

Quantum Toda Systems and Lax Pairs

N. Ganoulis

Physics Division, School of Technology, University of Thessaloniki, Greece

Abstract. We describe a general method for constructing a Lax pair representation of certain quantum mechanical systems that are integrable at the classical level. This is then used to find conserved quantities at the quantum level for the Toda systems.

1. Introduction

There is as fairly general method for constructing conserved quantities of classical mechanical systems in one and two dimensions (field-theories), that has been much studied and developed recently. In this method [1, 2] one starts with a Fundamental Poisson Bracket relation, that is, an expression for the Poisson bracket between the elements of a certain matrix, which is a function of the canonical variables of the dynamical system. One then finds, under certain conditions, that it is possible to get a family of conserved quantities, or Hamiltonians, in involution. A zero curvature condition for the “gauge potentials” (which are the auxiliary matrices constructed out of the canonical variables), plays an important role in this construction.

In the one dimensional case, this corresponds to showing that there is a Lax pair representation of the dynamical system, i.e. the classical equations of motion can be written in the form $\frac{dA}{dt} = [A, B]$, where A and B are matrices, functions of the canonical variables. The conserved quantities are then $\text{Tr } A^N$ for any power N .

In this paper we will develop a similar approach for a quantum mechanical system. We consider the case when the Fundamental Poisson Bracket (FPB) goes over directly into a commutator bracket. We then show that there is a Lax pair representation of the quantum system.

This method has been developed with a particular application in mind. We wanted to construct conserved quantities for the quantum mechanical Toda system [3]. This is done by making use of the Lie algebraic properties of our “quantum mechanical” Lax pair. Our approach works uniformly for all the finite classical Lie algebras.

A first proof of integrability of the quantum Toda lattice was given by Kostant in [6]. Goodman and Wallach [7] give a recursive procedure for constructing integrals of the classical and quantum system. The reader is also referred to the review article [11], where several explicit formulae for first integrals can be found.

What we believe is new in this article is the use of the P -operator as introduced in [1], to derive the quantum-mechanical conservation laws of the Toda system. As we discuss in the next section the method is quite general, so that we hope that it will be possible to apply it also to other systems. Finally this method gives us a simple algorithm for explicit construction of the quantum integrals which may be used in applications.

The techniques that we use are closely related to [4], who discuss the classical mechanics of Toda systems. We have been also motivated by some of the results of [5], who refers to construction of solutions in one and two dimensions. The quantization of Poisson brackets and the relation between classical and quantum R -matrices is considered in [12].

2. Quantum Commutators and the Lax Representation

In order to realise the FPB one starts with the canonical variables of the system (q_α, p_α) and uses an auxiliary vector space (e.g. a Lie algebra), in order to construct the A operator (in general some matrix function of p_α, q_α). The Poisson bracket between the elements of A is then given by the expression.

$$\{A \otimes A\} = [\mathbb{P}, A \otimes 1 + 1 \otimes A]. \quad (1)$$

We have here used the compact notation [1].

The tensor product between two matrices is defined as usual

$$(A \otimes B)_{ijkl} = A_{ij} B_{kl}, \quad (2)$$

and the product between tensors

$$(\mathbf{RS})_{ijkl} = \mathbf{R}_{ij'kl'} \mathbf{S}_{j'l'i}. \quad (3)$$

$\{A \otimes A\}$ stands for the PB of two matrix elements

$$\{A \otimes A\}_{ijkl} = \{A_{ij}, A_{kl}\},$$

\mathbb{P} is constant, independent of the canonical variables. The usual commutator product of \mathbb{P} is considered in (1).

Relation (1) seems to apply for a wide range of integrable systems [2] and the corresponding \mathbb{P} operator is known for several models. In the next section we will present the explicit form of A and \mathbb{P} as it applies in the case of the Toda system.

In this section we will consider the most simple case for quantum mechanics, when the PB directly goes over to a commutator bracket. That is

$$[A \otimes 1, 1 \otimes A] = i\hbar [\mathbb{P}, A \otimes 1 + 1 \otimes A] \quad (4)$$

with the same constant \mathbb{P} as in the classical relation (1). Note that A is now a certain matrix with elements that are functions of the quantum mechanical operators p_α, q_α .

