}(0)=\delta_0^b. \tag{1.57}$$

Equation (1.48) follows. □

We now calculate the curvature of the connection ${}^1\nabla$ on λ.

Theorem 1.9. *The curvature* $({}^1\nabla)^2$ *of* ${}^1\nabla$ *is given by*

$$({}^1\nabla)^2=2i\pi\left[\int_Z Td\left(\frac{-R^Z}{2i\pi}\right)\mathrm{Tr}\left[\exp\left(\frac{-L^\xi}{2i\pi}\right)\right]\right]^{(2)}. \tag{1.58}$$

Proof. The curvature of the connection ${}^0\nabla^b$ on (λ^b, U^b) is clearly given by

$$-\mathrm{Tr}_s^{K^b}((\nabla^b)^2). \tag{1.59}$$

Therefore, the curvature of ${}^1\nabla^b$ on (λ^b, U^b) is given by

$$-\mathrm{Tr}_s^{K^b}((\nabla^b)^2)-\bar{\partial}^B\partial^B\theta^{b\prime}(0).$$

Since by Theorem 1.8, $(\partial^B-\bar{\partial}^B)\theta^{b\prime}(0)=\delta_0^b$, we find that

$$\bar{\partial}^B\partial^B\theta^{b\prime}(0)=\tfrac{1}{2}d^B\delta_0^b. \tag{1.60}$$

The curvature of ${}^1\nabla^b$ on (λ^b, U^b) is thus given by

$$-\mathrm{Tr}_s^{K^b}((\nabla^b)^2)-\tfrac{1}{2}d^B\delta_0^b. \tag{1.61}$$

By Bismut and Freed [BF 1, Theorem 1.18] (especially [BF 1, Eq. (1.77)]) and by [BF 2, Theorem 1.21], we find that (1.61) is exactly the right-hand side of (1.58). The theorem is proved. □

Remark 1.10. Assume that the family D has index 0. Let U^0 be the open set in B,

$$U^0=\{y\in B; D_y \text{ is invertible}\}.$$

If $y\in B$, for $a>0$ small enough so that $[0,a]\cap\mathrm{Sp}(D_y^2)=\phi$, $\lambda^a\simeq C$. The sections $1\in\lambda^a$ patch together into a non-zero holomorphic section of λ over U^0 which we note $T(\bar{\partial})$.

If $y\in U^0$, set

$$\theta_y^0(s)=-\mathrm{Tr}_s[N_V[D^2]^{-s}],\qquad \tau(\bar{\partial})=\exp\{-\tfrac{1}{2}\theta^{0\prime}(0)\}. \tag{1.62}$$

$\tau(\bar{\partial})$ is exactly the Ray-Singer analytic torsion [RS] of the complex $(E,\bar{\partial})$. Clearly

$$\|T(\bar{\partial})\|=\tau(\bar{\partial}), \tag{1.63}$$

and so on U^0

$$\bar{\partial}^B\partial^B\,\mathrm{Log}\|T(\bar{\partial})\|^2=2i\pi\left[\int_Z Td\left(\frac{-R^Z}{2i\pi}\right)\mathrm{Tr}_s\left[\exp\left(\frac{-L^\xi}{2i\pi}\right)\right]\right]^{(2)}. \tag{1.64}$$

f) Identification of the Metrics and Connections on λ and λ'

Recall that in Bismut and Freed [BF 2] a metric and a unitary connection were constructed on λ', whose curvature is exactly given by (1.58). This fact, together with the proof of Theorem 1.9, suggests that λ and λ' are in fact the same bundles. We now explicitly construct an isometry from λ into λ'.

We first briefly recall the construction of [BF 1, 2].

Definition 1.11. For $\mathrm{Re}(s)>\ell$, $b>0$, $y\in U$, set

$$\zeta_y^b(s)=\tfrac{1}{2}\mathrm{Tr}[(D_y^2)^{-s}Q^b], \tag{1.65}$$

or equivalently

$$\zeta_y^b(s)=\frac{1}{2\Gamma(s)}\int_0^{+\infty} u^{s-1}\,\mathrm{Tr}[\exp(-uD^2)Q^b]du.$$

ζ_y^b extends into a meromorphic function on C which is holomorphic at $s=0$. Set

$$\tau'(D_+^{(b,+\infty)})=\exp\{-\tfrac{1}{2}\zeta^{b\prime}(0)\}. \tag{1.66}$$

Definition 1.12. $\|\ \|'^b$ denotes the metric on (λ'^b, U^b) given by

$$\|\ \|'^b=|\ |'^b\tau'(D_+^{(b,+\infty)}).$$

Let ${}^1\nabla'^b$ be the connection on (λ'^b, U^b):

$${}^1\nabla'^b={}^0\nabla^b+\tfrac{1}{2}d^B\,\mathrm{Log}\,\tau'^2(D_+^{(b,+\infty)})-\tfrac{1}{2}\delta_0^b. \tag{1.67}$$

From [BF 2, Proposition 1.11 and Theorems 1.14 and 1.21] we find:

Theorem 1.13. *The metrics $\|\ \|'^b$ and the connections ${}^1\nabla'^b$ on (λ'^b, U^b) patch together into a smooth metric $\|\ \|'$ on λ' and a smooth connection ${}^1\nabla'$ on λ' which is unitary with respect to $\|\ \|'$. The curvature of ${}^1\nabla'$ is exactly*

$$2i\pi\left[\int_Z Td\left(\frac{-R^Z}{2i\pi}\right)\mathrm{Tr}\left[\exp\frac{-L^\xi}{2i\pi}\right]\right]^{(2)}. \tag{1.68}$$

We now identify λ and λ'.

Definition 1.14. Over U^b, j^b denotes the linear isomorphism from λ^b into λ'^b given by

$$s\in\lambda^b\to j^b(s)=\frac{\tau(\bar\partial^{(b,+\infty)})}{\tau'(D_+^{(b,+\infty)})}\,i^b(s)\in\lambda'^b. \tag{1.69}$$

We now have:

Theorem 1.15. *The C^∞ isomorphisms $j^b:\lambda^b\to\lambda'^b$ patch together into a C^∞ isomorphism $j:\lambda\to\lambda'$. j maps the metric $\|\ \|$ on the metric $\|\ \|'$ and the connection ${}^1\nabla$ on the connection ${}^1\nabla'$.*

Proof. To show that the j^b patch together, we must prove that for $0<b<c$, on $U^b\cap U^c$, if $s\in\lambda^b$, then

$$j^b(s)\otimes\det D_+^{(b,c)}=j^c(s\otimes T(\bar\partial^{(b,c)})). \tag{1.70}$$

Clearly

$$i^c(s\otimes T(\bar\partial^{(b,c)}))=i^b(s)\otimes i^{(b,c)}(T(\bar\partial^{(b,c)}))\,. \tag{1.71}$$

To simplify notation, we write $|\ |'$ instead of $|\ |'^{(b,c)}$.

By [BGS 1, Proposition 1.5], we know that

$$i^{(b,c)}(T(\bar\partial^{(b,c)}))=\frac{\tau(\bar\partial^{(b,c)})}{|\det D_+^{(b,c)}|'}\det D_+^{(b,c)}\,. \tag{1.72}$$

Using (1.71), (1.72), we find that

$$j^c(s\otimes T(\bar\partial^{(b,c)}))=\frac{\tau(\bar\partial^{(b,c)})}{|\det D_+^{(b,c)}|'}\,\frac{\tau(\bar\partial^{(c,+\infty)})}{\tau'(D_+^{(c,+\infty)})}\,i^b(s)\otimes\det D_+^{(b,c)}\,.$$

By (1.37) and also by [BF 1, Eq. (1.39)], we know that

$$\begin{aligned}\tau(\bar\partial^{(b,+\infty)})&=\tau(\bar\partial^{(b,c)})\tau(\bar\partial^{(c,+\infty)})\,,\\ \tau'(D_+^{(b,+\infty)})&=|\det D_+^{(b,c)}|'\tau'(D_+^{(c,+\infty)})\,.\end{aligned} \tag{1.73}$$

Using (1.71)–(1.73), we find that (1.70) holds. Recall that i^b is an isometry from $(\lambda^b,|\ |^b)$ into $(\lambda'^b,|\ |'^b)$. Also if $s\in\lambda^b$,

$$\|j^b(s)\|'^b=\frac{\tau(\bar\partial^{(b,+\infty)})}{\tau'(D_+^{(b,+\infty)})}\|i^b(s)\|'^b\,. \tag{1.74}$$

Therefore j maps $\|\ \|$ into $\|\ \|'$. Also i^b maps the connection ${}^0\nabla^b$ on λ^b on the connection ${}^0\nabla^b$ on λ'^b. If s is a C^∞ section of λ^b on U^b, we find that

$${}^0\nabla^b j^b(s)=[d^B(\operatorname{Log}\tau(\bar\partial^{(b,+\infty)})-\operatorname{Log}\tau'(D_+^{(b,+\infty)}))]j^b(s)+j^b({}^0\nabla^b s)\,, \tag{1.75}$$

and so

$${}^1\nabla'^b j^b(s)=j^b({}^1\nabla^b s)+\tfrac12[(\bar\partial^B-\partial^B)\operatorname{Log}\tau^2(\bar\partial^{(b,+\infty)})-\delta_0^b]j_b(s)\,. \tag{1.76}$$

By Theorem 1.8, we find that:

$${}^1\nabla'^b j^b(s)=j^b({}^1\nabla^b s)\,. \tag{1.77}$$

The theorem is proved. □

g) The Holomorphic Structure Does not Depend on the Kähler Metric in the Fibers

We now again assume that B is a complex manifold. Assume that (π,g^Z,T^HM) and (π,T_1^HM,g_1^Z) are two Kähler fibration structures on TZ with associated closed $(1,1)$ forms $\omega=\omega^Z+\omega^H$ and $\omega_1=\omega_1^Z+\omega_1^H$. We again consider one single holomorphic Hermitian vector bundle ξ over M.

By Theorem 1.3, we can construct two holomorphic Hermitian vector bundles λ and λ_1 over B, associated with (TZ,T^HM,g^Z) and (TZ,T_1^HM,g_1^Z). For $y\in B$, let $H_y^0(E),\dots,H_y^\ell(E)$ be the cohomology groups associated with the restriction of the vector bundle ξ to the fiber Z_y. For every $y\in B$, we have the canonical identifications

$$\lambda_y \simeq (\det H^0_y(E))^{-1} \otimes \det H^1_y(E) \otimes \dots ,$$
$$\lambda_{1,y} \simeq (\det H^0_y(E))^{-1} \otimes \det H^1_y(E) \otimes \dots . \tag{1.78}$$

Therefore for every $y \in B$, there is a canonical isomorphism:

$$\phi_y : \lambda_y \to \lambda_{1,y} .$$

Consider the double chain complex:

$$\begin{tikzcd} & 0 & 0 & \\ 0 \arrow[r] & E^\ell \arrow[u] \arrow[r, "i"] & E^\ell \arrow[u] \arrow[r] & 0 \\ & \vdots \arrow[u, "\bar\partial"'] & \vdots \arrow[u, "\bar\partial"'] & \\ 0 \arrow[r] & E^0 \arrow[u, "\bar\partial"'] \arrow[r, "i"'] & E^0 \arrow[u, "\bar\partial"'] \arrow[r] & 0 \\ & 0 \arrow[u] & 0 \arrow[u] & \end{tikzcd} . \tag{1.79}$$

Here i is the identity mapping. In the first column, $E^0, \dots, E^\ell$ are endowed with the metric corresponding to g^Z, in the second column they are endowed with the metric corresponding to g^Z_1. The lines of (1.79) are obviously acyclic, and so ϕ is the Knudsen-Mumford section of $\lambda^{-1} \otimes \lambda_1$ [KM].

Theorem 1.16. *The map ϕ is a smooth holomorphic isomorphism from λ into λ_1.*

Proof. At this stage, it is not even clear that ϕ is smooth. The statement to be proven is clearly local on B. Let U be an open set in B, which is relatively compact. There is $\varepsilon > 0$ such that for any $y \in U$, $c \in]-\varepsilon, 1+\varepsilon[$, if $g^Z_{c,y}$ is given by

$$g^Z_{y,c} = (1-c) g^Z_y + c g^Z_{1,y} , \tag{1.80}$$

then $\{g^Z_{y,c}\}$ is a smooth family of Kähler metrics.

For $c \in]-\varepsilon, 1+\varepsilon[$, set

$$\omega_c = (1-c)\omega + c\omega_1 . \tag{1.81}$$

The restriction ω^Z_c of ω_c to TZ induces the Kähler metric g^Z_c. Let $T^H_c M$ be the orthogonal of TZ with respect to ω_c. By [BGS 2, Theorem 1.5], $(\pi, g^Z_c, T^H_c M)$ is a Kähler fibration, and ω^Z_c is an associated $(1,1)$ form. For every $c \in]-\varepsilon, 1+\varepsilon[$ we can construct the holomorphic Hermitian line bundle λ_c over U associated with $(\pi, T^H_c M, g^Z_c)$. Of course $\lambda_0 = \lambda$.

For $y \in U$, $c \in]-\varepsilon, 1+\varepsilon[$, let $\bar\partial^*_{y,c}$ be the adjoint of $\bar\partial_y$ with respect to g^Z_c. Then $D_{y,c} = \bar\partial_y + \bar\partial^*_{y,c}$ is a smooth family of first order elliptic operators. By proceeding as in [BF 1] and in Sect. 1b), we can construct a C^∞ line bundle λ^{tot} over $U \times]-\varepsilon, 1+\varepsilon[$ which coincides with λ_c over $U \times \{c\}$. λ^{tot} restricted to $U \times \{c\}$ is endowed with a connection ${}^1\nabla^c$ which depends smoothly on c. We now define a connection ${}^1\nabla$ on λ^{tot}. If s is a smooth section of λ^{tot}, if $Y \in T_y B$, set

$$({}^1\nabla_Y s)(y,c) = {}^1\nabla_Y^c s(y,c).$$

To define ${}^1\nabla$, we now only need to define ${}^1\nabla_{\frac{\partial}{\partial c}}$.

The bundles $K_{y,c}^{b,p}$, $K_{y,c}^b$, $\lambda_{y,c}^{\mathrm{tot},b}$ ... are taken as in Sect. 1b), simply replacing everywhere y by y, c. We otherwise use the same notations as in Sect. 1b). Note that $K_{y,c}^b$ is a vector subspace of E_y (which does not depend on c), which is endowed with the Hermitian product defined in [BGS 2, Eq. (1.38)] associated with g_c^Z. Let $P_{y,c}^b$ be the orthogonal projection operator from E_y on $K_{y,c}^b$.

If s is a smooth section of $\lambda^{\mathrm{tot},b}$ over

$$V^b = \{(y,c) \in U \times]-\varepsilon, 1+\varepsilon[, b \notin \operatorname{spec} D_{y,c}^2\},$$

set

$${}^1\nabla^b_{\frac{\partial}{\partial c}} s = P_{y,c}^b \frac{\partial}{\partial c} s. \tag{1.82}$$

Similarly for $0<b<b'$, we define ${}^1\nabla^{(b,b')}_{\frac{\partial}{\partial c}}$ acting on $\lambda^{\mathrm{tot}(b,b')}$ by simply replacing $P_{y,c}^b$ in (1.82) by the orthogonal projection $P_{y,c}^{(b,b')}$ from E_y on $K_{y,c}^{(b,b')}$. We claim that the operators ${}^1\nabla^b_{\frac{\partial}{\partial c}}$ patch together into a smooth differential operator of order 1 on λ^{tot}.

If $0<b<b'$, if s is a smooth section of λ^b over $V^b \cap V^{b'}$, we have

$${}^1\nabla^{b'}_{\frac{\partial}{\partial c}}(s \otimes T(\bar\partial^{(b,b')})) = {}^1\nabla^b_{\frac{\partial}{\partial c}} s \otimes T(\bar\partial^{(b,b')}) + s \otimes {}^1\nabla^{(b,b')}_{\frac{\partial}{\partial c}} T(\bar\partial^{(b,b')}). \tag{1.83}$$

Fix $c \in]-\varepsilon, 1+\varepsilon[$. For $|c'-c|$ small enough, $P_{y,c'}^{(b,b')}$ is one to one from $K_{y,c'}^{(b,b')}$ into $K_{y,c'}^{(b,b')}$. Also $P_{y,c'}^{(b,b')}$ commutes with $\bar\partial$. It is thus obvious that

$$T(\bar\partial_{y,c'}^{(b,b')}) = P_{y,c'}^{(b,b')} T(\bar\partial_{y,c}^{(b,b')}). \tag{1.84}$$

Since $(P_{y,c}^{(b,b')})^2 = P_{y,c}^{(b,b')}$, $\frac{\partial}{\partial c} P_{y,c}^{(b,b')}$ maps $K_{y,c}^{(b,b')}$ in its orthogonal. Using (1.83), we find that

$${}^1\nabla^{(b,b')}_{\frac{\partial}{\partial c}} T(\bar\partial^{(b,b')}) = 0.$$

From (1.83), (1.84), we get

$${}^1\nabla^{b'}_{\frac{\partial}{\partial c}}(s \otimes T(\bar\partial^{(b,b')})) = {}^1\nabla^b_{\frac{\partial}{\partial c}} s \otimes T(\bar\partial^{(b,b')}). \tag{1.85}$$

Equation (1.85) exactly means that the operators ${}^1\nabla^b_{\frac{\partial}{\partial c}}$ patch together into an operator ${}^1\nabla_{\frac{\partial}{\partial c}}$ on λ. So λ^{tot} is now endowed with a (non-unitary) connection ${}^1\nabla$.

Let $K_{y,c}^0$ be the eigenspace corresponding to the eigenvalue 0 of $D_{y,c}^2$. For every $y \in B$, K^0 is a vector bundle over $\{y\} \times]-\varepsilon, 1+\varepsilon[$. Let $\lambda_{y,c}^{\mathrm{tot},0}$ denote the corresponding determinant fiber. Clearly $\lambda_{y,c}^{\mathrm{tot},0}$ is canonically isomorphic to $\lambda_{y,c}^{\mathrm{tot}}$. Also for every $y\in U$, $\lambda_{y,\cdot}^{\mathrm{tot},0}$ is a smooth vector bundle over $]-\varepsilon,1+\varepsilon[$.

