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Continuum Limit of a Hierarchical SU(2) Lattice Gauge Theory in 4 Dimensions

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Abstract. We study nonperturbative renormalizability of a $d=4$ hierarchical SU(2) gauge model that realizes Migdal's recursion relation as an exact renormalization group transformation. A continuum limit of effective actions is shown to exist as the scaling limit, both for initial Wilson and heat kernel actions. These limit effective actions exhibit ultraviolet asymptotic freedom and provide a strictly positive string tension.

1. Introduction

The study of nonperturbative renormalization of gauge theories is presently one of the major efforts within constructive quantum field theory. To employ for this aim the framework of lattice quantization [1] looks very appealing – in spite of its obvious shortcomings – since it preserves the characterizing property of such theories: local gauge invariance. Moreover Wilson's [2] renormalization group approach to remove the ultraviolet cutoff introduced by the lattice, turned out to be an extraordinarily attractive suggestion. From a physical point of view the 4-dimensional Yang-Mills theory is of central interest. Although there is important progress in the rigorous construction of this theory [3, 4] a complete solution has not yet been achieved. The study of the ultraviolet behaviour of this asymptotically free theory by exact renormalization group transformations allows us to make use of a perturbation expansion but requires i) a control of higher order contributions, ii) a nonperturbative treatment of the "large fields," and iii) a control of the nonlocal effective actions generated. Shortly after the pioneering work of Wilson [1] on lattice gauge theories, Migdal [5] proposed a recursion relation considered as an approximate real space renormalization group transformation both for spin systems and lattice gauge theories. In the case of spin systems it was realized some time ago that this recursion relation holds exactly on hierarchical lattices [6], yet only recently Ito [7] presented hierarchical lattice gauge models with similar properties. In view of the enormous complexity to be mastered attacking the full $d=4$ nonabelian gauge theory, we pursue the modest aim to analyze the ultraviolet behaviour generated by the nonabelian Migdal recursion relation [5] in the critical dimension. Its treatment involves the aforementioned problems i), ii) but releases from iii), since the virtue of this recursion relation – and clearly its deficiency! – is to strictly reproduce local actions. The price to pay is violation of reflection positivity and of the full lattice symmetry.

Hence we investigate a sequence of "Gibbs factors" $g^{(n)}$ on a nonabelian group G recursively generated from an initial one by the transformation

$$(\mathscr{T} g)(u) = \left\{ \frac{g^{*r}(u)}{g^{*r}(e_0)} \right\}^r \tag{1.1}$$

with $u, e_0 \in G$, e_0 the unit element. g^{*r} denotes the multiple convolution product of r factors. In the case of a hierarchical gauge model {spin system} r is the number of plaquettes {bonds} united by a block spin transformation, hence the minimal value for r is 4 {2}. The equal powers of ordinary and convolution products in (1.1) reflect critical dimension, i.e. dimension 4 {2} in gauge models {spin systems}. As already emphasized by Migdal [5] this property rather than the particular value of r determines the qualitative behaviour of the solutions, which is confirmed by our rigorous analysis.

This article relies upon our previous work [8] – hereafter referred to as (I) – where Migdal-Kadanoff recursion relations for gauge groups $SU(N)$ and $U(N)$, analytically continued in the central angles, have been introduced. For the sake of technical simplicity only we restrict our present analysis to $G = SU(2)$. In Sect. 2 we briefly state properties of the analytically continued Gibbs factors derived in (I) and exhibit further functional relations (Proposition 1). These results are valid for an initial Gibbs factor of the Wilson (2.5) or of the heat kernel form (2.20). One should keep in mind that the real parameter β appearing in (2.5) is proportional to the inverse square of the bare Yang-Mills coupling constant. Our analysis covers large values of β, hence weak coupling. Section 3 contains our main result for a single iteration step of the analytically continued recursion relations (3.1) and (3.8), formulated in Theorem 1 and in its Corollary, respectively. We devise an analyticity technique, inspired by Gawędzki and Kupiainen's treatment of the hierarchical ϕ_4^4 model [9]. In contradistinction to scalar models no "large fields" in the proper sense occur in our case due to the compact manifold; furthermore there is no independent dominating gaussian fluctuation measure. In order to develop a controlled perturbation expansion however, we have to divide the group manifold into a properly chosen neighbourhood of the unit element and its complement. The former, called "small field region," allows a convergent perturbation expansion, whereas the latter one is controlled by a simple nonperturbative bound. Our induction assumptions (A_1)–(A_3) are a precise formulation of this method. For large values of β, (3.3), Theorem 1 states in the case $r = 2$ that these assumptions are reproduced with a change in β of order $\mathcal{O}(1)$. Under repeated iterations β eventually decreases, allowing us to proceed until we leave the weak coupling domain. Thus β acts as a scale parameter. It is noteworthy that in the power series expansion of $\ln h(z)$, see (A_3), when explicitly controlling the flow of β to order β^{-1} (included), it suffices to perform for β a third order ("two loop"), for λ a second order ("one loop") and for σ a first order ("tree") calculation. The Corollary contains the generalization to the case $r = 4$, iterating the $r = 2$ result appropriately. Such a procedure could be extended to $r=2^m$, $m\in\mathbb{N}$. The proof of Theorem 1 is given in Sects. 4 (small field) and 5 (nonperturbative bound). In Sect. 6 we show existence of continuum limits of effective actions on a sequence of lattices with spacings 2^{-n}, $n\in\mathbb{N}_0$, emerging from repeated subdivision of the unit lattice. Gawędzki and Kupiainen's work on the continuum limit of the ultraviolet asymptotically free ϕ_4^4 model with negative coupling constant [10] indicates that it should suffice to prescribe the cutoff dependence of the bare marginal coupling coefficient to third order in a perturbative solution of the recursion relations (3.7). This holds in our case. The particular form of these coupled equations requires some analysis (Proposition 2). Among other properties, we prove in Theorem 2 both for initial Wilson and heat kernel action, due to the chosen cutoff dependence of the initial β, that for any fixed lattice ("scale") the sequence of effective Gibbs factors indexed by the cutoff, and considered as functions of the analytically continued central angle, forms a normal family of holomorphic functions. Hence convergent subsequences exist. This result is then sharpened in Theorem 3, the main result of this investigation: there is a common subsequence of cutoffs implying convergence of effective Gibbs factors for all scales. Hence there is ultraviolet asymptotic freedom for the running $[\beta^{(-n)}]^{-1}$. Moreover, employing a recent result of Ito [11], we deduce for the continuum limits of these effective actions a strictly positive string tension (Theorem 4). Finally we exhibit in the Appendix some properties of the heat kernel Gibbs factor required by our induction assumptions. A proof of uniqueness of the continuum limit actions, allowing us to avoid the restriction to subsequences of cutoffs, might be achieved adapting the method of iterating differences of coupling constants as performed in [10]. Moreover, a lower bound for the string tension should be obtainable. Work in this direction is in progress.

2. General Properties of the Analytically Continued Gibbs Factors

In the sequel we consider exclusively the (gauge) group $G=SU(2)$. The Gibbs factor attached to a plaquette in the case of a gauge model or to a link in the case of a (chiral) spin system after n iterations of the renormalization group transformation is denoted by $g^{(n)}(u)$. It is a real valued, positive function of $u\in G$ and has the following properties, see (I): (2.2), (2.5), (2.6), with e_0 the unit element of G and $u,v\in G$,

$$g^{(n)}(e_0)=1\,, \tag{2.1}$$

$$g^{(n)}(vuv^{-1})=g^{(n)}(u)\,, \tag{2.2}$$

$$g^{(n)}(u^{-1})=g^{(n)}(u)\,, \tag{2.3}$$

valid for $n\in\mathbb{N}_0$. Usually $g^{(0)}(u)$ is chosen as a function of positive type; this property iterates too, see (I).

We first treat Migdal's recursion relation for the simplest critical setting, $r=q=2$ in (I): (2.10 M),

$$g^{(n+1)}(e^{-ix\sigma_3}u)=\frac{1}{\mathcal{N}}\left\{\int dv\, g^{(n)}\left(e^{-\frac{i}{2}x\sigma_3}uv^{-1}\right)g^{(n)}\left(e^{-\frac{i}{2}x\sigma_3}v\right)\right\}^2. \tag{2.4}$$

In (2.4) we introduced an additional translation on the group in the "direction" given by the diagonal Pauli matrix σ_3, with $x \in \mathbb{R}$. Moreover dv is the Haar measure on G and $\mathcal{N}$ determined by (2.1).

Starting with the Wilson action, $\beta \in \mathbb{R}_+$,

$$g^{(0)}(u) = \exp\{\beta(\text{trace } u - 2)\}, \tag{2.5}$$

we proved inductively in (I): (Proposition 1), that the functions

$$\tilde{g}^{(n)}(u, x) := g^{(n)}(e^{-ix\sigma_3} u), \qquad n \in \mathbb{N}_0 \tag{2.6}$$

can be analytically continued:

$$\tilde{g}^{(n)}(u, x) \to \tilde{g}^{(n)}(u, z), \qquad z = x + iy;\ x, y \in \mathbb{R}, \tag{2.7}$$

and the functions $\tilde{g}^{(n)}(u, z)$ are (i) continuous in u for fixed $z \in \mathbb{C}$, (ii) entire and real analytic in $z \in \mathbb{C}$ for fixed $u \in G$.

From these properties we can furthermore infer the following functional relations, with $u \in G$, $z \in \mathbb{C}$, $x' \in \mathbb{R}$,

$$\tilde{g}^{(n)}(e^{-ix'\sigma_3} u, z) = \tilde{g}^{(n)}(u, x' + z), \tag{2.8}$$

$$\tilde{g}^{(n)}(u, z) = \tilde{g}^{(n)}(u^{-1}, -z). \tag{2.9}$$

For fixed x', u all the functions appearing in (2.8) and (2.9) are holomorphic in z. The relations are obvious identities for $z \in \mathbb{R}$ [use (2.3) in the case of (2.9)] and thus valid for $z \in \mathbb{C}$ due to uniqueness of the analytic continuation.

In particular the functions

$$h^{(n)}(z) := \tilde{g}^{(n)}(e_0, z), \qquad n \in \mathbb{N}_0 \tag{2.10}$$

are entire holomorphic in $z \in \mathbb{C}$, periodic with period 2π and even functions due to (2.9),

$$h^{(n)}(z + \pi) = h^{(n)}(z - \pi), \tag{2.11}$$

$$h^{(n)}(z) = h^{(n)}(-z). \tag{2.12}$$

The functions $h^{(n)}(x)$ are the Gibbs factors, which are class functions, (2.2), expressed in terms of the central angle.

The analytical properties can be further exploited. Parametrizing $u \in SU(2)$ by $u = u_0\sigma_0 + i\mathbf{u} \cdot \boldsymbol{\sigma}$ with $\{u_0, \mathbf{u}\} \in \mathbb{S}^3$, the unit matrix σ_0 and the Pauli matrices σ_k, $k = 1, 2, 3$, we arrive at the

Proposition 1. *For* $z \in \mathbb{C}$, $|z| < \frac{1}{4}$, *we have*

$$\tilde{g}^{(n)}(u, z) = \begin{cases} h^{(n)}(\theta), & \textit{for} \quad u_0 > -\frac{1}{2} \\ h^{(n)}(\pi - \check{\theta}), & \textit{for} \quad u_0 < \frac{1}{2}, \end{cases}$$

where

$$\theta^2 = \theta^2(u, z) = 4f(\tfrac{1}{2}\{1 - u_0 \cos z - u_3 \sin z\}), \tag{2.13}$$

$$\check{\theta}^2 = \check{\theta}^2(u, z) = 4f(\tfrac{1}{2}\{1 + u_0 \cos z + u_3 \sin z\}), \tag{2.14}$$

are defined by inversion of $\eta = \sin^2 \dfrac{\theta}{2}$ *as a holomorphic function of* θ^2, *i.e.* $\theta^2 = 4f(\eta)$.

Hence, with $|\eta|<1$,

$$f(\eta)=\eta[{}_2F_1(\tfrac{1}{2},\tfrac{1}{2};\tfrac{3}{2};\eta)]^2\,. \tag{2.15}$$

Moreover $\theta^2(u,z)$ *and* $\check{\theta}^2(u,z)$ *are continuous in* u *and holomorphic in* z *in their respective domains.*

Remark. Due to (2.11), (2.12) $h^{(n)}(\theta)$ and $h^{(n)}(\pi-\check{\theta})$ are even functions of θ and $\check{\theta}$, respectively; for notational convenience only we use $\theta,\check{\theta}$ instead of θ^2 and $\check{\theta}^2$ [we take the main branch of the square root in (2.13), (2.14)].