One would have to consider each model separately to determine whether (1) and (4) apply in classical and quantum mechanics respectively. The Toda systems that we are interested in, certainly satisfy relation (4) as it will be verified.

Classically the Hamiltonian is taken to be $H = \text{Tr} A^2$. We will also take this as our quantum Hamiltonian, which determines the time evolution of any operator through the Heisenberg equations. We will use this to construct a Lax pair. We can write (4) in a more compact form

$$[A \otimes 1 - i\hbar \mathbb{P}, 1 \otimes A + i\hbar \mathbb{P}] = 0. \quad (5)$$

This expression looks more interesting than (4). Note the interplay between the ordinary commutator of \mathbb{P} with A and the “quantum” commutator between the components of A . This form suggests to us that $A \otimes 1 - i\hbar \mathbb{P}$ acts as a single operator and we should therefore consider taking its powers. In particular it follows that $[(A \otimes 1 - i\hbar \mathbb{P})^2, 1 \otimes A + i\hbar \mathbb{P}] = 0$.

Since we want the Hamiltonian to enter we will take Tr_L of the above expression, where $\text{Tr}_L \mathbf{S} = \mathbf{S}_{ikb}$ that is the trace on the left indices of the tensors:

$$\text{Tr}_L [A \otimes 1 - i\hbar \mathbb{P}]^2, 1 \otimes A + i\hbar \mathbb{P}] + \text{Tr}_L [(A \otimes 1 - i\hbar \mathbb{P})^2, i\hbar \mathbb{P}] = 0.$$

The second term is zero, since we can make use of the cyclic property of the trace, so that terms $\text{Tr}_L [A^2 \otimes 1, \mathbb{P}] = 0$ and

$$\text{Tr}_L [(A \otimes 1) \mathbb{P}, \mathbb{P}] \text{ cancels } \text{Tr}_L [\mathbb{P} (A \otimes 1), \mathbb{P}].$$

We therefore get

$$\text{Tr}_L (A^2 \otimes 1 - 2i\hbar \mathbb{P} (A \otimes 1) - \hbar^2 \mathbb{P}^2), A] = 0. \quad (6)$$

The reader can convince himself that one can simply take the Tr_L inside the commutator, as we contract from tensor products to matrices. Using the Hamiltonian $H = \text{Tr} A^2$ the above expression can be brought to a form

$$\frac{i}{\hbar} [H, A] = [-2 \text{Tr}_L \mathbb{P} (A \otimes 1) + i\hbar \text{Tr}_L \mathbb{P}^2, A]. \quad (7)$$

The left-hand side is simply $\frac{dA}{dt}$ and we therefore interpret (7) as a Lax pair representation of our quantum mechanical system, with the two operators:

$$A \text{ and } B = -2 \text{Tr}_L \mathbb{P} (A \otimes 1) + i\hbar \text{Tr}_L \mathbb{P}^2, \quad \frac{dA}{dt} = [B, A]. \quad (8)$$

Note that B is constructed from the A operator. Furthermore B has the same form as in the classical Lax pair apart from the constant matrix $i\hbar \text{Tr}_L \mathbb{P}^2$. In other words if we take $\hbar \rightarrow 0$ and interpret A and B as matrix functions of the classical variables (p_α, q_α) we recover the classical equations of motion.

Up to this point we have kept our discussion quite general. At this point it is not clear which is the best way to proceed. Note that (8) although in form is the same as in the Classical Lax representation, it does not share all of its nice properties. In particular $\text{Tr} A^N$ are not constant in general, since we cannot use the cyclic property of the trace on $[B, A^N]$, because their matrix elements do not commute. But we observe the following: if we define the time ordered exponential

$u = T \exp \int B dt$, so that $B = \frac{du}{dt} u^{-1}$, we can easily see that $\frac{d}{dt} (u^{-1} A u) = 0$, if we use

Eq. (8). We would like to give a precise meaning to these conservation laws by studying particular models.