Let $\phi_{y,c}$ be the canonical isomorphism from $\lambda_{y,0}^{\mathrm{tot}}$ into $\lambda_{y,c}^{\mathrm{tot}}$. Clearly $\phi_{y,1}=\phi_y$.

Let $P_{y,c}^0$ be the orthogonal projection operator from E_y on $K_{y,c}^0$. $P_{y,c}^0$ is one to one from $K_{y,0}^0$ in $K_{y,c}^0$. Also if $s\in\lambda_{y,0}^{\mathrm{tot},0}$ and if $c,c'\in]-\varepsilon,1+\varepsilon[$,

$$\phi_{y,c'}s=P_{y,c'}^0P_{y,c}^0s\,. \tag{1.86}$$

Since $\frac{\partial}{\partial c}P_{y,0}^0$ maps $K_{y,c}^0$ in its orthogonal, using (1.86), we find that:

$${}^1\nabla\phi_{y,c}s=0\,. \tag{1.87}$$

Equivalently, if $s\in\lambda_{y,c}^{\mathrm{tot}}$, $\phi_{y,c}\,s$ is the parallel transport of s along $c\to(y,c)$ for the connection ${}^1\nabla$. Therefore, $\phi_{y,c}$ depends smoothly on (y,c). In particular $\phi_y=\phi_{y,1}$ is a smooth isomorphism from λ_y into $\lambda_{y,1}$.

We now prove that $\phi_{y,c}$ is a holomorphic section of $(\lambda_{y,0})^{-1}\otimes\lambda_{y,c}$. If $({}^1\nabla)^2$ is the curvature of ${}^1\nabla$, since $\phi_{y,c}(s)$ is the parallel transport of $s\in\lambda_{y,0}$ along $c\to(y,c)$, it is equivalent to prove that if $Y\in T^{(0,1)}B$, then

$$({}^1\nabla)^2\left(\frac{\partial}{\partial c},Y\right)=0\,. \tag{1.88}$$

Let $\tilde{\nabla}^c$ be the connection on E over B associated with the Kähler fibration $(\pi,g_{c'}^Z T_c^H M)$. By [BGS 2, Theorem 1.14], we know that

$$[\tilde{\nabla}^{c''},\bar\partial]=0\,,\qquad \frac{\partial}{\partial c}\bar\partial=0\,. \tag{1.89}$$

Using (1.89) and by proceeding as in the proof of Theorem 1.3, i.e. by eliminating the covariant derivatives of the projectors P^b, we find that for $b>0$ not in the spectrum of $D_{y,c}^2$:

$$({}^1\nabla)^2\left(\frac{\partial}{\partial c},Y\right)=\mathrm{Tr}_s\left[P^b\left(\frac{\partial}{\partial c}\tilde{\nabla}_Y^c\right)P^b\right]. \tag{1.90}$$

By (1.89), we have

$$\left[\frac{\partial}{\partial c}\tilde{\nabla}_Y^c,\bar\partial\right]=0\,. \tag{1.91}$$

By [BGS 1, Proposition 1.3] we find that

$$({}^1\nabla)^2\left(\frac{\partial}{\partial c},Y\right)=\mathrm{Tr}_s\left[P^0\left(\frac{\partial}{\partial c}\tilde{\nabla}_Y^c\right)P^0\right]. \tag{1.92}$$

We now calculate $\frac{\partial}{\partial c}\tilde{\nabla}_Y^c$ acting on E_y. Let ∇^c be the holomorphic Hermitian connection on $T^{(1,0)}Z$ associated with the metric g_c^Z. By [BGS 2, Theorem 1.7], the torsion tensor T_c associated with the Kähler fibration $(\pi,g_c^Z,T_c^H M)$ is of complex type $(1,1)$. We can assume that Y is a smooth section of $T^{(0,1)}B$. Let Y_c^H be the lift of Y in $T_c^{H(0,1)}M$. If Y' is a smooth section of $T^{(0,1)}Z$, since $T_c(Y_c^H,Y')=0$, we find that

$$\nabla_{Y_c^H}^c Y'-[Y_c^H,Y']=0\,. \tag{1.93}$$

Set

$$A = \frac{\partial}{\partial c} Y_c^H. \tag{1.94}$$

Clearly $A \in T^{(0,1)}Z$. From (1.93), we find that

$$\frac{\partial}{\partial c}(\nabla^c_{Y_c^H}) Y' = [A, Y']. \tag{1.95}$$

Let L^Z_A be the Lie derivative operator on Z associated with A, and d^Z be the exterior differentiation operator on Z. Clearly

$$L^Z_A = d^Z i_A + i_A d^Z. \tag{1.96}$$

Using (1.95), we find that if α is a smooth section of $\Lambda^p(T^{*(0,1)}Z)$, $\frac{\partial}{\partial c}(\nabla^c_{Y_c^H}\alpha)$ is the component of complex type $(0,p)$ of $L^Z_A\alpha$. Equivalently

$$\frac{\partial}{\partial c}(\nabla^c_{Y_c^H}\alpha) = (\bar\partial^Z i_A + i_A \bar\partial^Z)\alpha. \tag{1.97}$$

More generally, if h is a smooth section of $\Lambda(T^{*(0,1)}Z) \otimes \xi$, we deduce from (1.97) that

$$\frac{\partial}{\partial c}(\nabla^c_{Y_c^H} h) = (\bar\partial^Z i_A + i_A \bar\partial^Z) h. \tag{1.98}$$

Since $\tilde\nabla^c_Y = \nabla^c_{Y_c^H}$, we find from (1.98) that:

$$\frac{\partial}{\partial c} \tilde\nabla^c_Y = \bar\partial^Z i_A + i_A \bar\partial^Z. \tag{1.99}$$

Take $\eta \in K^0_{y,c}$. Clearly

$$\bar\partial\eta = \bar\partial^*_{y,c}\eta = 0. \tag{1.100}$$

Therefore if $\eta, \eta' \in K^0_{y,c}$,

$$\sqrt{2}\left\langle \eta, \frac{\partial}{\partial c} \tilde\nabla_Y \eta' \right\rangle = \langle \eta, (i_A\bar\partial + \bar\partial i_A)\eta'\rangle = \langle \eta, i_A \bar\partial \eta'\rangle + \langle \bar\partial^*\eta, i_A\eta'\rangle = 0. \tag{1.101}$$

Using (1.92), (1.101), we find that (1.88) has been proved.

Since $\phi_y = \phi_{y,1}$, we have proved that ϕ is holomorphic from λ into λ_1. □

Remark 1.17. Theorem 1.16 exactly says that the holomorphic structure on the line bundle λ does not depend on the specific Kähler fibration structure which is considered.

h) Dependence of the Metric of λ on (g^Z, h^ξ)

We now calculate how the metric on λ depends upon the metrics (g^Z, h^ξ). For this we need only to consider a single fiber, so we assume that $B = \{y_0\}$ is reduced to a single point. Let $c \in R \to g_c^Z$ denote a smooth family of Kähler metrics on $T^{(1,0)}Z$ and $c \in R \to h_c^\xi$ a smooth family of Hermitian metrics on ξ. Let R_c^Z and L_c^ξ be the curvature of the Hermitian holomorphic connections on $(T^{(1,0)}Z, g_c^Z)$ and on (ξ, h_c^ξ). Let K_c denote the scalar curvature on Z for the metric g_c^Z.

Let λ^{tot} be the Hermitian line bundle over $\mathbb{R}$ constructed in the proof of Theorem 1.16. For every $c \in \mathbb{R}$, its fiber λ_c^{tot} is the determinant line λ_c associated to (g_c^Z, h_c^ξ). There is a canonical isomorphism of line bundles $\phi_c : \lambda_0 \to \lambda_c$, i.e. a section of $\lambda^{-1} \otimes \lambda_c$. Let $\|\phi_c\|$ be the norm of this non-zero section. We shall study how $\|\phi_c\|$ depends upon c.

For every $c \in \mathbb{R}$, let $*c$ be the complex star operator associated to the metric g_c^Z. It maps (p,q) forms on Z into $(\ell - p, \ell - q)$ forms, for $p, q \geqq 0$. The operator $(*c)^{-1} \dfrac{\partial(*c)}{\partial c}$ maps (p,q) forms into (p,q) forms. Therefore it induces an endomorphism of E^p and E. The following result is closely related to a result of Ray-Singer [RS, Theorem 2.1].

Theorem 1.18. *As $u \downarrow\downarrow 0$, for every $k \in \mathbb{N}$, there is an asymptotic expansion*

$$-\operatorname{Tr}_s\left[\left((*_c^{-1})\frac{\partial(*_c)}{\partial c} + (h_c^\xi)^{-1}\frac{\partial(h_c^\xi)}{\partial c}\right)\exp(-uD_c^2)\right] = \sum_{j=-\frac{n}{2}}^{k} M_{j,c}u^j + o(u^k). \tag{1.102}$$

Furthermore

$$\frac{\partial}{\partial c}\operatorname{Log}\|\phi_c\|^2 = M_{0,c}. \tag{1.103}$$

Proof. The existence of the asymptotic expansion above follows from Greiner [Gr, Theorem 1.6.1].

To simplify our computations, we shall now assume that h^ξ does not vary with c. Remember that $[A,B]$ denotes the supercommutator of A and B.

Let $\bar\partial_c^*$ be the adjoint of $\bar\partial$ for the metric g_c^Z, so that $D_c = \bar\partial + \bar\partial_c^*$. For $b \in \mathbb{R}$ we have (with $D = D_c$)

$$\begin{aligned}\frac{\partial}{\partial c}\operatorname{Tr}_s(\exp(-uD^2 + bN_V)) &= -u\operatorname{Tr}_s\left(\left[D, \frac{\partial D}{\partial c}\right]\exp(-uD^2 + bN_V)\right)\\ &= -u\operatorname{Tr}_s\left(\frac{\partial D}{\partial c}[D, \exp(-uD^2 + bN_V)]\right).\end{aligned} \tag{1.104}$$

Furthermore

$$\frac{\partial}{\partial b}\operatorname{Tr}_s(\exp(-uD^2 + bN_V))|_{b=0} = \operatorname{Tr}_s(N_V \exp(-uD^2)).$$

Moreover

$$\left[D, \frac{\partial}{\partial b}\exp(-uD^2 + bN_V)\right]\Bigg|_{b=0} = \frac{\partial}{\partial b}(\exp(-uD^2 + b[D, N_V]))|_{b=0}. \tag{1.105}$$

Using (1.104), (1.105), we get

$$\frac{\partial}{\partial c}\operatorname{Tr}_s(N_V \exp(-uD^2)) = -u\operatorname{Tr}_s\left(\frac{\partial D}{\partial c}\frac{\partial}{\partial b}\exp(-uD^2 + b[D, N_V])\right)\Bigg|_{b=0} \tag{1.106}$$

By [BGS 2, Theorem 2.6], we know that

$$[D_c, N_V] = -\bar{\partial} + \bar{\partial}_c^*. \tag{1.107}$$

Clearly

$$\bar{\partial}_{c'}^* = *_{c'}^{-1} * c \bar{\partial}_c^* *_c^{-1} * c'. \tag{1.108}$$

Set

$$Q_c = -(*_c^{-1}) \frac{\partial}{\partial c} *c. \tag{1.109}$$

From (1.107), (1.108), we find that

$$\frac{\partial}{\partial c} \bar{\partial}_c^* = -[\bar{\partial}_c^*, Q_c]. \tag{1.110}$$

As before, we will omit the subscript c. Using (1.106), (1.107), (1.110), we get

$$\begin{aligned} &\operatorname{Tr}_s\left(\frac{\partial D}{\partial c} \frac{\partial}{\partial b} \exp(-uD^2 + b[D, N_V])\right)\bigg|_{b=0} \\ &= \operatorname{Tr}_s\left(Q\left[\bar{\partial}^*, \frac{\partial}{\partial b} \exp(-uD^2 + b(-\bar{\partial} + \bar{\partial}^*))\right]\right)\bigg|_{b=0}. \end{aligned} \tag{1.111}$$

By proceeding as in (1.105), since $[\bar{\partial}^*, D^2] = 0$, we get

$$\begin{aligned} &\left[\bar{\partial}^*, \frac{\partial}{\partial b} \exp(-uD^2 + b(-\bar{\partial} + \bar{\partial}^*))\right]\bigg|_{b=0} \\ &= \frac{\partial}{\partial b} \exp(-uD^2 - bD^2)|_{b=0} = \frac{\partial}{\partial u} \exp(-uD^2). \end{aligned} \tag{1.112}$$

From (1.106), (1.111), (1.112), we get

$$\frac{\partial}{\partial c} \operatorname{Tr}_s(N_V \exp(-uD^2)) = -u \frac{\partial}{\partial u} \operatorname{Tr}_s(Q \exp(-uD^2)). \tag{1.113}$$

We can find $b > 0$ such that D_0^2 has no eigenvalue in $]0, b]$. The dimension of $\operatorname{Ker} D_c^2$ does not depend on c. Therefore for c close enough to 0, D_c^2 has no eigenvalue in $]0, b]$. In the sequel, c will be chosen in this way.

For $0 \leq p \leq \ell$, set $K_c^{0,p} = \operatorname{Ker} D_c^2 \cap E^p$, $K_c^0 = \bigoplus_0^\ell K_c^{0,p}$. Let P_c^0 be the orthogonal projection operator from E on K_c^0, for the Hermitian product on E associated with (g_c^Z, h^ξ).

For $\operatorname{Re}(s)$ large enough we have

$$\theta_c^b(s) = \frac{-1}{\Gamma(s)} \int_0^\infty u^{s-1} [\operatorname{Tr}_s(N_V \exp(-uD_c^2)) - \operatorname{Tr}_s^{K_c^0}(N_V)] du. \tag{1.114}$$

Using (1.113), and the fact that $\operatorname{Tr}_s^{K_c^0}(N_V)$ does not depend on c, we find for $\operatorname{Re}(s)$ large enough

$$\frac{\partial}{\partial c} \theta_c^b(s) = \frac{1}{\Gamma(s)} \int_0^\infty u^s \frac{\partial}{\partial u} \operatorname{Tr}_s(Q_c \exp(-uD_c^2)) du. \tag{1.115}$$

When $u\uparrow+\infty$, $\mathrm{Tr}_s(Q_c(\exp(-uD_c^2)-P_c^0))$ decays exponentially. Integrating by parts in (1.115), we obtain, for $\mathrm{Re}(s)$ large enough:

$$\frac{\partial}{\partial c}\theta_c^b(s)=\frac{-s}{\Gamma(s)}\int_0^\infty u^{s-1}\,\mathrm{Tr}_s(Q_c(\exp(-uD_c^2)-P_c^0))du\,. \tag{1.116}$$

Using (1.102), we find

$$\frac{\partial}{\partial c}(\theta_c^{b\prime}(0))=-M_{0,c}+\mathrm{Tr}_s(Q_cP_c^0)\,. \tag{1.117}$$

The line bundle λ_c is canonically identified to

$$\lambda_c^0=(\det K_c^{0,0})^{-1}\otimes(\det K_c^{0,1})\otimes\ldots.$$

Under this identification, ϕ_c sends $s\in\lambda_0^0$ to $\phi_c(s)=P_c^0(s)$.

Let $\langle\,,\,\rangle_c$ be the Hermitian product on E attached to g_c^Z and h^ξ. If η and η' are forms in the kernel of D_0^2, we have, by definition:

$$\langle P_c^0\eta,P_c^0\eta'\rangle_c=\int_Z\langle P_c^0\eta\wedge{}^*cP_c^0\eta'\rangle_{h^\xi}\,. \tag{1.118}$$

The operator $\frac{\partial(P_c^0)}{\partial c}$ sends K_c^0 to its orthogonal complement for the metric g_c^Z. Therefore, from (1.118), we get:

$$\frac{\partial}{\partial c}\langle P_c^0\eta,P_c^0\eta'\rangle|_{c=0}=\left(\int_Z\left\langle P_c^0\eta\wedge\frac{\partial}{\partial c}({}^*c)P_c^0\eta'\right\rangle_{h^\xi}\right)\Bigg|_{c=0}=-\langle P_0^0\eta,Q_0P_0^0\eta'\rangle\,. \tag{1.119}$$

If $|\ |$ is the L^2 metric on λ_c^0 induced from (E,g_c^Z), we get from (1.119)

$$\left(\frac{\partial}{\partial c}\mathrm{Log}|\phi_c(s)|^2\right)\Bigg|_{c=0}=\mathrm{Tr}_s(Q_0P_0^0)\,. \tag{1.120}$$

By (1.34), we have,

$$\|\phi_c(s)\|^2=|\phi_c(s)|^2\exp(-\theta_c^{b\prime}(0))\,. \tag{1.121}$$

From (1.117), (1.120), and (1.121) we find

$$\left(\frac{\partial}{\partial c}\mathrm{Log}\,\|\phi_c\|^2\right)\Bigg|_{c=0}=M_{0,0}\,. \tag{1.122}$$

In general, if $c,c'\in\mathbb{R}$, the canonical isomorphism $\phi_{c,c'}$ from λ_c to $\lambda_{c'}$ satisfies the equality

$$\phi_{c'}=\phi_{c,c'}\phi_c\,.$$

We thus find that

$$\frac{\partial}{\partial c}\mathrm{Log}\,\|\phi_c\|^2=\frac{\partial}{\partial c'}\mathrm{Log}\,\|\phi_{c,c'}\|_{c'=c}^2\,.$$

We can then use (1.122) to get (1.103) for arbitrary c. When h^ξ also depends on c, the computations are essentially the same and are left to the reader. $\square$

Our aim is now to compute $M_{0,0}$. Set

$$Q=-\left(*^{-1}\frac{\partial *}{\partial c}\right)_{c=0}. \tag{1.123}$$

We first compute Q in terms of Clifford multiplication operators. Let ω_c^Z be the Kähler form associated with the metric g_c^Z, i.e., if $X, Y\in TZ$,

$$\omega_c^Z(X,Y)=\langle X, J^Z Y\rangle_{g_c^Z}.$$

Let $e_1,\dots,e_n$ be an orthonormal basis of TZ for the metric g_0^Z. In the sequel the Clifford multiplication is done with respect to the metric g_0^Z.
Set

$$\dot{\omega}_c=\frac{\partial}{\partial c}(\omega_c^Z). \tag{1.124}$$

We will write $\dot{\omega}$ instead of $\dot{\omega}_0$. Also to avoid possible confusions, we will temporarily adopt the notation $i=\sqrt{-1}$.