Proof of Proposition 1. Since the Gibbs factors are class functions, (2.2), we have for $x\in\mathbb{R}$,

$$\tilde{g}^{(n)}(u,x)=g^{(n)}(e^{-ix\sigma_3}u)=g^{(n)}(e^{-i\theta\sigma_3})=h^{(n)}(\theta) \tag{2.16}$$

with the central angle θ and its complement $\check{\theta}=\pi-\theta$ given by

$$\cos\theta=u_0\cos x+u_3\sin x=-\cos\check{\theta}\,. \tag{2.17}$$

This implies the claim of the proposition for $z=x$ real, $|x|<\frac{1}{4}$. Since $\theta^2(u,z)$ and $\check{\theta}^2(u,z)$ are holomorphic in z, $|z|<\frac{1}{4}$, and $h^{(n)}(\theta)$, $h^{(n)}(\pi-\check{\theta})$ entire and real analytic in θ^2 and $\check{\theta}^2$, respectively, existence and uniqueness of the analytic continuation to $|z|<\frac{1}{4}$ conclude the proof.

Remark. We observe that the symmetry (2.8) is reflected in a corresponding symmetry of (2.13), (2.14). Combining this property with Proposition 1 leads to a representation of $\tilde{g}(u,z)$, $u\in G$, $z\in\mathbb{C}$ with $|\mathrm{Im}\,z|<d<\frac{1}{4}$, in terms of $h(\theta)$, $|\mathrm{Im}\,\theta|<d$, which vice versa can be applied to define $\tilde{g}(u,z)$ for a given 2π-periodic and even function $h(\theta)$, being holomorphic in $|\mathrm{Im}\,\theta|<d$. It is important to note that $|\mathrm{Im}\,\theta(u,z)|\leqq|\mathrm{Im}\,z|$ in the domain of θ. This is deduced from the map

$$\{\theta:|\mathrm{Im}\,\theta|=t\}\xrightarrow{\omega=\cos\theta}\left\{\omega:\left(\frac{\mathrm{Re}\,\omega}{\cosh t}\right)^2+\left(\frac{\mathrm{Im}\,\omega}{\sinh t}\right)^2=1\right\}$$

for arbitrary $t>0$, taking into account $\cos\theta=u_0\cos z+u_3\sin z$ due to (2.17). A similar consideration yields $|\mathrm{Im}\,\check{\theta}(u,z)|\leqq|\mathrm{Im}\,z|$ in the domain of $\check{\theta}$. We further note that (with our choice of the square root) $0\leqq\mathrm{Re}\,\theta(u,z)<\frac{5}{6}\pi$, $0\leqq\mathrm{Re}\,\check{\theta}(u,z)<\frac{5}{6}\pi$ in their respective domains.

For later convenience we define particular subsets of G. Let $z\in\mathbb{C}$, $|z|<\frac{1}{4}$, and $\varrho\in\mathbb{R}_+$, $\varrho<\frac{1}{4}$. Then

$$\mathscr{G}[z,\varrho]:=\{u\in G:u_0>0\quad\text{and}\quad|\theta^2(u,z)|<\varrho^2\}\,. \tag{2.18}$$

Moreover by $\chi(\cdot\,;z,\varrho)$ we denote the characteristic function of this set, i.e.

$$\chi(u;z,\varrho)=\begin{cases}1 & \text{for}\quad u\in\mathscr{G}[z,\varrho]\\ 0 & \text{otherwise}\,.\end{cases} \tag{2.19}$$

Besides the Wilson action (2.5) we consider the heat kernel action (HK) in the initial Gibbs factor

$$g^{(0)}_{\mathrm{HK}}(u)=\mathscr{N}\sum_{j=0,\frac{1}{2},1,\dots}(2j+1)e^{-\frac{(j+\frac{1}{2})^2}{\gamma}}\chi_j(u)\,, \tag{2.20}$$

with $\gamma \in \mathbb{R}_+$, the characters χ_j of $SU(2)$ and a normalization factor $\mathcal{N}$ fixed by (2.1). The real-valued function $g_{\mathrm{HK}}^{(0)}$ is positive on G, see Appendix.

Expressing the characters $\chi_j(u)$ as polynomials of trace u, [12], the analytic continuation of (2.20) is given by

$$\tilde{g}_{\mathrm{HK}}^{(0)}(u,z)=\mathcal{N}\sum_{l=1}^{\infty} l e^{-\frac{l^2}{4\gamma}} \sum_{k=0}^{\left[\frac{l-1}{2}\right]} (-1)^k \binom{l-1-k}{k} \{\mathrm{trace}(e^{-iz\sigma_3}u)\}^{l-1-2k}. \tag{2.21}$$

For $a \in \mathbb{R}_+$ we denoted by $[a]$ the integer part of a. A little exercise shows that $\tilde{g}_{\mathrm{HK}}^{(0)}(u,z)$ is for fixed u an entire holomorphic function of $z \in \mathbb{C}$. Hence, due to our derivation in (I), the functions $\tilde{g}_{\mathrm{HK}}^{(n)}(u,z)$, $n \in \mathbb{N}_0$, also have the general analytic properties exhibited in this section, originally derived for the Wilson action.

3. Weak Coupling Analysis of the Analytically Continued Recursion Relation: Inductive Assumptions and Main Single Step Result

The functions $\tilde{g}^{(n)}(u,z)$, (2.7), are inductively defined by the analytically continued recursion relation (2.4) of Migdal,

$$\tilde{g}^{(n+1)}(u,z)=\frac{1}{\mathcal{N}}\left\{\int dv\, \tilde{g}^{(n)}\left(uv^{-1},\frac{z}{2}\right)\tilde{g}^{(n)}\left(v,\frac{z}{2}\right)\right\}^2. \tag{3.1}$$

In order to simplify the notation we introduce for a general $n \in \mathbb{N}_0$,

$$g(u)\equiv g^{(n)}(u), \tilde{g}\equiv \tilde{g}^{(n)}, h\equiv h^{(n)}, \tag{3.2}$$
$$g'(u)\equiv g^{(n+1)}(u), \quad \text{likewise } \tilde{g}', h'.$$

Let $\beta \in \mathbb{R}_+$ be defined by the power series expansion around $z=0$,

$$h(z)=1-\beta z^2+\mathcal{O}(z^4). \tag{3.3}$$

We establish and inductively reproduce assumptions on the function $\tilde{g}(u,z)$ in the weak coupling region

$$\beta \geqq \underline{\beta}, \underline{\beta} \in \mathbb{R}_+ \text{ sufficiently large}. \tag{3.4}$$

Our assumptions fall into two parts: i) existence of the logarithm of $h(z)$ in the "small field" region,

$$|z|<\beta^{-\alpha}, \quad \text{with } \tfrac{3}{7} \leqq \alpha < \tfrac{1}{2}(\text{fixed}), \tag{3.5}$$

together with bounds for the first three coefficients and for the remainder of its Taylor series; ii) a nonperturbative bound on $\tilde{g}(u,z)$. We might allow for $\frac{3}{8}<\alpha<\frac{1}{2}$; the choice (3.5) simplifies our analysis, however.

After these introductory remarks let us state our assumptions on the analytically continued Gibbs factor $\tilde{g}(u,z)$.

(A_1) *General Properties.* $\tilde{g}(u,z)$ is continuous in $u \in G$ for fixed $z \in \mathbb{C}$ and entire holomorphic in z for fixed u. Moreover $g(u)=\tilde{g}(u,0)$ is real and positive, satisfying (2.1)–(2.3).

(A_2) *Nonperturbative Bound.* For u "far from the unit element," which is the nonperturbative region, we assume the bound, see (2.18):

For $y\in\mathbb{R}$, $|y|<\frac{\kappa}{2}\beta^{-\alpha}$ and $u\in G\backslash\mathscr{G}[iy,\beta^{-\alpha}]$,

$$|\tilde{g}(u,iy)|<\exp\{\beta y^2-p\beta^{1-2\alpha}\},$$

with $1<\kappa<1+\varepsilon$ ($\varepsilon>0$ small) and $\frac{1}{2}<p<1-\frac{\kappa^2}{4}$.

(A_3) *Small Field.* In the domain $z\in\mathbb{C}$, $|z|<\beta^{-\alpha}$, there exists a holomorphic function $V(z)$ such that

$$\begin{aligned}&h(z)=\exp\{-V(z)\},\\&V(z)=\beta z^2+\tfrac{1}{2}\lambda z^4+\tfrac{1}{3}\sigma z^6+\tilde{V}(z),\\&\tilde{V}(z)=\mathcal{O}(z^8).\end{aligned}$$

$\beta\in\mathbb{R}_+$ is large and $|\beta^{-1}\lambda|$, $|\beta^{-1}\sigma|$ are bounded by constants. Moreover the bound $|\tilde{V}(z)|<D\beta^{-2}$ holds in the domain, with a constant D.

Remarks. From (A_1) follow the properties for $\tilde{g}(u,z)$ stated in Sect. 2. Due to the relation, $z=x+iy$ with $x,y\in\mathbb{R}$,

$$\tilde{g}(u,z)=\tilde{g}(e^{-ix\sigma_3}u,iy), \tag{3.6}$$

implied by (2.8), we consider in (A_2) purely imaginary values of z without restricting the generality.

The bound (A_2) is sufficient to estimate the nonperturbative contributions to $h'(z)$ in the small field domain $|z|<\kappa\beta^{-\alpha}$ which arise in (3.1) (with $u=e_0$) from v outside an $\mathcal{O}(\beta^{-\alpha})$ neighbourhood of e_0, as exponentially small.

We formulate our main result for one iterative step in

Theorem 1. *Given $\underline{\beta}\in\mathbb{R}_+$ sufficiently large the assumptions* (A_1)–(A_3) *for* $\tilde{g}(u,z)$ *with $\beta\geqq\underline{\beta}$ imply* (A_1)–(A_3) *to hold for its transform* $\tilde{g}'(u,z)$, (3.1), *with primed couplings β', λ', σ' and remainder $\tilde{V}'$ associated with $\tilde{g}'$. Moreover these couplings satisfy*

$$\begin{aligned}&\beta'=\beta-\tfrac{1}{6}+\tfrac{5}{4}\lambda\beta^{-1}+\tfrac{1}{72}\beta^{-1}-\tfrac{5}{24}\lambda\beta^{-2}-\tfrac{25}{16}\lambda^2\beta^{-3}+\tfrac{35}{16}\sigma\beta^{-2}+\mathcal{O}(\beta^{-2+2\alpha}),\\&\lambda'=\tfrac{1}{4}\lambda-\tfrac{7}{360}-\tfrac{11}{16}\lambda^2\beta^{-2}+\tfrac{7}{8}\sigma\beta^{-1}+\mathcal{O}(\beta^{-2+4\alpha}),\\&\sigma'=\tfrac{1}{16}\sigma+\mathcal{O}(\beta^{-2+6\alpha}).\end{aligned} \tag{3.7}$$

In order to prove this Theorem we recall that the reproduction of the general properties (A_1) is proven in (I) and quoted in Sect. 2. The reproduction of the properties (A_2) and (A_3) for $\tilde{g}'$ is deferred to Sects. 5 and 4, respectively.

Theorem 1 is a consequence of the recursion relation (3.1), which corresponds physically to a two-dimensional chiral spin system. In the case of a lattice gauge model in 4 dimensions with smallest block length ($l=2$) we have to deal instead of (3.1) with the recursion relation – see (2.1), (2.10 M) of (I) with $q=r=4$ –

$$\tilde{g}'(u,z)=\frac{1}{\mathcal{N}}\left\{\int dv_1\,dv_2\,dv_3\,\tilde{g}\left(uv_1^{-1},\frac{z}{4}\right)\tilde{g}\left(v_1v_2^{-1},\frac{z}{4}\right)\tilde{g}\left(v_2v_3^{-1},\frac{z}{4}\right)\tilde{g}\left(v_3,\frac{z}{4}\right)\right\}^4. \tag{3.8}$$

We can reduce its analysis to the preceding one. As a prerequisite we observe that we might have introduced an arbitrary constant factor $E \in \mathbb{R}_+$ in the definition of the domains occurring in the induction assumptions, viz. $|y| < \frac{\kappa}{2}(E\beta)^{-\alpha}$ and $u \notin \mathscr{G}[iy, (E\beta)^{-\alpha}]$ in (A_2) with p replaced by $pE^{-2\alpha}$ in the bound, and $|z| < (E\beta)^{-\alpha}$ in (A_3). This change would not affect Theorem 1, as follows from its proof.