3. The Quantum Mechanical Toda Molecule

In the simplest case this describes a finite chain of particles with interactions which vary exponentially with their separation. We will be using the mathematical language of Lie algebras [8] in order to describe these systems, since the algebraic structure underlies most of their integrability properties.

The Hamiltonian written in terms of the canonical variables p_α, q_α is

$$H = \sum_{\alpha, \beta} (\lambda_\alpha \cdot \lambda_\beta) p_\alpha p_\beta + \sum_\alpha \frac{1}{\alpha^2} \exp\left(\sum_\beta K_{\alpha\beta} q_\beta\right). \quad (9)$$

The sum is over the set Δ of simple roots. The number of simple roots is equal to the rank l of the Lie algebra, and they form a basis for the root space.

$K_{\alpha\beta}$ is the Cartan matrix. It encodes the structure constants of the algebra $K_{\alpha\beta} = \frac{2\alpha \cdot \beta}{\beta \cdot \beta} \alpha, \beta \in \Delta$.

λ_α are the fundamental weights.

The reader can check that the matrix $(\lambda_\alpha \cdot \lambda_\beta)$ is the inverse of $\frac{4\alpha \cdot \beta}{\alpha^2 \beta^2} = \frac{2}{\alpha^2} K_{\alpha\beta}$. It then follows from (9) that the classical equations of motion are

$$\ddot{q}_\alpha = -\exp\left(\sum_\beta K_{\alpha\beta} q_\beta\right). \quad (10)$$

This is the non-linear system for l variables known as the Toda equation.

We will be using the Chevalley basis for the Lie algebra $[H_\alpha, H_\beta] = 0$,

$$\begin{aligned} [H_\alpha, E_{\pm\beta}] &= \pm K_{\beta\alpha} E_{\pm\beta}, \quad \alpha, \beta \in \Delta, \\ [E_\alpha, E_{-\beta}] &= \delta_{\alpha\beta} H_\alpha, \end{aligned} \quad (11)$$

and normalize the Killing form by

$$\text{Tr}(H_\alpha H_\beta) = \frac{4\alpha \cdot \beta}{\alpha^2 \beta^2}, \quad \text{Tr}(E_\alpha E_{-\beta}) = \frac{2\delta_{\alpha\beta}}{\alpha^2}. \quad (12)$$

We will use (9) as our quantum Hamiltonian with canonical commutation relations

$$[p_\alpha, q_\beta] = -i\hbar \delta_{\alpha\beta}, \quad [p_\alpha, p_\beta] = 0, \quad [q_\alpha, q_\beta] = 0. \quad (13)$$

It is also useful to evaluate

$$\left[p_\alpha, e^{\sum_\gamma K_{\beta\gamma} q_\gamma} \right] = -i\hbar K_{\beta\alpha} e^{\sum_\gamma K_{\beta\gamma} q_\gamma}. \quad (14)$$

Then the expression for the Lax operator is given with

$$A = \sum_{\alpha, \beta} (\lambda_\alpha \cdot \lambda_\beta) H_\alpha p_\beta + \frac{1}{2} \sum_\alpha (E_\alpha + E_{-\alpha}) e^{1/2 \sum_\beta K_{\alpha\beta} q_\beta} \quad (15)$$

with $E_{\pm\alpha}, H_\alpha$ generators of the algebra.

It is easy to check that $H = \text{Tr} A^2$. We next evaluate the commutator using (14),

$$\begin{aligned} [A \otimes 1, 1 \otimes A] &= \sum_{\alpha, \beta, \gamma} \frac{1}{2} (\lambda_\alpha \cdot \lambda_\beta) (H_\alpha \otimes (E_\gamma + E_{-\gamma}) - (E_\gamma + E_{-\gamma}) \otimes H_\alpha) [p_\beta, e^{1/2 K_\gamma \delta q_\delta}] \\ &= -\frac{i\hbar}{8} \sum_\alpha (H_\alpha \otimes (E_\alpha + E_{-\alpha}) - (E_\alpha + E_{-\alpha}) \otimes H_\alpha) e^{1/2 K_\alpha \delta q_\delta}. \end{aligned}$$

The above expression can be now expressed using the \mathbf{P} operator, as it is done in the classical case. We refer the reader to [4], where the properties of this operator are discussed. It is given

$$\mathbf{P} = -\frac{1}{4} \sum_{\alpha} \alpha^2 (E_{\alpha} \otimes E_{-\alpha} - E_{-\alpha} \otimes E_{\alpha}), \quad \alpha \in \Phi^+. \quad (16)$$

The sum is over all positive roots Φ^+ and (not only the simple ones).