Proposition 1.19. *The following identity holds*

$$Q=\frac{\sqrt{-1}}{4}\dot{\omega}(e_i,e_j)c(e_i)c(e_j)+\tfrac{1}{4}\dot{\omega}(e_i,J^Ze_i). \tag{1.125}$$

Proof. Let $w_1,\dots,w_\ell$ be an orthonormal basis of $T^{(1,0)}Z$ for the metric g_0^Z, $\bar{w}_1,\dots,\bar{w}_\ell$ the conjugate basis in $T^{(0,1)}Z$. Denote by $\langle\,,\rangle$ the scalar product $\langle\,,\rangle_{g_0^Z}$. We get

$$Q=-\left\langle (g_0^Z)^{-1}\left(\frac{\partial g_c^Z}{\partial c}\right)_{c=0} w_i,\bar{w}_j\right\rangle i_{\bar{w}_i}\bar{w}^j\wedge=-\sqrt{-1}\dot{\omega}(w_i,\bar{w}_j)i_{\bar{w}_i}\bar{w}^j\wedge$$
$$=\frac{\sqrt{-1}}{2}\dot{\omega}(w_i,\bar{w}_j)c(\bar{w}_i)c(w_j).$$

Therefore

$$Q=\frac{\sqrt{-1}}{4}(\dot{\omega}(w_i,\bar{w}_j)c(\bar{w}_i)c(w_j)+\dot{\omega}(\bar{w}_i,w_j)c(w_i)c(\bar{w}_j))+\frac{\sqrt{-1}}{2}\dot{\omega}(\bar{w}_i,w_i). \tag{1.126}$$

Also

$$\frac{\sqrt{-1}}{2}\dot{\omega}(\bar{w}_i,w_i)=\tfrac{1}{2}\dot{\omega}(\bar{w}_i,J^Zw_i)=\tfrac{1}{4}\dot{\omega}(e_i,J^Ze_i). \tag{1.127}$$

From (1.126), (1.127) we deduce (1.125). □

Let now $D=D_0$ and let ∇ be the holomorphic Hermitian connection on $\Lambda(T^{*(0,1)}Z)\otimes\xi$ associated to the metrics (g_0^Z,h^ξ). We also introduce odd Grassmann variables da and $d\bar{a}$ which verify the same assumption as in [BGS 2, Sect. 2f)]. In particular they anticommute with $c(e_i)$.

Theorem 1.20. *The following identity holds*

$$\frac{\partial}{\partial u}(u\operatorname{Tr}_s(Q\exp(-uD^2)))$$
$$=\operatorname{Tr}_s\left(\exp\left(-uD^2-\sqrt{\frac{u}{2}}daD-\sqrt{\frac{u}{2}}d\bar{a}[D,Q]+dad\bar{a}Q\right)\right)^{dad\bar{a}}. \tag{1.128}$$

Proof. For $b\in\mathbb{R}$, we have:

$$\begin{aligned}\frac{\partial}{\partial u}&\mathrm{Tr}_s(\exp(-uD^2+ubQ))\\&=\mathrm{Tr}_s((bQ-D^2)\exp(-uD^2+ubQ))\\&=b\,\mathrm{Tr}_s(Q\exp(-uD^2+ubQ))-\tfrac{1}{2}\mathrm{Tr}_s([D,D]\exp(-uD^2+ubQ))\\&=b\,\mathrm{Tr}_s(Q\exp(-uD^2+ubQ))-\tfrac{1}{2}\mathrm{Tr}_s(D[D,\exp(-uD^2+ubQ)]).\end{aligned}\qquad(1.129)$$

By differentiating this equality at $b=0$ and proceeding as in (1.105), we get

$$\begin{aligned}\frac{\partial}{\partial u}&(u\,\mathrm{Tr}_s(Q\exp(-uD^2)))\\&=\mathrm{Tr}_s(Q\exp(-uD^2))-\tfrac{1}{2}\mathrm{Tr}_s\left(D\frac{\partial}{\partial b}\exp(-uD^2+ub[D,Q])\right)_{b=0}.\end{aligned}\qquad(1.130)$$

Clearly

$$\mathrm{Tr}_s(Q\exp(-uD^2))=\mathrm{Tr}_s(\exp(-uD^2+dad\bar{a}Q))^{dad\bar{a}},\qquad(1.131)$$

and using Dunhamel's formula, we get

$$\begin{aligned}-\tfrac{1}{2}&\mathrm{Tr}_s\left(D\frac{\partial}{\partial b}\exp(-uD^2+ub[D,Q])\right)_{b=0}\\&=\mathrm{Tr}_s\left(\exp\left(-uD^2-\sqrt{\frac{u}{2}}daD-\sqrt{\frac{u}{2}}d\bar{a}[D,Q]\right)\right)^{dad\bar{a}}.\end{aligned}$$

By (1.130), (1.131), (1.128) follows. □

We now prove a generalized Lichnerowicz formula which extends [BGS 2, Theorem 2.15]. We use the notation

$$L_c'^{\xi}=L_c^{\xi}+\tfrac{1}{2}\mathrm{Tr}(R_c^Z)I,$$

where R_c^Z is the curvature of the holomorphic connection on $(T^{(1,0)}Z,g_c^Z)$.

Theorem 1.21. *For any $u>0$, the following identity holds:*

$$\begin{aligned}-uD^2&-\sqrt{\frac{u}{2}}daD-\sqrt{\frac{u}{2}}d\bar{a}[D,Q]+dad\bar{a}Q\\&=u\left(\nabla_{e_i}-\frac{1}{2\sqrt{2u}}dac(e_i)-\frac{d\bar{a}}{2\sqrt{2u}}\sqrt{-1}\,\dot{\omega}(e_k,e_i)c(e_k)\right)^2\\&\quad+\frac{dad\bar{a}}{4}\dot{\omega}(e_j,J^Ze_j)-\sqrt{\frac{u}{2}}d\bar{a}\frac{c(e_i)}{4}\nabla_{e_i}\dot{\omega}(e_j,J^Ze_j)\\&\quad-\frac{uK}{4}-\frac{u}{2}c(e_i)c(e_j)\otimes L_0'^{\xi}(e_i,e_j).\end{aligned}\qquad(1.132)$$

Proof. Using (1.125), we get

$$[D,Q]=\sqrt{-1}[D,\tfrac{1}{4}\dot{\omega}(e_i,e_j)c(e_i)c(e_j)]+\tfrac{1}{4}c(e_i)\nabla_{e_i}\dot{\omega}(e_j,J^Ze_j).\qquad(1.133)$$

Recall that the metrics g_c^Z are Kähler, i.e. the differential forms ω_c^Z are closed. The same is true for $\dot{\omega}$. Since ∇ has no torsion, we find that for $i \neq j$, $i \neq k$, $j \neq k$

$$\nabla_{e_i}\dot{\omega}(e_j, e_k) + \nabla_{e_j}\dot{\omega}(e_k, e_i) + \nabla_{e_k}\dot{\omega}(e_i, e_j) = 0,$$

and so

$$[D, \tfrac{1}{4}\dot{\omega}(e_i, e_j)c(e_i)c(e_j)] = -\tfrac{1}{2}\nabla_{e_i}\dot{\omega}(e_i, e_j)c(e_j) + \dot{\omega}(e_i, e_j)c(e_i)\nabla_{e_j}. \tag{1.134}$$

Using Lichnerowicz's formula as in [BGS 2, Theorem 2.15], we immediately obtain (1.132). □

The following theorem is the main result of this section:

Theorem 1.22. *For any $j \leqq -2$ we have $M_{j,c} = 0$.* Furthermore

$$M_{-1,c} = \left(\frac{1}{2\pi i}\right)^{\ell} \int_Z \frac{i}{2}\dot{\omega}_c Td(-R_c^Z)\operatorname{Tr}[\exp(-L^{\xi})], \tag{1.135}$$

and

$$\frac{\partial}{\partial c}\operatorname{Log}\|\phi_c\|^2 = \left(\frac{1}{2\pi i}\right)^{\ell} \int_Z \frac{\partial}{\partial b}\left[Td\left(-R_c^Z - b(g_c^Z)^{-1}\frac{\partial g_c^Z}{\partial c}\right) \times \operatorname{Tr}\left[\exp\left(-L^{\xi} - b(h_c^{\xi})^{-1}\frac{\partial h_c^{\xi}}{\partial c}\right)\right]\right]_{b=0}. \tag{1.136}$$

Proof. By Theorem 1.18, we only need to show this theorem when $c=0$. We first assume that h_c^{ξ} does not vary with c.

Using Proposition 1.19 and the methods of [B 1, Sect. 4], we find that

$$\lim_{u \downarrow\downarrow 0} u\operatorname{Tr}_s(Q\exp(-uD^2)) = \left(\frac{1}{2\pi i}\right)^{\ell} \int_Z \frac{i}{2}\dot{\omega}Td(-R_0^Z)\operatorname{Tr}[\exp(-L^{\xi})]. \tag{1.137}$$

Therefore $M_{j,0} = 0$ for $j \leqq -2$ and $M_{-1,0}$ is given by the right-hand side of (1.135). We now calculate $M_{0,0}$. As $u \downarrow\downarrow 0$ we have the expansion

$$\frac{\partial}{\partial u}(u\operatorname{Tr}_s(Q\exp(-uD^2))) = M_{0,0} + O(u). \tag{1.138}$$

Using (1.128) and (1.138) we find that

$$\lim_{u \downarrow\downarrow 0} \operatorname{Tr}_s\left(\exp\left(-\frac{u}{2}D^2 - \tfrac{1}{2}\sqrt{u}\,da D - \tfrac{1}{2}d\bar{a}\sqrt{u}[D, Q] + da\,d\bar{a}Q\right)\right)^{da\,d\bar{a}} = M_{0,0}.$$

Also

$$\dot{\omega}(e_i, J^Z e_i) = 2\sqrt{-1}\,\dot{\omega}(\bar{w}_i, w_i) = -2\operatorname{Tr}\left[(g_c^Z)^{-1}\frac{\partial g_c^Z}{\partial c}\right]_{c=0}. \tag{1.139}$$

In Theorem 1.21, the operator on the right-hand side of (1.132) is exactly of the same type as the curvature of the Levi-Civita superconnection [B 1, Sect. 3]. Therefore, we can use the methods of [B 1, Theorem 4.12] to calculate $M_{0,0}$. Take $x_0 \in Z$ and let $T_u(\cdot, \cdot)$ be the smooth kernel of

$$\exp\left(-\frac{u}{2}D^2 - \frac{\sqrt{u}}{2}da D - \frac{\sqrt{u}}{2}d\bar{a}[D, Q] + da\,d\bar{a}Q\right)$$

with respect to the volume form η of Z for the metric g_0^Z. Let w^1 be an Euclidean Brownian bridge in $T_{x_0}Z$ (for the metric g_0^Z) with $w_0^1 = w_1^1 = 0$ and let P_1 be the probability law of w^1 on $C([0,1]; T_{x_0}Z)$.

As in [B 1] we shall use in an essential way the fact that $dac(e_j)$, $d\bar{a}c(e_k)$ and $dad\bar{a}$ span a Heisenberg algebra. If γ and γ' are forms on Z, we write $\gamma \underset{n}{=} \gamma'$ if γ and γ' have the same component of degree n. Set

$$U = \left[(g_c^Z)^{-1} \frac{\partial g^Z}{\partial c} \right]_{c=0}.$$

Using Theorem 1.21 and proceeding as in [B 1, Theorem 4.12], that is using formula (1.132) and doing the formal changes required to use (1.132) instead of [B 1, Theorem 3.6], we get

$$\begin{aligned} &\lim_{u \downarrow\downarrow 0} \operatorname{Tr}_s(T_u(x_0, x_0))\eta(x_0) \\ &\underset{n}{=} \left(\frac{1}{2\pi i}\right)^{\ell} \int \exp\left\{ \tfrac{1}{2} \int_0^1 \langle R_0^Z(\cdot,\cdot) w^1, dw^1 \rangle + \tfrac{1}{2} i dad\bar{a} \int_0^1 \dot{\omega}(w^1, dw^1) \right. \\ &\quad \left. - \tfrac{1}{2} dad\bar{a} \operatorname{Tr}[U] - \frac{d\bar{a}}{\sqrt{2}} i \int_0^1 \nabla_{w_u^1} \dot{\omega}(e_k, dw_u^1) dx^k + \frac{d\bar{a}}{2\sqrt{2}} d^Z \operatorname{Tr}[U] \right\} \\ &\quad \times dP_1(w^1) \operatorname{Tr}[\exp(-L'^{\xi})]. \end{aligned} \tag{1.140}$$

The left-hand side of (1.140) is an even form. Therefore it does not contain any multiple of $d\bar{a}$. We thus find

$$\begin{aligned} &\lim_{u \downarrow\downarrow 0} \operatorname{Tr}_s(T_u(x_0, x_0))\eta(x_0) \\ &\underset{n}{=} \left(\frac{1}{2\pi i}\right)^{\ell} \int \exp\left\{ \tfrac{1}{2} \int_0^1 \langle R_0^Z(\cdot,\cdot) w^1, dw^1 \rangle + \tfrac{1}{2} i dad\bar{a} \int_0^1 \dot{\omega}(w^1, dw^1) \right\} dP_1(w^1) \\ &\quad \times \exp\{ -\tfrac{1}{2} dad\bar{a} \operatorname{Tr}[U] - \tfrac{1}{2} \operatorname{Tr} R_0^Z \} \operatorname{Tr}[\exp(-L^{\xi})]. \end{aligned} \tag{1.141}$$

Clearly

$$\int_0^1 \dot{\omega}(w^1, dw^1) = \int_0^1 \langle w^1, J^Z U dw^1 \rangle.$$

Therefore

$$\begin{aligned} &\left(\lim_{u \downarrow\downarrow 0} \operatorname{Tr}_s(T_u(x_0, x_0))\eta(x_0) \right)^{dad\bar{a}} \\ &= \left(\frac{1}{2\pi i}\right)^{\ell} \int \frac{\partial}{\partial b} \left(\exp\left\{ \tfrac{1}{2} \int_0^1 \langle (R_0^Z - ibJ^Z U) w^1, dw^1 \rangle \right\} \right. \\ &\quad \times \exp\{ -\tfrac{1}{2} b \operatorname{Tr}(U) - \tfrac{1}{2} \operatorname{Tr} R_0^Z \} \Big)_{b=0} dP_1(w^1) \operatorname{Tr}[\exp(-L^{\xi})]. \end{aligned} \tag{1.142}$$

Using the same notation as in the proof of [BGS 2, Theorem 2.16], we find that:

$$\exp\left\{ \tfrac{1}{2} \int_0^1 \langle R_0^Z - ibJ^Z U w^1, dw^1 \rangle \right\} dP_1(w_1) = A'(R_0^Z + bU). \tag{1.143}$$

From (1.141)–(1.143) we find that

$$\lim_{u\downarrow\downarrow 0}[\mathrm{Tr}_s(T_u(x_0,x_0))\eta(x_0))]^{da d\bar a}$$
$$\underset{n}{=}\left(\frac{1}{2\pi i}\right)^{\ell}\frac{\partial}{\partial b}[(A'(R_0^Z+bU)\exp\{-\tfrac{1}{2}\mathrm{Tr}\,R_0^Z-\tfrac{1}{2}b\,\mathrm{Tr}(U)\})_{b=0}$$
$$\times\mathrm{Tr}[\exp(-L^{\xi})]=\left(\frac{1}{2\pi i}\right)^{\ell}\frac{\partial}{\partial b}(Td(-R_0^Z-bU))_{b=0}\,\mathrm{Tr}[\exp(-L^{\xi})]. \tag{1.144}$$

By proceeding as in [B 1, Sect. 4], we also find that:

$$\lim_{u\downarrow\downarrow 0}-\mathrm{Tr}_s\left[(h_c^{\xi})^{-1}\frac{\partial h_c^{\xi}}{\partial c}\exp(-uD^2)\right]$$
$$=\left(\frac{1}{2\pi i}\right)^{\ell}\int_Z Td(-R^Z)\,\mathrm{Tr}\left[\left(-(h_c^{\xi})^{-1}\frac{\partial h_c^{\xi}}{\partial c}\right)\exp(-L_c^{\xi})\right]$$
$$=\left(\frac{1}{2\pi i}\right)^{\ell}\int_Z Td(-R_c^Z)\frac{\partial}{\partial b}\left[\mathrm{Tr}\exp\left(-L_c^{\xi}-b(h_c^{\xi})^{-1}\frac{\partial h_c^{\xi}}{\partial c}\right)\right]_{b=0}. \tag{1.145}$$

The theorem follows from (1.138), (1.144), and (1.145). □

Let g^Z, g'^Z be two Kähler metrics on $T^{(0,1)}Z$, and h^{ξ}, h'^{ξ} two Hermitian metrics on ξ. Consider a smooth family of metrics $c\in[0,1]\to(g_c^Z,h_c^{\xi})$ on $T^{(1,0)}Z$ and ξ such that:

$$(g_0^Z,h_0^{\xi})=(g^Z,h^{\xi})\quad\text{and}\quad(g_1^Z,h_1^{\xi})=(g'^Z,h'^{\xi}).$$

By the results of [BGS 1, Sect. 1e)], the form

$$\alpha=\left(\frac{1}{2\pi i}\right)^{\ell}\int_0^1\frac{\partial}{\partial b}\left[Td\left(-R_c^Z-b(g_c^Z)^{-1}\frac{\partial g_c^Z}{\partial c}\right)\mathrm{Tr}\left[\exp\left(-L_c^{\xi}-b(h_c^{\xi})^{-1}\frac{\partial h_c^{\xi}}{\partial c}\right)\right]\right]_{b=0} \tag{1.146}$$

defines an element in P/P' which depends only on (g^Z,h^{ξ}) and (g'^Z,h'^{ξ}).