Defining now

$$[\mathscr{T}_{(2)}\tilde{g}]^{1/2}(u,z) = \frac{\int dv\, \tilde{g}\left(uv^{-1}, \frac{z}{2}\right)\tilde{g}\left(v, \frac{z}{2}\right)}{\int dv [\tilde{g}(v,0)]^2}, \tag{3.9}$$

$$\mathscr{T}_{(2)}\tilde{g} = \{[\mathscr{T}_{(2)}\tilde{g}]^{1/2}\}^2, \tag{3.10}$$

and writing for $\tilde{g}'$ defined by (3.8) $\tilde{g}'(u,z) = (\mathscr{T}_{(4)}\tilde{g})(u,z)$, we have

$$\mathscr{T}_{(4)}\tilde{g} = \{\mathscr{T}_{(2)}[\mathscr{T}_{(2)}\tilde{g}]^{1/2}\}^2 \tag{3.11}$$

connecting the recursion relations (3.8) and (3.1). It remains to trace the domains.

For the map $[\mathscr{T}_{(2)}\tilde{g}]^{1/2}$ we consider small field domains with $E=1$. Then Theorem 1 implies, denoting there $\beta' = \beta'(\beta, \lambda, \sigma, \ldots)$ etc.,

$$\begin{gathered} \beta_1 = \tfrac{1}{2}\beta'(\beta, \lambda, \sigma, \ldots), \\ |z| < (2\beta_1)^{-\alpha} : [\mathscr{T}_{(2)}\tilde{g}]^{1/2}(e_0, z) = \exp\{-\beta_1 z^2 + \mathcal{O}(z^4)\}. \end{gathered} \tag{3.12}$$

In the succeeding application of $\mathscr{T}_{(2)}$ we use small field domains with $E=2$, then

$$\begin{gathered} \beta_2 = \beta'(\beta_1, \lambda_1, \sigma_1, \ldots), \\ |z| < (2\beta_2)^{-\alpha} : (\mathscr{T}_{(2)}[\mathscr{T}_{(2)}\tilde{g}]^{1/2})(e_0, z) = \exp\{-\beta_2 z^2 + \mathcal{O}(z^4)\}. \end{gathered} \tag{3.13}$$

Hence, from (3.11),

$$\begin{gathered} \beta'' := 2\beta'(\tfrac{1}{2}\beta'(\beta, \lambda, \sigma), \tfrac{1}{2}\lambda'(\beta, \lambda, \sigma), \tfrac{1}{2}\sigma'(\beta, \lambda, \sigma), \ldots), \\ |z| < (\beta'')^{-\alpha} : (\mathscr{T}_{(4)}\tilde{g})(e_0, z) = \exp\{-\beta'' z^2 + \mathcal{O}(z^4)\}. \end{gathered} \tag{3.14}$$

Thus we obtained the important

Corollary. *Theorem 1 is valid too in the case of the recursion relation* (3.8) *provided we replace the transformation* (3.7) *of the coupling coefficients by*

$$\begin{aligned} \beta' &= \beta - \tfrac{1}{2} + \tfrac{15}{8}\lambda\beta^{-1} + \tfrac{1}{48}\beta^{-1} - \tfrac{5}{16}\lambda\beta^{-2} - \tfrac{285}{64}\lambda^2\beta^{-3} + \tfrac{315}{64}\sigma\beta^{-2} + \mathcal{O}(\beta^{-2+2\alpha}), \\ \lambda' &= \tfrac{1}{16}\lambda - \tfrac{7}{160} - \tfrac{33}{128}\lambda^2\beta^{-2} + \tfrac{21}{64}\sigma\beta^{-1} + \mathcal{O}(\beta^{-2+4\alpha}), \\ \sigma' &= (\tfrac{1}{16})^2\sigma + \mathcal{O}(\beta^{-2+6\alpha}). \end{aligned} \tag{3.15}$$

The initial (analytically continued) Gibbs factor $\tilde{g}^{(0)}(u,z)$ both with Wilson action (2.5) and with heat kernel action (2.20) satisfies (A_1) as shown in (I) and in Sect. 2, respectively. The properties (A_2) and (A_3), stated for the weak coupling region $\beta \geqq \underline{\beta}$ with large $\underline{\beta}$, are easily seen to hold in the case of the Wilson action. For the heat kernel action they are derived in the Appendix.

4. Reproduction of the Small Field Assumptions

In this section we analyze $h'(z)=\tilde{g}'(e_0, z)$ in the region $z\in\mathbb{C}$, $|z|<(\beta')^{-\alpha}<\kappa\beta^{-\alpha}$. Let ξ, ξ' be fixed positive numbers with

$$0<\xi'<\xi\leqq\tfrac{1}{4}. \tag{4.1}$$

Denoting by $\chi(v)$ the characteristic function of $\mathcal{G}[0, \xi'\beta^{-\alpha}]\subset G$, see (2.19), i.e.

$$\chi(v)=\chi(v; 0, \xi'\beta^{-\alpha}), \tag{4.2}$$

we write, recalling (3.1) and the symmetry (2.9),

$$h'(z)=\left\{K(z)\frac{1+L(z)}{1+L(0)}\right\}^2, \tag{4.3}$$

with

$$K(z)=\frac{\int dv\,\chi(v)\,\tilde{g}\left(v,\frac{z}{2}\right)\tilde{g}\left(v,-\frac{z}{2}\right)}{\int dv\,\chi(v)\,g(v)^2} \tag{4.4}$$

and

$$L(z)=\frac{\int dv[1-\chi(v)]\,\tilde{g}\left(v,\frac{z}{2}\right)\tilde{g}\left(v,-\frac{z}{2}\right)}{\int dv\,\chi(v)\,\tilde{g}\left(v,\frac{z}{2}\right)\tilde{g}\left(v,-\frac{z}{2}\right)}. \tag{4.5}$$

The denominator in (4.4) is obviously positive. Since the following analysis will show that $K(z)$ does not vanish in $|z|<2(1-\xi)\beta^{-\alpha}$, the denominator in (4.5) is non-zero, too. Moreover, $K(z)$ and $L(z)$ are even and holomorphic in $|z|<2(1-\xi)\beta^{-\alpha}$ due to (A_1).

We first analyze $K(z)$ in the extended domain,

$$z\in\mathbb{C}, |z|<2(1-\xi)\beta^{-\alpha}. \tag{4.6}$$

For β sufficiently large, (4.1), (4.6) and $\chi(v)=1$ imply with (2.13),

$$\left|\theta^2\left(v, \pm\frac{z}{2}\right)\right|<\beta^{-2\alpha}. \tag{4.7}$$

Thus we can use the assumption (A_3) together with Proposition 1 leading to

$$K(z)=\frac{\int dv\,\chi(v)\exp\{-V(\theta_+)-V(\theta_-)\}}{\int dv\,\chi(v)\exp\{-2V(\theta_0)\}} \tag{4.8}$$

with the definitions $\theta_\pm:=\theta\left(v, \pm\frac{z}{2}\right)$ and $\theta_0:=\theta(v,0)$.

We rewrite $K(z)$ defining the probability measure on G,

$$\langle\mathcal{F}\rangle:=\frac{\int\mathcal{F}\,d\mu}{\int d\mu}, \tag{4.9}$$

$$d\mu=e^{-2V(\theta_0)}\chi(v)\,dv, \tag{4.10}$$

hence

$$K(z)=\langle\exp\{-V(\theta_+)-V(\theta_-)+2V(\theta_0)\}\rangle. \tag{4.11}$$

Introducing the notations,

$$\begin{aligned} w&=1-v_0,\\ \Delta&=1-\cos\frac{z}{2},\\ a&=w+\Delta-w\Delta,\\ b&=v_3\sin\frac{z}{2}, \end{aligned} \tag{4.12}$$

we recall from Proposition 1

$$1-\cos\theta_\pm=a\mp b. \tag{4.13}$$

It is convenient to expand $V(z)$ in powers of $1-\cos z$. From the definition in (A_3) and with the coefficients

$$s=\tfrac{1}{3}\beta+2\lambda,\qquad t=\tfrac{4}{45}\beta+\tfrac{2}{3}\lambda+\tfrac{8}{3}\sigma, \tag{4.14}$$

we obtain in $|z|<\beta^{-\alpha}$,

$$V(z)=2\beta(1-\cos z)+s(1-\cos z)^2+t(1-\cos z)^3+\tilde{\tilde{V}}(z). \tag{4.15}$$

The difference $\tilde{V}-\tilde{\tilde{V}}$ resulting from the orders $z^8, z^{10}, \ldots$ of the convergent series of $(1-\cos z)^k$ for $k=1,2,3$ is estimated from the leading term as $\mathcal{O}(\beta^{1-8\alpha})$ in $|z|<\beta^{-\alpha}$. Thus from (A_3) follows

$$|\tilde{\tilde{V}}(z)|<D\beta^{-2}+\mathcal{O}(\beta^{1-8\alpha}),\quad \text{in}\quad |z|<\beta^{-\alpha}, \tag{4.16}$$

where $1-8\alpha<-2$ due to (3.5).

Employing now (4.13) and (4.15), we obtain

$$d\mu=\exp\{-4\beta w-2sw^2-2tw^3-2\tilde{\tilde{V}}(\theta_0)\}\,\chi(v)\,dv, \tag{4.17}$$

and

$$\begin{aligned} V(\theta_+)+V(\theta_-)-2V(\theta_0)&=4\beta(a-w)+2s(a^2+b^2-w^2)+2t(a^3+3ab^2-w^3)\\ &\quad+\tilde{\tilde{V}}(\theta_+)+\tilde{\tilde{V}}(\theta_-)-2\tilde{\tilde{V}}(\theta_0). \end{aligned} \tag{4.18}$$

From (4.12) we calculate, using $w=\mathcal{O}(\beta^{-2\alpha})$ and $\Delta=\mathcal{O}(\beta^{-2\alpha})$,

$$\begin{aligned} a^2&=w^2+\Delta^2+2w\Delta-2(w^2\Delta+w\Delta^2)+\mathcal{O}(\beta^{-8\alpha}),\\ a^3&=w^3+\Delta^3+3(w\Delta^2+w^2\Delta)+\mathcal{O}(\beta^{-8\alpha}),\\ b^2&=v_3^2\Delta(2-\Delta),\\ ab^2&=2wv_3^2\Delta+2v_3^2\Delta^2+\mathcal{O}(\beta^{-8\alpha}). \end{aligned} \tag{4.19}$$

Inserting (4.19) into (4.18) we first observe that the pure powers of Δ give, because of (4.15) with z replaced by $\frac{z}{2}$, the leading terms in the expansion of $2V\left(\frac{z}{2}\right)$, with the remainder of order z^8 being $\mathcal{O}(\beta^{1-8\alpha})$ in the small field domain (4.6). Moreover we define

$$\begin{aligned}\zeta = & -4\beta w\Delta + 2s\Delta(2w + 2v_3^2 - 2w^2 - 2w\Delta - v_3^2\Delta) \\ & + 6t\Delta(w\Delta + w^2 + 2wv_3^2 + 2v_3^2\Delta),\end{aligned} \tag{4.20}$$

and write (4.18) in the form

$$V(\theta_+) + V(\theta_-) - 2V(\theta_0) = \frac{\beta}{2}z^2 + \frac{\lambda}{16}z^4 + \frac{\sigma}{96}z^6 + \zeta + R(v, z). \tag{4.21}$$

Due to (4.18), (4.19) we have for $v \in \operatorname{supp}\chi(v)$ and z in the domain (4.6)

$$R(v, z) = \tilde{\tilde{V}}(\theta_+) + \tilde{\tilde{V}}(\theta_-) - 2\tilde{\tilde{V}}(\theta_0) + \mathcal{O}(\beta^{1-8\alpha}) \tag{4.22}$$

and $R(v, 0) = 0$. Using the decomposition (4.21) in (4.11) we arrive at

$$\left.\begin{aligned} & K(z) = \tilde{K}(z)\exp\left\{-\frac{\beta}{2}z^2 - \frac{\lambda}{16}z^4 - \frac{\sigma}{96}z^6\right\} \\ & \tilde{K}(z) = \langle \exp\{-\zeta - R(v, z)\}\rangle . \end{aligned}\right\} \tag{4.23}$$

In (4.23) we split

$$\tilde{K}(z) = K^{(1)}(z) + K^{(2)}(z), \tag{4.24}$$

$$K^{(1)}(z) = \langle (1 - \zeta + \tfrac{1}{2}\zeta^2)\exp(-R)\rangle , \tag{4.25}$$

$$K^{(2)}(z) = \langle (\exp(-\zeta) - 1 + \zeta - \tfrac{1}{2}\zeta^2)\exp(-R)\rangle , \tag{4.26}$$

where $K^{(1)}$ contains the (convergent) perturbation expansion for the effective action "to two loops for the β-vertex."

