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On the Connection Between Quantum Fields and von Neumann Algebras of Local Operators

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Abstract. The relationship between a standard local quantum field and a net of local von Neumann algebras is discussed. Two natural possibilities for such an association are identified, and conditions for these to obtain are found. It is shown that the local net can naturally be so chosen that it satisfies the Special Condition of Duality. The notion of an intrinsically local field operator is introduced, and it is shown that such an operator defines a local net with which the field is locally associated. A regularity condition on the field is formulated, and it is shown that if this condition holds, then there exists a unique local net with which the field is locally associated if and only if the field algebra contains at least one intrinsically local operator. Conditions under which a field and other fields in its Borchers class are associated with the same local net are found, in terms of the regularity condition mentioned.

1. Introduction

In the attempts to formulate a mathematically satisfactory theory of particles consistent with special relativity and incorporating the notion of locality, two main approaches stand out. One of these is the general theory of (finite-component) local quantum fields [21, 28] and the other is the algebraic relativistic quantum theory [16, 1, 17, 7]. In the latter theory the primary object of interest is a net of algebras of local observables, and experience has shown that such a theory provides a suitable framework for the analysis of the general structure of a relativistically covariant, local quantum theory. Quantum field theory deals with operator-valued distributions and algebras of closable, but in general unbounded operators. The study of such objects entails considerable technical difficulties involving domain of definition questions. In spite of this, the notion of a local

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quantum field is attractive in many respects. It has been easier to imagine the formulation of a genuine dynamical principle in terms of fields than in terms of bounded operator algebras, and the notion of a field is also basic in the perturbation-theoretic approach for the explicit computation of physically interesting quantities.

It is of obvious interest to explore the connection between finite-component quantum fields and nets of local algebras. It is the purpose of this paper to discuss the nature of the connection and to present results which amount to a substantial reduction of the apparent complexity of the problem. In particular we give an answer to the following question of principle: when does a quantum field have a net of local algebras to which it is associated, and what is the mathematical nature of this association? Moreover, we show how local nets can be constructed from the fields if these satisfy certain additional conditions. We do not discuss the converse problem – the construction of local fields from local nets – but we note here that considerable progress has been made recently on this question [13, 18, 27, 29].

In the formulation of Haag and others [16, 1, 17, 7], a *net* of local algebras is a specific assignment $\mathcal{O}\to\mathfrak{A}(\mathcal{O})$ of a C^*-algebra $\mathfrak{A}(\mathcal{O})$ to each $\mathcal{O}$ in a suitable set $\mathscr{R}$ of subsets of Minkowski space. This association is required to satisfy certain well-known minimum conditions of isotony, locality, and Poincaré-covariance. The framework is very general, but we shall here be concerned solely with the vacuum representation of such a net. The algebras are then algebras of operators on a Hilbert space $\mathscr{H}$, and for our purposes it is convenient to assume that all the algebras of the net are von Neumann algebras. The Hilbert space carries a strongly continuous, unitary representation $\lambda\to U(\lambda)$ of (the universal covering group of) the Poincaré group, and the Poincaré-covariance of the net is then expressed in an obvious fashion in terms of this representation.

In this paper we shall be concerned with a particular kind of local net, introduced in [2] and called an *AB-system.* The set $\mathscr{R}$ then consists of all (closed) double cones, all (open) causal complements of these, and certain wedge-shaped regions which are bounded by two non-parallel characteristic planes. In Sect. 2 we give the technical definition of an *AB*-system, and we discuss the features of such a net which make it particularly suited as the object with which the fields can be related. In the interest of simplicity, the discussion in this paper is confined to a standard quantum field theory [21, 28] of a single, irreducible hermitian scalar field, with the exception of Sect. 6 in which we discuss the case of two local and relatively local fields. The generalizations of our considerations to the case of an arbitrary number of *finite*-component quantum fields is straightforward. In the presence of Fermi-fields the conditions of locality and duality have to be appropriately modified, but this does not affect the essence of our reasoning. The generalization to fields of the kind considered by Jaffe [19], which are strictly localizable but not necessarily operator-valued *tempered* distributions, also seems to present no essential difficulties.

A natural and desirable relationship between a field and a net of local von Neumann algebras is the following. For any $\mathcal{O}\in\mathscr{R}$ all, or at least a "sufficiently large" subset of, the operators in the algebra of averaged field operators which are associated with $\mathcal{O}$ have closed extensions affiliated with $\mathfrak{A}(\mathcal{O})$ (in the sense of von Neumann [25] – see Sect. 2). In Definition 2.4 we give a precise formulation of this idea, and we identify two specific possibilities for the association, which we call Scenario G and Scenario A. The first of these is the "better" scenario in the sense that *every* operator in the algebra $\mathscr{P}_0(R)$ of field operators associated with any $R \in \mathscr{R}$ has a closed extension affiliated with the corresponding von Neumann algebra of a local net. In Scenario A such an association obtains only for a *subset* of the set of field operators. In both cases the association implies that the net is a local, TCP-covariant AB-system, which satisfies the *special condition of duality*. This latter condition, which emerged in [2], seems to be characteristic for theories of finite-component fields. It is a specific form of the well-known duality condition for local theories: see Definition 2.1 for a precise statement. The reader can regard Definition 2.4 as a description of goals of this paper. In Theorems 2.7 and 2.8 we show that some seemingly much less restrictive conditions on the field actually imply the scenarios in Definition 2.4. Our interpretation of Theorem 2.8 is that if the field is locally associated with a local net in *any* reasonable sense, then at least Scenario A must obtain.

In Sect. 3 we continue the discussion of the nature of the association. As a preliminary we first state and prove, in Lemma 3.1, a principle akin to the Reeh-Schlieder Principle. On the basis of the lemma we find, in the form of Theorem 3.2, conditions under which the vacuum vector Ω is cyclic and separating for a local algebra associated with a double cone. In Theorem 3.3 we present results concerning the existence of (local) selfadjoint extensions of symmetric field operators.

The entire system of local von Neumann algebras might very well be "generated" by a single averaged field operator, and in Sect. 4 we discuss how this can come about. In Definition 4.1 we introduce the notion of an *intrinsically local* operator X_s in the algebra of field operators. Somewhat loosely stated, the closure of X_s, relative to a subdomain (determined by X_s itself) of the usual domain of the field operators, generates a von Neumann algebra which is *locally* associated with the same double cone K_s to which X_s "belongs." In Theorem 4.6 we show that an intrinsically local operator X_s defines a local AB-system such that at least Scenario A obtains. If the intrinsically local operator is of the form $X_s = \varphi[f_s]$, i.e., is *linear* in the field, stronger conclusions can be drawn, as shown in Theorem 4.8. Here f_s is a real test function with support in K_s, and such that its Fourier transform vanishes nowhere. These premises imply a uniqueness of the local AB-system, as stated in Theorem 4.8.

In Sects. 5 and 6 we discuss quantum fields which satisfy a certain regularity condition, which is essentially that there exists an α, with $1 > \alpha \geqq 0$, such that for every test function f, $\varphi[f]\exp(-H^\alpha)$ is a bounded operator: here H is the Hamiltonian. See Definition 5.1 for a precise statement. Conditions of this general type have been considered before, and most examples of massive fields which have been constructed are known to satisfy such a condition [9, 15, 18].

In Theorems 5.5 and 5.6 we present the consequences of such a regularity condition. We thus show that if f_s is any test function of compact support, with a Fourier transform which vanishes nowhere, then $\varphi[f_s]$ is intrinsically local if and only if it has *some* closed extension affiliated with a double-cone algebra of *some* local net. Moreover, *if* $\varphi[f_s]$ is intrinsically local, then the closure of $\varphi[f_s]$ generates a *unique* local AB-system for which Scenario G also obtains, i.e., *every* operator in the algebra of field operators has a closed extension affiliated with the appropriate von Neumann algebra in the AB-system. Hence, if $\varphi[f_s]$ is intrinsically local, so is $\varphi[f]$ for any other (real) test function f which satisfies the same premises as f_s. The assumption of the regularity condition, which we call a *generalized H-bound*, thus has remarkable implications for the connection between fields and local nets. The problem of determining whether *all* field operators, linear or multi-linear, have closed extensions affiliated with the algebras of some local net is seemingly a formidable problem. The observation that it reduces to the study of just one single operator amounts to a substantial simplification. The results in Sect. 4 are of interest from this same point of view, but the results with a generalized H-bound are much stronger.

In addition, Theorem 5.5 makes explicit the following significant result. If the quantum field satisfies the regularity condition, then *either* there exists a (unique) local AB-system such that Scenario G applies to the field and the system, *or* there is nothing even remotely resembling a net of local algebras with which the quantum field can be in any sense locally associated.

In Sect. 6 we extend these considerations to the Borchers class of the "original" field $\varphi(x)$. In Theorem 6.1 we show that if there is an intrinsically local operator X_s in the algebra of the averaged field operators $\varphi[f]$, and if $\psi(x)$ is a field in the Borchers class which satisfies a generalized H-bound, then Scenario G obtains for the field $\psi(x)$ and the AB-system generated by X_s, i.e., every (local) element in the field-operator algebra generated by $\psi(x)$ has a closed extension affiliated with the AB-system. Furthermore, if f is a real test function of compact support, with a non-vanishing Fourier transform, then $\psi[f]$ is intrinsically local, and its closure generates the same *unique* AB-system as X_s. These results have some obvious potential applications, which we discuss in Sect. 6. In particular they are relevant for the theory of Wick polynomials of a massive free field. The question of what local algebras such a Wick polynomial generates has been discussed much earlier [23], and these earlier results now emerge rather naturally within our theory.

The conclusions reached in this paper, taken in conjunction with recent results concerning the reconstruction of fields locally associated with a net of local algebras from limits of sequences of operators from the algebras [13, 18, 27, 29], suggest that local AB-systems satisfying the special condition of duality are likely to play an important role in quantum field theories which fit into the framework of algebraic relativistic local theories.

2. Some Generalities About Local Nets Associated with a Local Quantum Field

We consider a theory of a single irreducible local hermitian scalar field $\varphi(x)$, and we adhere to all the standard assumptions and conventions as described in Chap. III of the monograph by Streater and Wightman [28]. For any subset R of Minkowski space $\mathscr{M}$ we define $\mathscr{P}_0(R)$ as the smallest unital *-algebra which contains the averaged field operator $\varphi[f]$ for every test function f with $\operatorname{supp}(f)\subset R$. The elements $X\in\mathscr{P}_0(\mathscr{M})$ are regarded as defined on a domain customarily denoted by D_1, which arises when the algebra generated by all averaged linear *and* multilinear field operators acts on the vacuum vector Ω. The star-operation referred to above is hermitian conjugation $\varphi[f]\to\varphi[f]^\dagger \equiv\varphi[f]^*\restriction D_1$, and since $\varphi(x)$ is hermitian we have $\varphi[f]^\dagger=\varphi[f^*]$. In what follows the overbar will *always* be used to denote the closure $\bar{X}$ of any $X \in \mathscr{P}_0(\mathscr{M})$, *as defined on* D_1. We define $D_0 = \mathscr{P}_0(\mathscr{M})\Omega$, and we assume that the subdomain D_0 of D_1 is dense in the Hilbert space $\mathscr{H}$. It is well-known that the closures of any $X \in \mathscr{P}_0(\mathscr{M})$ on D_0 and D_1 are the same: $\bar{X} = (X \upharpoonright D_0)^{**}$.

The Hilbert space $\mathscr{H}$ carries a strongly continuous unitary representation $\lambda \to U(\lambda)$ of the Poincaré group P. For the elements of the subgroup of translations we also employ the notation $T(x) = U(I, x)$. This subgroup is subject to the usual spectrum condition. The canonical TCP-operator is denoted Θ_0, and we have $\Theta_0 \varphi(x) \Theta_0 = \varphi(-x)$, $\Theta_0 \Omega = \Omega$ and $\Theta_0^2 = I$.

The discussion in this paper depends critically on some results in [2], which we shall now review very briefly. We define the wedge-regions W_R and W_L in Minkowski space by

$$W_R = \{x | x^3 > |x^4|\}, \qquad W_L = \{x | x^3 < -|x^4|\}.$$

The vector Ω is cyclic and separating for $\mathscr{P}_0(W_R)$ and $\mathscr{P}_0(W_L)$. It was shown in [2] that

$$JV(i\pi)X\Omega = X^\dagger \Omega, \qquad JV(-i\pi)Y\Omega = Y^\dagger \Omega, \tag{2.1}$$

for all $X \in \mathscr{P}_0(W_R)$, all $Y \in \mathscr{P}_0(W_L)$. Here J is the antiunitary involution $J = U(\pi_3, 0)\Theta_0$, where π_3 denotes the rotation by angle π about the 3-axis. The operators $V(i\pi)$ and $V(-i\pi)$ are positive selfadjoint operators obtained by analytic continuation of the unitary operators $V(t) = U(v_3(t), 0)$ which represent the one-parameter abelian group of velocity transformations in the 3-direction. The parametrization is so chosen that the action of $v_3(t)$ on the rest state of a (classical) particle leads to a state of velocity $\tanh(t)$ in the 3-direction. It was also shown in [2] that $\mathscr{P}_0(W_R)\Omega$ is a core for $V(i\pi)$, and that $\mathscr{P}_0(W_L)\Omega$ is a core for $V(-i\pi)$.

