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A Mathematical Classification of the One-Dimensional Deterministic Cellular Automata

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Abstract. We propose a theoretical classification of one-dimensional deterministic cellular automata in two types, type S and type O. This classification is connected with the phenomenological classification of S. Wolfram.

1. Introduction

This paper is devoted to theoretical investigations concerning the classification of one-dimensional deterministic cellular automata [1, 2]. Such an automaton is described (see in Sect. 2 for more precise definitions) by an evolution of the form $x_i(t+1) = f(x_{i-r_L}(t), x_{i-r_L+1}(t), \ldots, x_{i+r_R}(t))$, where for each lattice site $i \in \mathbb{Z}$ and discrete time $t, x_i(t)$ belongs to a fixed finite set E, (e.g. $\mathbb{Z}/p\mathbb{Z}$).

Among all the possible properties of an automaton which one may investigate, there is a natural one which we call surjectivity. We say that an automaton is surjective if for any finite configuration $(y_i, y_{i+1}, \ldots, y_{i+n})$ and any time $t > 0$, one can find initial conditions $(x_j(0))$ such that $x_{i+k}(t) = y_{i+k}$ for $k \in \{0, \ldots, n\}$. It is obvious that an automaton is surjective if and only if any finite configuration has at least one antecedent. The main result of this paper is that a deterministic one-dimensional automaton is surjective if and only if all finite configurations have the same number of antecedents. This result is typical of one-dimensional automata and means that, for such systems, surjective is equivalent to equiprobability of finite configurations, given equiprobable initial conditions.

We thus classify automata in two types: surjective automata will be said of type S, the other ones of type O.

Let us comment on the connection between this classification and the one introduced by Wolfram: In [3], S. Wolfram gave a phenomenological classification of such automata based on computer experiments. Class I consists of automata such that for $t \geqq T$ the values $x_i(t)$ do not depend on the initial conditions $x_j(0)$. Class II consists of automata such that for $t \geqq T$, $x_i(t)$ only depends on the initial values $x_j(0)$ at a finite number, say $m+1$, of adjacent sites with m independent of t, i.e. $x_i(t) =$

$F(\{x_j(0)\},\ j\in\mathscr{F}_{t,i})$, where $\mathscr{F}_{t,i}=\{j(t,i),\ j(t,i)+1,\ldots,j(t,i)+m\}$. In contrast to class I and class II, class III and class IV are only defined phenomenologically; class III corresponds to an apparently chaotic behavior, whereas class IV (i.e. the "other ones") corresponds to a behavior which exhibits some complex organization.

It is worth noticing here that class II and III are distinguished by the fact that no information is gained as the system evolves in contrast with the class IV and with the class I which in the extreme case where complete information is obtained after a finite time. In this respect, it could be natural to consider class I as a trivial subclass of class IV and class II as a trivial subclass (with $r_L+r_R=0$) of class III, (there are some subtleties concerning class II which we refrain to discuss here). In any case, it is clear that "interesting systems" lie in class IV and, in this respect, it is important to have an explicit mathematical characterization of class III. Type S is the correct mathematical definition.

The plan of the paper is the following. In Sect. 2 we introduce our conventions; in Sect. 3 we prove the main result (Theorem 3.2) of the paper; in Sect. 4 we describe a typical family of surjective automata.

2. Definitions and Notations

Let E be a finite set and let $E^{\mathbb{Z}^d}$ be the set of all maps from the d-dimensional lattice $\mathbb{Z}^d$ to E. Consider a dynamical system $\mathscr{A}$ with discrete time t and configuration space $E^{\mathbb{Z}^d}$; given a configuration $x(t)\in E^{\mathbb{Z}^d}$ of $\mathscr{A}$, the value of $x(t)$ at site $i\in\mathbb{Z}^d$ will be denoted by $x_i(t)\in E$. $\mathscr{A}$ is *deterministic* if the configuration $x(t+1)$ of $\mathscr{A}$ at time $t+1$ only depends on its configuration $x(t)$ at time t, i.e. if the dynamical law is of the form $x(t+1)=F_t(x(t))$ where, for each time t, F_t is a map of $E^{\mathbb{Z}^d}$ in itself; we are aware that this definition may be too restrictive for some purpose but it is convenient to use this definition here. A deterministic system $\mathscr{A}$ will be called *time-translational invariant* if F_t is independent of t, (i.e. $F_t=F$); it will be called *translational invariant* if the F_t are invariant by the lattice translations, finally $\mathscr{A}$ will be called *local* if, for any $i\in\mathbb{Z}^d$, the value $F_{t,i}(x)\in E$ of F_t at site i only depends on the values $x_{j_1(i)},\ldots,x_{j_{k_i}(i)}\in E$ of x at a finite number of adjacent sites $j_1(i),\ldots,j_{k_i}(i)\in\mathbb{Z}^d$. In the following we call *d-dimensional deterministic automaton with values in E* a deterministic system $\mathscr{A}$ of the above type which is time-translational invariant, translational invariant and local.