Due to $|e^{-\zeta} - 1 + \zeta - \frac{1}{2}\zeta^2| < |\zeta|^3$, for $|\zeta| < 3$, the remainder $K^{(2)}(z)$ is estimated uniformly in the region (4.6) as

$$|K^{(2)}(z)| \leqq \langle |\zeta|^3\rangle \exp\{4D\beta^{-2} + \mathcal{O}(\beta^{1-8\alpha})\} = \mathcal{O}(\beta^{3-12\alpha}), \tag{4.27}$$

where we use (4.16) to bound R and $s = \mathcal{O}(\beta)$, $t = \mathcal{O}(\beta)$, $w = \mathcal{O}(\beta^{-2\alpha})$, $\Delta = \mathcal{O}(\beta^{-2\alpha})$ in (4.20), thus avoiding a more careful estimate of the expectation. Our choice $\alpha \geqq \frac{3}{7}$ ensures $3 - 12\alpha < -2$, which is good enough.

In order to evaluate (4.25) we need the "moments"

$$I_{n,m}(z) = \langle w^n v_3^{2m} \exp\{-R(v, z)\}\rangle . \tag{4.28}$$

First we split the measure conveniently. The Haar measure dv in the neighbourhood of $v = e_0$ ($|\mathbf{v}|$ small) reads

$$dv = \frac{1}{2\pi^2}\frac{d^3\mathbf{v}}{1 - w}, 1 - w = v_0 = \sqrt{1 - \mathbf{v}^2}. \tag{4.29}$$

We put in the correct normalization constant which, of course, drops out in the expectations. Using (4.29) in (4.17), the measure $d\mu$ can be written

$$d\mu = \frac{1}{2\pi^2}[1 - 2sw^2 + \mathcal{O}(\beta^{2-8\alpha})]\exp\{-w(4\beta - 1)\}\,\chi(v)\,d^3\mathbf{v}. \tag{4.30}$$

We used $w=\mathcal{O}(\beta^{-2\alpha})$ due to the restriction $\chi(v)=1$, see (4.2), and $s=\mathcal{O}(\beta)$, $t=\mathcal{O}(\beta)$. Finally we expand w in terms of $\mathbf{v}^2$ consistently within the desired order, and obtain

$$d\mu=\frac{1}{2\pi^2}[1-\tfrac{1}{2}(\beta+s)(\mathbf{v}^2)^2+\mathcal{O}(\beta^{2-8\alpha})]\exp\{-\tfrac{1}{2}(4\beta-1)\mathbf{v}^2\}\,\chi(v)\,d^3\mathbf{v}\,. \quad (4.31)$$

The order term is $\mathcal{O}\left(\beta^{-1-\frac{3}{7}}\right)$, and therefore not needed explicitly. A simple calculation gives

$$\begin{aligned}&\int d^3\mathbf{v}\,\chi(v)\,(\mathbf{v}^2)^n v_3^{2m}\exp\{-\tfrac{1}{2}(4\beta-1)\mathbf{v}^2\}\\&=\left(\frac{2\pi}{4\beta-1}\right)^{3/2}\cdot\frac{(2(n+m)+1)!!}{2m+1}(4\beta-1)^{-n-m}+\mathcal{O}(\exp\{-2(\xi')^2\beta^{1-2\alpha}\}),\end{aligned} \quad (4.32)$$

with exponentially small orders that estimate the effect of extending the integration to all of $\mathbb{R}^3$; these depend on (n,m), of course.

Employing (4.31), (4.32) and the expansion $w=\frac{1}{2}\mathbf{v}^2+\frac{1}{8}\mathbf{v}^4+\mathcal{O}(\beta^{-6\alpha})$, we calculate the expectations (4.28) explicitly for the index pairs (0, 0), (0, 1), (1, 0) and (0, 2), (1, 1), (2, 0) to order β^{-2}, with remainders $\mathcal{O}(\beta^{1-8\alpha})$ and $\mathcal{O}(\beta^{-3})$, respectively. At $z=0$, where $R(v,0)=0$, we obtain

$$I_{0,0}(0)=1\,, \quad (4.33)$$

$$I_{0,1}(0)=\tfrac{1}{4}\beta^{-1}-\tfrac{3}{32}\beta^{-2}-\tfrac{5}{32}s\beta^{-3}+\mathcal{O}(\beta^{1-8\alpha})\,, \quad (4.34)$$

$$I_{0,2}(0)=\tfrac{3}{16}\beta^{-2}+\mathcal{O}(\beta^{-3})\,, \quad (4.35)$$

$$I_{1,0}(0)=\tfrac{3}{8}\beta^{-1}-\tfrac{3}{128}\beta^{-2}-\tfrac{15}{64}s\beta^{-3}+\mathcal{O}(\beta^{1-8\alpha})\,, \quad (4.36)$$

$$I_{2,0}(0)=\tfrac{15}{64}\beta^{-2}+\mathcal{O}(\beta^{-3})\,, \quad (4.37)$$

$$I_{1,1}(0)=\tfrac{5}{32}\beta^{-2}+\mathcal{O}(\beta^{-3})\,. \quad (4.38)$$

The higher moments $I_{n,m}(0)=\mathcal{O}(\beta^{-n-m})$ are not needed explicitly.

In order to estimate the difference $I_{n,m}(z)-I_{n,m}(0)$, we use for R of (4.22) the bound (4.16)

$$|e^{-R}-1|\leqq 4D\beta^{-2}+\mathcal{O}(\beta^{1-8\alpha})\,, \quad (4.39)$$

since the arguments $\theta=\theta_\pm,\theta_0$ in (4.22) are restricted to the domain $|\theta|<\beta^{-\alpha}$. Thus we deduce

$$\begin{aligned}|I_{n,m}(z)-I_{n,m}(0)|&\leqq\langle w^n v_3^{2m}|e^{-R(v,z)}-1|\rangle\\&\leqq I_{n,m}(0)\{4D\beta^{-2}+\mathcal{O}(\beta^{1-8\alpha})\}\,.\end{aligned} \quad (4.40)$$

We are now ready to evaluate $K^{(1)}(z)$, (4.25),

$$\begin{aligned}K^{(1)}(z)&=I_{0,0}(z)+\Delta\{4(\beta-s)I_{1,0}(z)-4sI_{0,1}(z)-12tI_{1,1}(z)+(4s-6t)I_{2,0}(z)\}\\&\quad+\Delta^2\{(4s-6t)I_{1,0}(z)+(2s-12t)I_{0,1}(z)\\&\quad+8(\beta-s)^2I_{2,0}(z)+8s^2I_{0,2}(z)-16(\beta-s)sI_{1,1}(z)+\mathcal{O}(\beta^{-1})\}\\&\quad+\Delta^3\mathcal{O}(1)+\Delta^4\mathcal{O}(1)\,.\end{aligned} \quad (4.41)$$

The order estimates in (4.40), (4.41) are valid uniformly in the domain (4.6). From (4.41) and the bounds (4.27), (4.40) together with $s=\mathcal{O}(\beta)$, $t=\mathcal{O}(\beta)$ due to (4.14) and (A_3), we conclude that for sufficiently large β the function $\tilde{K}(z)$, (4.23) is holomorphic in the extended domain (4.6), $|z|<2(1-\xi)\beta^{-\alpha}$, and bounded there by $|1-\tilde{K}(z)|=\mathcal{O}(\beta^{-2\alpha})$, so $\ln\tilde{K}(z)$ is holomorphic in the same domain with

$$\ln\tilde{K}(z)=K^{(1)}(z)-1-\tfrac{1}{2}(K^{(1)}(z)-1)^2+\mathcal{O}(\beta^{3-12\alpha}). \tag{4.42}$$

Hence $\ln K(z)$, (4.23), is holomorphic in the domain (4.6), too, and has the power series representation

$$-\ln K(z)=\beta_1 z^2+\tfrac{1}{2}\lambda_1 z^4+\tfrac{1}{3}\sigma_1 z^6+\tilde{V}_1(z), \tag{4.43}$$

where $\tilde{V}_1(z)$ is the sum of the even powers higher than six. The contribution of $\ln\tilde{K}$ to the coefficients of the power series (4.43) is derived from (4.41), the expectations (4.33)–(4.38) and the bounds (4.40). The couplings λ, σ are reintroduced by (4.14) and Δ from (4.12) has to be Taylor-expanded at $z=0$.

Moreover the powers of z^2 emerging from $I_{n,m}(z)-I_{n,m}(0)$ and z-dependent order terms are controlled by the Cauchy estimate for derivatives using the bound (4.40). After some calculation we find

$$\begin{aligned}
\beta_1&=\tfrac{1}{2}\beta-\tfrac{1}{12}+\tfrac{5}{8}\lambda\beta^{-1}+\tfrac{1}{144}\beta^{-1}-\tfrac{5}{48}\lambda\beta^{-2}-\tfrac{25}{32}\lambda^2\beta^{-3}+\tfrac{35}{32}\sigma\beta^{-2}+\mathcal{O}(\beta^{-2+2\alpha}),\\
\lambda_1&=\tfrac{1}{8}\lambda-\tfrac{7}{720}-\tfrac{11}{32}\lambda^2\beta^{-2}+\tfrac{7}{16}\sigma\beta^{-1}+\mathcal{O}(\beta^{-2+4\alpha}),\\
\sigma_1&=\tfrac{1}{32}\sigma+\mathcal{O}(\beta^{-2+6\alpha}).
\end{aligned} \tag{4.44}$$

The leading order in the Cauchy estimates for the coefficients of the power series $\tilde{V}_1(z)$ results from $I_{0,0}(z)-1$, bounded by (4.40) in the extended domain (4.6). Taking the nonleading parts into account by a factor $1+\delta_1$, with $\delta_1=\mathcal{O}(\beta^{5-12\alpha})$, we obtain the following bound on $\tilde{V}_1(z)$ in the restricted domain $|z|<(\beta')^{-\alpha}$, $\beta'=\beta+\mathcal{O}(1)$

$$|\tilde{V}_1(z)|\leqq\tfrac{1}{2}D(\beta')^{-2}\frac{\left(\frac{\beta}{\beta'}\right)^{8\alpha}(1+\delta_1)}{32(1-\xi)^8\left[1-(2-2\xi)^{-2}\left(\frac{\beta}{\beta'}\right)^{2\alpha}\right]}. \tag{4.45}$$

With our choice $\xi\leqq\frac{1}{4}$ the bound is smaller than $\frac{1}{2}D(\beta')^{-2}$ for β large enough.