For any subset R of Minkowski space we denote by R_λ the image of R under the Poincaré-transformation λ, and by R^c the *causal complement* of R, i.e., the set of all points of $\mathscr{M}$ strictly spacelike relative to R. We define $\mathscr{W} = \{W_{R,\lambda} | \lambda \in P\}$ as the set of all wedge-regions Poincaré-equivalent to W_R (and to W_L), and we denote by $\mathscr{K}$ the set of all closed double cones K with a non-empty interior. For any $K \in \mathscr{K}$ we have $K = \cap \{W | W \in \mathscr{W}, W \supset K\}$, and for the (open) causal complement we have $K^c = \cup \{W | W \in \mathscr{W}, W \subset K^c\}$.

The notion of an *AB-system* of von Neumann algebras was introduced and discussed in [2] and [3]. It is a particular kind of local net, with special properties of interest for this paper: the admittedly awkward term is used because *some* name is necessary to distinguish this kind of local net from other kinds of local nets. Some general properties of an AB-system were discussed in [30], and we shall here quote some definitions and results from this paper.

Definition 2.1. a) An AB-system is a set $\{\mathscr{A}(W), \mathscr{B}(K), \mathscr{A}(K^c)\}$ of von Neumann algebras such that to every $W \in \mathscr{W}$ corresponds an algebra $\mathscr{A}(W)$ and to every $K \in \mathscr{K}$ correspond two algebras $\mathscr{B}(K)$ and $\mathscr{A}(K^c)$, the correspondence being such that the following conditions hold:

$$U(\lambda)\mathscr{A}(W)U(\lambda)^{-1} = \mathscr{A}(W_\lambda), \quad \text{all} \quad \lambda \in P, \quad W \in \mathscr{W}; \tag{2.2a}$$

$$\mathscr{A}(W) \supset \mathscr{A}(W_i), \quad \text{whenever} \quad W \supset W_i; \tag{2.2b}$$

$$\mathscr{B}(K) = \cap \{\mathscr{A}(W) | W \in \mathscr{W}, W \supset K\}, \qquad \mathscr{A}(K^c) = \{\mathscr{A}(W) | W \in \mathscr{W}, W \subset K^c\}'' \tag{2.2c}$$

b) The AB-system is said to be *local* if and only if

$$\mathcal{A}(W)' \supset \mathcal{A}(\bar{W}^c), \quad \mathcal{B}(K)' \supset \mathcal{A}(K^c), \tag{2.3}$$

for all $W \in \mathcal{W}$, $K \in \mathcal{K}$. The set (algebra) $\mathcal{U}$ of operators is defined by

$$\mathcal{U} = \cup\{\mathcal{B}(K) | K \in \mathcal{K}\}. \tag{2.4}$$

If the AB-system is local the set $\mathcal{U}$ will be called the *set of all (strictly) local operators*.

c) The AB-system is said to be *generated by its $\mathcal{B}$-algebras* if and only if

$$\mathcal{A}(W) = \{\mathcal{B}(K) | K \in \mathcal{K}, K \subset W\}''. \tag{2.5}$$

If the AB-system is furthermore local it is said to be *generated by its local operators*.

d) The AB-system is said to be *TCP-covariant* if and only if

$$\Theta_0 \mathcal{A}(W) \Theta_0^{-1} = \mathcal{A}(-W), \quad \Theta_0 \mathcal{B}(K) \Theta_0^{-1} = \mathcal{B}(-K), \tag{2.6}$$

for all $W \in \mathcal{W}$, $K \in \mathcal{K}$. We here use the notation $-R = \{-x | x \in R\}$.

e) The AB-system is said to satisfy the *condition of duality* if and only if

$$\mathcal{A}(W)' = \mathcal{A}(\bar{W}^c), \quad \mathcal{B}(K)' = \mathcal{A}(K^c), \tag{2.7}$$

for all $W \in \mathcal{W}$, $K \in \mathcal{K}$.

f) The AB-system is said to satisfy the *special condition of duality* if and only if the following conditions hold [in which case the conditions (2.6) and (2.7) also trivially hold]:

$$\mathcal{A}(W_L) = \mathcal{A}(W_R)' = J\mathcal{A}(W_R)J. \tag{2.8}$$

The vector Ω is cyclic and separating for $\mathcal{A}(W_R)$. The linear manifold $\mathcal{A}(W_R)\Omega$ is a core for $V(i\pi)$, and

$$JV(i\pi)A\Omega = A^*\Omega, \quad \text{all} \quad A \in \mathcal{A}(W_R). \tag{2.9}$$

These definitions, which correspond to Definition 1 in [30], involve a certain amount of obvious redundancy. The relation at right in (2.3) thus follows from the relation at left [and the general relations (2.2)], and likewise the relation at right in (2.7) follows from the relation at left. It is important to note that the AB-system is *completely* determined, through (2.2c), by the wedge-algebras $\mathcal{A}(W)$, and the conditions (2.2a) and (2.2b) then imply conditions of covariance and a variety of conditions of isotony for *all* the algebras of the AB-system. For instance, $U(\lambda)\mathcal{B}(K)U(\lambda)^{-1} = \mathcal{B}(K_\lambda)$ for all $K \in \mathcal{K}$, $\lambda \in P$. Since the relationships in question are quite obvious, it is hardly necessary to present a complete list. We note here that the condition of duality is stronger than the condition of locality: the former implies the latter. The special condition of duality is stronger still in that it also implies TCP-covariance.

In the context of quantum field theory we expect that a relevant AB-system is local, and that it has the property that it is generated by its local operators in the sense that (2.5) holds. For a truly "local" theory the set $\mathcal{U}$ in (2.4) of all local operators ought to be "sufficiently large," which reasonably means that this set is irreducible.

Any system of local operators can be embedded in a natural way in a local AB-system. We state the matter as follows.

Lemma 2.2. *For each $K \in \mathscr{K}$, let $\mathscr{G}(K)$ be a selfadjoint set, i.e., $A^* \in \mathscr{G}(K)$ if $A \in \mathscr{G}(K)$, of bounded operators, and let $\mathscr{G} = \cup\{\mathscr{G}(K) | K \in \mathscr{K}\}$. We assume that the operators in $\mathscr{G}$ are local in the sense that $U(\lambda)\mathscr{G}(K_1)U(\lambda)^{-1} \subset \mathscr{G}(K_2)'$ for any two $K_1, K_2 \in \mathscr{K}$ and any $\lambda \in P$ such that $K_{1,\lambda}$ is spacelike relative to K_2. Then:*

a) *There exists a local AB-system $\{\mathscr{A}(W), \mathscr{B}(K), \mathscr{A}(K^c)\}$, said to be generated by $\mathscr{G}$, such that*

$$\mathscr{A}(W) = \{U(\lambda)\mathscr{G}(K)U(\lambda)^{-1} | K \in \mathscr{K},\ \lambda \in P,\ K_\lambda \subset W\}''. \tag{2.10}$$

This AB-system is generated by its local operators, and it satisfies the condition $\mathscr{G}(K) \subset \mathscr{B}(K)$ for all $K \in \mathscr{K}$.

b) *The following five conditions are equivalent:* 1) *The AB-system is irreducible;* 2) *The algebra $\mathscr{U}$ in* (2.4) *is irreducible;* 3) *The set $\cup\{U(\lambda)\mathscr{G}U(\lambda)^{-1} | \lambda \in P\}$ is irreducible;* 4) *Ω is a cyclic vector for $\mathscr{U}$, and* 5) *Ω is a cyclic vector for $\mathscr{A}(W_R)$.*

c) *If the AB-system defined by* (2.10) *satisfies the condition of duality it is the only AB-system for which the inclusion relations $\mathscr{G}(K) \subset \mathscr{B}(K)$ hold for all K.*

For the proof, which is almost totally trivial, we refer to [30] (see in particular Theorem 2). Note here that $K \to \mathscr{G}(K)$ is *not* assumed to be a local net, nor is it assumed that this mapping satisfies the conditions of Poincaré-covariance or isotony. It might thus well happen that $\mathscr{G}(K)$ is empty for all but one single $K_0 \in \mathscr{K}$, and furthermore it could happen that $\mathscr{G}(K_0)$ consists of just one single (selfadjoint) operator.

In the terminology of [30] the set $\mathscr{G}$ is a "primary set of local operators." A particular example of such a primary set is a local net $\mathcal{O} \to \mathfrak{A}(\mathcal{O})$ defined on the set of all open double cones $\mathcal{O}$. We define $\mathscr{G}(\bar{\mathcal{O}}) = \mathfrak{A}(\mathcal{O})$, and we then have $\mathfrak{A}(\mathcal{O}) \subset \mathscr{B}(\bar{\mathcal{O}})$. Furthermore it is easily seen that $\mathscr{A}(W) = \{\mathfrak{A}(\mathcal{O}) | \mathcal{O} \subset W\}''$.

In order to study the possible local association of a quantum field with a local AB-system we must first discuss a notion of *affiliation* due to von Neumann [25]. Let Q be any *closed* linear operator, with the polar decomposition $Q = VP$. In this paper we shall denote by $a(Q)$ the von Neumann algebra generated by the partial isometry V and the spectral projections of the non-negative selfadjoint operator P. The operator Q is said to be *affiliated with* a von Neumann algebra $\mathscr{A}$ if and only if $a(Q) \subset \mathscr{A}$. This condition is equivalent to the condition

$$QA \supset AQ, \quad \text{for all} \quad A \in \mathscr{A}'. \tag{2.11}$$

In the following we shall say that a bounded operator A *commutes in the strong sense* with a closed operator Q if and only if $A \in a(Q)'$, which is thus equivalent to the conditions that $QA \supset AQ$ *and* $QA^* \supset A^*Q$.

Let $\mathscr{P}_0$ be an algebra (over the complex field) of closable operators defined on a common dense invariant domain D, and let $\mathscr{P}_0$ be a *hermitian algebra* in the sense that for each $X \in \mathscr{P}_0$ the domain of X^* includes D, and such that $X^\dagger$ is contained in $\mathscr{P}_0$, where $X^\dagger \equiv X^* \upharpoonright D$. We shall say that a bounded operator A *commutes weakly with $\mathscr{P}_0$ on D* if and only if

$$\langle X^\dagger \phi' | A\phi'' \rangle = \langle A^*\phi' | X\phi'' \rangle$$

for all $X \in \mathscr{P}_0$, and all $\phi', \phi'' \in D$. We note that this condition, which is equivalent to the condition

$$X^{\dagger *} A \supset AX\,, \quad \text{all} \quad X \in \mathscr{P}_0\,, \tag{2.12}$$

implies that A^* also commutes weakly with $\mathscr{P}_0$ on D. The set of all bounded operators A which commute weakly with $\mathscr{P}_0$ on D is the *weak commutant* of $\mathscr{P}_0$. It is a weakly closed, linear manifold, closed under the star-operation, but it is in general *not* an algebra. We also note that the condition (2.12) implies that $X^{\dagger *} A \supset AX^{**}$, which condition should be contrasted with the condition (2.11).

For later reference we state the following simple

Lemma 2.3. *Let $\mathscr{P}_0$ be a hermitian algebra of closable operators on a common dense invariant domain D. Then:*

a) *A von Neumann algebra $\mathscr{A}$ has the property that each $X \in \mathscr{P}_0$ has a closed extension X_e affiliated with $\mathscr{A}$ and such that $X_e^* \supset X^\dagger$ if and only if every $A \in \mathscr{A}'$ commutes weakly with $\mathscr{P}_0$ (on D), in which case we say that $\mathscr{A}'$ commutes weakly with $\mathscr{P}_0$ (on D).*

b) *Suppose that $\mathscr{A}'$ commutes weakly with $\mathscr{P}_0$. Let $D_e = \mathrm{span}\{\mathscr{A}'D\}$. Then D_e is included in the domain of X^* for every $X \in \mathscr{P}_0$, and for each such X the operator $e(X) = X^{\dagger *} \restriction D_e \supset X$ is a well-defined closable operator, with the property that its closure $e(X)^{**}$ is affiliated with $\mathscr{A}$. The set $\{e(X) | X \in \mathscr{P}_0\}$ is a hermitian algebra on D_e, and the mapping $X \to e(X)$ is a *-representation of the algebra $\mathscr{P}_0$ such that $X^* \supset e(X)^* \supset e(X^\dagger) \supset X^\dagger$ for all $X \in \mathscr{P}_0$.*

For the simple proof, and for further elaborations on this theme, we refer to [3] (see in particular Lemma 10 in Sect. V), and also to the papers of Powers [26] and of Jørgensen [20].

In the above the operator $e(X)^{**}$ is closed extension of X which is affiliated with $\mathscr{A}'$. It should be noted that this does *not* mean that the closure X^{**} of X relative to the original domain D is also affiliated with $\mathscr{A}$. Nor is the possibility excluded that X has other closed extensions besides $e(X)^{**}$ which are also affiliated with $\mathscr{A}$.

We now continue the discussion of our local field theory.

Definition 2.4. Let $\varphi(x)$ be an irreducible local hermitian scalar field, subject to the general conditions stated in the beginning of this section. Let $\{\mathscr{A}(W), \mathscr{B}(K), \mathscr{A}(K^c)\}$ be a local, *TCP*-covariant *AB*-system which satisfies the special condition of duality and which is generated by its local operators. The following two possibilities for a local association of the field with the *AB*-system are hereby identified:

Scenario G. For each $K \in \mathscr{K}$ every $X \in \mathscr{P}_0(K)$ commutes weakly on D_1 with every $A \in \mathscr{A}(K^c)$, and every $X \in \mathscr{P}_0(K^c)$ commutes weakly on D_1 with every $A \in \mathscr{B}(K)$. For each $W \in \mathscr{W}$ every $X \in \mathscr{P}_0(W)$ commutes weakly on D_1 with every $A \in \mathscr{A}(\bar{W}^c)$. Equivalently stated, every $X \in \mathscr{P}_0(K)$, $\mathscr{P}_0(K^c)$, respectively $\mathscr{P}_0(W)$, has a closed extension X_e affiliated with $\mathscr{B}(K)$, $\mathscr{A}(K^c)$, respectively $\mathscr{A}(W)$, and such that $X_e^* \supset X^\dagger$.