We therefore conclude that Eq. (4) is satisfied and the analysis of our previous section applies. To determine B of Eq. (8) we need to evaluate

$$\begin{aligned} \mathrm{Tr}_L \mathbf{P}(A \otimes 1) &= -\frac{1}{8} \sum_{\beta \in \Delta} \sum_{\alpha \in \Phi^+} \exp\left(\frac{1}{2} \sum_{\gamma} K_{\beta\gamma} q_{\gamma}\right) \alpha^2 \mathrm{Tr}_L (E_{\alpha} (E_{\beta} + E_{-\beta}) \otimes E_{-\alpha} \\ &\quad - E_{-\alpha} (E_{\beta} + E_{-\beta}) \otimes E_{\alpha}) = -\frac{1}{4} \sum_{\beta} e^{1/2 \sum_{\gamma} K_{\beta\gamma} q_{\gamma}} (E_{-\beta} - E_{\beta}), \end{aligned}$$

and the constant term

$$\begin{aligned} \mathrm{Tr}_L \mathbf{P}^2 &= -\frac{1}{16} \sum_{\alpha, \beta} \alpha^2 \beta^2 ((\mathrm{Tr} E_{\alpha} E_{-\beta}) E_{-\alpha} E_{\beta} + (\mathrm{Tr} E_{-\alpha} E_{\beta}) E_{\alpha} E_{-\beta}) \\ &= -\frac{1}{8} \sum_{\alpha} \alpha^2 (E_{\alpha} E_{-\alpha} + E_{-\alpha} E_{\alpha}). \end{aligned}$$

But one can show that the Casimir operator of the algebra is

$$C = \sum_{\alpha, \beta \in \Delta} (\lambda_{\alpha} \lambda_{\beta}) H_{\alpha} H_{\beta} + \sum_{\alpha \in \Phi^+} \frac{\alpha^2}{2} (E_{\alpha} E_{-\alpha} + E_{-\alpha} E_{\alpha}).$$

This of course commutes with every other generator. We can therefore drop it and write for B in (8),

$$B = \frac{1}{2} \sum_{\beta} e^{1/2 \sum_{\gamma} K_{\beta\gamma} q_{\gamma}} (E_{-\beta} - E_{\beta}) + \frac{i\hbar}{4} \sum_{\alpha, \beta} (\lambda_{\alpha} \cdot \lambda_{\beta}) H_{\alpha} H_{\beta}. \quad (17)$$

Note that A is Hermitian and B antihermitian. The constant term in B also appears in [5]. Our derivation makes clear the origin of this term.

4. Conservation Laws from the Quantum Lax Pair

It is most suitable to write the Lax equation in the form

$$\left[\frac{i}{\hbar} H - B, A \right] = 0 \quad (18)$$

with H the Hamiltonian and A, B as given from (15) and (17) respectively. We now observe how the above equation is transformed by a ‘‘gauge transformation.’’ Define

$$g = \exp\left(-\frac{1}{2} \sum_{\alpha} q_{\alpha} H_{\alpha}\right), \quad (19)$$

where H_{α} are the Cartan subalgebra generators. Consequently

$$g p_{\alpha} g^{-1} = p_{\alpha} - \frac{i\hbar}{2} H_{\alpha}. \quad (20)$$

Therefore the Hamiltonian becomes

$$gHg^{-1} = H - i\hbar \sum_{\alpha, \beta} (\lambda_\alpha \cdot \lambda_\beta) p_\alpha H_\beta - \frac{\hbar^2}{4} \sum_{\alpha, \beta} (\lambda_\alpha \cdot \lambda_\beta) H_\alpha H_\beta. \quad (21)$$