It remains to estimate the nonperturbative contribution $L(z)$, (4.5), to $h'(z)$, (4.3), in the restricted region $z\in\mathbb{C}$, $|z|<(\beta')^{-\alpha}$. The denominator of (4.5) is obtained from the result (4.43) for $K(z)$ and from the measure $d\mu$, (4.31), together with the bounds assumed in (A_3)

$$\begin{aligned}
&\left|\int dv\,\chi(v)\,\tilde{g}\left(v,\frac{z}{2}\right)\tilde{g}\left(v,-\frac{z}{2}\right)\right|=|K(z)|\int d\mu\\
&=\exp\{-\beta_1(x^2-y^2)+\mathcal{O}(\beta^{1-4\alpha})\}\cdot\mathrm{const}\,\beta^{-3/2}\{1+\mathcal{O}(\beta^{1-4\alpha})\}\\
&=\exp\left\{-\frac{\beta}{2}(x^2-y^2)+\mathcal{O}(\ln\beta)\right\},
\end{aligned} \tag{4.46}$$

where $z=x+iy; x,y\in\mathbb{R}$, $|z|<(\beta')^{-\alpha}$. With the characteristic functions [see (2.19)]

$$\chi_\pm:=\chi\left(v;\pm\frac{z}{2},\beta^{-\alpha}\right), \tag{4.47}$$

we decompose the numerator of $L(z)$ as follows, replacing where possible $\tilde{g}\left(v, \pm\frac{z}{2}\right)$ by $h(\theta_\pm)$, $\theta_\pm = \theta\left(v, \pm\frac{z}{2}\right)$, due to Proposition 1,

$$\int dv[1-\chi(v)]\,\tilde{g}\left(v, \frac{z}{2}\right)\tilde{g}\left(v, -\frac{z}{2}\right) = \sum_{k=1}^{4} I^{(k)}, \tag{4.48}$$

$$I^{(1)} = \int dv[1-\chi(v)]\,\chi_+\chi_- h(\theta_+)h(\theta_-), \tag{4.49}$$

$$I^{(2)} = \int dv[1-\chi(v)]\,[1-\chi_+]\,\chi_-\tilde{g}\left(v, \frac{z}{2}\right)h(\theta_-), \tag{4.50}$$

$$I^{(3)} = \int dv[1-\chi(v)]\,\chi_+[1-\chi_-]\,h(\theta_+)\tilde{g}\left(v, -\frac{z}{2}\right), \tag{4.51}$$

$$I^{(4)} = \int dv[1-\chi(v)]\,[1-\chi_+]\,[1-\chi_-]\,\tilde{g}\left(v, \frac{z}{2}\right)\tilde{g}\left(v, -\frac{z}{2}\right). \tag{4.52}$$

These integrals are estimated using in the small field region $|\theta| < \beta^{-\alpha}$ the assumption (A_3), i.e.

$$h(\theta) = \exp\{-\beta\theta^2 + \mathcal{O}(\beta^{1-4\alpha})\}. \tag{4.53}$$

Moreover, due to the symmetry property (2.8), we have

$$\left.\begin{aligned} \tilde{g}\left(v, \pm\frac{z}{2}\right) &= \tilde{g}\left(e^{\mp i\frac{x}{2}\sigma_3}v, \pm i\frac{y}{2}\right), \\ \theta^2\left(v, \pm\frac{z}{2}\right) &= \theta^2\left(e^{\mp i\frac{x}{2}\sigma_3}v, \pm i\frac{y}{2}\right), \ \text{(where defined)} \end{aligned}\right\} \tag{4.54}$$

such that

$$\chi_\pm = \chi\left(e^{\mp i\frac{x}{2}\sigma_3}v;\ \pm i\frac{y}{2}, \beta^{-\alpha}\right). \tag{4.55}$$

Recalling $\left|\frac{y}{2}\right| < \frac{1}{2}(\beta')^{-\alpha} < \frac{\kappa}{2}\beta^{-\alpha}$, we can thus use in the supports of $1-\chi_\pm$ the bound of assumption (A_2), which yields

$$\left|\tilde{g}\left(v, \pm\frac{z}{2}\right)\right| \leq \exp\left\{-p\beta^{1-2\alpha} + \beta\left(\frac{y}{2}\right)^2\right\}. \tag{4.56}$$

Treating $I^{(1)}$ first we easily deduce from (4.12), (4.13) for $|\theta_\pm| < \beta^{-\alpha}$ and $|z| < (\beta')^{-\alpha}$,

$$\theta_+^2 + \theta_-^2 = 4w + \tfrac{1}{2}z^2 + \mathcal{O}(\beta^{-4\alpha}). \tag{4.57}$$

The support of $1-\chi(v)$ implies

$$w = 1 - v_0 \geq \tfrac{1}{2}(\xi'\beta^{-\alpha})^2 + \mathcal{O}(\beta^{-4\alpha}). \tag{4.58}$$

Hence we obtain

$$\begin{aligned} |I^{(1)}| &\leq \int dv[1-\chi(v)]\,\chi_+\chi_-\ \exp\left\{-4\beta w - \frac{\beta}{2}(x^2-y^2) + \mathcal{O}(\beta^{1-4\alpha})\right\} \\ &< \exp\left\{-(\xi')^2\beta^{1-2\alpha} - \frac{\beta}{2}(x^2-y^2)\right\}. \end{aligned} \tag{4.59}$$

From (4.54) and (2.14) we derive for $|\theta_\pm|<\beta^{-\alpha}$ and $|z|<(\beta')^{-\alpha}$,

$$\mathrm{Re}\,\theta_\pm^2=2\left[1-\left(e^{\mp i\frac{x}{2}\sigma_3}v\right)_0\right]\cosh\frac{y}{2}-\left(\frac{y}{2}\right)^2+\mathcal{O}(\beta^{-4\alpha}), \tag{4.60}$$

and hence, due to (4.53)

$$|h(\theta_\pm)|<\exp\left\{\beta\left(\frac{y}{2}\right)^2+\mathcal{O}(\beta^{1-4\alpha})\right\}. \tag{4.61}$$

Together with (4.56) we infer the bounds

$$\begin{aligned}|I^{(2)}|,|I^{(3)}|&<\exp\left\{-p\beta^{1-2\alpha}+\beta\left(\frac{y}{2}\right)^2\right\}\cdot\exp\left\{\beta\left(\frac{y}{2}\right)^2+\mathcal{O}(\beta^{1-4\alpha})\right\}\\&<\exp\left\{-\tfrac{1}{2}(p-\tfrac{1}{2})\beta^{1-2\alpha}-\frac{\beta}{2}(x^2-y^2)\right\},\end{aligned} \tag{4.62}$$

due to $|x|<(\beta')^{-\alpha}$ and $p-\frac{1}{2}>0$ in (A_2). Furthermore, due to $|x|<(\beta')^{-\alpha}$,

$$|I^{(4)}|<\exp\left\{-(p-\tfrac{1}{2})\beta^{1-2\alpha}-\frac{\beta}{2}(x^2-y^2)\right\}. \tag{4.63}$$

In the bounds on $I^{(k)}$ we absorbed orders $\mathcal{O}(\beta^{1-4\alpha})$ or $\mathcal{O}(\beta^{-2\alpha})$ by part of the negative $\mathcal{O}(\beta^{1-2\alpha})$ terms in the exponent, possible for β large enough. Collecting pieces we obtain from (4.3), (4.46), (4.48) and (4.59), (4.62), (4.63), the final result for the iterated Gibbs factor in the small field region,

$$h'(z)=K(z)^2\{1+\mathcal{O}(e^{-\frac{1}{4}\delta_2\beta^{1-2\alpha}})\}, \tag{4.64}$$

with

$$\delta_2=\min[p-\tfrac{1}{2},(\xi')^2].$$

It implies holomorphy of $-\ln h'(z)$ in $|z|<(\beta')^{-\alpha}$ with couplings β',λ',σ' and remainder $\tilde{V}'(z)$ receiving exponentially small corrections to the contributions $2\beta_1$, $2\lambda_1$, $2\sigma_1$, and $2\tilde{V}_1(z)$ coming from $(K(z))^2$.

Thus we proved (3.7) and the reproduction of (A_3), where the bounds on λ', σ' are immediately read off from (3.7). We still have to prove (A_2) for $\tilde{g}'(u,z)$, which is done in the following section.

5. Inductive Reproduction of the Nonperturbative Bound

From (3.1) we have the recursion relation

$$\tilde{g}'(u,iy)=\left\{\frac{J(u,iy)}{M}\right\}^2 \tag{5.1}$$

with the convolutions

$$J(u,iy)=\int dv\,\tilde{g}\left(uv^{-1},\frac{i}{2}y\right)\tilde{g}\left(v,\frac{i}{2}y\right), \tag{5.2}$$

$$M=\int dv[g(v)]^2. \tag{5.3}$$

In order to reproduce the bound (A_2) for $\tilde{g}'$ we have to consider

$$y\in\mathbb{R},|y|<\frac{\kappa}{2}(\beta')^{-\alpha},\quad u\in G\backslash\mathcal{G}[iy,(\beta')^{-\alpha}]. \tag{5.4}$$

From the assumptions (A_2) and (A_3) we can easily deduce

$$M=\mathrm{const}\,\beta^{-3/2}\{1+\mathcal{O}(\beta^{1-4\alpha})\}. \tag{5.5}$$

Introducing the characteristic functions, (2.19),

$$\chi_1:=\chi\left(uv^{-1};\frac{i}{2}y,\beta^{-\alpha}\right), \tag{5.6}$$

$$\chi_2:=\chi\left(v;\frac{i}{2}y,\beta^{-\alpha}\right), \tag{5.7}$$

and the corresponding squares of the central angles, (2.13),

$$\theta_1^2:=\theta^2\left(uv^{-1},\frac{i}{2}y\right), \tag{5.8}$$

$$\theta_2^2:=\theta^2\left(v,\frac{i}{2}y\right), \tag{5.9}$$

we decompose (5.2) as follows:

$$J(u,iy)=\sum_{k=1}^{4}J^{(k)}, \tag{5.10}$$

$$J^{(1)}:=\int dv\,\chi_1\chi_2 h(\theta_1)\,h(\theta_2), \tag{5.11}$$

$$J^{(2)}:=\int dv(1-\chi_1)\,\chi_2\,\tilde{g}\left(uv^{-1},\frac{i}{2}y\right)h(\theta_2), \tag{5.12}$$

$$J^{(3)}:=\int dv\,\chi_1(1-\chi_2)\,h(\theta_1)\,\tilde{g}\left(v,\frac{i}{2}y\right), \tag{5.13}$$

$$J^{(4)}:=\int dv(1-\chi_1)(1-\chi_2)\,\tilde{g}\left(uv^{-1},\frac{i}{2}y\right)\tilde{g}\left(v,\frac{i}{2}y\right). \tag{5.14}$$

In writing this decomposition we used in the small field region $|\theta^2|<\beta^{-2\alpha}$ the representation for $\tilde{g}$ given in Proposition 1. In this region we can use (4.53).

From the definition (2.13) we deduce for $|\theta_j^2|<\beta^{-2\alpha}, j=1,2$, and $|y|<\frac{1}{2}\kappa(\beta')^{-\alpha}$,

$$\mathrm{Re}\,\theta_1^2=2[1-(uv^{-1})_0]-\tfrac{1}{4}y^2+\mathcal{O}(\beta^{-4\alpha}), \tag{5.15}$$

$$\mathrm{Re}\,\theta_2^2=2[1-v_0]\qquad -\tfrac{1}{4}y^2+\mathcal{O}(\beta^{-4\alpha}). \tag{5.16}$$

We now derive upper bounds on the individual terms of the decomposition (5.10). With (4.53) and (5.15), (5.16) follows for $J^{(1)}$,

$$\begin{aligned}|J^{(1)}|&<\exp\left\{\frac{\beta}{2}y^2+\mathcal{O}(\beta^{1-4\alpha})\right\}\int dv\chi_1\chi_2\exp\{-2\beta[2-(uv^{-1})_0-v_0]\}\\ &<\exp\left\{\frac{\beta}{2}y^2+\mathcal{O}(\beta^{1-4\alpha})\right\}\int dv\exp\{-\beta[4-\mathrm{trace}(uv^{-1})-\mathrm{trace}\,v]\}\\ &<\mathrm{const}\exp\left\{\frac{\beta}{2}y^2-4\beta\right\}\frac{I_1(2\beta\sqrt{2(1+u_0)})}{\beta\sqrt{2(1+u_0)}},\end{aligned} \tag{5.17}$$

where I_1 is the modified Bessel function. Since $x^{-1}I_1(x)$ increases monotonously with $x\in\mathbb{R}_+$, we can in (5.17) replace u_0 by $\sup(u_0)$ compatible with the region (5.4). It is found on the boundary

$$(\beta')^{-2\alpha}=|\theta^2(u,iy)|=|2(1-u_0)-y^2-2iyu_3|+\mathcal{O}((\beta')^{-4\alpha}) \tag{5.18}$$

for $(u_3)^2=1-(u_0)^2$. With the ansatz $u_0=\cos\psi$, $\psi=\mathcal{O}(\beta^{-\alpha})$, we obtain

$$(\beta')^{-4\alpha}=(\psi^2+y^2)^2+\mathcal{O}(\beta^{-6\alpha}), \tag{5.19}$$

and hence

$$\sup(u_0)=1-\tfrac{1}{2}\{(\beta')^{-2\alpha}-y^2\}+\mathcal{O}(\beta^{-4\alpha}). \tag{5.20}$$