Furthermore Ω is cyclic and separating for $\mathscr{B}(K)$ for all $K \in \mathscr{K}$, and $\mathscr{A}(W) = \{\mathscr{B}(K_\lambda) | \lambda \in P,\ K_\lambda \subset W\}''$ for any $K \in \mathscr{K}$.

Scenario A. 1) To each subset $R \subset \mathscr{M}$ corresponds a unital *-algebra $\mathscr{P}_{os}(R) \subset \mathscr{P}_0(R)$, where the correspondence $R \to \mathscr{P}_{os}(R)$ satisfies the conditions of isotony and Poincaré-covariance, i.e., $\mathscr{P}_{os}(R_0) \supset U(\lambda)\mathscr{P}_{os}(R)U(\lambda)^{-1}$ whenever $R_0 \supset R_\lambda$. Furthermore $\mathscr{P}_{os}(R)$ is the smallest *-algebra which contains $\mathscr{P}_{os}(K)$ for all $K \subset R$, $K \in \mathscr{K}$. In particular this applies for $R = \mathscr{M}$, and every element $X \in \mathscr{P}_{os}(\mathscr{M})$ is thus a *local* operator in the sense that $X \in \mathscr{P}_{os}(K)$ for some $K \in \mathscr{K}$.

2) The linear manifold $D_{os} \equiv \mathscr{P}_{os}(\mathscr{M})\Omega$ is *dense* in $\mathscr{H}$.

3) For each $K \in \mathscr{K}$ every $X \in \mathscr{P}_{os}(K)$ commutes weakly on D_{os} with every $A \in \mathscr{A}(K^c)$, and every $X \in \mathscr{P}_{os}(K^c)$ commutes weakly on D_{os} with every $A \in \mathscr{B}(K)$. For each $W \in \mathscr{W}$ every $X \in \mathscr{P}_{os}(W)$ commutes weakly on D_{os} with every $A \in \mathscr{A}(\bar{W}^c)$. Equivalently stated, for every $X \in \mathscr{P}_{os}(K)$, $\mathscr{P}_{os}(K^c)$, respectively $\mathscr{P}_{os}(W)$, there exists a closed extension X_e of $X \upharpoonright D_{os}$ such that $X_e^* \supset X^\dagger \upharpoonright D_{os}$ and such that X_e is affiliated with $\mathscr{B}(K)$, $\mathscr{A}(K^c)$, respectively $\mathscr{A}(W)$.

We regard the state of affairs described as Scenario G, which is clearly a subscenario of Scenario A, as the *good* situation. The demonstration that this situation always obtains in a quantum field theory would represent a very satisfactory resolution of what might be called the "selfadjointness problem of field theory." Positive solutions of the selfadjointness problem under a variety of special conditions on the field have been known for some time [5, 2, 3], but whether Scenario G obtains in general remains an open question, and so does the question of whether Scenario A might actually imply Scenario G. A field theory for which Scenario A (but not Scenario G) obtains could still be regarded as a physically acceptable *local* theory. Irrespective of what the actual situation may be, the totality of the statements in A is a useful theoretical stepping stone for the statement of intermediate results. The above definition is also a statement of goals for this paper: we shall show that with certain assumptions on the field it can be concluded that Scenario G, respectively A, obtains.

For both scenarios the statements in part b) of Lemma 2.3 should be kept in mind as a further elaboration of the description. The field-operator algebras (for a particular region) thus have specific extensions by hermitian algebras such that the closures of the extended operators are all affiliated with the corresponding von Neumann algebras of the AB-system.

In Theorems 2.7 and 2.8, which follow shortly, we show how the situations described in the above definition can arise. It will then be clear that if Scenario A does not obtain, then the field operators are not locally associated with *any* local von Neumann algebras at all. For the discussion of these theorems we need to review some further properties of AB-systems. We shall summarize miscellaneous relevant facts in the form of two (overloaded) "working lemmas" for later reference.

Lemma 2.5. a) *Suppose that an AB-system is generated by its $\mathscr{B}$-algebras, i.e., the relation* (2.5) *holds. Then the set $\mathscr{U}$ defined in* (2.4) *is irreducible if and only if $\mathscr{A}(W_R)\Omega$ is dense.*

b) *With the premise in* a) *above the algebra $\mathscr{A}(W_R)$ equals the strong closure of the set $\cup\{\mathscr{B}(K) | K \in \mathscr{K}, K \subset W_R\}$.*

c) *Suppose that for a local AB-system the linear manifold $\mathscr{A}(W_R)\Omega$ is dense in $\mathscr{H}$ and contained in the domain of $V(i\pi)$, and that furthermore the relation* (2.9) *holds. Then the AB-system satisfies the special condition of duality.*

d) *Let $\mathscr{A}(W_R)$ be the wedge-algebra corresponding to W_R in an AB-system which satisfies the special condition of duality.*

Suppose that $\mathscr{A}_1 \subset \mathscr{A}(W_R)$ is a von Neumann algebra such that $\mathscr{A}_1\Omega$ is dense in $\mathscr{H}$, and $V(t)\mathscr{A}_1 V(t)^{-1} = \mathscr{A}_1$ for all velocity transformations $V(t)$ in the 3-direction. Then $\mathscr{A}_1 = \mathscr{A}(W_R)$.

Suppose that $\mathscr{A}_2 \supset \mathscr{A}(W_R)$ is a von Neumann algebra such that $\mathscr{A}_2\Omega$ is contained in the domain of $V(i\pi)$, and such that the relation (2.9) holds for all $A \in \mathscr{A}_2$. Then $\mathscr{A}_2 = \mathscr{A}(W_R)$.

e) *Suppose that an AB-system satisfies the condition of duality and is generated by its local operators. Suppose that $\mathcal{O} \to \mathfrak{A}(\mathcal{O})$ is a local net defined for all open double cones $\mathcal{O}$, with the property that $\mathfrak{A}(\mathcal{O}) \supset \mathscr{B}(K)$ whenever $\mathcal{O} \supset K$. Here $\mathscr{B}(K)$ is the algebra of the AB-system corresponding to $K \in \mathscr{K}$. Then $\mathfrak{A}(\mathcal{O}) \subset \mathscr{B}(\bar{\mathcal{O}})$ for all open double cones $\mathcal{O}$.*

Proof. The assertion in a) follows by a standard argument in quantum field theory, based on the spectrum condition: see also [30]. The assertion in b) follows from (2.5) and the obvious geometrical fact that any *compact* subset of the (open) set W_R is contained in *some* $K \subset W_R$, $K \in \mathscr{K}$. For the assertions in c) and d), see Theorem 2 in [2]. The assertion in e) is a triviality. □

The first lemma reveals the rather tight structure of an *AB*-system. The lemma which follows has to do with the implications of the relations (2.1).

Lemma 2.6. a) *Let the linear manifold $D_L \subset \mathscr{H}$ be a core for $V(-i\pi)$. If X is a closable (linear) operator such that Ω is in the domains of X and X^*, and if*

$$\langle X^*\Omega|\phi\rangle = \langle JV(-i\pi)\phi|X\Omega\rangle\,, \quad \text{all} \quad \phi \in D_L\,, \tag{2.13}$$

then $X\Omega$ and $X^*\Omega$ are in the domain of $V(i\pi)$ and

$$JV(i\pi)X\Omega = X^*\Omega\,, \qquad JV(i\pi)X^*\Omega = X\Omega\,. \tag{2.14}$$

b) *Let $\mathscr{L}$ be a linear manifold of operators in $\mathscr{P}_0(W_L)$ such that $D_L = \mathscr{L}\Omega$ is dense in $\mathscr{H}$, and such that $V(t)\mathscr{L}V(t)^{-1} = \mathscr{L}$ for all t. Then D_L is a core for $V(-i\pi)$.*

If X is a closable operator such that Ω is in the domains of X and X^*, and if

$$\langle X^*\Omega|Y\Omega\rangle = \langle Y^\dagger\Omega|X\Omega\rangle\,, \quad \text{all} \quad Y \in \mathscr{L}\,, \tag{2.15}$$

then the relations (2.14) *hold. Furthermore the relation* (2.15) *holds for all $Y \in \mathscr{P}_0(W_L)$.*

c) *Let $\mathscr{L}$ satisfy the premises in* b). *Let $\mathscr{A}_R$ be a von Neumann algebra such that $\mathscr{A}_R\Omega$ is dense and such that $V(t)\mathscr{A}_R V(t)^{-1} = \mathscr{A}_R$ for all t. Suppose furthermore that* (2.15) *holds for all $X \in \mathscr{A}_R$.*

Then

$$\mathscr{A}_R' = J\mathscr{A}_R J\,, \tag{2.16}$$

and Ω is cyclic and separating for $\mathscr{A}_R$ and $\mathscr{A}_R'$. Furthermore $\mathscr{A}_R\Omega$ is a core for $V(i\pi)$ and $\mathscr{A}_R'\Omega$ is a core for $V(-i\pi)$, and

$$JV(i\pi)A\Omega = A^*\Omega\,, \qquad JV(-i\pi)B\Omega = B^*\Omega\,,$$

for all $A \in \mathscr{A}_R$, all $B \in \mathscr{A}_R'$.

d) *With the premises in* c) *above, suppose in addition that* $\mathscr{A}_R = \mathscr{A}(W_R)$ *is the wedge-algebra corresponding to* W_R *in a local AB-system. Then this system satisfies the special condition of duality.*

Remark. Analogous statements apply to the situations in which the objects associated with the "right wedge" W_R are interchanged with the corresponding objects associated with the "left wedge" W_L, in which case the roles of $V(i\pi)$ and $V(-i\pi)$ are also interchanged. We omit the explicit statements, which should be obvious.

Proof. The assertion in a) is a triviality. We consider b), and note that D_L is contained in the domain of $V(-i\pi)$, in view of (2.1). Since D_L is assumed dense, and since $V(t)D_L = D_L$ for all t, it follows that D_L is a core for $V(-i\pi)$. Taking into account (2.1) the remaining assertions in b) follow readily.

The assertions in c) are paraphrases of assertions in Theorem 2 in [2]. (From a mathematical point of view the conclusions can be regarded as standard results within the Tomita-Takesaki theory.) The assertion in d) follows from the result in c), and from part c) in Lemma 2.5. □

Part d) of this lemma describes how an extremely weak condition of "relative locality" between the field and a local AB-system implies the special condition of duality for the latter. This theme recurs in the two theorems which follow. The first of these is *almost* a special case of the second, and the two might have been combined into a single theorem. The reason for our approach is that the first theorem is particularly clean, and deserves an explicit statement, whereas the second may at first appear complicated and contrived. We hope that it will be palatable as a generalization of the first in the same sense that Scenario A is a generalization of Scenario G.

Theorem 2.7. *Let* $\varphi(x)$ *be a local, irreducible hermitian scalar field.*

a) *Suppose that there exists an AB-system* $\{\mathscr{A}(W), \mathscr{B}(K), \mathscr{A}(K^c)\}$ *such that the set* $\mathscr{U} = \cup\{\mathscr{B}(K) | K \in \mathscr{K}\}$ *is irreducible, and such that for each* $K \in \mathscr{K}$ *every* $A \in \mathscr{A}(K^c)$ *commutes weakly on* D_0 *with every averaged field* $\varphi[f]$ *for which* $\operatorname{supp}(f) \subset K$.

Then Scenario G in Definition 2.4 obtains for this AB-system, which means in particular that it is local, TCP-covariant, and satisfies the special condition of duality.

b) *Suppose, instead, that there exists a local net* $\mathcal{O} \to \mathfrak{A}(\mathcal{O})$ *of von Neumann algebras defined for all open double cones* $\mathcal{O}$, *and with the property that every operator* $X = \varphi[f]$, *with* $\operatorname{supp}(f) \subset \mathcal{O}$, *has a closed extension* $X_e \subset X^{\dagger *}$ *affiliated with* $\mathfrak{A}(\mathcal{O})$. *Then the AB-system defined through*

$$\mathscr{A}(W) = \{U(\lambda)\mathfrak{A}(\mathcal{O}_0)U(\lambda)^{-1} | \lambda \in P,\ \mathcal{O}_{0,\lambda} \subset W\}'' \tag{2.17}$$

for a particular non-empty $\mathcal{O}_0$ *is independent of the choice of* $\mathcal{O}_0$. *It satisfies the premises in part* a) *above, and hence all the conclusions apply. The relation* $\mathscr{B}(\bar{\mathcal{O}}) \supset \mathfrak{A}(\mathcal{O})$ *holds for all* $\mathcal{O}$.