The term proportional to \hbar is the same one that appears in A and the \hbar^2 is the constant term in B . Here it appears through a gauge transformation. We also have

$$gE_{\pm\alpha}g^{-1} = \exp\left(\mp 1/2 \sum_{\beta} K_{\alpha\beta} q_\beta\right) E_{\pm\alpha}. \quad (22)$$

Using (20) and (22) we can also calculate

$$\tilde{B} \equiv gBg^{-1} = \frac{1}{2} \sum_{\alpha} e^{\sum_{\beta} K_{\alpha\beta} q_\beta} E_{-\alpha} - \frac{1}{2} \sum_{\alpha} E_{\alpha} + \frac{i\hbar}{4} \sum_{\alpha, \beta} (\lambda_\alpha \lambda_\beta) H_\alpha H_\beta, \quad (23)$$

$$\tilde{A} \equiv gAg^{-1} = \sum_{\alpha, \beta} (\lambda_\alpha \cdot \lambda_\beta) H_\alpha p_\beta - \frac{i\hbar}{2} \sum_{\alpha, \beta} (\lambda_\alpha \cdot \lambda_\beta) H_\alpha H_\beta + \frac{1}{2} \sum_{\alpha} e^{\sum_{\beta} K_{\alpha\beta} q_\beta} E_{-\alpha} + \frac{1}{2} \sum_{\alpha} E_{\alpha}$$

Finally the reader can check that (18) can be written

$$\left[\frac{i}{\hbar} H - \sum_{\alpha} E_{-\alpha} e^{K_{\alpha\beta} q_\beta}, \tilde{A} \right] = 0. \quad (24)$$

We have arrived at this form after adding \tilde{A} to the left-hand side of the equation.

The above Eq. (24) also applies for any power n of $\tilde{A}^n = gA^n g^{-1}$. Also, it applies for any representation of the Lie algebra. In order to find operators which commute with the Hamiltonian H , consider the following properties of the generators $E_{-\alpha}$. In any finite dimensional representation of a Lie algebra there is a highest weight state $|\lambda\rangle$ and a lowest weight state $|\bar{\lambda}\rangle$, such that $\langle\lambda|E_{-\alpha} = 0$ and $E_{-\alpha}|\bar{\lambda}\rangle = 0$, for every positive root α .

By taking matrix elements of (24) between these states, we conclude

$$[H, \langle\lambda|\tilde{A}^n|\bar{\lambda}\rangle] = 0 \quad (25)$$

for every power n . These are the conserved quantities.

We therefore have a simple prescription for determining conservation laws: Starting with a Hamiltonian (9) for any simple Lie algebra we use a low dimensional representation for the generators $H_\alpha, E_{\pm\alpha}$ to express \tilde{A} in Eq. (23) as a matrix. Take powers of this matrix and find the matrix element between the highest and lowest vectors. One should perhaps note here that the highest state vectors $|\lambda\rangle$, have also been used to construct solutions of the classical equations of motion [9].

Since there is an ambiguity with the phases of the $E_{\pm\alpha}$, the matrix elements are not automatically hermitian. We show in the appendix that the operators $\langle\lambda|\tilde{A}^n|\bar{\lambda}\rangle$ come out to be hermitian (up to constant phase factors) when evaluated for any representation of the Lie algebras B_l, C_l, D_l (l even), E_7, E_8, F_4, G_2 . For the other simple Lie algebras we can always choose a representation (e.g. the adjoint) that satisfies the hermiticity condition.

In order to complete our results we will next have to show that these operators that commute with the Hamiltonian, also commute with one another.