Employing the asymptotic expansion of the modified Bessel function in (5.17) and using (5.20) together with the maximal value of $|y|$ due to (5.4) finally yields the bound

$$|J^{(1)}|<\mathrm{const}\,\beta^{-3/2}\exp\left\{\frac{\beta}{2}\left[y^2-p(\beta')^{-2\alpha}-\left(1-\frac{\kappa^2}{4}-p\right)(\beta')^{-2\alpha}\right]\right\}. \tag{5.21}$$

The integrals $J^{(2)}$, (5.12), and $J^{(3)}$, (5.13), can be treated simultaneously. In the small field region we use (4.53) together with (5.15) and (5.16) respectively, discarding the positive terms there. The function $\tilde{g}$ is bounded by the induction assumption (A_2). Integrating then over the whole group we obtain the bounds

$$\begin{aligned}|J^{(2)}|,|J^{(3)}|&<\exp\left\{\beta\left(\frac{y}{2}\right)^2+\mathcal{O}(\beta^{1-4\alpha})+\beta\left(\frac{y}{2}\right)^2-p\beta^{1-2\alpha}\right\}\cdot\int dv\\&<\exp\{\tfrac{1}{2}\beta[y^2-p\beta^{-2\alpha}-p\beta^{-2\alpha}+\mathcal{O}(\beta^{-4\alpha})]\}.\end{aligned} \tag{5.22}$$

A bound on $J^{(4)}$ follows directly from the induction assumption (A_2),

$$|J^{(4)}|<\exp\{\tfrac{1}{2}\beta[y^2-p\beta^{-2\alpha}-3p\beta^{-2\alpha}]\}. \tag{5.23}$$

Collecting these bounds (5.21)–(5.23) and incorporating there $\beta'=\beta+\mathcal{O}(1)$, we deduce from the decomposition (5.10),

$$|J(u,iy)|<\mathrm{const}\exp\{\tfrac{1}{2}[\beta'y^2-p(\beta')^{1-2\alpha}-\delta_3(\beta')^{1-2\alpha}]\} \tag{5.24}$$

with $\delta_3=\min\left\{p,1-\frac{\kappa^2}{4}-p\right\}=1-\frac{\kappa^2}{4}-p$. Hence due to $\delta_3>0$ and β' sufficiently large, (5.24) and (5.5) imply for the analytically continued Gibbs factor (5.1) the bound

$$|\tilde{g}'(u,iy)|<\exp\{\beta'y^2-p(\beta')^{1-2\alpha}\}, \tag{5.25}$$

valid in the domain (5.4). Thus we have reproduced the induction assumption (A_2) after one iteration, which concludes the proof of Theorem 1.

6. Effective Actions and Continuum Limit

The inductive scheme described by the assumptions (A_1)–(A_3) and the one step results of Theorem 1 and its corollary provide the basis to control the flow of Gibbs factors under the iterated renormalization group transformation.

We consider a sequence of successively finer lattices with spacings 2^{-N}, $N \in \mathbb{N}_0$, obtained by repeatedly subdividing the unit lattice in dimension $d=2$ or $d=4$. Thus we have $r=2\{4\}$ for $d=2$ spin systems $\{d=4$ gauge theories$\}$ of the hierarchical "Migdal type," which we define on these lattices. Taking a smallest spacing 2^{-N} which provides the (physical) UV-cutoff we describe "physics" at lower momentum scales on the lattices with spacing 2^{-n}, $n=N, N-1, \ldots, 0$ by iterating the Migdal block spin transformation, which implies the successive coarse-graining of the original (cutoff) lattice. Our aim is to control on a given lattice scale 2^{-n} the sequence of Gibbs factors, i.e. of effective actions, which is generated by this renormalization group from properly chosen initial ("bare") Gibbs factors at scale ($=$cutoff) 2^{-N}, in the continuum limit $N \to \infty$.

For this purpose we modify our notation of the Gibbs factors conveniently. Instead of $g^{(n)}(u)$, $n \in \mathbb{N}_0$, we use $g_N^{(-n)}(u)$, $N \in \mathbb{N}_0$ and $n \in \mathbb{Z}$ with $n \leqq N$. The subscript N denotes the initial (cutoff) lattice, with spacing 2^{-N}, whereas n denotes the lattice with spacing 2^{-n} reached after $N-n$ renormalization group transformations $\mathscr{T}$. Thus we consider the flow

$$g_N^{(-N)}(u) \xrightarrow{\mathscr{T}} g_N^{(-N+1)}(u) \xrightarrow{\mathscr{T}} \ldots \xrightarrow{\mathscr{T}} g_N^{(0)}(u). \tag{6.1}$$

Similarly we introduce the notations $\tilde{g}_N^{(-n)}$, $h_N^{(-n)}$, $\beta_N^{(-n)}$ etc.

As a first step we investigate the flow of the coupling coefficients $\beta_N^{(-n)}$, $\lambda_N^{(-n)}$, $\sigma_N^{(-n)}$ for a given cutoff N, with the initial Gibbs factor $g_N^{(-N)}$ chosen such that Theorem 1, respectively its corollary, apply (at least) N times, i.e. $\beta_N^{(0)} > \underline{\beta}$. Denoting for shortness $\beta_N^{(-n)}$ by β and $\beta_N^{(-n+1)}$ by β', similarly λ and λ', σ and σ', we recall the recursion relations (3.7), (3.15).

In order to control the flow under repeated application of these recursions we first diagonalize partially by suitable nonlinear transformations. In the case of $r=2$ they are defined by

$$\begin{aligned}
\hat{\beta} &:= \beta + \tfrac{5}{3}\lambda\beta^{-1} - \tfrac{5}{27}\lambda\beta^{-2} - \tfrac{31}{9}\lambda^2\beta^{-3} + \tfrac{35}{9}\sigma\beta^{-2}, \\
\hat{\lambda} &:= \lambda + \tfrac{7}{270} - \tfrac{11}{3}\lambda^2\beta^{-2} + \tfrac{14}{3}\sigma\beta^{-1}, \\
\hat{\sigma} &:= \sigma,
\end{aligned} \tag{6.2}$$

and similar definitions for $\hat{\beta}'$, $\hat{\lambda}'$, $\hat{\sigma}'$. In the case of $r=4$ we calculate $\hat{\beta} = \beta + 2\lambda\beta^{-1} + \mathcal{O}(\beta^{-1})$. We observe that within the weak coupling domain considered, i.e. $\beta \geqq \underline{\beta}$ (large enough) and $|\lambda\beta^{-1}|$, $|\sigma\beta^{-1}|$ bounded by constants, the transformation (6.2) has a unique inverse,

$$\begin{aligned}
\beta &= \hat{\beta} - \tfrac{5}{3}\hat{\lambda}\hat{\beta}^{-1} + \mathcal{O}(\hat{\beta}^{-1}), \\
\lambda &= \hat{\lambda} - \tfrac{7}{270} + \tfrac{11}{3}\hat{\lambda}^2\hat{\beta}^{-2} - \tfrac{14}{3}\hat{\sigma}\hat{\beta}^{-1} + \mathcal{O}(\hat{\beta}^{-1}), \\
\sigma &= \hat{\sigma}.
\end{aligned} \tag{6.3}$$

From (3.7) and (6.2) (and similarly in the case of $r=4$) we obtain the "decoupled" recursions

$$\begin{aligned}
\hat{\beta}' &= \hat{\beta} - c_0 - c_{-1}\hat{\beta}^{-1} + R_1, \qquad & |R_1| &< A_1\hat{\beta}^{-2+2\alpha}, \\
\hat{\lambda}' &= r^{-2}\hat{\lambda} + R_2, & |R_2| &< A_2\hat{\beta}^{-2+4\alpha}, \\
\hat{\sigma}' &= r^{-4}\hat{\sigma} + R_3, & |R_3| &< A_3\hat{\beta}^{-2+6\alpha},
\end{aligned} \tag{6.4}$$

where we denote remainders (which, of course, depend on the respective Gibbs factor) by $R_{1,2,3}$. The positive constants $A_{1,2,3}$ do not depend on the iteration step, and the parameters c_0, c_{-1} are

$$\begin{aligned} c_0 &= \tfrac{1}{6}, c_{-1} = \tfrac{1}{54} \quad \text{for} \quad r=2, \\ c_0 &= \tfrac{1}{2}, c_{-1} = \tfrac{1}{15} \quad \text{for} \quad r=4. \end{aligned} \tag{6.5}$$

From the positivity of c_0, implying that the marginal coefficient $\hat{\beta}$ decreases with increasing lattice spacing, we recognize UV-asymptotic freedom. The flow of β, as well as that of the contractive couplings λ and σ, is controlled by the flow of $\hat{\beta}$.

We define for $m \in \mathbb{N}_0$ and $\beta_0 \in \mathbb{R}_+$, $\beta_0 > \beta$,

$$B_m := \beta_0 + c_0 m + \frac{c_{-1}}{c_0} \ln\left\{1 + \frac{c_0 m}{\beta_0}\right\}. \tag{6.6}$$

Then one easily calculates with $d \in \mathbb{R}$, $|d| < S_1 \in \mathbb{R}_+$,

$$\begin{aligned} B_m + d - c_0 - \frac{c_{-1}}{B_m + d} - B_{m-1} &= d + P_{m-1}, \\ |P_{m-1}| &< C_1 (B_{m-1})^{-2+\varepsilon_1}. \end{aligned} \tag{6.7}$$

The positive constant C_1 does not depend on m and is uniform in d, $|d| < S_1$, and the small positive ε_1 absorbs $\ln(m)$ terms. The flow of the effective actions is controlled by

Proposition 2. *Choosing for all $N \in \mathbb{N}_0$ initial Gibbs factors $g_N^{(-N)}(u)$ satisfying* (A_1)–(A_3), *with initial values $\hat{\beta}_N^{(-N)} = B_N + b_N$ with B_N from* (6.6) *and $b_N = \mathcal{O}(B_N^{-1})$, the recursion relations* (6.4) *imply for $n \in \mathbb{N}_0$ uniformly in $N \geqq n$,*

$$\begin{aligned} \hat{\beta}_N^{(-n)} &= B_n + \mathcal{O}(B_n^{-1+2\alpha}), \\ \hat{\lambda}_N^{(-n)} &= r^{-2(N-n)} \hat{\lambda}_N^{(-N)} + \mathcal{O}(B_n^{-2+4\alpha}), \\ \hat{\sigma}_N^{(-n)} &= r^{-4(N-n)} \hat{\sigma}_N^{(-N)} + \mathcal{O}(B_n^{-2+6\alpha}). \end{aligned} \tag{6.8}$$

Proof. For $\hat{\beta} = B_m + d$, $d = \mathcal{O}(1)$, follows from (6.4) due to (6.7),

$$\begin{aligned} \hat{\beta}' &= B_{m-1} + d + \phi_{m-1}, \quad \text{with} \quad \phi_{m-1} \equiv R_1 + P_{m-1}, \\ |\phi_{m-1}| &< A_1 (B_m + d)^{-2+2\alpha} + C_1 (B_{m-1})^{-2+\varepsilon_1} < (1+\delta_5) A_1 (B_{m-1})^{-2+2\alpha}, \end{aligned}$$

with a constant $\delta_5 > 0$. Repeated iteration, starting with the initial values of the proposition, implies

$$\hat{\beta}_N^{(-n)} = B_n + b_N + \sum_{m=n}^{N-1} \phi_m, \; |\phi_m| < (1+\delta_5) A_1 B_m^{-2+2\alpha}. \tag{6.9}$$

It is crucial to observe that the respective values of d occurring in these iterations are uniformly bounded, since the series $\sum_{m=0}^{\infty} \phi_m$ converges absolutely. Hence an appropriate bound S_1 entering (6.7) and a constant δ_5 can be chosen on account of a sufficiently large value of β_0. Repeated iteration of the recursion relations for $\hat{\lambda}$ and $\hat{\sigma}$ together with the result obtained for $\hat{\beta}_N^{(-n)}$ yields

$$\hat{\lambda}_N^{(-n)} = r^{-2(N-n)}\hat{\lambda}_N^{(-N)} + \sum_{k=n+1}^{N} r^{-2(k-n-1)}\psi_k, \; |\psi_k| < A_2(1+\delta_6)B_k^{-2+4\alpha}, \tag{6.10}$$

$$\hat{\sigma}_N^{(-n)} = r^{-4(N-n)}\hat{\sigma}_N^{(-N)} + \sum_{k=n+1}^{N} r^{-4(k-n-1)}\omega_k, \; |\omega_k| < A_3(1+\delta_7)B_k^{-2+6\alpha}, \tag{6.11}$$

with suitably chosen constants δ_6, δ_7.