Furthermore, if $\mathcal{O} \to \mathfrak{A}_e(\mathcal{O})$ *is any local net such that* $\mathfrak{A}_e(\mathcal{O}) \supset \mathfrak{A}(\mathcal{O})$ *for all open double cones* $\mathcal{O}$, *and such that* $\mathfrak{A}_e$ *satisfies the condition of duality in the sense that* $\mathfrak{A}_e(\mathcal{O})' = \{\mathfrak{A}_e(\mathcal{O}_0) | \mathcal{O}_0 \subset \mathcal{O}^c\}''$, *then* $\mathfrak{A}_e(\mathcal{O}) = \mathscr{B}(\bar{\mathcal{O}})$ *for all* $\mathcal{O}$.

Remarks. a) The following differences in the premises for parts a) and b) should be noted. In part b) the locality of the net is assumed, whereas it is a consequence in part a). In part a) the irreducibility of the set $\cup\{\mathscr{B}(K)|K\in\mathscr{K}\}$ is assumed, and the locality of the AB-system then follows from this and from the other assumptions.

b) Part b) reveals the canonical nature of the AB-system as a local net. The original net need not satisfy the condition of duality, but it has an embedding into a "larger" net which is unique if it satisfies the condition of duality, in which case it satisfies the *special* condition of duality. This, of course, applies only to local nets which are related to *finite*-component quantum fields in the manner stated. Examples exist of local algebras satisfying duality but not the special condition of duality (see, e.g. [11]) and of local algebras associated with certain infinite-component (free) fields for which the special condition of duality does not hold.

Theorem 2.8. *Let $\varphi(X)$ be a local, irreducible hermitian scalar field. For each $K\in\mathscr{K}$, let the subset $\mathscr{F}(K)\subset\mathscr{P}_0(K)$ be a hermitian set of local operators, i.e., $X^\dagger\in\mathscr{F}(K)$ if $X\in\mathscr{F}(K)$. For any $R\subset\mathscr{M}$, let $\mathscr{P}_{0s}(R)$ be defined as the smallest unital *-algebra which contains $U(\lambda)\mathscr{F}(K)U(\lambda)^{-1}$ whenever $K_\lambda\subset R$, $K\in\mathscr{K}$. Hence $\mathscr{P}_{0s}(R)\subset\mathscr{P}_0(R)$. It is assumed that $D_{0s}=\mathscr{P}_{0s}(\mathscr{M})\Omega$ is dense in $\mathscr{H}$.*

a) *Suppose that there exists an AB-system $\{\mathscr{A}(W),\mathscr{B}(K),\mathscr{A}(K^c)\}$ such that the set $\mathscr{U}=\cup\{\mathscr{B}(K)|K\in\mathscr{K}\}$ is irreducible, and such that for each $K\in\mathscr{K}$ every $X\in\mathscr{F}(K)$ commutes weakly on D_{0s} with every $A\in\mathscr{A}(K^c)$.*

Then Scenario A in Definition 2.4 obtains for this AB-system and the algebra $\mathscr{P}_{0s}(\mathscr{M})$, which means in particular that the AB-system is local, TCP-covariant, and satisfies the special condition of duality.

b) *Suppose, instead, that there exists a local net $\mathscr{O}\to\mathfrak{A}(\mathscr{O})$ of von Neumann algebras defined for all open double cones $\mathscr{O}$, and with the property that for each $K\in\mathscr{K}$ there corresponds to every $X\in\mathscr{F}(K)$ a closed extension $X_e\subset(X^\dagger\restriction D_{0s})^*$ of $X\restriction D_{0s}$ which is affiliated with $\mathfrak{A}(\mathscr{O})$ for any $\mathscr{O}$ such that $\mathscr{O}\supset K$. Then the AB-system defined through*

$$\mathscr{A}(W)=\{\mathfrak{A}(\mathscr{O})|\mathscr{O}\subset W\}'' \tag{2.18}$$

satisfies the premises in part a) *above, and hence all the conclusions apply. The relation $\mathscr{B}(\bar{\mathscr{O}})\supset\mathfrak{A}(\mathscr{O})$ holds for all $\mathscr{O}$.*

Furthermore, if $\mathscr{O}\to\mathfrak{A}_e(\mathscr{O})\supset\mathfrak{A}(\mathscr{O})$ is any local net defined on all open double cones which satisfies the condition of duality in the sense explained in Theorem 2.7, then $\mathfrak{A}_e(\mathscr{O})=\mathscr{B}(\bar{\mathscr{O}})$ for all $\mathscr{O}$.

Remarks. a) The remarks following the statement of Theorem 2.7 also apply to the present theorem.

b) The set $\mathscr{F}=\cup\{\mathscr{F}(K)|K\in\mathscr{K}\}$ is thus a generating set for the system of algebras $\{\mathscr{P}_{0s}(R)|R\subset\mathscr{M}\}$. It is not assumed that the mapping $K\to\mathscr{F}(K)$ satisfies the conditions of isotony and Poincaré-covariance. In particular it could happen that $\mathscr{F}$ consists of just one single pair $X,X^\dagger$ of local operators.

c) The theorem does not assert that every algebra $\mathscr{B}(K)$ is nontrivial: it could thus happen that for "small" $K\in\mathscr{K}$ the algebra $\mathscr{B}(K)$ contains only multiples of the identity. See, however, part c) of Theorem 3.2 for specific results concerning this issue.

d) It will be advantageous to prove Theorem 2.8 first, after which we prove Theorem 2.7.

Proof of Theorem 2.8. 1) We first note that the algebra $\mathscr{P}_{0s}(\mathscr{M})$ is obviously Poincaré-invariant, and in particular it is translation-invariant. Every element of $\mathscr{P}_{0s}(\mathscr{M})$ is *local*, and by a standard argument in quantum field theory involving the Reeh-Schlieder Principle we conclude that $\mathscr{P}_{0s}(W)\Omega$ is dense $\mathscr{H}$ for any $W\in\mathscr{W}$ since $D_{0s}=\mathscr{P}_{0s}(\mathscr{M})\Omega$ was assumed dense.

2) We next note that it follows immediately from the definition of the algebras $\mathscr{P}_{0s}(R)$ in the statement of the theorem that these satisfy precisely the same general conditions as the algebras so denoted and described within Scenario A in Definition 2.4. In particular the mapping $R\to\mathscr{P}_{0s}(R)$ satisfies the conditions of isotony and Poincaré-covariance, and $\mathscr{P}_{0s}(R)$ is the smallest unital *-subalgebra of $\mathscr{P}_0(R)$ which contains $\mathscr{P}_{0s}(K)$ for all $K\subset R$, $K\in\mathscr{K}$.

3) We now assume the premises in part a) of the theorem. We consider a particular $K\in\mathscr{K}$. Since the AB-system satisfies the conditions of isotony and Poincaré-covariance, and since D_{0s} is Poincaré-invariant, it follows from the stated assumptions that every $A\in\mathscr{A}(K^c)$ commutes weakly on D_{0s} with every $X\in U(\lambda)\mathscr{F}(K_i)U(\lambda)^{-1}$ whenever $K_i\in\mathscr{K}$, $K_{i,\lambda}\subset K$. We now depend on the following simple principle. If $\mathscr{P}_s$ is a hermitian algebra of closable operators on a common dense invariant domain D_{0s}, and if a bounded operator A commutes weakly on D_{0s} with a (hermitian) generating set for this algebra, then A commutes weakly on D_{0s} with every $X\in\mathscr{P}_s$. We thus conclude that every $A\in\mathscr{A}(K^c)$ commutes weakly on D_{0s} with every $X\in\mathscr{P}_{0s}(K)$. By similar reasoning we conclude that every $A\in\mathscr{B}(K)$ commutes weakly on D_{0s} with every $X\in\mathscr{P}_{0s}(K^c)$, and that every $A\in\mathscr{A}(\bar{W}^c)$ commutes weakly with every $X\in\mathscr{P}_{0s}(W)$, for any $W\in\mathscr{W}$. We omit the details, which involve very simple geometrical considerations. We have thus established all the weak commutation relations between the AB-system and the algebra $\mathscr{P}_{0s}(\mathscr{M})$, as stated in Definition 2.4.

4) It is assumed in part a) that the set $\mathscr{U}$ is irreducible, and it follows, by part a) of Lemma 2.5, that $\mathscr{A}(W)\Omega$ is dense for any $W\in\mathscr{W}$. For all t we have $V(t)\mathscr{A}(W_R)V(t)^{-1}=\mathscr{A}(W_R)$ and $V(t)\mathscr{P}_{0s}(W_L)V(t)^{-1}=\mathscr{P}_{0s}(W_L)$. By 3) above, $\mathscr{A}(W_R)$ and $\mathscr{P}_{0s}(W_L)$ commute weakly on D_{0s}, and by 1) above, $\mathscr{P}_{0s}(W_L)\Omega$ is dense. Hence $\mathscr{A}_R=\mathscr{A}(W_R)$ and $\mathscr{L}=\mathscr{P}_{0s}(W_L)\subset\mathscr{P}_0(W_L)$ satisfy the premises in part c) of Lemma 2.6, and it follows that

$$JV(i\pi)A\Omega=A^*\Omega\,,\quad \text{all}\quad A\in\mathscr{A}(W_R)\,. \tag{2.19}$$

Similarly we have,

$$JV(-i\pi)B\Omega=B^*\Omega\,,\quad \text{all}\quad B\in\mathscr{A}(W_L)\,. \tag{2.20}$$

5) Let $K\in\mathscr{K}$, $K\subset W_R$, and let $W_1\in\mathscr{W}$, $W_1\subset W_R\cap K^c$: the three sets W_L, K, and W_1 are thus pairwise spacelike relative to each other. Let $Y\in\mathscr{P}_{0s}(W_L)$, $A\in\mathscr{B}(K)$, and $X\in\mathscr{P}_{0s}(W_1)$. Then X and Y commute on D_1, and hence on D_{0s}, and both operators commute weakly on D_{0s} with A. It follows that $\langle Y^\dagger\Omega|AX\Omega\rangle=\langle A^*\Omega|YX\Omega\rangle=\langle A^*\Omega|XY\Omega\rangle=\langle A^*X^\dagger\Omega|Y\Omega\rangle$. By part b) of Lemma 2.6, $\mathscr{P}_{0s}(W_L)$ is a core for $V(-i\pi)$, and since furthermore $Y^\dagger\Omega=JV(-i\pi)Y\Omega$ for all $Y\in\mathscr{P}_{0s}(W_L)$, it follows from the above that $AX\Omega$ is in the domain of $V(i\pi)$, and that

$$JV(i\pi)AX\Omega=A^*X^\dagger\Omega \tag{2.21}$$

for all $A\in\mathscr{B}(K)$, all $X\in\mathscr{P}_{0s}(W_1)$.

Let A and X be as above, and let $B \in \mathscr{A}(W_L)$. In view of (2.20) and (2.21) we have

$$\langle B\Omega | AX\Omega \rangle = \langle A^* X^\dagger \Omega | B^* \Omega \rangle . \tag{2.22}$$

Since $AB^* \in \mathscr{A}(\bar{W}_1^c)$ it follows from the result in step 3 that $\langle X^\dagger \Omega | AB^* \Omega \rangle = \langle BA^* \Omega | X\Omega \rangle$, and from this, and from (2.22) we then conclude that

$$\langle \Omega | [B^*, A] X\Omega \rangle = 0 . \tag{2.23}$$

Let $X_1, X_2 \in \mathscr{P}_{0s}(W_1)$. Setting $X = X_1^\dagger X_2$ in (2.23), and taking into account the fact that $[B^*, A] \in \mathscr{A}(\bar{W}_1^c)$, we obtain $\langle X_1 \Omega | [B^*, A] X_2 \Omega \rangle = 0$ for all $X_1, X_2 \in \mathscr{P}_{0s}(W_1)$, all $A \in \mathscr{B}(K)$ and all $B \in \mathscr{A}(W_L)$. Since $\mathscr{P}_{0s}(W_1)\Omega$ is dense it follows that $[B^*, A] = 0$, i.e., we have $\mathscr{B}(K) \subset \mathscr{A}(W_L)'$ for any $K \subset W_R$, $K \in \mathscr{K}$.

6) We define the *AB-system* $\{\mathscr{A}_0(W), \mathscr{B}_0(K), \mathscr{A}_0(K^c)\}$ through

$$\mathscr{A}_0(W) = \{\mathscr{B}(K) | K \in \mathscr{K},\ K \subset W\}'' , \tag{2.24}$$

and we then have $\mathscr{B}_0(K) = \mathscr{B}(K)$. Since $\mathscr{U}$ is irreducible it follows by part a) of Lemma 2.5 that $\mathscr{A}_0(W_R)\Omega$ is dense. By the result in step 5 above we have $\mathscr{B}_0(K) = \mathscr{B}(K) \subset \mathscr{A}(W_L)' \subset \mathscr{A}_0(W_L)'$ for any $K \subset W_R$, $K \in \mathscr{K}$ [since obviously $\mathscr{A}(W_L) \supset \mathscr{A}_0(W_L)$]. In view of (2.24) this implies that $\mathscr{A}_0(W_R) \subset \mathscr{A}_0(W_L)'$, and hence the AB-system defined through (2.24) is *local.* The relation (2.19) holds for any $A \in \mathscr{A}_0(W_R) \subset \mathscr{A}(W_R)$, and by part c) of Lemma 2.5 it follows that the AB-system defined through (2.24) satisfies the special condition of duality. By part d) of Lemma 2.5 we readily conclude that $\mathscr{A}(W_R) = \mathscr{A}_0(W_R)$, i.e., the two AB-systems are identical. We have thus proved that Scenario A obtains with the premises in part a) of the theorem.