In this paper we investigate properties of one-dimensional automata, so we now suppose that $\mathscr{A}$ is a deterministic one-dimensional automaton and we introduce some specific notations for this case. From the definitions, it follows that there are two numbers r_L and r_R in $\mathbb{Z}$ with $r_L+r_R\geqq 0$ and a map $f:E^{r_L+r_R+1}\to E$ such that the evolution of $\mathscr{A}$ is given by $x_i(t+1)=f(x_{i-r_L}(t),\ x_{i-r_L+1}(t),\ldots,x_{i+r_R}(t))$. f will be called *the structure function of* $\mathscr{A}$ and r_L and r_R will be called *the left range* and *the right range* of $\mathscr{A}$. An *interval* I will be a non-empty finite ordered set of adjacent sites $I=(i,i+1,\ldots,i+|I|-1)$ of $\mathbb{Z}$, its *length* $|I|$ being the number of elements of I; a *configuration over* I will be a map $\Omega:I\to E$. We denote by $\mathscr{F}$ the set of all configurations over all intervals and call *finite configurations* the elements of $\mathscr{F}$. Given two finite configurations Ω and Ω' over intervals I and I', we say that Ω is *an extension of* Ω' if $I'\subset I$ and if *the restriction* $\Omega\restriction I'$ *of* Ω *to* I' is Ω'. Let $\Omega=$

$(x_i, x_{i+1}, \ldots, x_{i+k})$ be a finite configuration over $I=(i, \ldots, i+k)$, we denote by $\partial_L \Omega$ and by $\partial_R \Omega$ the configurations (x_i) and (x_{i+k}) over the sites i and $i+k$ and we call *adjacent sites to* Ω the sites $i-1$ and $i+k+1$. Given two finite configurations $\Omega = (x_i, \ldots, x_{i+k})$ and $\Omega' = (x_{i+k+1}, \ldots, x_{i+k+m})$ over adjacent intervals, their *composition* $\Omega\Omega' = (x_i, \ldots, x_{i+k}, x_{i+k+1}, \ldots, x_{i+k+m})$ is again a well defined finite configuration. For $a \in \mathbb{Z}$, one has the obvious notion of translation by a of intervals and of finite configurations; we denote by $T_a : \mathcal{F} \to \mathcal{F}$ the corresponding map on finite configurations. Let us come back to $\mathcal{A}$ and let $\Omega = (x_i, \ldots, x_{i+k})$ be a finite configuration over the interval $I=(i, \ldots, i+k)$; *an antecedent of* Ω will be a configuration $\Omega' = (y_{i-r_L}, \ldots, y_{i+k+r_R})$ over the interval $(i-r_L, \ldots, i+k+r_R)$ such that one has $x_j = f(y_{j-r_L}, \ldots, y_{j+r_R})$ for any $j \in I$. Notice that the set of antecedents of a finite configuration may be empty; we say that the automaton $\mathcal{A}$ is *surjective* if any finite configuration has an antecedent.

In Sect. 4 we show that other definitions of surjectivity lead to problems.

3. Surjectivity

In this section $\mathcal{A}$ denotes a one-dimensional deterministic automaton with values in a set of p elements (e.g. $\mathbb{Z}/p\mathbb{Z}$) and $r = r_L + r_R$. We shall discuss the condition of surjectivity for $\mathcal{A}$.