It is straightforward to bound the sums in (6.9)–(6.11), e.g. by corresponding integrals, and therefrom obtain the estimates stated in the proposition, uniformly in N. □

From Proposition 2 we deduce a very precise picture of the flow of couplings, exhibiting general features of the transformation. Since $\hat{\lambda}_N^{(-N)}$ and $\hat{\sigma}_N^{(-N)}$ are restricted to be (at most) $\mathcal{O}(\beta_N^{(-N)}) = \mathcal{O}(B_N)$, the first terms in (6.8) contributing to $\hat{\lambda}, \hat{\sigma}$ are strongly damped, which we infer from the inequality, valid for $N \in \mathbb{N}_0$, $N > n$,

$$B_N < B_n\left(1 + \frac{c_{-1}}{c_0\beta_0}\right)\left(1 + \frac{c_0}{\beta_0}\right)^{N-n}, \tag{6.12}$$

and $1 + \dfrac{c_0}{\beta_0}$ being small compared to r.

Thus Proposition 2 implies for any scale $n \in \mathbb{N}_0$, uniformly in $N > n + 2(\ln r)^{-1}\ln(\beta_0 + c_0 n)$,

$$\hat{\lambda}_N^{(-n)} = \mathcal{O}(B_n^{-2+4\alpha}), \qquad \hat{\sigma}_N^{(-n)} = \mathcal{O}(B_n^{-2+6\alpha}), \qquad 6.13)$$

which entails due to (6.3), in the case of $r = 2$,

$$\begin{aligned} \lambda_N^{(-n)} &= \lambda_* + \mathcal{O}(B_n^{-2+4\alpha}), \qquad \lambda_* = -\tfrac{7}{270}, \\ \sigma_N^{(-n)} &= \mathcal{O}(B_n^{-2+6\alpha}), \\ \beta_N^{(-n)} &= \hat{\beta}_N^{(-n)} + \mathcal{O}(B_n^{-1}) = B_n + \mathcal{O}(B_n^{-1+2\alpha}). \end{aligned} \tag{6.14}$$

For $r = 4$ (6.14) holds with $\lambda_* = -\frac{7}{150}$.

Hence the couplings $\beta_N^{(-n)}$, $\lambda_N^{(-n)}$, $\sigma_N^{(-n)}$ at scale n are found within intervals determined by B_n, uniformly in the cutoff $N > n + 2(\ln r)^{-1}\ln(\beta_0 + c_0 n)$ and for arbitrary bare actions $g_N^{(-N)}(u)$ [satisfying (A_1)–(A_3)], provided we fix $\hat{\beta}_N^{(-N)}$ as stated in the proposition. These bounds are inherited by the continuum limit $N \to \infty$, if it exists. As a basis for its construction we formulate

Theorem 2. *Let the initial Gibbs factors $g_N^{(-N)}(u)$ for the "cutoffs" $N \in \mathbb{N}_0$ be given either by the Wilson action (2.5) with*

$$\beta \equiv \beta_N^{(-N)} = B_N + c, \; c = \tfrac{5}{18}\{\tfrac{1}{3}\} \quad \textit{for} \quad r = 2\{4\}$$

or by the heat kernel action (2.20) with $\gamma = B_N + \frac{1}{6}$.

Then holds, if β_0 in (6.6) is chosen sufficiently large:

1) The inductively defined functions $h_N^{(-n)}(z)$, $n \in \mathbb{N}_0$ with $n \leqq N$, which are the Gibbs factors on scale n, expressed as a function of (and analytically continued in) the central angle, are entire holomorphic in $z \in \mathbb{C}$, 2π-periodic and real positive for $z \in \mathbb{R}$.

2) *In the small field domains* $|z|<(\beta_N^{(-n)})^{-\alpha}$,

$$h_N^{(-n)}(z)=\exp\{-V_N^{(-n)}(z)\},$$

$$V_N^{(-n)}(z)=\beta_N^{(-n)}z^2+\tfrac{1}{2}\lambda_N^{(-n)}z^4+\tfrac{1}{3}\sigma_N^{(-n)}z^6+\tilde{V}_N^{(-n)}(z),$$

$$\beta_N^{(-n)}=B_n+\mathcal{O}(B_n^{-1+2\alpha})\qquad[+\mathcal{O}(1)],$$

$$\lambda_N^{(-n)}=\lambda_*+\mathcal{O}(B_n^{-2+4\alpha})\qquad[+\mathcal{O}(1)+r^{-2(N-n)}\hat{\lambda}_N^{(-N)}],$$

$$\sigma_N^{(-n)}=\mathcal{O}(B_n^{-2+6\alpha})\qquad[+r^{-4(N-n)}\hat{\sigma}_N^{(-N)}],$$

$$|\tilde{V}_N^{(-n)}(z)|<D(\beta_N^{(-n)})^{-2}.$$

The terms in brackets [...] *are rapidly damped for* N *large, see* (6.14), *and* $\lambda_*=-\frac{7}{270}\{-\frac{7}{150}\}$ *for* $r=2\{4\}$.

3) *For* $z=x+iy$, *with* $x,y\in\mathbb{R}$, $|z|>(\beta_N^{(-n)})^{-\alpha}$, $|x|\leqq\pi$, $|y|<\frac{\kappa}{2}(\beta_N^{(-n)})^{-\alpha}$,

$$|h_N^{(-n)}(z)|<\exp\{\beta_N^{(-n)}y^2-p(\beta_N^{(-n)})^{1-2\alpha}\}.$$

4) *The families*, $n\in\mathbb{N}_0$, $\{h_N^{(-n)}(z)\}_{N\geqq n}$ *are normal families of holomorphic functions in the domains*

$$\left\{z\in\mathbb{C}:|\mathrm{Im}\,z|<\frac{\kappa}{2}(\underline{\beta}^{(-n)})^{-\alpha}\right\}\quad \textit{with}\quad \underline{\beta}^{(-n)}:=\sup_{N\geqq n}\beta_N^{(-n)}=B_n+\mathcal{O}(1).$$

Proof. For the Wilson action (W) and the heat kernel action (HK) we have

$$\mathrm{W}:\quad \lambda_N^{(-N)}=-\tfrac{1}{3!}\beta_N^{(-N)},\ \sigma_N^{(-N)}=\tfrac{1}{5!}\beta_N^{(-N)},$$

$$\mathrm{HK}:\quad \beta_N^{(-N)}=\gamma-\tfrac{1}{6}+\mathcal{O}(\gamma^3e^{-4\pi^2\gamma}),$$

$$\lambda_N^{(-N)}=-\tfrac{1}{90}+\mathcal{O}(\gamma^5e^{-4\pi^2\gamma}),\ \sigma_N^{(-N)}=-\tfrac{1}{945}+\mathcal{O}(\gamma^7e^{-4\pi^2\gamma}).$$

Both their respective analytically continued Gibbs factors $\tilde{g}_N^{(-N)}(u,z)$, (2.7), satisfy the induction assumptions (A_1)–(A_3). Due to the chosen initial values we have $\beta_N^{(-N)}=B_N+\mathcal{O}(B_N^{-1})$ and deduce from Proposition 2 and (6.3) that for any given initial cutoff N the iterations, leading to lower energy scales, can be performed until the unit lattice is reached. Hence the properties (A_1)–(A_3) are valid for any $\tilde{g}_N^{(-n)}(u,z)$.

1) restates (A_1) and has already been commented on in Sect. 2.

2) restates (A_3) together with the values of the coupling coefficients determined by Proposition 2 and (6.3).

In proving 3) we suppress subscript N and superscript $(-n)$. From the definition (2.10) and the symmetry (2.8) we obtain $\forall z\in\mathbb{C}$

$$h(z)=\tilde{g}(e_0,z)=\tilde{g}(e^{-ix\sigma_3},iy).\tag{6.15}$$

Proposition 1 shows for $|z|$ sufficiently small $|\theta(e^{-ix\sigma_3},iy)|=|z|$. Property (A_2) then implies 3). In order to prove 4) we first observe for given cutoff N and scale n the bound, valid for $|y|<\frac{\kappa}{2}(\beta_N^{(-n)})^{-\alpha}$,

$$|h_N^{(-n)}(x+iy)|<\exp\left\{(1+\varepsilon')\left(\frac{\kappa}{2}\right)^2(\beta_N^{(-n)})^{1-2\alpha}\right\}.\tag{6.16}$$

It follows from 2), 3) and the periodicity of $h_N^{(-n)}(z)$; the small $\varepsilon'>0$ absorbs an order $\mathcal{O}((\beta_N^{(-n)})^{-2\alpha})$.

Now consider the family of a given scale n. Because of 2) there exists

$$\underline{\beta}^{(-n)}=B_n+\mathcal{O}(B_n^{-1+2\alpha})+\mathcal{O}(1). \tag{6.17}$$

In the strip $|\mathrm{Im}\,z|<\frac{\kappa}{2}(\underline{\beta}^{(-n)})^{-\alpha}$ the members of the family are holomorphic functions bounded by

$$N\geqq n:\ |h_N^{(-n)}(z)|<\exp\left\{(1+\varepsilon')\left(\frac{\kappa}{2}\right)^2(\underline{\beta}^{(-n)})^{1-2\alpha}\right\}. \tag{6.18}$$

This proves 4), [13], and completes the proof of Theorem 2.

Remark. Part 4) states the existence of a convergent subsequence of $(h_N^{(-n)}(z))_{N\geqq n}$ for fixed n, i.e. existence of continuum limits, at any scale, via subsequences of cutoffs $N_j\to\infty$. This result is sharpened in

Theorem 3. *Under the assumptions of Theorem 2 there exists a subsequence* $(N_j)_{j\in\mathbb{N}}$ *of cutoffs, with* $N_j<N_{j+1}$, *such that for any scale* $n\in\mathbb{N}_0$ *the sequence* $(h_{N_j}^{(-n)}(z))$ *converges,*

$$\lim_{\substack{j\to\infty\\ N_j\geqq n}} h_{N_j}^{(-n)}(z)=h^{(-n)}(z).$$

The convergence to the holomorphic function $h^{(-n)}(z)$ *is uniform in the strip* $|\mathrm{Im}\,z|\leqq\frac{1}{2}(\underline{\beta}^{(-n)})^{-\alpha}$. *These limit functions* $h^{(-n)}(z)$ *uniquely define class functions* $g^{(-n)}(u)$, $u\in G$, *and extensions* $\tilde{g}^{(-n)}(u,z)$, (2.7), *satisfying* (A_1). *The sequence* $g^{(-n)}(u)$, $n\in\mathbb{N}_0$, *defines continuum effective actions at all scales due to the property*

$$\mathcal{T}g^{(-n)}(u)=g^{(-n+1)}(u),\qquad n\in\mathbb{N},$$

where $\mathcal{T}$ *denotes the Migdal transformation* (1.1), *with* $r=2$ *or* 4.