According to [BGS 1, Theorems 1.27, 1.29 and Corollary 1.30], the component of degree (ℓ,ℓ) of α represents in P/P' the corresponding component of the Bott-Chern class

$$\widetilde{Td}(g^Z,g'^Z)ch(h^{\xi})+Td(g'^Z)\widetilde{ch}(h^{\xi},h'^{\xi}). \tag{1.147}$$

Let λ,λ' be the Hermitian determinant line fibers associated to the metrics (g^Z,h^{ξ}) and (g'^Z,h'^{ξ}). Let $\phi\in\lambda^{-1}\otimes\lambda'$ be the canonical isomorphism from λ into λ'.

Theorem 1.23. *The following identity holds:*

$$\|\phi\|^2=\exp\left\{\int_Z(\widetilde{Td}(g^Z,g'^Z)ch(h^{\xi})+Td(g'^Z)\widetilde{ch}(h^{\xi},h'^{\xi}))\right\}. \tag{1.148}$$

Proof. Since the space of Kähler metrics on TZ is convex we may assume that (g_c^Z) is a smooth family of Kähler metrics such that $g_0^Z=g^Z$, $g_1^Z=g'^Z$. The theorem then follows from Theorem 1.22. □

Remark 1.24. When g'^Z is not Kähler, α is still defined but we do not know what $\exp\left(\int_Z \alpha\right)$ represents. Also observe that when only the metric h^ξ is allowed to change, Theorem 1.23 is a special case of Theorems 2.4 and 2.8.

i) The Curvature of λ for the Quillen Metric: The General Case

We now work again under the general assumptions of [BGS 2, Sect. 1c)].

Our datas are then:

- The connected complex manifolds M and B.
- A smooth proper holomorphic map π $M \to B$, with connected fibers Z.
- A holomorphic Hermitian bundle ξ on M with metric h^ξ.

Definition 1.25. The fibration π will be said to be locally Kähler if there is an open covering $\mathscr{U}$ of B such that for any $U \in \mathscr{U}$, there exists a Kähler metric g_U on $\pi^{-1}(U)$.

Remark 1.26. Professors J. P. Demailly and N. J. Hitchin have pointed out to us that there are holomorphic fibrations whose fibers are Kähler, and which are not locally Kähler, a typical example being the fibration of the $K3$ surfaces over their moduli space. This last example is fully developed in Bingener [Bin, Example (3.9)].

From now on, we assume that π is locally Kähler.

For $U \in \mathscr{U}$, let g_U^Z be the Hermitian metric on TZ over $\pi^{-1}(U)$ induced by g_U. Let $T_U^H M$ be the orthogonal of TZ in TM for the metric g_U. Finally let $\omega_U = \omega_U^Z + \omega_U^H$ be the Kähler form of $(\pi^{-1}(U), g_U)$. By [BGS 2, Theorem 1.5], over U, $(\pi, g_U^Z, T_U^H M)$ is a Kähler fibration and ω_U is an associated $(1,1)$ form. By Theorem 1.3, we can define the holomorphic Hermitian line bundle λ_U over U. Note that the construction of λ_U involves g_U^Z explicitly. Also by Theorem 1.16, the canonical isomorphism $\phi_{U \cap V}: \lambda_U \to \lambda_V$ is an isomorphism of holomorphic line bundles.

Let now g^Z be any smooth Hermitian metric on TZ, which induces a Kähler metric on the fibers Z. By proceeding as in Sect. 1d), we construct a smooth Hermitian line bundle λ associated with the family of operators $\bar{\partial} + \bar{\partial}^*$ (where $\bar{\partial}^*$ is now calculated with respect to g^Z).

The metric on λ is of course the same as in Theorem 1.6. Namely for each $y \in B$, we endow λ_y with the corresponding Quillen metric associated with the metrics (g^{Z_y}, h^ξ).

Note that since in general, g^Z does not come from a Kähler metric on M, the construction of Sect. 1d) does not define a holomorphic structure on λ. However on $U \in \mathscr{U}$, we have a canonical smooth isomorphism $\phi_U: \lambda_U \to \lambda$. Therefore over U, λ inherits the holomorphic structure of λ_U. Since the canonical isomorphisms $\phi_{U \cap V}: \lambda_U \to \lambda_V$ are holomorphic, the holomorphic structures on λ are compatible. λ now becomes a holomorphic Hermitian line bundle over B.

Let ∇^Z be the holomorphic Hermitian connection on $T^{(1,0)}Z$ associated to the metric g^Z. Let R^Z be the curvature of ∇^Z.

Theorem 1.27. *The curvature of the holomorphic Hermitian connection ${}^1\nabla$ on λ is given by*

$$({}^1\nabla)^2 = 2i\pi\left[\int_Z Td\left(-\frac{R^Z}{2i\pi}\right)\mathrm{Tr}\left[\exp\left(-\frac{L^\xi}{2i\pi}\right)\right]\right]^{(2)}. \tag{1.149}$$

Proof. Take $U \in \mathcal{U}$. Let $R^{Z,U}$ be the curvature of the holomorphic Hermitian connection on $(T^{(1,0)}Z, g_U^Z)$. By Theorem 1.9, we know that the curvature of the holomorphic Hermitian connection on λ_U is given by:

$$2i\pi\left[\int_Z Td\left[-\frac{R^{Z,U}}{2i\pi}\right]\mathrm{Tr}\left[\exp\left(-\frac{L^\xi}{2i\pi}\right)\right]\right]^{(2)}. \tag{1.150}$$

Let α_U be the form in P defined in (1.146), associated with the metrics (g_U^Z, h^ξ) and (g^Z, h^ξ). By Theorem 1.23 we know that

$$\mathrm{Log}\,\|\phi_U\|^2 = \left[\int_Z \alpha_U\right]^{(0)}.$$

Also, by [BGS 1, Theorem 1.27], we know that

$$\bar\partial^M \partial^M \alpha_U = \frac{1}{(2i\pi)^\ell}\left[Td(-R^Z) - Td(-R^{Z,U})\right]\mathrm{Tr}[\exp(-L^\xi)]. \tag{1.151}$$

Also

$$\bar\partial^B \partial^B \mathrm{Log}\,\|\phi_U\|^2 = \left[\int_Z \bar\partial^M \partial^M \alpha_u\right]^{(2)}. \tag{1.152}$$

Now the curvature of ${}^1\nabla$ is given by

$$\left(\frac{1}{2i\pi}\right)^\ell\left[\int_Z Td\left(\frac{-R^{Z,U}}{2i\pi}\right)\mathrm{Tr}[\exp(-L^\xi)]\right]^{(2)} + \bar\partial^B \partial^B \mathrm{Log}\,\|\phi_u\|^2. \tag{1.153}$$

Using (1.150)–(1.153), the theorem follows. □

Remark 1.28. As we shall see in Theorems 2.12 and 3.14, the holomorphic structure on λ is exactly the holomorphic structure of Knudsen Mumford [KM].

It follows from Theorem 1.27 that under the assumptions of this theorem, if the complex $(E, \bar\partial)$ is everywhere acyclic, and if $\tau(\bar\partial)$ is the Ray-Singer analytic torsion of the complex $(E, \bar\partial)$ [RS], then

$$\bar\partial^B \partial^B \mathrm{Log}[\tau^2(\bar\partial)] = 2i\pi\left[\int_Z Td\left(\frac{-R^Z}{2i\pi}\right)\mathrm{Tr}\left[\exp\left(\frac{-L^\xi}{2i\pi}\right)\right]\right]^{(2)}. \tag{1.154}$$

2. $\lambda = \lambda^{\mathrm{KM}}$: An Analytic Proof

We prove here that if π is projective, λ and λ^{KM} are canonically isomorphic as holomorphic line bundles on B. We also establish [BGS 1, Theorem 0.3].

In a), we consider an acyclic exact sequence of holomorphic Hermitian vector bundles

$$0 \to \xi_0 \xrightarrow[v]{} \xi_1 \to \ldots \xrightarrow[v]{} \xi_m \to 0$$

and the associated holomorphic vector bundles $\lambda_0, \lambda_1, \ldots, \lambda_m$. We thus define a holomorphic non-zero section $T(\bar\partial + v)$ of the line bundle $\prod_1^m (\lambda_i)^{(-1)^i}$.

In b), we prove that $\|T(\bar\partial + v)\|$ is exactly given by the formula for $\|\sigma\|$ in [BGS 1, Theorem 0.3]. This is essentially a simple consequence of the results of [BGS 2].

In c), we prove that $T(\bar\partial+v)$ is multiplicative with respect to double complexes $(\xi_{i,j})$ on M. This is again proved by analytic methods.

In d), we prove by brute force that $T(\bar\partial+v)$ coincides with the Knudsen-Mumford section σ described before [BGS 1, Theorem 0.3]. We thus complete the proof of [BGS 1, Theorem 0.3].

In e), we prove that $\lambda\simeq\lambda^{\mathrm{KM}}$ when π is projective as a consequence of [BGS 1, Theorem 0.3].

In f), we complete the proof of [BGS 1, Theorem 0.1] when π is projective.

a) Infinite Determinants and Exact Sequences

We make the same assumptions as in Sect. 1a). In particular $(\pi, g^Z, T^H M)$ is still assumed to be a Kähler fibration, and ω is an associated $(1,1)$ form. Let ξ_0 be a holomorphic Hermitian vector bundle on M, with associated holomorphic Hermitian connection ∇^{ξ_0}. Let $L^{\xi_0}=(\nabla^{\xi_0})^2$ be the curvature of ∇^{ξ_0}. We use the notations of Sect. 1, except that we introduce the index 0 at every stage. The infinite dimensional complex E_0 is written

$$0\to E_0^0\xrightarrow[\bar\partial]{}\dots\xrightarrow[\bar\partial]{}E_0^\ell\to 0\,.$$

λ_0 denotes the holomorphic Hermitian line bundle on B associated with E_0, $\|\ \|_0$ is the metric of λ_0 and ${}^1\nabla_0$ the corresponding holomorphic Hermitian connection.

Let

$$\xi: 0\to\xi_0\xrightarrow[v]{}\xi_1\xrightarrow[v]{}\dots\xrightarrow[v]{}\xi_m\to 0$$

be an acyclic holomorphic chain complex of holomorphic Hermitian vector bundles over M, which starts at ξ_0, i.e. provides a resolution of ξ_0.

We will use the same notations as in [BGS 2, Sect. 2]. We thus have a double complex E

$$\begin{tikzcd}
 & 0 & 0 & & 0 & \\
0 \arrow[r] & E_0^\ell \arrow[r, "v"'] \arrow[u] & E_1^\ell \arrow[r] \arrow[u] & \dots \arrow[r, "v"'] & E_m^\ell \arrow[r] \arrow[u] & 0 \\
 & \vdots \arrow[u] & \vdots & & \vdots \arrow[u] & \\
 & E_0^1 \arrow[u] & \vdots & & E_m^1 \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & E_0^0 \arrow[r, "v"'] \arrow[u, "\bar\partial"'] & E_1^0 \arrow[r] \arrow[u, "\bar\partial"'] & \dots \arrow[r, "v"'] & E_m^0 \arrow[r] \arrow[u, "\bar\partial"'] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & & 0 \arrow[u] &
\end{tikzcd}$$

Also for $1\leqq j\leqq m$, we consider the holomorphic Hermitian line bundle λ_j, with metric $\|\ \|_j$ and holomorphic Hermitian connection ${}^1\nabla_j$. For $0\leqq i\leqq\ell$, $0\leqq j\leqq m$, let $H^i(E_j)$ be the i-th cohomology groups in the complex E_j. For $(y,a)\in B\times C$, we consider the double chain complex E with differential operator $\bar\partial+av$. The adjoint of $\bar\partial+av$ is $\bar\partial^*+\bar a v^*$. We can extend the construction of Sect. 1 in this situation.

Namely we construct the determinant bundle $\tilde{\lambda}$ over $B \times C$ associated with $(E, \bar{\partial} + av)$. $\tilde{\lambda}$ is a holomorphic Hermitian line bundle on $B \times C$, with metric $\| \ \|$, and holomorphic Hermitian connection ${}^1\nabla$. Let $\tilde{\lambda}^B$ be the restriction of $\tilde{\lambda}$ to B. Clearly as holomorphic Hermitian bundles on B, we have the identifications:

$$\tilde{\lambda}^B = \lambda_0 \otimes \lambda_1^{-1} \otimes \lambda_2 \dots .$$

Let ϱ be the projection $(y, a) \in B \times C \to y \in B$.

Proposition 2.1. *The curvature $({}^1\nabla)^2$ of the connection ${}^1\nabla$ on $\tilde{\lambda}$ is given by:*

$$({}^1\nabla)^2 = 2i\pi \varrho^* \left[\int_Z Td\left(\frac{-R^Z}{2i\pi}\right) \mathrm{Tr}_s\left[\exp \frac{-L^\xi}{2i\pi}\right]\right]^{(2)}. \tag{2.1}$$

Proof. The proof is the same as the proofs of [BGS 2, Theorem 2.2] and of Theorem 1.9. □

Equation (2.1) does not contain the variable a. This means in particular that for every $y \in B$, the connection ${}^1\nabla$ is flat on $\{y\} \times C$. Therefore we can identify the fibers $\{\tilde{\lambda}_{y,a}\}_{a \in C}$ with $\tilde{\lambda}_y^B$ using parallel transport along any C^∞ curve in $\{y\} \times C$ which connects (y, a) and $(y, 0)$. So we have identified the C^∞ Hermitian bundles over $B \times C$:

$$\tilde{\lambda} = \varrho^* \tilde{\lambda}^B .$$

Notice that $\varrho^* \tilde{\lambda}^B$ is naturally a holomorphic line bundle on $B \times C$.

Proposition 2.2. *The identification*

$$\tilde{\lambda} = \varrho^* \tilde{\lambda}^B \tag{2.2}$$

identifies the holomorphic structures of $\tilde{\lambda}$ and $\varrho^* \tilde{\lambda}^B$.

Proof. Let Y be a C^∞ section of TB. Since (2.1) does not contain da or $d\bar{a}$, we find that

$$\left[{}^1\nabla_Y, {}^1\nabla_{\frac{\partial}{\partial a}}\right] = \left[{}^1\nabla_Y, {}^1\nabla_{\frac{\partial}{\partial \bar{a}}}\right] = 0 . \tag{2.3}$$

From (2.3), we immediately deduce that if τ_0^a is the parallel transportation operator in $\{y\} \times C$ from (y, a) into $(y, 0)$, if σ is a section of $\tilde{\lambda}$ (i.e. $\sigma_{(y.a)} \in \tilde{\lambda}_{(y \cdot a)}$), then

$$\tau_0^a \, {}^1\nabla_Y \sigma = {}^1\nabla_Y [\tau_0^a \sigma] . \tag{2.4}$$

Similarly, since $\tilde{\lambda}$ is flat on $\{y\} \times C$, we have

$$\tau_0^a \, {}^1\nabla_{\frac{\partial}{\partial a}} \sigma = \frac{\partial}{\partial a} \tau_0^a \sigma , \qquad \tau_0^a \, {}^1\nabla_{\frac{\partial}{\partial \bar{a}}} \sigma = \frac{\partial}{\partial \bar{a}} \tau_0^a \sigma . \tag{2.5}$$

The proposition is proved. □

Clearly for any $a \in C$

$$\mathrm{Ind}(D + V^a)_+ = 0 . \tag{2.6}$$

Moreover for any $a \in C^*$, the complex $(E, \bar{\partial} + av)$ is acyclic. By Remark 1.10, over $B \times C^*$, we can define a non-zero holomorphic section of $\tilde{\lambda}$, which we note $T(\bar{\partial} + av)$.

With our conventions, for $(y, a) \in B \times C^*$, $T(\bar{\partial} + av) \in \tilde{\lambda}_y^B$.

Theorem 2.3. *The section $T(\bar{\partial}+v)$ is a non-zero holomorphic section of $\tilde{\lambda}^B$. It defines a holomorphic isomorphism from λ_0 into $\lambda_1 \otimes \lambda_2^{-1} \otimes \ldots$.*

Proof. By Proposition 2.2, as holomorphic bundles over $B \times C$, $\tilde{\lambda} = \varrho^* \tilde{\lambda}^B$. Since $T(\bar{\partial}+av)$ is a holomorphic section of $\tilde{\lambda}$ over $B \times C^*$, $T(\bar{\partial}+v)$ naturally defines a holomorphic section of $\tilde{\lambda}^B$. □

b) Evaluation of $\|T(\bar{\partial}+v)\|$

Our purpose is now to prove that $T(\bar{\partial}+v)$ coincides with the Knudsen Mumford section of $\tilde{\lambda}^B$. We only need to do this fiberwise, i.e. for one fixed $y_0 \in B$. In this subsection only, we will assume that B is reduced to one single point y_0. However, we still use the notation $\tilde{\lambda}$.

$T(\bar{\partial}+av)$ is then a section of the line bundle $\tilde{\lambda}$ over C^*.

Remember that in [BGS 1, Sect. 1c)], to the holomorphic Hermitian chain complex (ξ, v), we associate $\zeta'_\xi(0) \in P$. For $0 \leqq j \leqq m$, let χ_j be the Euler characteristic of E_j, i.e.

$$\chi_j = \sum_0^{\ell} (-1)^i \dim H^i(E_j). \tag{2.7}$$

Set

$$d = \sum_{j=1}^{m} (-1)^{j+1} j \chi_j. \tag{2.8}$$

Theorem 2.4. *$a^{-d} T(\bar{\partial}+av)$ is a non-zero parallel section of $\tilde{\lambda}$ over C^*. Moreover*

$$\operatorname{Log} \left\| \frac{T(\bar{\partial}+av)}{a^d} \right\|^2 = -\left(\frac{1}{2i\pi}\right)^{\ell} \int_Z Td(-R^Z) \zeta'_\xi(0). \tag{2.9}$$

Proof. We first assume that $a=1$. Since the complex $(E, \bar{\partial}+v)$ is acyclic, with the notations of [BGS 2, Definition 2.19], we have the equality

$$\operatorname{Log} \| T(\bar{\partial}+v) \|^2 = -\tilde{\zeta}'_E(0) \tag{2.10}$$

[remember that here $B = \{y_0\}$, so that the right-hand side of (2.10) is a real number].