5. Commuting Set of Operators

The idea is to modify the original commutation relation (4) in a way that it applies for the \tilde{A} operator. Let us first introduce the notation,

$$\begin{aligned}\mathbf{C}_\pm &= \sum_{\alpha \in \Phi^\pm} \left(\frac{\alpha^2}{2} \right) E_\alpha \otimes E_{-\alpha}, \\ \mathbf{C}_0 &= \sum_{\alpha, \beta \in \Delta} (\lambda_\alpha \cdot \lambda_\beta) H_\alpha \otimes H_\beta.\end{aligned}\tag{26}$$

The combination $\mathbf{C} = \mathbf{C}_+ + \mathbf{C}_- + \mathbf{C}_0$ is a Casimir-like operator and commutes [4] with any generator T :

$$[\mathbf{C}, 1 \otimes T + T \otimes 1] = 0.$$

This means that we can add any fraction of \mathbf{C} to the \mathbb{P} operator without affecting the fundamental commutation relation (4). Instead of \mathbb{P} we take

$$\mathbb{P} \rightarrow \mathbb{P} - \frac{1}{2} \mathbf{C} = -\frac{1}{2}(\mathbf{C}_+ - \mathbf{C}_-) - \frac{1}{2}(\mathbf{C}_+ + \mathbf{C}_- + \mathbf{C}_0) = -\mathbf{C}_+ - \frac{1}{2} \mathbf{C}_0,$$

and therefore

$$A \otimes 1 - i\hbar \mathbb{P} \rightarrow A \otimes 1 + \frac{i\hbar}{2} \mathbf{C}_0 + i\hbar \mathbf{C}_+ = (1 \otimes g^{-1})(A \otimes 1)(1 \otimes g) + i\hbar \mathbf{C}_+.\tag{27}$$

The last equation follows from the definition (19) of g and its property (20). To see this consider only the part of $A \otimes 1$ which does not commute with $(1 \otimes g)$:

$$\begin{aligned}(1 \otimes g^{-1}) \left(\sum_{\alpha, \beta \in \Delta} (\lambda_\alpha \cdot \lambda_\beta) H_\alpha p_\beta \otimes 1 \right) (1 \otimes g) &= \sum_{\alpha, \beta} (\lambda_\alpha \lambda_\beta) H_\alpha \otimes g^{-1} p_\beta g \\ &= \sum_{\alpha, \beta} (\lambda_\alpha \lambda_\beta) H_\alpha p_\beta \otimes 1 + \frac{i\hbar}{2} \sum_{\alpha, \beta} (\lambda_\alpha \lambda_\beta) H_\alpha \otimes H_\beta.\end{aligned}$$

The last term in the sum is precisely $\frac{i\hbar}{2} \mathbf{C}_0$ and this proves (27). We can therefore substitute (27) in Eq. (4) or (5) and take arbitrary powers n and m and arrive at the final expression:

$$[((1 \otimes g^{-1})(A \otimes 1)(1 \otimes g) + i\hbar \mathbf{C}_+)^n, ((g \otimes 1)(1 \otimes A)(g^{-1} \otimes 1) - i\hbar \mathbf{C}_+)^m] = 0.\tag{28}$$

We will need this expression rather than (5) for two reasons: first it includes g and this is desirable since we know that the conserved quantities are not given in terms of A but $\tilde{A} = gAg^{-1}$. Secondly in this last commutator we have \mathbf{C}_+ instead of the operator \mathbb{P} . \mathbf{C}_+ does not mix positive with negative roots; in fact we can find a tensor product of states which is annihilated by \mathbf{C}_+ :

$$\mathbf{C}_+ |\lambda\rangle \otimes 1 = 0 \quad \text{and} \quad \mathbf{C}_+ 1 \otimes |\bar{\lambda}\rangle = 0,$$

where $|\lambda\rangle$ and $|\bar{\lambda}\rangle$ are, as before, the highest and lowest weight states of a representation. Similarly $\langle \bar{\lambda}| \otimes 1 \mathbf{C}_+ = 0$ and $1 \otimes \langle \lambda| \mathbf{C}_+ = 0$. This is precisely the property that we want in order to remove \mathbf{C}_+ from (28) and find a commutation relation that no longer contains tensor products: We will take the matrix elements of (28) between the states $\langle \bar{\lambda}| \otimes \langle \lambda|$ on the left and $|\lambda\rangle \otimes |\bar{\lambda}\rangle$ on the right. This is completely analogous to the step that gave us Eq. (25).