7) We assume the premises in part b). The AB-system defined through (2.18) is then, by Lemma 2.2, *local*, and we have $\mathfrak{A}(\mathcal{O}) \subset \mathscr{B}(\bar{\mathcal{O}}) \subset \mathscr{A}(\bar{\mathcal{O}}^c)'$ for all open double cones $\mathcal{O}$. If $K \in \mathscr{K}$, $K \subset \mathcal{O}$, and $X \in \mathscr{F}(K)$, it follows from the above, and from the premises in part b), that X commutes weakly on D_{0s} with $\mathscr{A}(\bar{\mathcal{O}}^c)$. From the relation at right in (2.2c) in the Definition 2.1 of an AB-system it is obvious that the algebras $\mathscr{A}(K^c)$ are "continuous from the inside" in the sense that $\mathscr{A}(K^c)$ is equal to the weak (or strong) closure of the set $\cup\{\mathscr{A}(K_0^c) | K_0 \in \mathscr{K},\ \overline{K_0^c} \subset K^c\}$, and we thus conclude that every $X \in \mathscr{F}(K)$ commutes weakly on D_{0s} with every $A \in \mathscr{A}(K^c)$, for any $K \in \mathscr{K}$.

8) To show that *all* the premises in part a) are in fact implied by the premises in part b) it remains to be shown that the set $\mathscr{U} = \cup\{\mathscr{B}(K) | K \in \mathscr{K}\}$ is irreducible for the AB-system defined through (2.18). Let $Q \in \mathscr{U}'$, and hence $Q \in \mathscr{B}(K)'$ for all $K \in \mathscr{K}$. It follows that Q commutes weakly on D_{0s} with $\mathscr{P}_{0s}(K)$ for all K, and hence Q commutes weakly with $\mathscr{P}_{0s}(\mathscr{M})$ on D_{0s}. Since $D_{0s} = \mathscr{P}_{0s}(\mathscr{M})\Omega$ is dense, it follows by a standard argument in quantum field theory (based on the spectrum condition) that Q is a multiple of the identity. Hence $\mathscr{U}$ is irreducible. Thus all the premises in part a) obtain, and the conclusions follow.

9) Let $\mathfrak{A}_e$ be a "larger" (local) net which satisfies the condition of duality, as stated in the theorem. Applying all the earlier considerations to this net we obtain an AB-system which satisfies the special condition of duality, and in particular we obtain a wege-algebra $\mathfrak{A}_e(W_R) \supset \mathscr{A}(W_R)$ through the construction in (2.18). By part d) of Lemma 2.5 we have $\mathfrak{A}_e(W_R) = \mathscr{A}(W_R)$, i.e., the two AB-systems are identical. We have $\mathscr{B}(\bar{\mathcal{O}}) \supset \mathfrak{A}_e(\mathcal{O})$ for all open double cones, and since $\mathfrak{A}_e$ satisfies the condition of duality, equality must obtain. □

Proof of Theorem 2.7. 1) For each $K \in \mathscr{K}$ we define the hermitian set $\mathscr{F}(K) = \{\varphi[f] | f \in \mathscr{S}(R^4), \operatorname{supp}(f) \subset K\}$, and we define the algebras $\mathscr{P}_{0s}(R)$ in terms of the $\mathscr{F}(K)$ as in Theorem 2.8. Since $D_1 \supset D_0 \supset D_{0s} \equiv \mathscr{P}_{0s}(\mathscr{M})\Omega$, it follows that the premises in part a) of Theorem 2.7 imply the premises in part a) of Theorem 2.8. All the conclusions in part a) of Theorem 2.8 thus apply to the algebra $\mathscr{P}_{0s}(\mathscr{M})$ as defined above.

Likewise the premises in part b) of Theorem 2.7 imply the premises in part b) of Theorem 2.8, and the corresponding conclusions apply to the algebra $\mathscr{P}_{0s}(\mathscr{M})$ and the AB-system defined through (2.18).

2) The algebra $\mathscr{P}_{0s}(\mathscr{M})$ is smaller than the algebra $\mathscr{P}_0(\mathscr{M})$ since the former is generated by all operators $\varphi[f]$ for which f is of *compact* support, whereas the latter is generated by all $\varphi[f]$ with f unrestricted. If R is a *bounded* subset of $\mathscr{M}$ we have, of course, $\mathscr{P}_{0s}(R) = \mathscr{P}_0(R)$. We now appeal to a well-known consequence of the field being an operator-valued tempered distribution. If $X \in \mathscr{P}_0(R)$, for an arbitrary (open) $R \subset \mathscr{M}$, then there exists a sequence $\{X_n | n = 1, 2, \ldots,\}$ of operators in $\mathscr{P}_{0s}(R)$ such that the sequence $\{X_n\phi | n = 1, 2, \ldots,\}$ converges strongly to $X\phi$, for any $\phi \in D_1$. Furthermore, each $\phi \in D_1$ is the strong limit of a sequence of vectors in D_{0s}. From this we conclude that all the weak commutation relations between the elements of $\mathscr{P}_0(\mathscr{M})$ and the AB-system hold precisely as described within Scenario G in Definition 2.4, but with the provision that under the premises in part b) the AB-system is defined through (2.18), rather than through (2.17).

3) Let $\mathcal{O}_0$ be a non-empty open double cone, and let the AB-system $\{\mathscr{A}_0(W), \mathscr{B}_0(K), \mathscr{A}_0(K^c)\}$ be defined such that $\mathscr{A}_0(W)$ equals the right member in (2.17). Let $\mathscr{U}_0 = \cup\{\mathscr{B}_0(K) | K \in \mathscr{K}\}$. We shall show that $\mathscr{U}_0$ is irreducible. Let $Q \in \mathscr{U}_0'$, in which case $Q \in \mathscr{B}_0(K)'$ for all $K \in \mathscr{K}$. It follows from the premises that Q commutes weakly on D_1 with every field operator $\varphi[f]$ such that $\operatorname{supp}(f) \subset \mathcal{O}_{0,\lambda}$ for some $\lambda \in P$. This implies that Q commutes weakly (on D_1) with $\varphi[f]$ for *any* test function f, and it is well-known that this implies that Q is a multiple of the identity. Hence $\mathscr{U}_0$ is irreducible, and by part a) of Lemma 2.5 it follows that $\mathscr{A}_0(W_R)\Omega$ is dense. Since $\mathscr{A}_0(W_R) \subset \mathscr{A}(W_R)$, it follows from part d) of Lemma 2.5 that $\mathscr{A}_0(W_R) = \mathscr{A}(W_R)$, and hence the two AB-systems defined through (2.17) and (2.18) are identical.

4) Let $K \in \mathscr{K}$. From the above it follows at once that $\mathscr{A}(W) = \{\mathscr{B}(K_\lambda) | \lambda \in P, K_\lambda \subset W\}''$. It is well-known that Ω is cyclic for $\mathscr{P}_0(K)$, which implies that Ω is separating for $\mathscr{A}(K^c)$. This, in turn, implies that Ω is cyclic for $\mathscr{B}(K) = \mathscr{A}(K^c)'$. The vector Ω is trivially separating for $\mathscr{B}(K)$. We have now shown that Scenario G obtains under the premises of the theorem. □

Concerning Theorem 2.7 we note that it can well happen that the local net mentioned in part b) is "much smaller" than the AB-system. For any $\mathcal{O}$ the closed extensions of the field operators $\varphi[f] \in \mathscr{P}_0(\mathcal{O})$ which are affiliated with $\mathfrak{A}(\mathcal{O})$, and with $\mathscr{B}(\bar{\mathcal{O}})$, generate *locally* an algebra $\mathfrak{A}_m(\mathcal{O})$ which can be regarded as belonging to a *minimal* local net. It is known [22] that such a net $\mathfrak{A}_m$ need not satisfy the condition of duality, in which case $\mathfrak{A}_m(\mathcal{O})$ cannot equal $\mathscr{B}(\bar{\mathcal{O}})$. The local net is accordingly not unique, whereas the AB-system is uniquely determined by the particular closed extensions of the field operators.

Theorem 2.8 permits us to say the following. If Scenario A does *not* obtain, then there does not exist *any* irreducible subset of *local* operators in $\mathscr{P}_0(\mathscr{M})$ such that its elements have closed extensions affiliated in the manner stated with the algebras of *some* local net. At first the situation described in Scenario A may appear rather "special," but we now see that it corresponds to minimum requirements for the field to be locally associated with a local net.

Concerning Scenario G we emphasize that it is *not* said that the closed extensions of the field operators X which are affiliated with the local algebras of the AB-system are actually the closures $\bar{X}$ of the operators X as defined on D_1. It seems to us to be unreasonable to believe that this could be the case for *all* the local operators in $\mathscr{P}_0(\mathscr{M})$, but it might be the case of some subset of these, say the operators $\varphi[f]$ which are linear in the fields. We shall explore this possibility in Sect. 4.

3. A Principle Akin to the Reeh-Schlieder Principle. Further Discussion of the Properties of Local AB-Systems

In this section we shall discuss two general properties of AB-systems associated with quantum fields in the manner described in Definition 2.4. We first digress, and state and prove a lemma which is of particular interest for this paper, but which may also have other applications in quantum field theory.

Lemma 3.1. *Let* $\{\mathscr{A}(W), \mathscr{B}(K), \mathscr{A}(K^c)\}$ *be an AB-system with the property that* $\mathscr{A}(W_R)\Omega$ *is in the domain of* $V(i\pi)$. *In particular this applies to the case when the AB-system satisfies the special condition of duality. Let* $\{A_n | n=1,2,\ldots,N\}$ *be an N-tuplet of (bounded) operators contained in* $\mathscr{B}(K)$ *for some* $K \in \mathscr{K}$. *Let* $\{O_n | n=1,2,\ldots,N\}$ *be an N-tuplet of non-empty open subsets of the Poincaré group* P. *We write* $A_n(\lambda)=U(\lambda)A_nU(\lambda)^{-1}$ *for any* $\lambda \in P$. *Then:*

$$\begin{aligned}&\overline{\operatorname{span}}\{A_1(\lambda_1)A_2(\lambda_2)\ldots A_N(\lambda_N)\Omega | \lambda_n \in O_n, \text{ for } n=1,\ldots,N\}\\ &\quad=\overline{\operatorname{span}}\{A_1(\lambda_1)A_2(\lambda_2)\ldots A_N(\lambda_N)\Omega | \lambda_n \in P, \text{ for } n=1,\ldots,N\}, \qquad (3.1)\end{aligned}$$

where the overbars indicate strong closures.

Remark. The above assertion resembles the celebrated Reeh-Schlieder Principle [21, 28] (as applied to bounded operators). This principle asserts an identity such as (3.1) with P replaced by the translation subgroup and with the O_n being non-empty open subsets of this subgroup. Now it should be noted that the Reeh-Schlieder Principle applies to *any* N-tuplet of bounded operators, and it is a simple consequence of the spectrum condition for the translation group. In contrast with this, our conclusion is manifestly false for an *arbitrary* set of operators A_n. It can also be false for an N-tuplet of *local* operators if the additional domain conditions do not hold.

Proof. 1) The crucial step is the proof of the case $N=1$, which we now consider. We have to show that if for some vector ϕ the function $f(\lambda)=\langle\phi|U(\lambda)A_1\Omega\rangle$ satisfies the condition $f(\lambda)=0$ for $\lambda \in O_1$, then $f(\lambda)=0$ for all $\lambda \in P$. For a fixed ϕ, let P_0 be the largest open subset of P throughout which f vanishes. We then have $P_0 \supset O_1$, and, by the Reeh-Schlieder Principle, $P_0=(I,x)P_0$ for any $x \in \mathscr{M}$, i.e., P_0 is invariant under left multiplication by any element (I,x) of the translation subgroup of P.

2) Let $\lambda_1 \in P_0$ and let λ_2 be an arbitrary element of P. There then exists a translation $\lambda_x = (I, x)$ such that the image K_λ of K under the Poincaré transformation $\lambda = \lambda_2 \lambda_x \lambda_1$ is contained in W_R. Let $v_3(t)$ denote a velocity transformation in the 3-direction, as before. We can then write

$$\lambda_2^{-1} v_3(t) \lambda_2 \lambda_x \lambda_1 = \lambda_3 \lambda_2^{-1} v_3(t) \lambda_2 \lambda_1 , \tag{3.2}$$

where λ_3 is a *translation*. We temporarily regard λ_1, λ_2, and λ_x as fixed. The group element in (3.2) is in P_0 if and only if $\lambda_2^{-1} v_3(t) \lambda_2 \lambda_1 \in P_0$, which is certainly the case for sufficiently small t, say for $|t| < t_0$ for some $t_0 > 0$. We consider

$$h(t) = f(\lambda_2^{-1} v_3(t) \lambda_2 \lambda_x \lambda_1) = \langle U(\lambda_2)\phi | V(t) U(\lambda) A_1 \Omega \rangle \tag{3.3}$$

as a function of t (with λ and λ_2 fixed). We then have $h(t) = 0$ whenever $|t| < t_0$. Since $K_\lambda \subset W_R$, and hence $U(\lambda) A_1 U(\lambda)^{-1} \in \mathscr{A}(W_R)$, we conclude, in view of the special property of the AB-system, that $U(\lambda) A_1 \Omega$ is in the domain of $V(i\pi)$. Hence the function $h(t)$ in (3.3) has an analytic continuation to the strip $\pi > \mathrm{Im}(t) > 0$ in the complex t-plane, and we can thus conclude that $h(t) = 0$ for *all* real t. This means that $\lambda_2^{-1} v_3(t) \lambda_2 \lambda_1 \in P_0$ for *all* real t. Since λ_2 was arbitrary, and since λ_1 was an arbitrary element of P_0, we conclude that $\lambda_2^{-1} v_3(t) \lambda_2 P_0 = P_0$ for all real t, and all $\lambda_2 \in P$. From this it readily follows that $P_0 = P$, and we have thus proved the theorem for the case $N = 1$.