By [BGS 2, Theorem 2.21], we know that

$$\tilde{\zeta}'_E(0) = \left(\frac{1}{2\pi i}\right)^{\ell} \int_Z Td(-R^Z) \zeta'_\xi(0). \tag{2.11}$$

Equation (2.9) is proved for $a=1$. If $a \in C^*$, by [BGS 1, Sect. 1c)], we know that if v is changed into av, $\zeta'_\xi(0)$ is changed into

$$\zeta'_\xi(0) + 2 \operatorname{Log}|a| \operatorname{Tr}_s[N_H \exp(-L^\xi)].$$

Also by the Atiyah-Singer Index Theorem, for $0 \leqq j \leqq m$,

$$\chi_j = \left(\frac{1}{2\pi i}\right)^{\ell} \int_Z Td(-R^Z) \operatorname{Tr}[\exp(-L^{\xi_j})]. \tag{2.12}$$

Equation (2.9) immediately follows for every $a \neq 0$. Since $\frac{T(\bar{\partial}+av)}{a^d}$ is a holomorphic section of $\tilde{\lambda}$ over C^* whose norm is constant, it is parallel. □

Remember that we have identified by parallel transport the fibers $\tilde{\lambda}_a$ with $\tilde{\lambda}_0$. Using this identification, and also Theorem 2.4, we find for any $a \in C^*$,

$$T(\bar{\partial}+v) = \frac{T(\bar{\partial}+av)}{a^d}. \tag{2.13}$$

Assume *temporarily* that the cohomology groups of $E_0, \ldots, E_m$ are all 0. Let $\tau_j(\bar{\partial})$ be the Ray-Singer analytic torsion of the complex $(E_j, \bar{\partial})$ [RS].

From Theorem 2.4, we deduce

Theorem 2.5. *The following identity holds:*

$$\sum_{j=0}^{m} (-1)^j \operatorname{Log} \tau_j^2(\bar{\partial}) = -\left(\frac{1}{2i\pi}\right)^{\ell} \int_Z Td(-R^Z)\zeta'_{\xi}(0). \tag{2.14}$$

Proof. Clearly $d=0$. Moreover using the notations of Theorem 2.4, we know that since the complexes $(E_j, \bar{\partial})$ are acyclic, the section $T(\bar{\partial})$ of $\tilde{\lambda}$ is well defined and moreover that as $a \in C^* \to 0$, then $T(\bar{\partial}+av) \to T(\bar{\partial})$. Equation (2.14) is now a consequence of (1.63) and of Theorem 2.4. □

We now do not assume any more that any of the complexes E_j is acyclic.

c) Multiplicativity Properties of $T(\bar{\partial}+v)$

In [BGS 1, Sect. 1d)], we verified that $\zeta'_{\xi}(0)$ verifies certain additivity properties with respect to double complexes and exact sequences. We will verify that $T(\bar{\partial}+v)$ verifies the analogous multiplicativity properties. Of course we still assume that B is reduced to one single point $\{y_0\}$.

Assume first that ξ is a double holomorphic Hermitian chain complex over Z,

$$\begin{array}{ccccccc}
 & 0 & & & & 0 & \\
 & \uparrow & & & & \uparrow & \\
0 \longrightarrow & \xi_{m',0} & \underset{v}{\longrightarrow} & & & \xi_{m',m} & \longrightarrow 0\\
 & \vdots & & & & & \\
 & \uparrow{\scriptstyle v'} & & & & \uparrow{\scriptstyle v'} & \\
0 \longrightarrow & \xi_{1,0} & \underset{v}{\longrightarrow} & & & \xi_{1,m} & \longrightarrow 0 \\
 & \uparrow{\scriptstyle v'} & & & & \uparrow{\scriptstyle v'} & \\
0 & \xi_{0,0} & \underset{v}{\longrightarrow} & \xi_{0,1} & \underset{v}{\longrightarrow} \cdots \longrightarrow \cdots & \xi_{0,m} & \longrightarrow 0
\end{array}$$

We assume that the lines and columns of ξ are acyclic.

To the chain complex $\xi_{i,\cdot}$ $(0 \leqq i \leqq m')$ we can associate the corresponding determinant fiber $\tilde{\lambda}^B_{i,\cdot}$ and the non-zero section $T_{i,\cdot}(\bar{\partial}+v)$ of $\tilde{\lambda}^B_{i,\cdot}$ by the construction of Sect. 2b). Similarly to the chain complex $\xi_{\cdot,j}$ $(0 \leqq j \leqq m)$ we associate the determinant fiber $\tilde{\lambda}^B_{\cdot,j}$ and the non-zero section $T_{\cdot,j}(\bar{\partial}+v')$ of $\tilde{\lambda}^B_{\cdot,j}$. We have a canonical identification of determinant fibers

$$\prod_0^{m'}(\tilde{\lambda}^B_{i,\cdot})^{(-1)^i}=\prod_0^{m}(\tilde{\lambda}^B_{\cdot,j})^{(-1)^j}.$$

We now claim

Theorem 2.6. *The following identity holds:*

$$\prod_0^{m}(T_{i,\cdot}(\bar{\partial}+v))^{(-1)^i}=\prod_0^{m'}(T_{\cdot,j}(\bar{\partial}+v'))^{(-1)^j}. \tag{2.15}$$

Proof. We proceed very much as in the proof of [BGS 1, Theorem 1.20]. For $a\in C$, we consider the double chain complex $(\xi, v'+av)$, which is acyclic. Associated with the corresponding $\bar{\partial}$ complex, we construct the holomorphic Hermitian determinant bundle λ'' over C and a non-zero holomorphic section $T''(\bar{\partial}+v'+av)$.

By the proof of [BGS 1, Theorem 1.20] (and more precisely by [BGS 1, Eq. (1.75)]), we know that for any $a\in C$, if $(\zeta_\xi^{av+v})'(0)$ is the differential form on Z associated with $(\xi, av+v')$, then $(\zeta_\xi^{av+v})'(0)$ is constant in P/P'.

By Theorem 2.4, we find that $\|T''(\bar{\partial}+v'+av)\|$ does not depend on a. Therefore λ'' is a flat bundle, and $T''(\bar{\partial}+v'+av)$ is a parallel section of λ''. By trivializing λ'' by parallel transport, we get

$$T''(\bar{\partial}+v')=T''(\bar{\partial}+v+v'). \tag{2.16}$$

Interchanging the roles of v and v', we find that

$$T''(\bar{\partial}+v)=T''(\bar{\partial}+v+v'). \tag{2.17}$$

Equation (2.15) follows from (2.16), (2.17). □

Similarly let

$$\xi:0\to\xi_0\xrightarrow[v]{}\dots\xrightarrow[v]{}\xi_m\to 0,$$

$$\xi':0\to\xi_m\xrightarrow[v]{}\dots\xrightarrow[v]{}\xi_{m+m'}\to 0$$

be two holomorphic Hermitian acyclic chain complexes over Z, with ξ_m appearing in ξ and ξ' with the same metric. Let (ξ'', v'') be the holomorphic Hermitian acyclic chain complex

$$0\to\xi_0\to\dots\xrightarrow[v]{}\xi_{m-1}\xrightarrow[v^2]{}\xi_{m+1}\xrightarrow[v]{}\dots\to 0.$$

Let $\tilde{\lambda}^B, \tilde{\lambda}'^B, \tilde{\lambda}''^B$ be the determinant fibers associated with ξ, ξ', ξ''. One has the canonical identification

$$\tilde{\lambda}''^B=\tilde{\lambda}^B\otimes(\tilde{\lambda}'^B)^{(-1)^{m+1}}. \tag{2.18}$$

Let $T(\bar{\partial}+v)$, $T'(\bar{\partial}+v)$, $T''(\bar{\partial}+v)$ be the corresponding non-zero sections of $\tilde{\lambda}^B, \tilde{\lambda}'^B, \tilde{\lambda}''^B$.

Theorem 2.7. *The following identity holds:*

$$T''(\bar{\partial}+v'')=T(\bar{\partial}+v)\otimes[T'(\bar{\partial}+v)]^{(-1)^{m+1}}. \tag{2.19}$$

Proof. We first assume that $m'=2$. We use the double complex constructed in the proof of [BGS 1, Theorem 1.22], whose lines and columns are acyclic. We apply Theorem 2.6 in this situation. The lines or columns of the type $0 \to E_i \to E_i \to 0$ only give a trivial contribution. Theorem 2.7 is proved when $m'=2$. By splitting the exact sequence ξ' into short exact sequences, we obtain the theorem in the general case. $\square$

d) $T(\bar{\partial}+v)$ is the Knudsen-Mumford Section

We now again assume that we are under the assumptions of Sect. 2a). Let us briefly recall how the Knudsen-Mumford section of $\tilde{\lambda}$ is defined in [KM]. Assume first that ξ is a short exact sequence, i.e. that $m=2$. We then have a long exact sequence in cohomology.

$$\begin{tikzcd} \vdots & \vdots & \vdots \\ H^1(E_0) \arrow[r, "v"'] & H^1(E_1) \arrow[r, "v"'] & H^1(E_2) \\ 0 \arrow[r] & H^0(E_0) \arrow[r, "v"'] & H^0(E_1) \arrow[r, "v"'] & H^0(E_2) \arrow[ulll, "\delta"'] \end{tikzcd} \tag{2.20}$$

where δ is a coboundary operator.

By the construction of [BGS 1, Definition 1.1], to the exact sequence (2.20), we can canonically associate a non-zero section σ of $\otimes[\det H^i(E_j)]^{(-1)^{i+j+1}}$.

Also, for every j $(0 \leqq j \leqq 2)$, we have the canonical identification

$$\lambda_j = \bigotimes_{i=0}^{\ell} (\det H^i(E_j))^{(-1)^{i+1}}.$$

Therefore σ defines a non-zero section of $\tilde{\lambda}^B = \lambda_0 \otimes \lambda_1^{-1} \otimes \lambda_2$, which is the Knudsen-Mumford section.

When $m \geqq 2$, we split the sequence ξ into short exact sequences, and define σ multiplicatively.

Theorem 2.8. *The following identity holds*

$$T(\bar{\partial}+v) = \sigma. \tag{2.21}$$

Proof. Using Theorem 2.7, it is enough to prove the theorem for $m=2$. We will use the fact that for $a \neq 0$, by (2.13),

$$T(\bar{\partial}+v) = \frac{T(\bar{\partial}+av)}{a^d}, \tag{2.22}$$

and we will prove that

$$\lim_{\substack{a \to 0 \\ a \neq 0}} \frac{T(\bar{\partial}+av)}{a^d} = \sigma. \tag{2.23}$$

Incidentally, note that (2.22) makes sense because we used the parallel transport trivialization of $\tilde{\lambda}$. On the contrary, since $\frac{T(\bar{\partial}+av)}{a^d}$ is a section of $\tilde{\lambda}_a$ and σ a section of $\tilde{\lambda}_0$, (2.23) is a differential geometric statement, which can be verified locally.

Note that if $H^i(E_j)=0$ for $0\leqq i\leqq \ell$, $0\leqq j\leqq 2$, (2.23) is trivial. In fact $d=0$, and $T(\bar\partial)$ is well defined. In this case by (2.22), $T(\bar\partial+av)=T(\bar\partial)$ and tautologically $T(\bar\partial)=\sigma$.

So we now assume that not all the $H^i(E_j)$ are reduced to 0, so that 0 is in the spectrum of $D^2_{y_0}$. There is $b>0$ such that $D^2_{y_0}$ has no eigenvalue in $]0,b]$. We can choose $\varepsilon>0$ such that, for $|a|\leqq\varepsilon$, b is not an eigenvalue of $(D+V^a)^2_{y_0}$.

The double complex E has a total grading. For $|a|\leqq\varepsilon$, $0\leqq q\leqq 3\ell$, let K^q_a be the sum of eigenspaces of total degree q of $(D+V^a)^2_{y_0}$ for eigenvalues $<b$. Note that for $0<|a|<\varepsilon$, since $(E,\bar\partial+av)$ is acyclic, 0 is not an eigenvalue of $(D+V^a)^2$. Identifying the kernel of $D^2_{y_0}$ with $\bigoplus_{i=1}^{\ell} H^i(E_j)$, we find that

$$K^q_0=\bigoplus_{i+j=q} H^i(E_j)\,. \tag{2.24}$$

Also, the various K^q are smooth vector bundles on $(|a|<\varepsilon)$. Let P_a be the orthogonal projection operator from E_{y_0} on $K_a=\oplus K^q_a$. We can take $\varepsilon>0$ small enough so that P_a is one to one from K_0 into K_a and so P_0 is one to one from K_a into K_0.

Recall that we identify K_0 with $\oplus H^i(E_j)$. We first give a description of σ, by using [BGS 1, Definition 1.1]. Choose an element $\sigma^i_j\in \Lambda H^i(E_j)$ such that $v(\sigma^i_j)$ is non-zero in $\det(v(H^i(E_j)))$ for $j=0,1$, and $\delta(\sigma^i_2)$ is non-zero in $\det(\delta(H^i(E_2)))$. We get

$$\begin{aligned}\sigma=&(\sigma^0_0)^{-1}\otimes(v\sigma^0_0\wedge\sigma^0_1)\otimes(v\sigma^0_1\wedge\sigma^0_2)^{-1}\\&\otimes(\delta\sigma^0_2\wedge\sigma^1_0)\otimes(v\sigma^1_0\wedge\sigma^1_1)^{-1}\otimes(v\sigma^1_1\wedge\sigma^1_2)\otimes(\bar\partial\sigma^1_2\wedge\sigma^2_0)^{-1}\otimes\dots\,.\end{aligned} \tag{2.25}$$

Let $s^i_j\in\Lambda K^i_0(E_j)$ be the element representing σ^i_j. If $j=0,1$, $P_0v(s^i_j)\in\Lambda K^i_0(E_{j+1})$ represents $v(\sigma^1_j)$. Furthermore the representative of $\delta(\sigma^{i-1}_2)$ in $\Lambda K^i_0(E_0)$ can be described as follows:

Let $n_i=\deg(s^{i-1}_2)$, $\beta_i\in\Lambda^{n_i}E^{i-1}_1$, and $\alpha_i\in\Lambda^{n_i}E^i_0$ be such that

$$v(\beta_i)=s^{i-1}_2\,,\qquad \bar\partial(\beta_i)+v(\alpha_i)=0\,,\qquad \bar\partial(\alpha_i)=0\,.$$

Then $P_0(\alpha_i)$ represents $\delta(\sigma^{i-1}_2)$. Therefore, when $K^i_0(E_j)$ is identified with $H^i(E_j)$, σ is represented by

$$\begin{aligned}&(s^0_0)^{-1}\otimes(P_0vs^0_0\wedge s^0_1)\otimes(P_0vs^0_1\wedge s^0_2)^{-1}\\&\otimes(P_0\alpha_1\wedge s^1_0)\otimes(P_0vs^1_0\wedge s^1_1)^{-1}\otimes(P_0vs^1_1\wedge s^1_2)\\&\otimes(P_0\alpha_2\wedge s^2_0)^{-1}\otimes\dots\,.\end{aligned} \tag{2.26}$$

We now fix a such that $0<|a|<\varepsilon$ and we describe the canonical section $T(\bar\partial+av)$ considered as a section of $\bigotimes_{q\geqq 0}\det(K^q_a)^{(-1)^{q+1}}$. For this, according to [BGS 1, Definition 1.1], we have to find, for all $q\geqq 0$, a generator of $\det((\bar\partial+av)(K^q_a))$.

Note that P_a induces an isomorphism $\det(K^q_0)\simeq\det(K^q_a)$. Furthermore, for all i,j, since P_a commutes with $\bar\partial+av$:

$$\begin{aligned}(\bar\partial+av)(s^i_j)&=(av)(s^i_j)\,,\\(\bar\partial+av)(P_av(s^i_j))&=0\,,\\(\bar\partial+av)P_a(\alpha_i+a^{n_i}\beta_i)&=a^{2n_i}P_as^{i-1}_2\,.\end{aligned} \tag{2.27}$$

Set

$$\phi(i)=\deg(s_0^i)+\deg(s_1^{i-1})+2\deg(s_2^{i-1}).$$

Since $(av)(s_j^i)=a^{\deg(s_j^i)}v(s_j^i)$ and counting dimensions, we see that $\det((\bar\partial+av)(K_a^i))$ admits the non-zero generator

$$a^{\phi(i)}P_a(v(s_0^i)\wedge v(s_1^{i-1})\wedge s_2^{i-1})=(\bar\partial+av)(P_a(s_0^i)\wedge P_a(s_1^{i-1})\wedge P_a(\alpha_i+a^{n_i}\beta_i)). \quad (2.28)$$

One checks that

$$\sum_{i\geqq 0}(-1)^i\phi(i)=d.$$

Therefore $T(\bar\partial+av)$ is represented by:

$$\begin{aligned}&a^dP_a((s_0^0)^{-1}\otimes(vs_0^0\wedge s_1^0)\otimes(vs_1^0\wedge s_2^0)^{-1}\\&\otimes((\alpha_1+a^{n_1}\beta_1)\wedge s_0^1)\otimes(vs_0^1\wedge s_1^1)^{-1}\otimes(vs_1^1\wedge s_2^1)\\&\otimes((\alpha_2+a^{n_2}\beta_2)\wedge s_0^2)^{-1}\otimes\ldots).\end{aligned} \quad (2.29)$$

When $a\to 0$, the quotient $T(\bar\partial+av)/a^d$ tends to σ as represented by (2.26). □

Remark 2.9. It is striking that the proof does not involve estimates on the lowest eigenvalue of $(D+V^a)^2$.

Incidentally note that a priori, $T(\bar\partial+v)$ depended on the metrics (g^Z, h^ξ). A by-product of Theorem 2.7 is that it does not depend on these metrics. Note that such a result could have been easily obtained by the methods used in the proof of Theorem 1.16.

e) The Case where π is Projective: λ is the Knudsen-Mumford Determinant

We make the same assumptions and use the same notations as in Sect. 1a). In particular (π, g^Z, T^HM) is still assumed to be a Kähler fibration, and ξ is a holomorphic Hermitian vector bundle on M.

For $y\in B$, $H_y^0(E),\ldots,H_y^l(E)$ denote the cohomology groups of the complex $(E_y,\bar\partial_y)$. Let us *temporarily* assume that for $i\geqq 1$, $H^i(E)=0$. Clearly

$$H_y^0(E)=\{h\in E_y^0;\bar\partial_y h=0\}.$$

Then $H^0(E)$ is a smooth vector bundle on B, which is a subbundle of E^0. $H^0(E)$ inherits the Hermitian metric of E^0. Let P^0 be the orthogonal projection operator from E^0 on $H^0(E)$.