Proof. Consider first an arbitrary convergent subsequence of $(h_N^{(-n)})_{N\geqq n}$ at a given scale n; i.e. assume

$$\lim_{j\to\infty} h_{N_j}^{(-n)}(z)=h^{(-n)}(z). \tag{6.19}$$

The convergence is locally uniform, and thus $h^{(-n)}(z)$ holomorphic, in $|\mathrm{Im}\,z|<d_n:=\frac{\kappa}{2}(\underline{\beta}^{(-n)})^{-\alpha}$ due to the uniform bound (6.18). Part 1) of Theorem 2 implies that $h^{(-n)}$ is even, 2π-periodic and nonnegative for real arguments. Furthermore, the Fourier expansion $\sum_{m=0}^{\infty} a_m^{(-n)}\cos(mz)$ of $h^{(-n)}(z)$ is uniformly convergent in $|\mathrm{Im}\,z|\leqq d'<d_n$. Since $\cos mz$ is a polynomial in $w=\cos z$, this series defines a holomorphic function

$$H^{(-n)}(w)\quad\text{on}\quad \mathcal{D}_n:=\left\{w=w_1+iw_2\in\mathbb{C}:\frac{w_1^2}{\cosh^2 d_n}+\frac{w_2^2}{\sinh^2 d_n}<1\right\},$$

the image of $\{|\mathrm{Im}\,z|<d_n\}$ under $w=\cos z$, satisfying there

$$h^{(-n)}(z)=H^{(-n)}(\cos z). \tag{6.20}$$

Thus we can define, for $u\in G$ and $z\in\mathbb{C}$, $|\mathrm{Im}\, z|<d_n$, implying $u_0\cos z+u_3\sin z\in\mathscr{D}_n$,

$$\tilde{g}^{(-n)}(u,z)=H^{(-n)}(u_0\cos z+u_3\sin z), \tag{6.21}$$

which is clearly continuous in $u\in G$ and holomorphic in z, satisfying (2.8)–(2.10). Equation (6.21) is the unique "analytic continuation," (2.7), of $g^{(-n)}(u):=\tilde{g}^{(-n)}(u,0)=H^{(-n)}(u_0)$ which is the class function on G represented by $h^{(-n)}(x)$ as a function of the central angle. Note that

$$g^{(-n)}(u)=\lim_{\substack{j\to\infty\\(N_j\geqq n)}} g^{(-n)}_{N_j}(u) \tag{6.22}$$

as a uniform limit on G, due to (6.19).

The limit Gibbs factor $g^{(-n)}(u)$ has a zero set of Haar measure zero since this set consists at most of a finite number of conjugacy classes ($h^{(-n)}$ can only have discrete zeros). Moreover $g^{(-n)}(u)$ is of positive type, since the $g^{(-n)}_N(u)$ have this property.

The final steps of our proof rely on two general properties of the Migdal transformation $\mathscr{T}$, namely (i) $\mathscr{T}$ is continuous (with respect to the supremum norm) on $\mathfrak{F}:=\{g(u)$ continuous class function on $G, g(u)\geqq 0, g(e_0)=1\}$, and (ii) $\mathscr{T}$ is injective on $\mathfrak{F}_0:=\{g\in\mathfrak{F}, g$ is of positive type$\}$. Both (i) and (ii) are deduced from properties of $[\mathscr{T}]^{1/r}$, invoking for (ii) the character expansion of g.

We fix the subsequence N_j such that (6.19) holds for $n=0$, and prove (6.19) inductively for all $n\in\mathbb{N}$. Assume (6.19), and thus (6.22), at scale n. The sequence $(h^{(-n-1)}_{N_j}(z))$ of the preceding scale, being a normal family for $z\in\mathbb{C}$, $|\mathrm{Im}\, z|<\frac{\kappa}{2}(\beta^{(-n-1)})^{-\alpha}$ due to Theorem 2, has convergent subsequences, with corresponding limits (6.22) of the Gibbs factors $g^{(-n-1)}_N(u)$. Those are mapped onto $g^{(-n)}(u)$ due to (i); thus there is a unique limit $g^{(-n-1)}(u)=\lim_{j\to\infty} g^{(-n-1)}_{N_j}(u)$ due to (ii), which shows (6.19), (6.22) for $n+1$.

The property $\mathscr{T}g^{(-n-1)}(u)=g^{(-n)}(u)$ implies $g^{(-n)}(u)>0$ on G, since $g^{(-n-1)}(u)$ is nonnegative and vanishes at most on a set of Haar measure zero.

Finally the analytically extended version of $\mathscr{T}$, (3.1), (3.8), when applied to $\tilde{g}^{(-n-1)}(u,z)$ as given by (6.21), yields an analytic continuation of $\tilde{g}^{(-n)}(u,z)$ to the wider strip

$$z\in\mathbb{C}, |\mathrm{Im}\, z|<r\cdot\frac{\kappa}{2}(\beta^{(-n-1)})^{-\alpha}.$$

Iterating this argument, with $r^m(\beta^{(-n-m)})^{-\alpha}\xrightarrow[m\to\infty]{}\infty$, shows that $\tilde{g}^{(-n)}(u,z)$ is entire in $z\in\mathbb{C}$ for all n. $\square$

The continuum limit Gibbs factors $g^{(-n)}(u)$ respectively $h^{(-n)}(z)$ inherit small field properties at least in $|z|<\frac{\kappa}{2}(\beta^{(-n)})^{-\alpha}$ which are directly read off from Theorem 2, 2), omitting subscript N and the orders in brackets there. These results for the "running" couplings reflect UV-asymptotic freedom.

We can iterate $g^{(0)}(u)$, obtaining $g^{(1)}(u)=\mathscr{T}g^{(0)}(u)$ etc., and continue, thus reaching lower and lower momentum scales. Although we leave (eventually) the realm of our weak coupling analysis, Ito's nonperturbative analysis [11] can be applied, due to the properties stated in Theorem 3. It shows that $g^{(n)}(u)$ converges for $n\to\infty$ to the strong coupling fixed point $g\equiv 1$. Therefore the infinite volume limit of expectations within this hierarchical lattice theory can be performed in the strong coupling region. In the gauge model ($r=4$) the string tension can be derived from a sequence of suitably chosen Wilson loops, see [7, 8]; Ito's result [11] then implies

Theorem 4. *The continuum limit Gibbs factors $g^{(-n)}(u)$ of a hierarchical gauge model, as obtained in Theorem 3, yield a positive string tension.*

Appendix

In this appendix we exhibit some properties of the heat kernel (2.20). We denote (2.20) by $g(u)$ and by $h(\theta)$ and $\tilde{g}(u,z)$ the related functions as defined in Sect. 2, suppressing the parameter $\gamma\in\mathbb{R}_+$.

Lemma. *For $\gamma\in\mathbb{R}_+$,*

(i) $h(\theta)>0,\ \forall\theta\in\mathbb{R}$,

(ii) $h(\theta)$ *decreases monotonously in* $\theta\in(0,\pi)$,

(iii) $$-\ln h(\theta)=(\gamma-\tfrac{1}{6}+\varrho_1)\theta^2+\tfrac{1}{2}(-\tfrac{1}{90}+\varrho_2)\theta^4+\tfrac{1}{3}(-\tfrac{1}{945}+\varrho_3)\theta^6+\mathcal{O}(\theta^8)$$

with $\varrho_j=\mathcal{O}(\gamma^{2j+1}e^{-4\pi^2\gamma})$ *for* $\gamma\to\infty$,

(iv) *in* $\left\{|\theta|>\beta^{-\alpha},\ |\mathrm{Re}\,\theta|\leqq\pi,\ |\mathrm{Im}\,\theta|<\frac{\kappa}{2}\beta^{-\alpha}\right\}$,

$$|h(\theta)|<\exp\{\beta(\mathrm{Im}\,\theta)^2-p\beta^{1-2\alpha}\}$$

with a constant $p=\frac{5}{8}$ and $\beta:=\gamma-\frac{1}{6}+\varrho_1$ large enough. (κ and α are defined in Sect. 3.)

Proof. From (2.20) we obtain the entire holomorphic function

$$h(\theta)=\mathcal{N}\sum_{l=1}^{\infty}le^{-\frac{l^2}{4\gamma}}\frac{\sin l\theta}{\sin\theta}\tag{A.1}$$

with the positive normalization factor $\mathcal{N}$ determined by $h(0)=1$. Equation (A.1) can be expressed in terms of the theta-function ϑ_3, [14], writing

$$q=\exp\left\{-\frac{1}{4\gamma}\right\},\tag{A.2}$$

$$h(\theta)=-\frac{\mathcal{N}}{2}\frac{1}{\sin\theta}\frac{d}{d\theta}\left\{1+2\sum_{l=1}^{\infty}q^{l^2}\cos l\theta\right\}\tag{A.3}$$

$$=-\frac{\mathcal{N}}{2}\frac{1}{\sin\theta}\frac{d}{d\theta}\vartheta_3\left(\frac{\theta}{2},q\right).\tag{A.4}$$

From the product representation of ϑ_3, [14], we obtain

$$h(\theta)=\mathcal{N}q_0\sum_{n=1}^{\infty}q^{2n-1}\prod_{m\neq n}\{1+2q^{2m-1}\cos\theta+q^{4m-2}\}\,,\tag{A.5}$$

$$q_0=\prod_{n=1}^{\infty}(1-q^{2n})\,.\tag{A.6}$$

In (A.5) each factor is strictly positive $\forall\theta\in\mathbb{R}$ and decreases in $(0,\pi)$, thus proving (i), (ii).

Employing in (A.3) the Poisson summation formula we obtain, with $\delta>0$ small, in $\mathcal{D}=\{\theta\in\mathbb{C}:|\mathrm{Re}\,\theta|\leqq\pi-\delta\}$,

$$h(\theta)=\mathcal{N}'\frac{\theta}{\sin\theta}e^{-\gamma\theta^2}\left\{1+2\sum_{n=1}^{\infty}[\cosh 4\pi\gamma n\theta-2\pi n\theta^{-1}\sinh 4\pi n\gamma\theta]\right.$$
$$\left.\cdot\exp(-4\pi^2\gamma n^2)\right\},\tag{A.7}$$

and with $\check{\theta}=\pi-\theta$ in $\check{\mathcal{D}}=\{\check{\theta}\in\mathbb{C}:|\mathrm{Re}\,\check{\theta}|\leqq\pi-\delta\}$,

$$h(\theta)=\mathcal{N}'\frac{\check{\theta}}{\sin\check{\theta}}e^{-\gamma\check{\theta}^2}\sum_{n=1}^{\infty}\exp[-4\pi^2\gamma(n-\tfrac{1}{2})^2]$$
$$\cdot\{4\pi(n-\tfrac{1}{2})\check{\theta}^{-1}\sinh 4\pi\gamma(n-\tfrac{1}{2})\check{\theta}-2\cosh 4\pi\gamma(n-\tfrac{1}{2})\check{\theta}\}\,.\tag{A.8}$$

Both representations are explicitly holomorphic in their respective domains. For large values of γ the constant $\mathcal{N}'$ determined by normalization of (A.7) is

$$\mathcal{N}'=1+\mathcal{O}(\gamma\exp\{-4\pi^2\gamma\})\,.\tag{A.9}$$

From (A.7) we read off (iii). Moreover we deduce from (A.7) and (A.8) bounds, valid for $\gamma\to\infty$,

$$|h(\theta)|\leqq\exp\{-\gamma\,\mathrm{Re}\,\theta^2+\mathcal{O}(1)\},\qquad \theta\in\mathcal{D}\,,\tag{A.10}$$

$$|h(\theta)|\leqq\exp\{-\gamma\pi(\pi-2\delta)-\gamma\,\mathrm{Re}\,\check{\theta}^2+\mathcal{O}(\ln\gamma)\},\qquad |\mathrm{Re}\,\check{\theta}|\leqq\delta\,.\tag{A.11}$$

From these bounds follows (iv) observing that $p<1-\dfrac{\kappa^2}{4}+\mathcal{O}(\beta^{-1+2\alpha})$. □

Part (iv) of the lemma directly entails the induction assumption (A_2) for $\tilde{g}(u,z)$, (2.21), using Proposition 1 to represent $\tilde{g}$ by h. We have to consider arguments (u,iy) with

$$u\in G\backslash\mathcal{G}[iy,\beta^{-\alpha}],\qquad |y|<\frac{\kappa}{2}\beta^{-\alpha}\,.\tag{A.12}$$

For $u_0\leqq 0$ we have $\tilde{g}(u,iy)=h(\pi-\check{\theta}(u,iy))$ with $\check{\theta}$ defined in Sect. 2. We observe that $|\mathrm{Im}\,\check{\theta}|\leqq|y|$ and $0\leqq\mathrm{Re}\,\check{\theta}\leqq\frac{\pi}{2}$, which is implied by the remark following the proof of Proposition 1; thus $\pi-\check{\theta}$ is in the domain considered in (iv). For $u_0>0$ and $|\theta(u,iy)|>\beta^{-\alpha}$ we have $\tilde{g}(u,iy)=h(\theta(u,iy))$, bounded by (iv), too. Both pieces together establish the bound (A_2).

This method to derive (A_2) from a bound on $h(\theta)$ is easily applied to the Wilson Gibbs factor, too.

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