Evaluating the matrix element is straightforward, for example evaluating one of the products

$$((g \otimes 1)(1 \otimes A)(g^{-1} \otimes 1) - i\hbar \mathbf{C}_+)^m |\lambda\rangle \otimes |\bar{\lambda}\rangle = (|\lambda\rangle \otimes 1)(1 \otimes g_\lambda^{-1} |\bar{\lambda}\rangle),$$

where $g_\lambda = \langle \lambda | g | \lambda \rangle = \exp\left(\frac{1}{2} \sum_\alpha \frac{2\alpha \cdot \lambda}{\alpha^2} q_\alpha\right)$.

We omit all details here and just quote the final result for the matrix element of (28):

$$[\langle \bar{\lambda} | g^{-1} A^n g | \lambda \rangle, \langle \lambda | g A^m g^{-1} | \bar{\lambda} \rangle] = 0. \quad (29)$$

This proves our claim that we have a set of commuting operators. If we make use of the hermiticity property (see the Appendix), we can write this in the form

$$[\langle \lambda | \tilde{A}^n | \bar{\lambda} \rangle, \langle \lambda | \tilde{A}^m | \bar{\lambda} \rangle] = 0.$$

The first non-trivial example is $SU(3)$. After a suitable change of coordinates we can take as our Hamiltonian $H = p_1^2 + p_2^2 - p_1 p_2 + e^{q_1} + e^{q_2}$, with canonical commutation relations $[p_i, q_j] = -i\hbar \delta_{ij}$. Then in the fundamental representation

$$\tilde{A} = g A g^{-1} = 1/2 \begin{bmatrix} p_1 & 1 & 0 \\ e^{q_1} & p_2 - p_1 & 1 \\ 0 & e^{q_2} & -p_2 \end{bmatrix} - \frac{i\hbar}{6} \mathbf{1},$$

$\mathbf{1}$ is the 3×3 unit matrix. The matrix element between $\langle \lambda |$ and $|\bar{\lambda}\rangle$ corresponds to selecting the *top right* element of the matrix. Equation (25) tells us that for any power n of A , this matrix element commutes with the Hamiltonian. We find indeed that for $n=4$ we recover H , and for $n=5$ an operator

$$Q = p_1^2 p_2 - p_1 p_2^2 + p_2 e^{q_1} - p_1 e^{q_2}.$$

Higher powers of \tilde{A} do not produce any more constants.

In conclusion we have shown how to construct Lax pairs in quantum mechanical systems, and we used it to find conserved quantities for the Toda molecule, related to all classical or exceptional Lie algebras. We would like to mention that the same method also applies for the periodic Toda lattice. This system is known [10] to be related to the infinite dimensional Kac-Moody algebras. However the last part of our method will have to be modified, because we cannot find both a highest and a lowest weight state for the representations of these algebras. Progress will be reported in a subsequent publication.

Appendix

We will here discuss the hermiticity properties of the operators. The position and momentum operators q_a, p_a are hermitian, and for the Lie algebra generators we have

$$(E_{\pm a})^+ = E_{\mp a}, \quad (H_a)^+ = H_a.$$

Therefore $A^+ = A$ and $g^+ = g$, and the matrix elements

$$\langle \lambda | g A^n g^{-1} | \bar{\lambda} \rangle^+ = \langle \bar{\lambda} | g^{-1} A^n g | \lambda \rangle. \quad (\text{A.1})$$

We know [8] that for the root systems of the Lie algebras A_2, B_l, C_l, D_l (l even), E_7, E_8, F_4, G_2 there is a Weyl group element that transforms each root to its negative. This means that there is a group element s_0 with the following properties:

$$\begin{aligned} s_0 H_a s_0^{-1} &= H_{-a} = -H_a, \\ s_0 E_{\pm a} s_0^{-1} &= -E_{\mp a} \quad \text{with} \quad s_0^+ = s_0^{-1}, s_0^2 = 1. \end{aligned} \quad (\text{A.2})$$

s_0 permutes the weights, and it is easy to see that it takes the lowest weight state to the highest weight state:

$$s_0 |\bar{\lambda}\rangle = |\lambda\rangle \quad (\text{up to a phase}). \quad (\text{A.3})$$