Theorem 2.10. *Let ∇ be the connection on $H^0(E)$*

$$\nabla=P^0\tilde\nabla.$$

Then ∇ is unitary. There is a unique holomorphic structure on $H^0(E)$ such that ∇ is the associated holomorphic Hermitian connection. If U is an open set in B, a smooth section $y\in U\to h_y\in H_y^0(E)$ is holomorphic for ∇ if and only if h is holomorphic on $\pi^{-1}(U)$. The canonical isomorphism $\lambda\simeq(\det H^0(E))^{-1}$ is an isomorphism of holomorphic line bundles.

Proof. Clearly ∇ is unitary. By [BGS 2, Theorem 1.14], we know that

$$\tilde\nabla''\bar\partial=0. \quad (2.30)$$

If h is a smooth section of $H^0(E)$, $\bar{\partial}h=0$ and so using (2.30), we find

$$\bar{\partial}\tilde{\nabla}''h=0. \tag{2.31}$$

We thus find that

$$\nabla''h=\tilde{\nabla}''h. \tag{2.32}$$

By [BGS 2, Theorem 1.14], we know that the curvature $(\tilde{\nabla})^2$ of $\tilde{\nabla}$ is of complex type $(1,1)$. By (2.32) we find that $(\nabla'')^2=0$. Since ∇ is unitary, we also have $(\nabla')^2=0$. Therefore the curvature $(\nabla)^2$ of ∇ is of complex type $(1,1)$.

By [AHS, Theorem 5.1], there is a unique holomorphic structure on the vector bundle $H^0(E)$ such that ∇ is the associated holomorphic Hermitian connection.

Let h be a smooth section of $H^0(E)$ on U. Clearly h defines a smooth section of ξ on $\pi^{-1}(U)$. Since $\bar{\partial}h=0$, by [BGS 2, Theorem 2.8] we find that

$$\tilde{\nabla}''h=\bar{\partial}^M h. \tag{2.33}$$

Using (2.32), we get

$$\nabla''h=\bar{\partial}^M h. \tag{2.34}$$

By (2.34), it is now clear that $\nabla''h=0$ if and only if $\bar{\partial}^M h=0$.

Given $y_0\in B$, we can find $b>0$, such that $D^2_{y_0}$ restricted to E_{y_0} has no eigenvalue in $]0,b]$. Since $\operatorname{Ker} D^2\cap E=H^0(E)$ has constant rank, we can find an open neighborhood U of y_0 in B such that if $y\in B$, D^2_y restricted to E_y has no eigenvalue in $]0,b]$.

If $\{K^{b,p}\}_{0\leqq p\leqq \ell}$ are the vector bundles on U which were defined in Sect. 1b), we find that for $y\in U$

$$K^{b,0}=H^0(E), \qquad K^{b,p}=\{0\}; \qquad p\geqq 1. \tag{2.35}$$

Also in Definition 1.2, a connection ∇^b was defined on K^b. It is clear that $\nabla^b=\nabla$. Therefore, from Theorem 1.3, we find that the canonical isomorphism $\lambda\simeq(\det H^0(E))^{-1}$ is holomorphic. □

Remark 2.11. The first part of Theorem 2.10 would of course be obvious if E had been an ordinary finite dimensional holomorphic vector bundle.

We now formulate the main result of this section. Namely, we assume that the fibration π is *locally Kähler* in the sense of Definition 1.25.

We also make the following assumption:

(A) There exists a resolution

$$0\to\xi=\xi_0\underset{v}{\longrightarrow}\xi_1\underset{v}{\longrightarrow}\dots\underset{v}{\longrightarrow}\xi_m\to 0 \tag{2.36}$$

of ξ by holomorphic vector bundles on M such that, for every $y\in B, j=1,\dots,m$ and $i=1,\dots,\ell$, the cohomology group $H^i(Z_y,\xi_j)$ vanishes.

By [Q 3, Sect. 7.27], this hypothesis is satisfied if the map $\pi: M\to B$ is *projective*, i.e. admits a factorisation

$$\begin{tikzcd} M \arrow[rr, "j"] \arrow[dr, "\pi"'] & & P(E) \arrow[dl, "p"] \\ & B & \end{tikzcd}$$

where j is a closed immersion and $P(E) \xrightarrow[p]{} B$ is the projective space over B associated to a holomorphic vector bundle E on B. Note that if π is projective, then π is locally Kähler.

In [KM], Knudsen and Mumford defined a holomorphic line bundle over B

$$\lambda^{\mathrm{KM}} = (\det(R\pi_*\xi))^{-1} \tag{2.37}$$

For $0 \leqq j \leqq m$, if λ_j^{KM} is the Knudsen-Mumford line bundle associated to ξ_j, for every $y \in B$, we have a canonical isomorphism of the fibers

$$\lambda_{j,y}^{\mathrm{KM}} \simeq (\det H_y^0(E_j))^{-1} \otimes (\det H_y^1(E_j)) \otimes \dots . \tag{2.38}$$

Therefore for every $y \in B$, the fibers $\lambda_{j,y}^{\mathrm{KM}}$ and $\lambda_{j,y}$ are canonically isomorphic.

Theorem 2.12. *If π is locally Kähler and if assumption* (A) *is verified, the canonical isomorphism of the fibers $\lambda_y \simeq \lambda_y^{\mathrm{KM}}$ induces a smooth isomorphism of holomorphic line bundles.*

Proof. Since the statement of the theorem is local on the base B, we can as well assume that $(\pi, g^Z, T^H M)$ is a Kähler fibration.

We first prove the theorem when $\xi = \xi_j$ $(1 \leqq j \leqq m)$. Then $H^0(E_j)$ has constant rank. Also in Theorem 2.10, $H^0(E_j)$ has been endowed with a holomorphic structure. Moreover the characterization of the holomorphic sections of $H^0(E_j)$ shows that $H^0(E_j)$ is endowed with the same holomorphic structure as in Knudsen-Mumford [KM]. Also by [KM, Proposition 8] the identification $\lambda_j^{\mathrm{KM}} \simeq \det H^0(E_j)$ is an isomorphism of holomorphic line bundles. Using the final part of Theorem 2.10 we find that the canonical isomorphism $\lambda_j \simeq \lambda_j^{\mathrm{KM}}$ is holomorphic.

By [KM], the Knudsen-Mumford section σ of $\tilde{\lambda}^{\mathrm{KM}} = \prod_0^m (\lambda_j^{\mathrm{KM}})^{(-1)^j}$ is holomorphic. By Theorem 2.3, we know that $T(\bar{\partial} + v)$ is a holomorphic section of $\tilde{\lambda} = \prod_0^m (\lambda_j)^{(-1)^j}$. Also by Theorem 2.8, for every $y \in B$, $T(\bar{\partial} + v)_y \simeq \sigma_y$. Since for $j \geqq 1$, the canonical isomorphism $\lambda_j \simeq \lambda_j^{\mathrm{KM}}$ is holomorphic, it is now clear that the canonical isomorphism $\lambda_0 \simeq \lambda_0^{\mathrm{KM}}$ is also holomorphic. The theorem is proved. $\square$

Remark 2.13. In Sect. 3, we will establish an analogue of Theorem 2.9 which is valid for all the cohomology groups, and not only for $H^0(E)$. This will permit us to prove Theorem 2.12 even when assumption (A) is not verified.

f) A First Proof of Theorem 0.1

We now assume that the assumptions of Sect. 1 i) are verified, i.e. π defines a locally Kähler fibration. g^Z is still a smooth Hermitian metric on TZ, which induces a Kähler metric on the fibers Z. ξ is a holomorphic Hermitian vector bundle with metric h^ξ, which is such that assumption (A) is verified.

In Sect. 1 i), we have constructed a holomorphic line bundle λ endowed with the Quillen metric associated to (g^Z, h^ξ).

From Theorems 1.27 and 2.12 we obtain the result of [BGS 1, Theorem 0.1].

Theorem 2.14. *The canonical isomorphisms of fibers $\lambda_y \simeq \lambda_y^{\mathrm{KM}}$ is a smooth holomorphic isomorphism of line bundles over B. The curvature of the holomorphic*

Hermitian connection on $\lambda \simeq \lambda^{\mathrm{KM}}$ associated with the Quillen metric of λ is given by

$$2i\pi\left[\int_Z Td\left(\frac{-R^Z}{2i\pi}\right)\mathrm{Tr}\left[\exp-\frac{L^\xi}{2i\pi}\right]\right]^{(2)}. \tag{2.39}$$

3. $\lambda = \lambda^{\mathrm{KM}}$: A Sheaf Theoretic Proof

In this chapter, we give a proof of $\lambda \simeq \lambda^{\mathrm{KM}}$ valid for any locally Kähler fibration, and we obtain [BGS 1, Theorem 0.1] in the general case.

In a) we recall some facts about the Knudsen-Mumford theory of determinant line bundles [KM] and indicate how these extend to the category of smooth and analytic sheaves.

In b) we show that λ and λ^{KM} are isomorphic as smooth line bundles. This follows from Theorem 3.5, which gives two descriptions of the smooth sheaves of cohomology of a family of vector bundles.

In c) we prove that the smooth isomorphism $\lambda \simeq \lambda^{\mathrm{KM}}$ preserves the holomorphic structure, by comparing the $\bar{\partial}$ operators.

In d) we give a proof of [BGS 1, Theorem 0.1.]

a) Determinants of Perfect Complexes

In this section, we outline the basic properties of determinants to be used later on. In particular, given a complex manifold B, we want to know which results of Knudsen-Mumford [KM] (which is written for schemes) extend to the categories of coherent analytic sheaves and modules over the sheaf of C^∞ functions.

Let $\mathcal{O}_B$ be the sheaf of holomorphic functions on B, and $\mathcal{O}_B^\infty$ the sheaf of C^∞ functions. We write P_B (respectively P_B^∞) for the category of locally free $\mathcal{O}_B$ (respectively $\mathcal{O}_B^\infty$) modules of finite rank. Let L_B (respectively L_B^∞) be the category of $\mathcal{O}_B$ (respectively $\mathcal{O}_B^\infty$) line bundles (more precisely graded line bundles (L, α), where L is a line bundle and $\alpha: B \to Z$ is a continuous map as in [KM]; but we shall forget about α, as indicated in [BGS 1, Remark 1.2]).

Let $L\,\mathrm{is}_B$, $L\,\mathrm{is}_B^\infty$, $P\,\mathrm{is}_B$, and $P\,\mathrm{is}_B^\infty$ be the corresponding categories of isomorphisms.

The determinant functors

$$\det: P\,\mathrm{is}_B \to L\,\mathrm{is}_B$$

and

$$\det_\infty: P\,\mathrm{is}_B^\infty \to L\,\mathrm{is}_B^\infty,$$

are given by maximal exterior powers. If

$$0 \to F' \xrightarrow{\alpha} F \xrightarrow{\beta} F'' \to 0$$

is a short exact sequence in P_B, there is an isomorphism

$$i^*(\alpha, \beta): \det F' \otimes \det F'' \xrightarrow{\sim} \det F$$

given locally by the formula

$$\begin{aligned} &i^*(\alpha,\beta)(x_1 \wedge \ldots \wedge x_p \wedge \beta y_1 \wedge \ldots \wedge \beta y_q) \\ &= \alpha x_1 \wedge \ldots \wedge \alpha x_p \wedge y_1 \wedge \ldots \wedge y_q. \end{aligned}$$

A similar isomorphism $i^*_\infty(\alpha,\beta)$ exists for $\det_\infty$. These satisfy compatibility properties given by Proposition 1 in [KM]. Furthermore, there is a canonical isomorphism

$$\det_\infty\left(F\underset{\mathcal{O}_B}{\otimes}\mathcal{O}_B\right)\simeq\det(F)\underset{\mathcal{O}_B}{\otimes}\mathcal{O}_B^\infty .$$

One can also define the determinant $\det(V)$ of a vector bundle V with sheaf of sections $\Gamma(V)$ in a such a way that $\det\Gamma(V)=\Gamma(\det V)$.

Let $C^\cdot_B$ (respectively $C^{\cdot\infty}_B$) be the category of bounded complexes in P_B (respectively P^∞_B). By the same method as in [KM, Theorem 1], we have

Proposition 3.1. *There is one, and, up to canonical isomorphism only one determinant functor*

$$(f,i): C^\cdot\mathrm{is}_B\to L\,\mathrm{is}_B \qquad (\textit{respectively } (f,i)_\infty: C^\cdot\mathrm{is}^\infty_B\to L\,\mathrm{is}^\infty_B)$$

satisfying the conditions of [KM, *Definition 1*].

Remark 3.2. Proposition 2 of [KM] still holds, with affine open sets being replaced by small open sets (or Stein open sets in the analytic case). Again

$$(f,i)_\infty\left(K\underset{\mathcal{O}_B}{\otimes}\mathcal{O}_B^\infty\right)\simeq\left(((f,i)(K))\underset{\mathcal{O}_B}{\otimes}\mathcal{O}_B^\infty\right)$$

if K is in $C^\cdot\mathrm{is}_B$.

Let $R=\mathcal{O}_B$ or $\mathcal{O}_B^\infty$. A complex $F^\cdot$ of R-modules is called *perfect* if, locally on B, there exists a quasi-isomorphism

$$G^\cdot\to F^\cdot$$

with $G^\cdot$ a bounded complex of locally free R-modules. As in [KM] we may extend the functor (f,i) [respectively $(f,i)_\infty$] to the category of perfect R-modules. The key lemma is:

Lemma 3.3. *Let $G^\cdot$ be a bounded complex of locally free R-modules ($R=\mathcal{O}_B$ or $\mathcal{O}_B^\infty$), and $F^\cdot$ an acyclic complex of R-modules. If*

$$f: G^\cdot\to F^\cdot$$

is a map of complexes, there exists an open cover $B=\bigcup_\alpha U_\alpha$ of B and nullhomotopies $h_\alpha: G^\cdot_{|U_\alpha}\to F^\cdot_{|U_\alpha}$ such that $h_\alpha d+dh_\alpha=f|U_\alpha$.

Proof. The assertion being true when B is reduced to a point, for every $y\in B$ we can find an homotopy $h_y: G^\cdot_y\to F^\cdot_y$ between 0 and f_y. Since each G^i has finite rank and $G^\cdot$ is bounded, h_y extends to a map h in a neighborhood U of y. The identity $dh+hd=f$ is true in some (possibly smaller) neighborhood of x. □

From this lemma, we obtain a modified version of [KM, Proposition 4], where affine schemes are replaced by sufficiently small open sets. The results analogous to Theorem 2 in [KM], and the remark after it, are true.

Let $\pi: M\to B$ be a proper map of complex spaces. By a theorem of Grauert [G], if F is a coherent sheaf of $\mathcal{O}_M$-modules, for all $i\geqq 0$, the $\mathcal{O}_B$-module $R^i\pi_*F$ is coherent. If $i>\dim(M)$, then $R^i\pi_*F=0$. The functor $R\pi_*$ maps the derived category of $\mathcal{O}_M$-modules to the derived category of $\mathcal{O}_B$-modules, and sends coherent sheaves to complexes with coherent cohomology. If B is a complex manifold, for every $y \in B$, the local ring $\mathcal{O}_{B,y}$ is regular, hence all coherent analytic sheaves on B are perfect, and more generally any complex with bounded coherent cohomology is perfect. Hence we obtain (see [KM, p. 46]):

Theorem 3.4. *Let $\pi: M \to B$ be a proper morphism of complex spaces, with B a complex manifold. Then to every complex F of $\mathcal{O}_M$-modules with bounded coherent cohomology, we can associate a (graded) invertible holomorphic sheaf* $\det(R\pi_* F)$ *on B. For every true triangle on M,*

$$O \to F \to G \to H \to 0,$$

we have an isomorphism

$$\det(R\pi_* F) \otimes \det(R\pi_* H) \simeq \det(R\pi_* G),$$

which is functorial with respect to isomorphisms of true triangles.

For F a coherent sheaf on M we define

$$\lambda^{\mathrm{KM}}(F) = \det(R\pi_* F)^{-1},$$

the Knudsen-Mumford determinant attached to F and π. Proposition 8 of [KM] is also true in our case. In particular, if $R^k\pi_* F$ is locally free for all k, we get

$$\lambda^{\mathrm{KM}}(F) = \bigotimes_{k \geqq 0} \det(R^k \pi F)^{(-1)^{k+1}}.$$

When $R^k \pi_* F = 0$ for every $k > 0$ we have

$$\lambda^{\mathrm{KM}}(F) = \det(R^0 \pi_* F)^{-1}.$$

Let now $\pi: M \to B$ be a smooth proper map of complex manifolds and ξ a finite dimensional complex vector bundle over M. Let

$$E^{\cdot}: E^0 \xrightarrow{\bar\partial} E^1 \xrightarrow{\bar\partial} E^2 \xrightarrow{\bar\partial} \dots \to E^l$$

be the complex of smooth vector bundles over B considered in [BGS 2, Definition 1.10]. Call $\mathcal{E}^{\cdot}$ the complex of $\mathcal{O}_B^{\infty}$-modules given by smooth sections of $E^{\cdot}$. We know from Sect. 1b) that, given $y \in B$, we can find an open neighborhood U of y and a smooth complex K^b of finite dimensional vector bundles on U which is quasi-isomorphic to $E^{\cdot}$ on U. Therefore $\mathcal{E}^{\cdot}$ is a perfect complex of $\mathcal{O}_B^{\infty}$-modules. By what we said above (before Lemma 3.3.), we can define an invertible sheaf $\det(\mathcal{E}^{\cdot})$ over B. This is precisely the sheaf of C^{∞} sections of the line bundle λ defined in Definition 1.1.

b) The Knudsen-Mumford Determinant is Smoothly Isomorphic to λ

Let M be a complex manifold. We write $\mathcal{D}_M^{\cdot}$ for the sheaf of Dolbeault complexes on M. For each $p \geqq 0$, $\mathcal{D}_M^p$ is the sheaf of C^{∞} sections of the vector bundle $\Lambda^p T^{*(0,1)} M$. The differential $\bar\partial^M$ is $\mathcal{O}_M$ linear. By [Go, II.3.7], if F is any sheaf of $\mathcal{O}_M$-modules, the complex

$$\mathcal{D}_M^{\cdot}(F) = F \bigotimes_{\mathcal{O}_M} \mathcal{D}_M^{\cdot}$$

is a resolution of F since $\mathcal{O}_M^\infty$ hence each $\mathcal{D}_M^p$ is a flat $\mathcal{O}_M$-module [M]. Furthermore, $\mathcal{D}_M^p(F)$ is fine by [Go, II.3.7.3]. Therefore, for any map $\pi: M \to B$, the object $R\pi_* F$ in the derived category of $\mathcal{O}_B$-modules is canonically isomorphic to $\pi_*(\mathcal{D}_M^\cdot(F))$, i.e. $R^p\pi_* F$ is the sheaf associated to the presheaf on B:

$$U \mapsto H^p(\mathcal{D}_M^\cdot(F)(\pi^{-1}(U))),$$

or

$$R^p\pi_* F = \mathcal{H}^p(\pi_* \mathcal{D}_M^\cdot(F)),$$

Note that $\pi_* \mathcal{D}_M^\cdot(F)$ is a complex of $\mathcal{O}_B$-modules.