3.1 Lemma. *Let $N_{\min}$ be the greatest integer such that the number of antecedents of any finite configuration is greater than or equal to $N_{\min}$ and let $\mathcal{F}_{\min}$ denote the set of finite configurations with exactly $N_{\min}$ antecedents. Then any finite configuration which extends a configuration of $\mathcal{F}_{\min}$ is in $\mathcal{F}_{\min}$.*

Proof. It is clearly sufficient to show that, if $\Omega \in \mathcal{F}_{\min}$, then any configuration extending Ω over one site (on the left or the right) is in $\mathcal{F}_{\min}$. Now, given an adjacent site to Ω, there are p configurations extending Ω over this site. Let $N_1, \ldots, N_p$ be the numbers of antecedents of each of these configurations. The total number of antecedents of all these configurations is the number of extensions over one site of the antecedents of Ω. Thus one has $\sum_{i=1}^{i=p} N_i = pN_{\min}$ which implies $N_i = N_{\min}$ for any i since $N_i \geqq N_{\min}$. □

Notice that the lemma is trivial for $N_{\min} = 0$.

3.2 Theorem. *The following conditions for $\mathcal{A}$ are equivalent:*

(i) *Every finite configuration has an antecedent,* (i.e. $N_{\min} \geqq 1$).

(ii) *There is a smallest integer $N_{\max}$ such that the number of antecedents of any finite configuration is smaller than or equal to $N_{\max}$.*

(iii) *Every finite configuration has exactly p^r antecedents* (i.e. $N_{\max} = N_{\min} = p^r$).

Proof. (i)$\Rightarrow$(ii). Assume that (i) is satisfied and let Φ be an arbitrary finite configuration starting at $\varphi = \partial_L \Phi$; let $\bar{\varphi}$ be an antecedent of φ and N be the number of antecedents $\bar{\Phi}$ of Φ which extend $\bar{\varphi}$. By translation invariance of $\mathcal{F}_{\min}$, one can find a $\Omega \in \mathcal{F}_{\min}$ with an antecedent $\bar{\Omega}$ such that $\bar{\Omega}\bar{\Phi}$ is a configuration. Now, for any $\bar{\Phi}$ extending $\bar{\varphi}$ as above, $\bar{\Omega}\bar{\Phi}$ is an antecedent of a fixed configuration Ψ which extends both Ω and Φ; so $\Psi \in \mathcal{F}_{\min}$, by Lemma 3.1, and therefore $N \leqq N_{\min}$. Thus the number of antecedents of Φ extending $\bar{\varphi}$ is less than $N_{\min}$ which implies, since the number of antecedents $\bar{\varphi}$ of φ is less than p^{r+1}, that the number of antecedents of Φ is less than $p^{r+1}N_{\min}$. So there is a $N_{\max}$ and $N_{\max} \leqq p^{r+1}N_{\min}$.

(ii)$\Rightarrow$(iii). Let $\mathscr{F}_{\max}$ be the set of finite configurations with exactly $N_{\max}$ antecedents; we claim that, by the same argument as in 3.1, any finite configuration which extends a configuration of $\mathscr{F}_{\max}$ is in $\mathscr{F}_{\max}$. Indeed, let $\Omega \in \mathscr{F}_{\max}$, choose an adjacent site and consider the p configurations extending Ω over this site; the number of antecedents of all these configurations is $pN_{\max} = \sum_{i=1}^{i=p} N_i$, where $N_1, \ldots, N_p$ are the number of antecedents of each of these configurations. This implies $N_i = N_{\max}$ for all i since $N_i \leqq N_{\max}$. $\mathscr{F}_{\max}$ and $\mathscr{F}_{\min}$ are both translation invariant so, for any element of $\mathscr{F}_{\max}$ (respectively $\mathscr{F}_{\min}$) one can find an element of $\mathscr{F}_{\min}$ (respectively $\mathscr{F}_{\max}$) and a finite configuration extending both. This implies $\mathscr{F}_{\min} = \mathscr{F}_{\max}$ and therefore any finite configuration has exactly $N_{\min} = N_{\max}$ antecedents, and then one necessarily has $N_{\min} = N_{\max} = p^r$.