3) The validity of the identity (3.1) for any $N > 1$ is now easily proved by induction on N. We write $\lambda_k = \lambda_1 \lambda_k'$ for $k = 2, 3, \ldots, N$, and assume that (3.1) holds for any $(N-1)$-tuplet. We then apply the result in step 2 to the variable λ_1, keeping the λ_k' fixed, and it readily follows that (3.1) holds for any N-tuplet. □

We can now draw some interesting conclusions.

Theorem 3.2. *Let $\{\mathscr{A}(W), \mathscr{B}(K), \mathscr{A}(K^c)\}$ be an AB-system which satisfies the special condition of duality.*

a) *Suppose that for some $K_i \in \mathscr{K}$ the algebra $\mathscr{B}(K_i)$ satisfies the condition that the set $\mathscr{G}_i = \cup \{U(\lambda) \mathscr{B}(K_i) U(\lambda)^{-1} | \lambda \in P\}$ is irreducible. Then Ω is cyclic (and separating) for every algebra $\mathscr{B}(K)$ such that $K_{i,\lambda}$ is contained in the interior of K for some $\lambda \in P$.*

b) *The vector Ω is cyclic for $\mathscr{B}(K)$ for every $K \in \mathscr{K}$ if and only if $\mathscr{A}(W) = \{\mathscr{B}(K_\lambda) | \lambda \in P, K_\lambda \subset W\}''$ for every $K \in \mathscr{K}$.*

c) *Assume Scenario A in Definition 2.4, and suppose that there exists, for a particular $K_i \in \mathscr{K}$, a hermitian subset $\mathscr{N}_i \subset \mathscr{P}_{0s}(K_i)$ such that Ω is a cyclic vector for the hermitian algebra $\mathscr{P}_i$, defined as the smallest unital *-algebra which contains $U(\lambda) \mathscr{N}_i U(\lambda)^{-1}$ for all $\lambda \in P$. Then K_i satisfies the premises in part* a) *above.*

Proof. 1) Suppose that K_i satisfies the condition in part a) above, and suppose that $K \in \mathscr{K}$ is such that $K_{i,\lambda}$ is contained in the interior of the (closed) set K. There then exists a non-empty open subset O_i of P such that $K_{i,\lambda} \subset K$ for all $\lambda \in O_1$. This means that the product of any number of operators of the form $U(\lambda_n) A_n U(\lambda_n)^{-1}$, with $A_n \in \mathscr{B}(K_i)$ and $\lambda_n \in O_i$, is an element of $\mathscr{B}(K)$. It follows at once from Lemma 3.1 that $\overline{\mathscr{B}(K)\Omega} = \mathscr{H}$.

2) The assertion in part b) follows trivially from the result in part a), in view of part a) of Lemma 2.5.

3) We consider part c), with $\mathcal{G}_i$ defined as in part a). Let $Q \in \mathcal{G}_i'$, and hence $Q \in \mathcal{B}(K_{i,\lambda})' = \mathcal{A}(K_{i,\lambda}^c)$ for all $\lambda \in P$. It follows that, for each $\lambda \in P$, the set $U(\lambda)\mathcal{N}_i U(\lambda)^{-1}$ commutes weakly with Q on D_{0s}, and hence $\mathcal{P}_i$ commutes weakly with Q on D_{0s}. Since $\mathcal{P}_i\Omega \subset D_{0s}$ was assumed dense, we conclude, as in step 8 in the proof of Theorem 2.8, that Q is a multiple of the identity. Hence $\mathcal{G}_i$ is irreducible, and K_i thus satisfies the premises in part a), as asserted. □

We note that it is a feature of Scenario G in Definition 2.4 that $\mathcal{B}(K)\Omega$ is dense for every $K \in \mathcal{K}$. Part c) of the above theorem can thus be regarded as an amendment to the description of Scenario A.

We next consider the question of selfadjoint extensions of symmetric local operators in the algebra $\mathcal{P}_0(\mathcal{M})$ of field operators.

Theorem 3.3. *For any $K \in \mathcal{K}$, let $\mathcal{B}(K)$ be the von Neumann algebra corresponding to K in a local AB-system, and suppose that Ω is cyclic for $\mathcal{B}(K)$ for any K. Let Q be a closed symmetric operator affiliated with $\mathcal{B}(K)$, for a particular $K \in \mathcal{K}$, and suppose that Q is not maximal-symmetric. Let $K_0 \in \mathcal{K}$ be such that K is contained in the interior of K_0. Then Q has a selfadjoint extension Q_e affiliated with $\mathcal{B}(K_0)$.*

Proof. 1) Let F_+, respectively F_-, be the selfadjoint projections onto the deficiency subspaces $\mathcal{H}_+$, respectively $\mathcal{H}_-$, of the operator Q, such that $Q^*F_+ = iF_+$ and $Q^*F_- = -iF_-$. We then have $F_+ \in a(Q) \subset \mathcal{B}(K)$ and $F_- \in a(Q) \subset \mathcal{B}(K)$, where $a(Q)$ denotes the von Neumann algebra generated by Q. Since Q was assumed not to be maximal-symmetric we also have $F_+ > 0$, $F_- > 0$.

2) Although it is known [8, 24] that the wedge-algebras $\mathcal{A}(W)$ are Type III_1 factors, it is not known in general whether the double cone algebras $\mathcal{B}(K)$ are factors, nor is it known what type they are (but see [14] for partial information). However, it was shown by Borchers [6] that the local algebras have properties which justify the statement that they are *almost* Type III factors. With the stated premises it thus follows from Borchers' work that there then exist isometries V_+ and V_- in $\mathcal{B}(K_0)$ such that $V_+V_+^* = F_+$, $V_-V_-^* = F_-$, and $V_+^*V_+ = I = V_-^*V_-$. We define the partial isometry V by $V = V_-V_+^*$, and we then have $F_- = VV^*$, $F_+ = V^*V$, $\mathcal{H}_- = V\mathcal{H}_+$, and $V \in \mathcal{B}(K_0)$.

3) Let $D(Q)$, respectively $D(Q^*)$, denote the domain of Q, respectively Q^*. We define a dense linear manifold D_e by

$$D_e = \{\phi + (I+V)\phi_+ \mid \phi \in D(Q),\ \phi_+ \in \mathcal{H}_+\}.$$

We then have $D(Q^*) \supset D_e \supset D(Q)$, and the operator $Q_e = Q^* \restriction D_e$ is selfadjoint.

4) For any $\phi \in D(Q)$ and any $\psi \in \mathcal{H}$ we have

$$Q_e(\phi + (I+V)F_+\psi) = Q\phi + i(I-V)F_+\psi.$$

Let $A \in \mathcal{B}(K_0)' \subset \mathcal{B}(K)'$. We then have $AQ\phi = QA\phi$, $AV = VA$, and $AF_+ = F_+A$, and hence $AD_e \subset D_e$, and, by a simple computation,

$$AQ_e(\phi + (I+V)F_+\psi) = Q_eA(\phi + (I+V)F_+\psi).$$

This means that A commutes in the strong sense with Q_e, and since this holds for all $A \in \mathcal{B}(K_0)'$, it follows that Q_e is affiliated with $\mathcal{B}(K_0)$. □

If $\mathscr{B}(K)$ were actually a Type *III* factor we could conclude, by an obvious modification of the above reasoning, that Q has a selfadjoint extension affiliated with $\mathscr{B}(K)$ itself. (See also in this connection [20].) If $\mathscr{B}(K)$ is not a Type *III* factor, our theorem does not decide this question, but it says that a selfadjoint extension can be found which is affiliated with the algebra of a *slightly* larger region.

The circumstances noted in step 2 of the proof imply that all deficiency spaces of closed symmetric operators affiliated with $\mathscr{B}(K)$ are either empty or infinite-dimensional. Such an operator Q thus has an infinite number of *local* selfadjoint extensions, unless it is maximal-symmetric. If Q is not maximal-symmetric, it also has *non*-local selfadjoint extensions, which can be constructed by replacing the partial isometry V in the above proof by $V=V_- U V_+^*$, where U is a suitably chosen unitary operator *not* contained in any local algebra.

Let us here note that because of the TCP-covariance of any local quantum field theory there always exists a great multitude of operators in $\mathscr{P}_0(\mathscr{M})$ for which selfadjoint extensions are guaranteed to exist. Let $K \in \mathscr{K}$ be symmetric with respect to the origin, and let X be a symmetric operator in $\mathscr{P}_0(K)$. Then $X_s = X + \Theta_0 X \Theta_0$ is also in $\mathscr{P}_0(K)$, and since it is symmetric, and since it commutes with the antiunitary involution Θ_0, it has at least one selfadjoint extension.

The above theorem is of obvious interest in the situations when either Scenario G or A obtains. In the case of Scenario A it is important to note the statements about the domains of the operators which have closed extensions affiliated with the local algebras. If $X \in \mathscr{P}_{0s}(K)$, then $X \restriction D_{0s}$ has a closed extension affiliated with $\mathscr{B}(K)$, but if Scenario G does *not* obtain, it might happen that X, *as defined on* D_0, has no closed local extension at all.

4. On the Association of a Single (Unbounded) Field Operator with Local von Neumann Algebras

As we have seen in the preceding sections a local AB-system associated with a quantum field has a rather tight structure. Such a system is generated by a variety of sub-algebras, and in this section we want to consider the possibility that the AB-system is generated by a sub-algebra defined in terms of a *single* local operator in $\mathscr{P}_0(\mathscr{M})$. We begin with a definition.

Definition 4.1. Let $K_s \in \mathscr{K}$, and let $X_s = X_s^\dagger \in \mathscr{P}_0(K_s)$. For any subset $R \subset \mathscr{M}$ we denote by $\mathscr{P}_{0s}(R)$ the smallest unital *-algebra which contains $U(\lambda) X_s U(\lambda)^{-1}$ whenever $K_{s,\lambda} \subset R$.

We shall say that the hermitian operator X_s is *intrinsically local* (and locally associated with K_s) if and only if the following two conditions hold:

a) The linear manifold $D_{0s} = \mathscr{P}_{0s}(\mathscr{M})\Omega$ is dense in $\mathscr{H}$.

b) The von Neumann algebra $a_s \equiv a((X_s \restriction D_{0s})^{**})$ generated by the closure of the restriction of X_s to D_{0s} is locally associated with K_s in the sense that $U(\lambda) a_s U(\lambda)^{-1}$ commutes with a_s whenever $K_{s,\lambda}$ is spacelike relative to K_s.

For the discussion in this paper we find it convenient to restrict the notion of intrinsic locality to hermitian operators, as stated above, but it is clear that generalizations to non-hermitian operators, and to *sets* of operators, could be considered.

It follows at once from Lemma 2.2 that an intrinsically local operator generates a local AB-system: we define $\mathscr{G}(K_s)=a_s$ and $\mathscr{G}(K)=\emptyset$ if $K \neq K_s$. We shall now study the properties of such a system. The main results are presented as Theorems 4.6 and 4.8. Since the proofs involve many considerations of detail we shall proceed to the goal in a step-wise fashion.

Proposition 4.2. *Let $K_s \in \mathscr{K}$ and suppose that $X_s = X_s^\dagger \in \mathscr{P}_0(K_s)$ is intrinsically local in the sense of Definition 4.1. Let the notation be as in this definition, and let an AB-system $\{\mathscr{A}(W), \mathscr{B}(K), \mathscr{A}(K^c)\}$ be defined through*

$$\mathscr{A}(W)=\{U(\lambda)a_s U(\lambda)^{-1} | \lambda \in P,\ K_{s,\lambda} \subset W\}''. \tag{4.1}$$

Then Scenario A in Definition 2.4 obtains for this AB-system and the algebra $\mathscr{P}_{0s}(\mathscr{M})$. If $K \in \mathscr{K}$ is such that $K_{s,\lambda}$ is contained in the interior of K for some $\lambda \in P$, then Ω is cyclic for $\mathscr{B}(K)$.

Proof. The relation (4.1) corresponds to (2.10) in Lemma 2.2, and the AB-system defined above is thus local. We define $\mathscr{F}(K_s)=\{X_s\}$, $\mathscr{F}(K)=\emptyset$ if $K \neq K_s$, and define $\mathfrak{A}(\mathscr{O}) \equiv \mathscr{B}(\bar{\mathscr{O}})$ for all open double cones $\mathscr{O}$. The premises in part b) of Theorem 2.8 are then satisfied, and hence all the conclusions follow, i.e., Scenario A obtains. If we select $\mathscr{N}_s \equiv \{X_s\} \subset \mathscr{P}_{0s}(K_s)$ the premises in part c) of Theorem 3.2 are satisfied, and hence Ω is cyclic for any $\mathscr{B}(K)$ such that $K_{s,\lambda}$ is contained in the interior of K for some $\lambda \in P$. □

Proposition 4.3. *Let the premises and notation be as in Proposition 4.2. Suppose that A is a bounded operator such that*

$$\langle X^\dagger \phi' | A\phi''\rangle = \langle A^*\phi' | X\phi''\rangle \tag{4.2}$$

for all $\phi', \phi'' \in D_{0s}$, and every X of the form $X=U(\lambda)X_sU(\lambda)^{-1}$, where $\lambda \in P$ is such that $K_{s,\lambda} \subset W_L$. Then $A \in \mathscr{A}(W_R)$.