Now suppose that $\pi: M \to B$ is a smooth and proper map of complex manifolds which is locally Kähler in the sense of Definition 1.25. Let TZ be the relative tangent bundle (on M). The relative Dolbeaut complex $\mathcal{D}_Z^\cdot$ is such that $\mathcal{D}_Z^p$ is the sheaf of C^∞ sections of $\Lambda^p(T^{*(0,1)}Z)$ on M; its differential $\bar\partial^Z$ is the $\bar\partial$-operator along the fibers. Note that $\bar\partial^Z$ is both $\mathcal{O}_M$ and $\mathcal{O}_B^\infty$-linear. When F is any sheaf of $\mathcal{O}_M$-modules, define $\mathcal{D}_Z^\cdot(F) = F \bigotimes_{\mathcal{O}_M} \mathcal{D}_Z^\cdot$. Taking its direct image under π we get a complex $\pi_* \mathcal{D}_Z^\cdot(F)$ of sheaves of $\mathcal{O}_B^\infty$-modules. For every $p \geqq 0$, we write $\mathcal{H}_{\bar\partial}^p(F)$ for the cohomology of $\pi_* \mathcal{D}_Z^\cdot(F)$. This is a sheaf of $\mathcal{O}_B^\infty$-modules, while $R^p\pi_* F$ is a sheaf of $\mathcal{O}_B$-module. Notice that when F is the sheaf of sections of the holomorphic bundle $\xi, \pi_* \mathcal{D}_Z^\cdot(F) = \mathcal{E}^\cdot$ [cf. end of a)]. The natural map $T_M^* \to T_Z^*$ induces a map of complexes $\mathcal{D}_M^\cdot \to \mathcal{D}_Z^\cdot$. Hence, on B, for every $\mathcal{O}_M$-module F, we get a map

$$\pi_* \mathcal{D}_M^\cdot(F) \to \pi_* \mathcal{D}_Z^\cdot(F), \tag{3.1}$$

which induces a map on cohomology sheaves

$$R^p\pi_* F \to \mathcal{H}_{\bar\partial}^p(F),$$

and, by extending scalars,

$$\varrho_p: (R^p\pi_* F) \bigotimes_{\mathcal{O}_B} \mathcal{O}_B^\infty \to \mathcal{H}_{\bar\partial}^p(F), \qquad p \geqq 0.$$

Theorem 3.5. *Suppose that $\pi: M \to B$ is a proper smooth map between complex manifolds and that F is the sheaf of holomorphic sections of a holomorphic vector bundle ξ on M. Then, for all $p \geqq 0$, the map ϱ_p is an isomorphism.*

Proof. To prove the theorem it is enough to show that ϱ_p is an isomorphism locally on B. Therefore we shall fix $y \in B$ and restrict our attention to arbitrarily small open neighborhoods of y.

By [G] we know that the sheaves $R^p\pi_* F$ are coherent. In particular, the stalk $(R^p\pi_* F)_y$ is a finitely generated $\mathcal{O}_{B,y}$ module. The local ring $\mathcal{O}_{B,y}$ is regular, hence any finitely generated $\mathcal{O}_{B,y}$ has a finite resolution by finite rank free $\mathcal{O}_{B,y}$-modules.

We recall a standard result from homological algebra.

Lemma 3.6. *Let R be a regular local ring, and $A^\cdot$ a bounded complex of R-modules whose cohomology groups $H^p(A)$, $p \geqq 0$, are finitely generated. Then there is a bounded complex of finitely generated free R-modules $P^\cdot$ and a quasi-isomorphism*

$$\varphi: P^\cdot \to A^\cdot.$$

Proof. We proceed by induction on the length of $A^\cdot$. If this length is zero $A^\cdot \simeq A^k$ is concentrated in a single degree k, hence $A^k \simeq H^k(A)$ is finitely generated, and it has a resolution $P^\cdot \to A^k$ of the required form since R is regular.

Assume now that the length of $A^\cdot$ is positive and let $n = \sup\{k \mid A^k \neq 0\}$. Choose a free R-module of finite rank P^n mapping onto $H^n(A^\cdot) = A^n/d(A^{n-1})$. Since P^n is free, the map $P^n \to H^n$ lifts to a map $\varphi^n : P^n \to A^n$. Consider the complex

$$\tilde{A}^\cdot : \ldots \to A^{n-3} \xrightarrow{d} A^{n-2} \xrightarrow{(d,0)} A^{n-1} \oplus P^n \xrightarrow{d-\varphi^n} A^n \to 0 .$$

From the exact sequence of complexes $0 \to A^\cdot \to \tilde{A}^\cdot \to P^n \to 0$ we get, in cohomology,

$$0 \to H^{n-1}(A^\cdot) \to H^{n-1}(\tilde{A}^\cdot) \to P^n \to H^n(A^\cdot) \to 0 ,$$

and

$$H^p(\tilde{A}^\cdot) \simeq H^p(A^\cdot)$$

if $p < n-1$. Hence $\tilde{A}^\cdot$ satisfies the hypotheses of the Lemma, but $H^n(\tilde{A}^\cdot) = 0$. Let now

$$\bar{A}^\cdot = \ker\left(\tilde{A}^\cdot \to \left(A^n \xrightarrow{\mathrm{id}} A^n\right)\right),$$

i.e.

$$\bar{A}^\cdot : \ldots A^{n-3} \to A^{n-2} \to \ker(A^{n-1} \oplus P^n \to A^n) \to 0 .$$

The obvious map $\bar{A}^\cdot \to \tilde{A}^\cdot$ is a quasi-isomorphism, so $\bar{A}^\cdot$ has finitely generated cohomology, and its length is smaller than the length of $A^\cdot$. Hence there is a quasi-isomorphism

$$\psi : Q^\cdot \to \bar{A}^\cdot ,$$

with $Q^\cdot$ a bounded complex of finitely generated free R-modules. We may assume $Q^p = 0$ if $p \geqq n$. The induced map $Q^\cdot \to \tilde{A}^\cdot$ is also a quasi-isomorphism:

$$\begin{tikzcd}
\ldots \arrow[r] & A^{n-2} \arrow[r] & A^{n-1} \oplus P^n \arrow[r, "d-\varphi^n"] & A^n \arrow[r] & 0 \\
\ldots \arrow[r] \arrow[u] & Q^{n-2} \arrow[r] \arrow[u, "\psi^{n-2}"'] & Q^{n-1} \arrow[r] \arrow[u, "\alpha\oplus\beta"'] & 0 \arrow[r] \arrow[u] & 0
\end{tikzcd} .$$

Since $Q^n = 0$ we have $d \circ \alpha = \varphi^n \circ \beta$. So we get a map of complexes

$$\begin{tikzcd}
\ldots \arrow[r] & A^{n-2} \arrow[r, "d"] & A^{n-1} \arrow[r, "d"] & A^n \arrow[r] & 0 \\
\ldots \arrow[r] \arrow[u] & Q^{n-2} \arrow[r] \arrow[u, "\psi^{n-2}"'] & Q^{n-1} \arrow[r, "\beta"'] \arrow[u, "\alpha"'] & P^n \arrow[r] \arrow[u, "\varphi^n"'] & 0
\end{tikzcd}$$

which is easily seen to be a quasi-isomorphism. □

Returning to the proof of Theorem 3.5, we may use Lemma 3.6 to construct a map of complexes of $\mathcal{O}_{B,y}$-modules

$$\varphi_y : \mathcal{P}_y^\cdot \to \pi_* \mathcal{D}_M^\cdot(F)_y ,$$

which is a quasi-isomorphism and such that $\mathcal{P}_y^\cdot$ is a bounded complex of finitely generated bree $\mathcal{O}_{B,y}$-modules. By the finiteness of $\mathcal{P}_y^\cdot$, the map φ_y extends from the stalk at y to a neighborhood of y, giving a map of complexes of $\mathcal{O}_B$-modules (B small enough)

$$\varphi: \mathscr{P}^{\cdot} \to \pi_* \mathscr{D}_M^{\cdot}(F).$$

Since both complexes have bounded coherent cohomology, and φ_y is a quasi-isomorphism, the map φ is a quasi-isomorphism in a neighborhood of y. We extend scalars and compose with the map (3.1) to get a map of complexes of $\mathcal{O}_B^\infty$-modules

$$\varphi^\infty: \mathscr{P}^{\cdot} \bigotimes_{\mathcal{O}_B} \mathcal{O}_B^\infty \to \pi_* \mathscr{D}_Z^{\cdot}(F) = \mathscr{E}^{\cdot}.$$

Let $P^{\cdot}$ be the complex of holomorphic vector bundles on B whose sheaf is $\mathscr{P}^{\cdot}$. The map φ^∞ is induced by a smooth map of complexes of vector bundles

$$\varphi: P^{\cdot} \to E^{\cdot}.$$

Lemma 3.7. *The fiber*

$$\varphi_y: P_y^{\cdot} \to E_y^{\cdot}$$

of φ at y is a quasi-isomorphism.

Proof. Let $j: y \to B$ and $i: Z_y = \pi^{-1}(y) \to M$ be the inclusions. We have a commutative diagram

$$\begin{tikzcd} Z_y \arrow[r, "i"] \arrow[d, "\pi"] & M \arrow[d, "\pi"] \\ y \arrow[r, "j"] & B \end{tikzcd}.$$

Let C_y be the constant sheaf with stalk C at y. Since $P_y^{\cdot}$ is equal to $\mathscr{P}^{\cdot} \bigotimes_{\mathcal{O}_B} j_* C_y$, it is quasi-isomorphic to $\pi_* \mathscr{D}_M^{\cdot}(F) \bigotimes_{\mathcal{O}_B} j_* C_y$ [using the fact that $\pi_* \mathscr{D}_M^{\cdot}(F)$ is flat over $\mathcal{O}_B$]. Now, since $i_* \mathcal{O}_{Z_y} = \mathcal{O}_M \bigotimes_{\mathcal{O}_B} \pi^* j_* C_y$, we get

$$\pi_* \mathscr{D}_M(F) \bigotimes_{\mathcal{O}_B} j_* C_y = \pi_* \mathscr{D}_M \left(F \bigotimes_{\mathcal{O}_M} i_* \mathcal{O}_{Z_y} \right).$$

Furthermore, if F_y is the sheaf of holomorphic sections of ξ_y over Z_y, we have

$$F \bigotimes_{\mathcal{O}_M} i_*(i^* F) = i_*(F_y).$$

On the other hand

$$E_y^{\cdot} = \pi_* \mathscr{D}_Z^{\cdot}(F) \bigotimes_{\mathcal{O}_B^\infty} j_* C_y = \pi_* i_* (\mathscr{D}_{Z_y}^{\cdot}(F_y)).$$

Hence it is enough to show that the map

$$\pi_* (\mathscr{D}_M^{\cdot}(i_* F_y)) \to \pi_* i_* \mathscr{D}_{Z_y}^{\cdot}(F_y) = j_* \pi_* \mathscr{D}_{Z_y}^{\cdot}(F_y)$$

is a quasi-isomorphism. The target of this map computes $j_* R\pi_*(F_y)$, and its domain computes $R\pi_* i_*(F_y)$. Both i and j are finite maps, so $Ri_* = i_*$ and $Rj_* = j_*$, and we get a chain of quasi-isomorphisms

$$R\pi_* i_*(F_y) \simeq R\pi_* Ri_*(F_y) \simeq R(\pi_* i_*)(F_y) \simeq R(j_*\pi_*)(F_y) \simeq j_* R\pi_*(F_y). \quad \square$$

This lemma just means that, to compute the cohomology of a coherent sheaf on Z_y, one may use Dolbeault resolutions on M or on Z_y.

We want to pass from the fiberwise statement of Lemma 3.7 to a global one, saying that φ is a quasi-isomorphism in a neighborhood of y. Since $(B, \mathcal{O}_B^\infty)$ is not an Oka space, this does not follow directly. Using the notations of Sect. 1b), over the open set U^b, we can introduce the subcomplex

$$K^b \subset E^\cdot$$

of (1.1) and the corresponding $\mathcal{O}_B^\infty$*-module* $\mathcal{K}^b \subset \mathcal{E}^\cdot$ (for $b > 0$ well chosen and B small enough).

Lemma 3.8. *The inclusion $\mathcal{K}^b \subset \mathcal{E}^\cdot$ is a quasi-isomorphism of complexes of $\mathcal{O}_B^\infty$-modules.*

Proof. Over the open set U^b defined in Sect. 1b), the projector $P^b: E^\cdot \to K^b$ is a smooth family of regularizing operators [B 1, Proposition 2.13]. The family of operators $D^2 = (\bar\partial + \bar\partial^*)^2$ has an inverse G on the orthogonal complement of K^b. Since D^2 commutes with P^b, $\bar\partial$ and $\bar\partial^*$, we have

$$\bar\partial\bar\partial^* G(1-P^b) + \bar\partial^* G(1-P^b)\bar\partial = 1 - P^b.$$

Hence $\bar\partial^* G(1-P^b)$ is a homotopy between 1 and P^b, which depends smoothly on the base point. Therefore the inclusions $\mathcal{K}^b \subset \mathcal{E}^\cdot$ and $K^b \subset E^\cdot$ are quasi-isomorphisms. $\square$

We can now complete the proof of Theorem 3.5. On B, we have maps of complexes of $\mathcal{O}_B^\infty$-modules:

$$\varphi^\infty : \mathcal{P}^\cdot \underset{\mathcal{O}_B}{\otimes} \mathcal{O}_B^\infty \to \mathcal{E}^\cdot,$$

and

$$\psi : \mathcal{K}^b \to \mathcal{E}^\cdot.$$

Since by Lemma 3.8, ψ is a quasi-isomorphism and $\mathcal{P}^\cdot$ is bounded and locally free, we can lift φ^∞ to a map

$$\tilde\varphi : \mathcal{P}^\cdot \underset{\mathcal{O}_B}{\otimes} \mathcal{O}_B^\infty \to \mathcal{K}^b,$$

with $\psi \circ \tilde\varphi$ homotopic to φ^∞.

On the fibers over y we know that ψ_y and φ_y^∞ are quasi-isomorphisms (Lemma 3.7 and Lemma 3.8) therefore $\tilde\varphi_y$ is a quasi-isomorphism. Since $P^\cdot$ and K^b are bounded complexes of finite dimensional C^∞ bundles on B, the map $\tilde\varphi$ induces, in a neighborhood of y, a quasi-isomorphism

$$P^\cdot \to K^b$$

and, for all $p \geqq 0$, the map ϱ_p is an isomorphism. $\square$

Corollary 3.9. *Let λ^{KM} be the holomorphic line bundle on B whose sheaf of holomorphic sections is $\lambda^{KM}(\xi)=\det(R\pi_*\xi)^{-1}$. As smooth bundles, λ^{KM} and λ are canonically isomorphic. In particular the Quillen metric is smooth on λ^{KM}.*

Proof. From Theorem 3.5, we deduce that the $\mathcal{O}_B^\infty$*-modules* $\mathcal{H}_{\bar\partial}^p(F)$ are perfect. Therefore by [KM, Proposition 8] there is a canonical isomorphism

$$\det(\mathcal{E}^\cdot)\simeq \bigotimes_{p\geq 0}\det\mathcal{H}_{\bar\partial}^p(F))^{(-1)^{p+1}}.$$

We know that the left-hand side is the sheaf of smooth sections of λ, and, by Theorem 3.5, the right-hand side is canonically isomorphic to

$$\lambda^{KM}(\xi)\bigotimes_{\mathcal{O}_B}\mathcal{O}_B^\infty. \quad \square$$

c) Comparison of the Complex Structures

Let $p>0$ and let $\mathcal{H}_{\bar\partial}^p(F)=H^p(\mathcal{E}^\cdot)$ the cohomology of the complex $\mathcal{E}^\cdot=\pi_*\mathcal{D}_Z^\cdot(F)$ of $\mathcal{O}_B^\infty$-modules. We shall define a map

$$\bar\partial_B:\mathcal{H}_{\bar\partial}^p(F)\to T^{*(0,1)}B\bigotimes_{\mathcal{O}_B^\infty}\mathcal{H}_{\bar\partial}^p(F),$$

such that $\bar\partial_B^2=0$ and, if $f\in\mathcal{O}_B^\infty$ and $h\in H^p(\mathcal{E}^\cdot)$,

$$\bar\partial_B(fh)=f\bar\partial_B(h)+\bar\partial^B(f)\otimes h. \tag{3.2}$$

Let $p\geq 0$ and let $\alpha\in E^p$ be a section of $\Lambda^p(T^{*(0,1)}Z)\otimes\xi$ such that $\bar\partial^Z(\alpha)=0$. Choose any lift $\tilde\alpha\in\Lambda^pT^{*(0,1)}M\otimes\xi$ of α. The form $\bar\partial^M(\tilde\alpha)$ will map to zero in $\Lambda^{p+1}T^{*(0,1)}Z\otimes\xi$. From the exact sequence

$$0\to\pi^*T^{*(0,1)}B\to T^{*(0,1)}M\to T^{*(0,1)}Z\to 0,$$

we get a map

$$\begin{aligned}\varrho:\ker\{&\Lambda^{p+1}(T^{*(0,1)}M)\otimes\xi\to\Lambda^{p+1}(T^{*(0,1)}Z)\otimes\xi\}\\ &\to\pi^*T^{*(0,1)}B\otimes\Lambda^p(T^{*(0,1)}Z)\otimes\xi.\end{aligned}$$

We define $\bar\partial_B(\alpha)=\varrho(\bar\partial^M(\tilde\alpha))$.