(iii)$\Rightarrow$(i) is a triviality. □

3.3 Remarks. a) (i) means that the automaton is surjective and (iii) means that the finite configurations are "equiprobable," the equivalence of these properties suggests that surjectivity is the "good mathematical characterization" of automata of class III, [3], (for r "really" bigger than or equal to 2).

b) The above results are typically one-dimensional. One verifies, by inspection, that the proof of Lemma 3.1 breaks down for automata on the d-dimensional lattice for $d \geqq 2$.

3.4 Definition. An automaton satisfying the equivalent conditions (i), (ii) and (iii) of Theorem 3.2 will be said to be of *type S*; an automaton which is not of type S will be said to be of *type O*. Type S corresponds obviously to class III plus some trivial class II systems whereas type O corresponds to class IV plus class I systems.

4. Examples: Quasi-Linear Automata

Let $\mathscr{A}$ be a one-dimensional automaton with values in $\mathbb{Z}/p\mathbb{Z}$ and suppose that the structure function f of $\mathscr{A}$ is of the form $f(x_{i-r_L}, \ldots, x_{i+r_R}) = \varphi(x_{i-r_L}, \ldots, x_{i+r_R-1}) + \pi(x_{i+r_R})$ (respectively $f(x_{i-r_L}, \ldots, x_{i+r_R}) = \pi(x_{i-r_L}) + \varphi(x_{i-r_L+1}, \ldots, x_{i+r_R})$), where $\pi \in \mathscr{G}_p$ is a permutation of $\mathbb{Z}/p\mathbb{Z}$. Then $\mathscr{A}$ is obviously a surjective automaton since any antecedent $\bar{\Omega}$ of a finite configuration Ω can be uniquely extended to an antecedent of an extension over one site on the right (respectively on the left) of the original configuration Ω; indeed if Ω is extended as Ωx (respectively $x\Omega$) then there is a unique $\bar{x}$ such that $\bar{\Omega}\bar{x}$ (respectively $\bar{x}\bar{\Omega}$) is an antecedent of Ωx (respectively $x\Omega$).

In the case $p = 2$, where $E = \mathbb{Z}/2\mathbb{Z}$, $\pi(x)$ is either x of $1 + x$, i.e. one may suppose that $\pi(x) = x$ by absorbing the eventual constant 1 in the definition of φ. We say that such an automaton with $\pi = \mathrm{Id}_E$ is right linear (respectively left linear) or quasi-linear (to cover both cases). It can be shown that, for $E = \mathbb{Z}/2\mathbb{Z}$ and $r = r_L + r_R < 4$, all surjective automata are quasi-linear (i.e. either right linear or left linear).

It is clear that surjectivity, as defined above, implies (and is equivalent to) surjectivity for global configurations, i.e. any global configuration has an antecedent. However, surjectivity of an automaton of type S may be lost if one restricts attention to a subclass of global configurations. For instance, let us consider the automaton of type S with values in $\mathbb{Z}/2\mathbb{Z}$ defined by $x_i(t+1) = x_i(t) + x_{i+1}(t)$; since $f(0,0) = 0$, it is natural to restrict attention to global configurations with finite supports, i.e. configurations $x = (x_i)$ such that $x_i = 0$ except for a finite number of sites $i \in \mathbb{Z}$. Now, let us consider the configuration x defined by $x_i = 0 \, \forall i \neq 0$ and $x_0 = 1$; this configuration has two antecedents y^L and y^R, with $y_i^L = 1$ for $i \leqq 0$, $y_i^L = 0$ for $i \geqq 1$ and $y_i^R = 0$ for $i \leqq 0$, $y_i^R = 1$ for $i \geqq 1$, but none of these configurations has a finite support. Thus one loses the surjectivity if one restricts attention to such configurations.

5. Conclusion

We gave a theoretical classification of one-dimensional deterministic cellular automata (type S and type O) which is connected with the phenomenological classification of S. Wolfram. Related with this classification there are several problems which we did not discuss here. One of these problems is to give an analytical characterization of the structure functions of automata of type S. Another very important problem is to find a subclassification for automata of type O, since these are candidates to describe self-organizing systems. These problems and some other ones connected with group invariance are still in investigation.

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