Proof. We define $\mathscr{F}_L = \{U(\lambda)X_sU(\lambda)^{-1} | \lambda \in P,\ K_{s,\lambda} \subset W_L\}$. The set $\mathscr{F}_L$ generates $\mathscr{P}_{0s}(W_L)$ in the sense that $\mathscr{P}_{0s}(W_L)$ is the smallest unital *-algebra which contains $\mathscr{F}_L$. By (4.2) the set $\mathscr{F}_L$ commutes weakly with A on D_{0s}, and since $\mathscr{F}_L D_{0s} \subset D_{0s}$, we conclude that $\mathscr{P}_{0s}(W_L)$ commutes weakly with A on D_{0s}, i.e., the relation (4.2) holds for *all* $X \in \mathscr{P}_{0s}(W_L)$. Since $V(t)\mathscr{P}_{0s}(W_L)V(t)^{-1} = \mathscr{P}_{0s}(W_L)$, and since $\mathscr{P}_{0s}(W_L)\Omega$ is dense, it follows from part b) of Lemma 2.6 that $A\Omega$ and $A^*\Omega$ are in the domain of $V(i\pi)$, and that

$$JV(i\pi)A\Omega = A^*\Omega\,. \tag{4.3}$$

2) Every $X \in \mathscr{F}_L$ is intrinsically local. Let $\lambda \in P$ be such that $K_{s,\lambda} \subset W_L$, and hence $X=U(\lambda)X_sU(\lambda)^{-1} \in \mathscr{F}_L$. The closed operator $\hat{X} \equiv (X \restriction D_{0s})^{**} = U(\lambda)(X_s \restriction D_{0s})^{**}U(\lambda)^{-1}$ is affiliated with $U(\lambda)a_sU(\lambda)^{-1} \subset \mathscr{B}(K_{s,\lambda})$, and hence with $\mathscr{A}(W_L)$. It follows that $\hat{X}$ commutes *strongly* with $\mathscr{A}(W_R)=\mathscr{A}(W_L)'$. The relation (4.2) implies (since X is actually hermitian) that $\langle \hat{X}\phi'_e | A\phi''_e\rangle = \langle A^*\phi'_e | \hat{X}\phi''_e\rangle$ for any ϕ'_e and ϕ''_e in the domain $D(\hat{X})$ of $\hat{X}$. In particular, this relation holds for $\phi'_e = A_1\phi'$ and $\phi''_e = A_2\phi''$, for any $A_1, A_2 \in \mathscr{A}(W_R)$ and any $\phi', \phi'' \in D_{0s}$. With this choice we thus have

$$\langle A_1X\phi' | AA_2\phi''\rangle = \langle \hat{X}A_1\phi' | AA_2\phi''\rangle = \langle A^*A_1\phi' | \hat{X}A_2\phi''\rangle = \langle A^*A_1\phi' | A_2X\phi''\rangle\,,$$

since $\hat{X}$ commutes strongly with A_1 and A_2. From the equality of the first and fourth members we conclude that the operator $A_1^* A A_2$ satisfies the *same* premises as the operator A, for any $A_1, A_2 \in \mathscr{A}(W_R)$. In view of (4.3) we then have

$$JV(i\pi)A_1^* A A_2 \Omega = A_2^* A^* A_1 \Omega \tag{4.4}$$

for all $A_1, A_2 \in \mathscr{A}(W_R)$.

3) Let $B \in \mathscr{A}(W_L)$. We then have $JV(-i\pi)B^*\Omega = B\Omega$, and from this relation and from (4.4) it readily follows that $\langle B^*\Omega | A_1^* A A_2 \Omega\rangle = \langle A_2^* A^* A_1 \Omega | B\Omega\rangle$. Since B commutes with A_1 and A_2 we conclude that $\langle A_1\Omega | [B, A] A_2\Omega\rangle = 0$. Since $\mathscr{A}(W_R)\Omega$ is dense it follows that $[B, A] = 0$, and hence $A \in \mathscr{A}(W_L)' = \mathscr{A}(W_R)$. □

This proposition in effect establishes a uniqueness property of the local AB-system generated by a single intrinsically local operator. One may ask whether it holds more generally, within Scenario A. This could be the case, but the above proof does not apply without the assumption of intrinsic locality: the reasoning in step 2 depended *critically* on the fact that D_{0s} is a core for the closed extension of $X_s \restriction D_{0s}$ which is affiliated with $\mathscr{B}(K_s)$. If X_s is intrinsically local, the extension is simply $(X_s \restriction D_{0s})^{**}$.

Proposition 4.4. *Let the premises and notation be as in Proposition 4.2. Then*

$$\langle X^\dagger \phi' | B\phi''\rangle = \langle B^* \phi' | X\phi''\rangle \tag{4.5}$$

for all $\phi', \phi'' \in D_{0s}$, all $B \in \mathscr{A}(W_L)$, and all $X \in \mathscr{P}_0(W_R)$.

Proof. 1) Let $K \in \mathscr{K}$, $K \subset W_L$, and let $W_1 \in \mathscr{W}$, $W_1 \subset W_L \cap K^c$: the three sets K, W_1, *and* W_R are thus pairwise spacelike separated from each other. Let $Y \in \mathscr{P}_{0s}(W_1)$ and $B_k \in \mathscr{B}(K)$. By the same reasoning as in step 5 of the proof of Theorem 2.8 we conclude that $JV(-i\pi)B_k Y\Omega = B_k^* Y^\dagger \Omega$. Let $X \in \mathscr{P}_0(W_R)$, in which case we have $JV(i\pi)X\Omega = X^\dagger\Omega$. It follows that

$$\langle B_k Y\Omega | X\Omega\rangle = \langle X^\dagger \Omega | B_k^* Y^\dagger \Omega\rangle . \tag{4.6}$$

2) We define $\mathscr{F}_{L1} = \{U(\lambda)X_s U(\lambda)^{-1} | \lambda \in P,\ K_{s,\lambda} \subset W_1\}$. Every $Y_1 \in \mathscr{F}_{L1}$ is hermitian and intrinsically local, and the operator $\hat{Y}_1 \equiv (Y_1 \restriction D_{0s})^{**}$ is affiliated with $\mathscr{A}(W_1)$. For such a Y_1 we have $\hat{Y}_1 B_k Y_2 \Omega = B_k Y_1 Y_2 \Omega$, if $B_k \in \mathscr{B}(K)$ and $Y_2 \in \mathscr{P}_{0s}(W_1)$. Since $\hat{Y}_1 \subset \bar{Y}_1$ (where $\bar{Y}_1$ denotes the closure of Y_1 relative to D_1), we have $\hat{Y}_1^* \supset Y_1^* \supset Y_1$, and hence $\hat{Y}_1^* X Y_3 \Omega = Y_1^* X Y_3 \Omega = Y_1 X Y_3 \Omega = X Y_1 Y_3 \Omega$ for any $X \in \mathscr{P}_0(W_R)$ and any $Y_3 \in \mathscr{P}_{0s}(W_1) \subset \mathscr{P}_0(W_L)$. We thus have $\langle B_k Y_1 Y_2 \Omega | X Y_3 \Omega\rangle = \langle \hat{Y}_1 B_k Y_2 \Omega | X Y_3 \Omega\rangle = \langle B_k Y_2 \Omega | X Y_1 Y_3 \Omega\rangle$. Since this holds for any $Y_1 \in \mathscr{F}_{L1}$, and since $\mathscr{F}_{L1}$ generates $\mathscr{P}_{0s}(W_1)$, it follows that $\langle B_k Y_b^\dagger Y_a \Omega | X\Omega\rangle = \langle B_k Y_a \Omega | X Y_b \Omega\rangle$ and $\langle X^\dagger \Omega | B_k^* Y_a^\dagger Y_b \Omega\rangle = \langle X^\dagger Y_a \Omega | B_k^* Y_b \Omega\rangle$ for any Y_a, $Y_b \in \mathscr{P}_{0s}(W_1)$. From these relations, and from (4.6) with $Y = Y_b^\dagger Y_a$, we obtain

$$\langle B_k Y_a \Omega | X Y_b \Omega\rangle = \langle X^\dagger Y_a \Omega | B_k^* Y_b \Omega\rangle \tag{4.7}$$

for any Y_a, $Y_b \in \mathscr{P}_{0s}(W_1)$, any $B_k \in \mathscr{B}(K)$, and any $X \in \mathscr{P}_0(W_R)$.

3) We write $T(x) = U(I, x)$ for the translations in P. It follows from (4.7) that the relation

$$\langle B_k Y_a \Omega | X T(x) Y_b \Omega\rangle = \langle X^\dagger Y_a \Omega | B_k^* T(x) Y_b \Omega\rangle \tag{4.8}$$

holds for all $T(x)$ with x in some non-empty open subset of $\mathcal{M}$ (since there is an open subset of translations which map W_1 into itself). Both members in (4.8) can be continued analytically to the forward imaginary tube, and it follows that (4.8) holds for *all* x. This means that (4.7) holds for all $B_k \in \mathcal{B}(K)$, all $X \in \mathcal{P}_0(W_R)$, *all* $Y_b \in \mathcal{P}_{0s}(\mathcal{M})$, and all $Y_a \in \mathcal{P}_{0s}(W_1)$. By similar reasoning we then conclude that (4.7) also holds for *all* $Y_a \in \mathcal{P}_{0s}(\mathcal{M})$. Hence (4.5) holds as stated when $B = B_k \in \mathcal{B}(K)$, for any $K \subset W_L$, $K \in \mathcal{K}$. By part b) of Lemma 2.5 it then follows that (4.5) holds for *all* $B \in \mathcal{A}(W_L)$. □

This proposition thus establishes a certain property of relative locality. That it is rather weak can be seen if we compare it with the following more desirable, but purely hypothetical situation: The relation (4.5) holds for all $X \in \mathcal{P}_0(K)$, all $B \in \mathcal{A}(K^c)$, and all $\phi', \phi'' \in D_0$, for any $K \in \mathcal{K}$. The reason why our proof of Proposition 4.4, which is concerned with *wedge*-regions, cannot be trivially extended to the case of double cones is that we depend in an essential manner on the relation (2.1) which refers specifically to wedge-regions. We do not have available an analogous relation for double-cone regions at this time, and it is even possible that no such analog of the same generality exists.

It is tempting to believe that any two intrinsically local operators generate the same AB-system. We note here that the attempt to draw such a conclusion directly from Propositions 4.3 and 4.4 founders on the domain restrictions for ϕ' and ϕ'' in (4.5). There are, however, special cases in which progress is possible, as we shall see later.

Proposition 4.5. *Let the premises and notation be as in Proposition 4.2. Let $K_{s,\Theta} = -K_s$. Then the operator $X_{s\Theta} = \Theta_0 X_s \Theta_0^{-1} \in \mathcal{P}_0(K_{s,\Theta})$ is an intrinsically local hermitian operator (in the sense of Definition 4.1), and it generates the same AB-system as X_s.*

Proof. It is trivial that $X_{s\Theta}$ is hermitian and intrinsically local. By Proposition 4.2 the AB-system generated by X_s is TCP-covariant (since it satisfies the special condition of duality), and we thus have $\Theta_0 \mathcal{B}(K_s) \Theta_0^{-1} = \mathcal{B}(K_{s,\Theta})$. Since $a_s = a((X_s \upharpoonright D_{0s})^{**}) \subset \mathcal{B}(K_s)$ it follows that $a((X_{s\Theta} \upharpoonright \Theta_0 D_{0s})^{**}) = \Theta_0 a_s \Theta_0^{-1} \subset \mathcal{B}(K_{s,\Theta})$. This implies that $\mathcal{A}(W) = \{U(\lambda)\Theta_0 a_s \Theta_0^{-1} U(\lambda)^{-1} | \lambda \in P,\ K_{s,\Theta,\lambda} \subset W\}''$, and hence X_s and $X_{s\Theta}$ generate the same AB-system [through (4.1)]. □

We shall now summarize the facts concerning an intrinsically local operator as follows.