(For more details about the s_0 we refer the reader to the second paper of [9].) We can use s_0 to write (A.1),

$$\langle \bar{\lambda} | g^{-1} A^n g | \lambda \rangle = \langle \lambda | s_0 g^{-1} A^n g | \lambda \rangle = \langle \lambda | s_0 g^{-1} A^n g s_0^{-2} | \lambda \rangle.$$

But g and A transform under conjugation with s_0 as follows:

$$s_0 g s_0^{-1} = g^{-1} \quad \text{and} \quad s_0 A s_0^{-1} = -A.$$

We conclude that $\langle \lambda | g A^n g^{-1} | \bar{\lambda} \rangle^+ = (\text{phase}) \langle \lambda | g A^n g^{-1} | \bar{\lambda} \rangle$. This is the hermiticity property that we have used. We should note that when we have one of the other Lie algebras we can still construct an s_0 which satisfies (A.3). This s_0 takes every simple root to the negative of another simple root related by an automorphism of the Dynkin diagram. Instead of (A.2) we have

$$\begin{aligned} s_0 H_a s_0^{-1} &= -H_a, \quad a, a' \in A, \\ a &\rightarrow a' \text{ an automorphism of Dynkin diagram.} \end{aligned}$$

Then everything that we have said applies unchanged, provided that we consider only representations with highest weight, which does not change under automorphisms of the Dynkin diagram.

Acknowledgements. I would like to thank D.I. Olive, L. Ferreira, and P. Mansfield for discussions.

References

1. Fadeev, L.D.: Integrable models in 1 + 1 dimensional quantum field theory. In: Les Houches Summer School Proceedings, Vol. 39. Amsterdam, New York: North-Holland 1984
2. Kulish, P.P., Sklyanin, E.K.: Quantum spectral transform method, recent developments. In: Integrable quantum field theories. Lecture Notes in Physics, Vol. 151. Berlin, Heidelberg, New York: Springer 1982
3. Toda, M.: Theory of non-linear lattices. Springer Series in Solid State Physics, Vol. 20. Berlin, Heidelberg, New York: Springer 1981
4. Olive, D.I., Turok, N.: Algebraic structure of Toda systems. Nucl. Phys. **B 220** [FS8], 491–507 (1983)
5. Mansfield, P.: Solution of Toda systems. Nucl. Phys. **B 208**, 277–300 (1982)
6. Kostant, B.: Quantization and representation theory of Lie groups. London Math. Soc. Lecture Notes Series **34**, 287 (1979)
7. Goodman, R., Wallach, N.R.: Classical and quantum-mechanical systems of Toda lattice type. I. Commun. Math. Phys. **83**, 355–386 (1982)

8. Humphreys, J.E.: Introduction to Lie algebras and representation theory. Berlin, Heidelberg, New York: Springer 1972
9. Leznov, A.N., Saveliev, M.V.: Representation theory and integration of nonlinear spherically symmetric equations to gauge theories. *Commun. Math. Phys.* **74**, 111–118 (1980)
Ganoulis, N., Goddard, P., Olive, D.: Self-dual monopoles and Toda molecules. *Nucl. Phys. B* **205**, 601–636 (1982)
10. Olive, D., Turok, N.: Local conserved densities and zero curvature conditions for Toda lattice field theories. *Nucl. Phys. B* **257**, 277–301 (1985)
11. Olshanetsky, M.A., Perelomov, A.M.: Quantum integrable systems related to Lie algebras. *Phys. Rep.* **94**, 313–404 (1983)
12. Semenov-Tjan-Shanskii: *Zapiski of seminars LOMI, in geometry, Lie groups and mechanics*, YI, p. 228 (1984)

Communicated by Ya. G. Sinai

Received February 20, 1986; in revised form June 25, 1986