Proposition 3.10. *The map $\bar\partial_B$ induces a morphism $\mathcal{H}_{\bar\partial}^p(F)\to T^{*(0,1)}B\bigotimes_{\mathcal{O}_B^\infty}\mathcal{H}_{\bar\partial}^p(F)$ satisfying* (3.2). *Its kernel contains $\varrho_p(R^p\pi_*F)$.*

Proof. First notice that $\bar\partial^Z(\bar\partial_B(\alpha))=\varrho((\bar\partial^M)^2\alpha)=0$. To see that the class of $\bar\partial_B(\alpha)$ does not depend on the choice of $\tilde\alpha$, let $\tilde\alpha'$ be another lift of α. Then $\tilde\alpha-\tilde\alpha'$ is in the image of the map

$$\pi^*T^{*(0,1)}B\otimes\Lambda^{p-1}(T^{*(0,1)}M)\to\Lambda^p(T^{*(0,1)}M),$$

say $\tilde\alpha-\tilde\alpha'=\beta\otimes\gamma$. We get

$$\varrho(\bar\partial^M(\tilde\alpha-\tilde\alpha'))=\varrho\bar\partial^M(\beta\otimes\gamma)=\varrho(\bar\partial^M\beta\otimes\gamma-\beta\otimes\bar\partial^M\gamma).$$

We have

$$\bar\partial^M\beta\otimes\gamma\in\Lambda^2\pi^*T^{*(0,1)}B\otimes\Lambda^{p-1}T^{*(0,1)}M,$$

(since $\Lambda^{\cdot}\pi^* T^{*(0,1)}B$ is stable under $\bar\partial^M$), hence $\varrho(\bar\partial^M\beta\otimes\gamma)=0$. Therefore

$$\varrho(\bar\partial^M(\tilde\alpha-\tilde\alpha'))=\varrho(\beta\otimes\bar\partial^M\gamma)=\bar\partial^Z\varrho(\beta\otimes\gamma),$$

and the class of $\bar\partial_B(\alpha)$ in $\mathscr{H}^p_{\bar\partial}(F)\bigotimes_{\mathcal{O}^\infty_B} T^{*(0,1)}B$ does not depend on the choice of $\tilde\alpha$.

Using $(\bar\partial^M)^2=0$ we get $\bar\partial_B^2=0$ and (3.2) is easily shown. Finally $\varrho_p(R^p\pi_*F)$ is killed by $\bar\partial_B$, since it is represented by sections $\tilde\alpha$ in $\Lambda^p(T^{*(0,1)}M)\otimes\xi$ such that $\bar\partial^M\tilde\alpha=0$. $\square$

Let us now consider a Kähler fibration (π,g^Z,T^HM) as in [BGS 2, Sect. 1c], with associated (1,1) form $\omega=\omega^H+\omega^Z$. According to [BGS 2, Theorem 1.14], we have an operator

$$\tilde V'':\mathscr{E}^{\cdot}\to T^{*(0,1)}B\bigotimes_{\mathcal{O}^\infty_B}\mathscr{E}^{\cdot}$$

such that $(\tilde V'')^2=0$, which commutes with $\bar\partial^Z$, hence induces a complex structure on $H^p(\mathscr{E}^{\cdot})$.

Theorem 3.11. *For every $p\geqq 0$, the operators $\tilde V''$ and $\bar\partial_B$ on $H^p(\mathscr{E}^{\cdot})$ coincide.*

Proof. It is enough to consider the case where ξ is trivial. Let us first recall the definition of $\tilde V''$ in [BGS 2, Definition 1.13]. The form ω induces maps

$$\lrcorner\,\omega: T^{(0,1)}M\to T^{*(1,0)}M$$

and

$$\lrcorner\,\omega: T^{(1,0)}M\to T^{*(0,1)}M$$

which, by definition of a Kähler fibration, induce an isomorphism $\lrcorner\,\omega^Z$ from $T^{(0,1)}Z$ to $T^{*(1,0)}Z$. If η is a section of $T^{(0,1)}B$, its horizontal lift η^H is the unique section of $T^{(0,1)}M$ such that $\eta^H\lrcorner\,\omega$ has image zero in $T^{*(1,0)}Z$. In other words

$$T^{H(0,1)}M=\ker\left(T^{(0,1)}M\xrightarrow{\lrcorner\,\omega}T^{*(1,0)}M\to T^{*(1,0)}Z\right).$$

If $\alpha\in T^{*(0,1)}Z$ there is a unique element $\beta\in T^{(1,0)}Z$ such that $\alpha=\beta\lrcorner\,\omega^Z$. We defined

$$\tilde V''_\eta(\alpha)=V''_{\eta^H}(\beta)\lrcorner\,\omega^Z,$$

where V'' is the standard $\bar\partial^M$ operator on the holomorphic bundle $T^{(1,0)}Z$.

Let us simplify our notations by setting

$$A^p(M)=C^\infty \text{ sections of } \Lambda^pT^{(1,0)}M,$$

$$B^p(M)=C^\infty \text{ sections of } \Lambda^pT^{*(0,1)}M,$$

$$A^p(Z)=C^\infty \text{ sections of } \Lambda^pT^{(1,0)}Z,$$

$$B^p(Z)=C^\infty \text{ sections of } \Lambda^pT^{*(0,1)}Z.$$

In particular $B^p(Z)=E^p$ in the notations of 3.a). The key step in the proof of Theorem 3.11 is the following Lemma.

Lemma 3.12. *For every section η of $T^{(0,1)}B$ the composite map*

$$A^1(Z) \xrightarrow{\nabla''_{\eta^H}} A^1(Z) \xrightarrow{\lrcorner\,\omega^Z} B^1(Z)$$

is equal to the composite

$$A^1(Z) \hookrightarrow A^1(M) \xrightarrow{\lrcorner\,\omega} B^1(M) \xrightarrow{\bar\partial_M} B^2(M) \xrightarrow{i_{\eta^H}} B^1(M) \to B^1(Z).$$

Proof. This is a statement about C^∞ sections of bundles on M, therefore we can work locally on M and assume that $M = Z \times B$, with holomorphic coordinates $z_1, \dots, z_{n/2|i}, y_1, \dots, y_m$. By linearity we can assume that $\eta = \dfrac{\partial}{\partial \bar y_b}$ for some b. We omit the summation signs and write

$$\omega = s^{ij} dz_i \wedge d\bar z_j + t^{i\ell} dz_i \wedge d\bar y_\ell + u^{kj} d\bar z_j \wedge dy_k + v^k_\ell dy_k \wedge d\bar y_\ell .$$

Since ω^Z the metric (s^{ij}) is positive definite. Since ω is closed we have

$$\frac{\partial s^{ij}}{\partial \bar z_p} = \frac{\partial s^{ip}}{\partial \bar z_j} \quad \text{and} \quad \frac{\partial t^{i\ell}}{\partial \bar z_p} = \frac{\partial s^{ip}}{\partial \bar y_\ell}. \tag{3.3}$$

Let $\beta = \zeta^i \dfrac{\partial}{\partial z_i} \in A^1(Z)$. We shall compare its two images in $B^1(Z)$. We can first write

$$\eta^H = f_p \frac{\partial}{\partial \bar z_p} + \frac{\partial}{\partial \bar y_b},$$

where the f_p's are determined by the condition

$$s^{ip} f_p + t^{ib} = 0 \quad \forall i = 1, \dots, n/2, \tag{3.4}$$

which expresses the fact that $\eta^H \lrcorner\, \omega$ maps to zero in $B^1(Z)$. Since

$$\nabla''(\beta) = d\bar z_p \otimes \frac{\partial \zeta^i}{\partial \bar z_p} \frac{\partial}{\partial z_i} + d\bar y_\ell \otimes \frac{\partial \zeta^i}{\partial \bar y_\ell} \frac{\partial}{\partial z_i},$$

we get

$$\nabla''_{\eta^H}(\beta) = \frac{\partial \zeta^i}{\partial \bar z_p} f_p \frac{\partial}{\partial z_i} + \frac{\partial \zeta^i}{\partial \bar y_b} \frac{\partial}{\partial z_i},$$

and

$$\nabla''_{\eta^H}(\beta) \lrcorner\, \omega^Z = \left(\frac{\partial \zeta^i}{\partial \bar z_p} f_p + \frac{\partial \zeta^i}{\partial \bar y_b} \right) s^{ij} d\bar z_j . \tag{3.5}$$

On the other hand,

$$\begin{aligned} i_{\eta^H} \bar\partial^M(\beta \lrcorner\, \omega) &= i_{\eta^H} \bar\partial^M(\zeta^i s^{ij} d\bar z_j + \zeta^i t^{i\ell} d\bar y_\ell) \\ &= i_{\eta^H} \Big(\frac{\partial}{\partial \bar z_k} (\zeta^i s^{ij}) d\bar z_k \wedge d\bar z_j + \frac{\partial}{\partial \bar z_k} (\zeta^i t^{i\ell}) d\bar z_k \wedge d\bar y_\ell \\ &\quad + \frac{\partial}{\partial \bar y_\ell} (\zeta^i s^{ij}) d\bar y_\ell \wedge d\bar z_j + \frac{\partial}{\partial \bar y_k} (\zeta^i t^{i\ell}) d\bar y_k \wedge d\bar y_\ell \Big). \end{aligned}$$

We want to restrict this form to $B^1(Z)$ so we need only compute the coefficient of $d\bar{z}_j$. Since

$$\eta^H = f_p \frac{\partial}{\partial \bar{z}_p} + \frac{\partial}{\partial \bar{y}_b},$$

we get that the image of $i_{\eta^H}\bar{\partial}^M(\beta \lrcorner \omega)$ in $B^1(Z)$ is equal to

$$-\frac{\partial}{\partial \bar{z}_k}(\zeta^i t^{ib}) d\bar{z}_k + \frac{\partial}{\partial \bar{y}_b}(\zeta^i s^{i,j}) d\bar{z}_j + \left(\frac{\partial}{\partial \bar{z}_p}(\zeta^i s^{ij}) - \frac{\partial}{\partial \bar{z}_j}(\zeta^i s^{ip})\right) f_p d\bar{z}_j .$$

After reindexing, we obtain

$$\begin{aligned}
&\left[\frac{\partial}{\partial \bar{y}_b}(\zeta^i s^{ij}) - \frac{\partial}{\partial \bar{z}_j}(\zeta^i t^{ib}) + \left(\frac{\partial}{\partial \bar{z}_p}(\zeta^i s^{ij}) - \frac{\partial}{\partial \bar{z}_j}(\zeta^i s^{ip})\right) f_p\right] d\bar{z}_j \\
&\quad = \left[\frac{\partial}{\partial \bar{y}_b}(\zeta^i s^{ij}) - \frac{\partial}{\partial \bar{z}_j}(\zeta^i t^{ib}) + \left(\frac{\partial \zeta^i}{\partial \bar{z}_p} s^{ij} - \frac{\partial \zeta^i}{\partial \bar{z}_j} s^{ip}\right) f_p\right] d\bar{z}_j
\end{aligned}$$

(since ω^Z is closed under $\bar{\partial}^Z$)

$$= \left[\frac{\partial \zeta^i}{\partial \bar{y}_b} s^{ij} - \frac{\partial \zeta^i}{\partial \bar{z}_j} t^{ib} + \frac{\partial \zeta^i}{\partial \bar{z}_p} s^{ij} f_p + \frac{\partial \zeta^i}{\partial \bar{z}_j} \partial \bar{z}_j t^{ib}\right] d\bar{z}_j$$

[by (3.3) and (3.4)]

$$= \left[\frac{\partial \zeta^i}{\partial \bar{y}_b} s^{ij} + \frac{\partial \zeta^i}{\partial \bar{z}_p} s^{ij} f_p\right] d\bar{z}_j . \tag{3.6}$$

By (3.5) and (3.6) the lemma follows. □

Let now $\alpha \in B^p(Z)$ with $p \geqq 1$. By linearity we can assume that

$$\alpha = (\beta_1 \lrcorner \omega^Z) \wedge \ldots \wedge (\beta_p \lrcorner \omega^Z)$$

with $\beta_j \in A^1(Z)$, $j = 1, \ldots, p$. A lift $\tilde{\alpha} \in B^p(M)$ of α is

$$\tilde{\alpha} = (\beta_1 \lrcorner \omega) \wedge \ldots \wedge (\beta_p \lrcorner \omega) = (\beta_1 \wedge \ldots \wedge \beta_p) \lrcorner \omega^p .$$

To get the theorem, we just need to show that $i_{\eta^H}\bar{\partial}^M(\tilde{\alpha})$ and $\nabla_{\eta^H}(\beta_1 \wedge \ldots \wedge \beta_p) \lrcorner \omega^p$ have the same restriction in $B^{p+1}(Z)$. By Leibnitz rule we get

$$\begin{aligned}
\nabla_{\eta^H}(\beta_1 \wedge \ldots \wedge \beta_p) \lrcorner \omega^p &= \sum_j (\beta_1 \wedge \ldots \wedge \nabla_{\eta^H}(\beta_j) \wedge \ldots \wedge \beta_p) \lrcorner \omega^p \\
&= \sum_j (\beta_1 \lrcorner \omega) \wedge \ldots \wedge (\nabla_{\eta^H}(\beta_j) \lrcorner \omega) \wedge \ldots \wedge (\beta_p \lrcorner \omega) .
\end{aligned}$$

On the other hand,

$$\begin{aligned}
i_{\eta^H}\bar{\partial}^M(\tilde{\alpha}) &= i_{\eta^H}\Big(\sum_j (-1)^{j+1} (\beta_1 \lrcorner \omega) \wedge \ldots \wedge (\bar{\partial}^M(\beta_j \lrcorner \omega)) \wedge \ldots \wedge (\beta_p \lrcorner \omega)\Big) \\
&= \sum_j (\beta_1 \lrcorner \omega) \wedge \ldots \wedge i_{\eta^H}\bar{\partial}^M(\beta_j \lrcorner \omega) \wedge \ldots \wedge (\beta_p \lrcorner \omega) + (*) ,
\end{aligned}$$

where $(*)$ is a sum of terms of type

$$(\beta_1 \lrcorner \omega) \wedge \ldots \wedge i_{\eta^H}(\beta_j \lrcorner \omega) \wedge \ldots \wedge \bar{\partial}^M(\beta_k \lrcorner \omega) \wedge \ldots \wedge (\beta_p \lrcorner \omega) ,$$

with $j \neq k$. By definition

$$i_{\eta^H}(\beta_j \lrcorner \omega) = \omega(\beta_j, \eta^H) = 0 .$$

Finally, by Lemma 3.12, we know that $\nabla_{\eta^H}(\beta_j) \lrcorner\, \omega$ and $i_{\eta^H}\bar{\partial}^M(\beta_j \lrcorner\, \omega)$ have the same restriction in $B^1(Z)$. This implies the result. □

Remark 3.13. Theorem 3.11 is also a trivial corollary of [BGS 2, Theorem 2.8].

Theorem 3.14. *The smooth isomorphism $\lambda^{\mathrm{KM}} \simeq \lambda$ is an isomorphism of holomorphic line bundles.*

Proof. Since $R^p\pi_* F$ is a coherent $\mathcal{O}_B$-module [G], there is a dense open set in B where it is locally free. Therefore, to prove the corollary, we can assume, that, for every $p \geqq 0$, $R^p\pi_* F$ is locally free. If we replace B by a small neighborhood of any point y in B we can assume that there is a real number $b > 0$ such that D^2 has no eigenvalue in $]0, b]$. Therefore $K^b \subset E^{\cdot}$ consists of harmonic forms. For every $p \geqq 0$ there is an orthogonal decomposition (Hodge decomposition)

$$E^p = \mathrm{Im}(\bar{\partial}^Z) \oplus K_p^b \oplus \mathrm{Im}(\bar{\partial}^{Z*})$$

and $K_p^b = \ker(\bar{\partial}^Z) \cap \ker(\bar{\partial}^{Z*})$. The restriction of the orthogonal projection $P^b: E^p \to K_p^b$ to $\ker(\bar{\partial}^Z)$ factors through an isomorphism $\bar{P}^b: H^p(E^{\cdot}) \to K_p^b$. This isomorphism is, by definition, compatible with the action of $\tilde{\nabla}''$. From Proposition 3.10 and Theorem 3.11 we know that $\varrho_p(R^p\pi_* F)$ lies in the kernel of $\tilde{\nabla}''$ and, since the ranks are the same, $\varrho_p(R^p\pi_* F) = \mathrm{Ker}(\tilde{\nabla}'')$. On $H^p(E)$ the complex structure induced by ϱ_p (using Theorem 3.5) is thus the same as the one given by $\tilde{\nabla}''$. Since the canonical isomorphism $\lambda^{\mathrm{KM}} \simeq \lambda$ of Corollary 3.9 is induced by ϱ_p, $p \geqq 0$, it is therefore compatible with the holomorphic structures of these line bundles. □

d) Conclusion

By Theorem 3.14 and Theorem 1.27, we have now completed the proof of Theorem 0.1 in the introduction of [BGS 1]. So let $\pi: M \to B$ be a smooth proper morphism of complex manifolds. Assume that π is locally Kähler. Let ξ be a hermitian holomorphic vector bundle on M. Choose any Kähler metric g^Z on the fibers of π, which varies smoothly with the base point.

Theorem 0.1. *Under the above hypotheses, the curvature of the Quillen metric on the Knudsen-Mumford determinant is*

$$(2i\pi)\left[\int_Z Td(-R^Z/2i\pi)\,\mathrm{Tr}\exp(-L^\xi/2i\pi)\right]^{(2)}.$$

Theorems 0.2 and 0.3 in the introduction of [BGS 1] follow from [BGS 1, Corollary 1.30], Theorem 3.14, Theorems 1.23, 2.4, and 2.8.

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