Theorem 4.6. *Suppose that for some $K_s \in \mathcal{K}$ there exists a hermitian, intrinsically local operator $X_s \in \mathcal{P}_0(K_s)$, in the sense of Definition 4.1. Let $\mathcal{P}_{0s}(R)$ and D_{0s} be as in Definition 4.1, and let $a_s = a((X_s \upharpoonright D_{0s})^{**})$ be the von Neumann algebra generated by the closure of $X_s \upharpoonright D_{0s}$. Let $\{\mathcal{A}(W), \mathcal{B}(K), \mathcal{A}(K^c)\}$ be the AB-system generated by X_s in the sense that*

$$\mathcal{A}(W) = \{U(\lambda) a_s U(\lambda)^{-1} | \lambda \in P,\ K_{s,\lambda} \subset W\}''.$$

Then:

a) *Scenario A in Definition 2.4 obtains for this AB-system and the algebra $\mathcal{P}_{0s}(\mathcal{M})$, and in particular the AB-system is local and TCP-covariant, and it satisfies the special condition of duality.*

b) *The operator* $X_{s\Theta} \equiv \Theta_0 X_s \Theta_0^{-1}$ *is hermitian and intrinsically local, and it generates the same AB-system as* X_s.

c) *If* $K \in \mathscr{K}$ *is such that* $K_{s,\lambda}$ *is contained in the interior of* K *for some* $\lambda \in P$, *then* Ω *is cyclic and separating for* $\mathscr{B}(K)$.

d) *Let* $W \in \mathscr{W}$. *A bounded operator* B *is an element of* $\mathscr{A}(W)$ *if and only if it commutes weakly on* D_{0s} *with every* $X \in \mathscr{P}_{0s}(\bar{W}^c)$, *which is the case if and only if* B *commutes weakly on* D_{0s} *with every operator* $U(\lambda)X_s U(\lambda)^{-1}$ *such that* $\lambda \in P$, $K_{s,\lambda} \subset \bar{W}^c$.

e) *Let* $K \in \mathscr{K}$. *A bounded operator* B *is an element of* $\mathscr{B}(K)$ *if and only if it commutes weakly on* D_{0s} *with every* $X \in \mathscr{P}_{0s}(K^c)$, *which is the case if and only if* B *commutes weakly on* D_{0s} *with every operator* $U(\lambda)X_s U(\lambda)^{-1}$ *such that* $\lambda \in P$, $K_{s,\lambda} \subset K^c$.

f) *Let* $W \in \mathscr{W}$. *Then* $\mathscr{P}_0(W)$ *commutes weakly on* D_{0s} *with* $\mathscr{A}(\bar{W}^c)$.

g) *Let* $\{\mathscr{A}_1(W), \mathscr{B}_1(K), \mathscr{A}_1(K^c)\}$ *be an AB-system, and let* $\mathscr{P}_{0s1}(\mathscr{M})$ *be a sub-algebra of* $\mathscr{P}_0(\mathscr{M})$ *such that Scenario A obtains for this sub-algebra and this AB-system. If* $X_s \in \mathscr{P}_{0s1}(K_s)$, *then the AB-system is identical with the one generated by* X_s.

Proof. 1) The assertion in a) follows from Proposition 4.2, and the assertion in b) follows from Proposition 4.5. The assertion in c) follows from Proposition 4.2, and the assertion in d) follows from Proposition 4.3 (and from the Poincaré-covariance of the *AB*-system). The assertion in f) follows from Proposition 4.4 and Poincaré-covariance.

2) We consider the assertion in e). Let $K \in \mathscr{K}$, and suppose that B is a bounded operator which commutes weakly on D_{0s} with every operator $U(\lambda)X_s U(\lambda)^{-1}$ such that $\lambda \in P$, $K_{s,\lambda} \subset K^c$: in particular this is the case if B commutes weakly on D_{0s} with $\mathscr{P}_{0s}(K^c)$. Let $W \in \mathscr{W}$ be such that $K \subset W$. Then B also commutes weakly on D_{0s} with every operator $U(\lambda)X_s U(\lambda)^{-1}$ such that $\lambda \in P$, $K_{s,\lambda} \subset \bar{W}^c$, and hence, by part d), we have $B \in \mathscr{A}(W)$. Since this holds for every $W \supset K$, $W \in \mathscr{W}$, it follows that $B \in \mathscr{B}(K)$. The converse statement is an aspect of Scenario A in Definition 2.4.

3) We consider the assertion in g). Let $D_{0s1} \equiv \mathscr{P}_{0s1}(\mathscr{M})\Omega$. If $X_s \in \mathscr{P}_{0s1}(K_s)$ we have $D_{0s} \subset D_{0s1}$, and it is a feature of Scenario A that for any $W \in \mathscr{W}$, every $B \in \mathscr{A}_1(W)$ commutes weakly on D_{0s1} with $\mathscr{P}_{0s1}(\bar{W}^c)$. Since obviously $\mathscr{P}_{0s1}(\bar{W}^c) \supset \mathscr{P}_{0s}(\bar{W}^c)$ we conclude, by part d) of the present theorem, that $B \in \mathscr{A}(W)$. Hence $\mathscr{A}_1(W) \subset \mathscr{A}(W)$, and since both AB-systems satisfy the special condition of duality it follows that they are identical. □

The scenario described in the above theorem can be regarded as an "improvement" of Scenario A, which derives from the circumstance that the algebra $\mathscr{P}_{0s}(\mathscr{M})$ contains an intrinsically local operator. In particular the association of such an operator with a local AB-system is unique. Note, however, the domain conditions in part f) of the theorem. The statement in f) is equivalent to the statement that for every $X \in \mathscr{P}_0(W)$ there exists a closed extension X_e of $X \restriction D_{0s}$ which is affiliated with $\mathscr{A}(W)$ and such that $(X^\dagger \restriction D_{0s})^* \supset X_e$, but from this it does not follow (as far as we can see) that X *as defined on* D_1 also has a closed extension affiliated with $\mathscr{A}(W)$.

The domain considerations simplify substantially if the operator X_s in the theorem is *linear* in the field $\varphi(x)$, and we shall now study this case. We first consider a technical preliminary.

Lemma 4.7. *Let $K_s \in \mathscr{K}$ be a fixed double cone, and let $f_s(x)$ be a real test function with support in K_s and such that its Fourier transform $\tilde{f}_s(p) \neq 0$ for all p. Let $X_s \equiv \varphi[f] = X_s^\dagger$, and let D_{0s} be defined as in Definition 4.1 in terms of X_s. Then:*

a) *D_{0s} is dense in $\mathscr{H}$.*

b)
$$\bar{Y} = (Y \upharpoonright D_{0s})^{**} \tag{4.9}$$

for all $Y \in \mathscr{P}_0(\mathscr{M})$, where $\bar{Y}$ is the closure of Y as defined on D_1.

Proof. 1) If the functions $f(x)$ and $g(x)$ are elements of the test function space $\mathscr{S}(\mathbb{R}^4)$, then their convolution $f * g \in \mathscr{S}(\mathbb{R}^4)$, and for a fixed g the linear mapping $f \to f * g$ of $\mathscr{S}(\mathbb{R}^4)$ into itself is continuous in the test function space topology relevant for tempered distributions. Furthermore, if $f_s \in \mathscr{S}(\mathbb{R}^4)$ and if $\{g_k | k = 1, \ldots, n\}$ is any fixed n-tuplet of elements of $\mathscr{S}(\mathbb{R}^4)$, then the element $(f_s * g_1) \otimes (f_s * g_2) \otimes \ldots \otimes (f_s * g_n)$ of $\mathscr{S}(\mathbb{R}^{4n})$ is the image of the element $f_s \otimes f_s \otimes \ldots \otimes f_s$ of $\mathscr{S}(\mathbb{R}^{4n})$ under a linear mapping of $\mathscr{S}(\mathbb{R}^{4n})$ into itself which is *continuous* in the test function space topology.

2) Let $Y \in \mathscr{P}_0(\mathscr{M})$, and let ϕ be any vector. Let $\{g_k\}$ be an n-tuplet of test functions in $\mathscr{S}(\mathbb{R}^4)$ as above. We write $h_k = f_s * g_k$ for $k = 1, \ldots, n$, and $X_s(x) = T(x) X_s T(x)^{-1}$ for the translate of X_s by x. In view of what was said above, and in view of the nature of the quantum field as an operator-valued tempered distribution, it follows that

$$\begin{aligned} &\int d(x_1) \ldots d(x_n) g_1(x_1) \ldots g_n(x_n) \langle \phi | Y X_s(x_1) X_s(x_2) \ldots X_s(x_n) \Omega \rangle \\ &\quad = \langle \phi | Y \varphi[h_1] \varphi[h_2] \ldots \varphi[h_n] \Omega \rangle , \end{aligned} \tag{4.10}$$

where the integral at left makes good sense as a Riemann integral, since the function $\langle \phi | Y X_s(x_1) X_s(x_2) \ldots X_s(x_n) \Omega \rangle$ is a jointly continuous function of the variables $(x_1, x_2, \ldots, x_n)$.

3) From our crucial assumption that $\tilde{f}_s(p) \neq 0$ for *all* p it readily follows that the set $\{f_s * g | g \in \mathscr{S}(\mathbb{R}^4)\}$ is dense in $\mathscr{S}(\mathbb{R}^4)$ in the test function space topology. From this we conclude that the span of Ω and all vectors of the form $\varphi[h_1] \ldots \varphi[h_n] \Omega$, where n is an arbitrary positive integer, and where $h_k = f_s * g_k$ for arbitrary elements $g_k \in \mathscr{S}(\mathbb{R}^4)$, is a dense sub-manifold of D_0, and hence dense in $\mathscr{H}$.

We consider (4.10) in the special case $Y = I$. If the vector ϕ is orthogonal to D_{0s}, the left member in (4.10) vanishes, and in view of what was said above we conclude that $\phi = 0$, i.e., D_{0s} is dense in $\mathscr{H}$, as asserted.

4) We consider (4.10) for an arbitrary $Y \in \mathscr{P}_0(\mathscr{M})$. Let $\hat{Y} = Y \upharpoonright D_{0s}$, and let ϕ be in the domain of $\hat{Y}^*$. Since $X_s(x_1) \ldots X_s(x_n) \Omega \in D_{0s}$ we conclude from (4.10) that

$$\langle \hat{Y}^* \phi | \varphi[h_1] \varphi[h_2] \ldots \varphi[h_n] \Omega \rangle = \langle \phi | Y \varphi[h_1] \varphi[h_2] \ldots \varphi[h_n] \Omega \rangle ,$$

where the h_k are constructed as in step 2). Hence every vector of the form $\varphi[h_1] \varphi[h_2] \ldots \varphi[h_n] \Omega$ is in the domain of $\hat{Y}^{**}$ which, of course, also includes Ω. Since the span of the elements $h_1 \otimes h_2 \otimes \ldots \otimes h_n \in \mathscr{S}(\mathbb{R}^{4n})$ is dense in $\mathscr{S}(\mathbb{R}^{4n})$ in the test function space topology, we conclude that D_0 is included in the domain of $\hat{Y}^{**}$, and that $\hat{Y}^{**} \supset Y$. Since $Y \supset \hat{Y}$ the relation (4.9) follows, and D_{0s} is thus a core for $\bar{Y}$. □

We remark here that it is easy to construct a great abundance of test functions $f_s(x)$ which satisfy the premises of the lemma.

The facts stated in the lemma permit an improvement of Theorem 4.6 in the special case that X_s is *linear* in the field.

Theorem 4.8. *Let $K_s \in \mathscr{K}$, and let $f_s(x)$ be a real test function with support in K_s, and such that its Fourier transform satisfies the condition $\tilde{f}_s(p) \neq 0$ for all p. Let $X_s \equiv \varphi[f_s] = X_s^\dagger$, and let the notation in general be as in Definition 4.1 and Theorem 4.6. Suppose furthermore that $U(\lambda)a_s U(\lambda)^{-1}$ commutes with a_s for all $\lambda \in \boldsymbol{P}$ such that $K_{s,\lambda}$ is spacelike relative to K_s. Then:*

a) *$\bar{X}_s = (X_s \upharpoonright D_{0s})^{**}$, and hence $a_s = a(\bar{X}_s)$.*

b) *The operator X_s is intrinsically local, in the sense of Definition 4.1, and hence all the conclusions in Theorem 4.6 apply to the algebra $\mathscr{P}_{0s}(\mathscr{M})$ and the AB-system generated by X_s, in the sense described in Theorem 4.6. In particular Scenario A obtains.*

c) *Furthermore, for any $K \in \mathscr{K}$, $\mathscr{P}_{0s}(K)$ commutes weakly on D_1 with $\mathscr{A}(K^c)$, and $\mathscr{P}_0(K^c)$ commutes weakly on D_1 with $\mathscr{B}(K)$. For any $W \in \mathscr{W}$, $\mathscr{P}_0(W)$ commutes weakly on D_1 with $\mathscr{A}(\bar{W}^c)$. Equivalently stated: If $X \in \mathscr{P}_{0s}(K)$, then X has a closed extension X_e affiliated with $\mathscr{B}(K)$, and such that $X^{\dagger *} \supset X_e \supset \bar{X}$, and similarly $X \in \mathscr{P}_0(K^c)$ has a closed extension affiliated with $\mathscr{A}(K^c)$. If $X \in \mathscr{P}_0(W)$, then X has a closed extension X_e affiliated with $\mathscr{A}(W)$, and such that $X^{\dagger *} \supset X_e \supset \bar{X}$.*

d) *Suppose that $X_r = X_r^\dagger \in \mathscr{P}_0(K_r)$, for some $K_r \in \mathscr{K}$, is another intrinsically local operator, not necessarily linear in the field. Then X_r generates the same AB-system as X_s.*

Proof. 1) The relation (4.9) in Lemma 4.7 applies in particular to X_s, and hence the assertions in a) follow. D_{0s} is dense, by the same lemma, and hence X_s is intrinsically local, and the assertions in b) follow.

2) The assertions in c) follow readily from Lemma 4.7 and the corresponding weak commutation relations asserted in Theorem 4.6. The salient point is that the domain D_{0s} can be replaced by the domain D_1 in the statements of the weak commutation relations between the elements of $\mathscr{P}_0(\mathscr{M})$ and the elements of the algebras of the AB-system, in view of (4.9). We can then conclude that $\mathscr{B}(K)$ commutes weakly with $\varphi[f]$ on D_1, for any f with $\mathrm{supp}(f) \subset W$ when $W \subset K^c$. It follows that $\mathscr{B}(K)$ commutes weakly on D_1 with $\varphi[f]$ for *any* f with $\mathrm{supp}(f) \subset K^c$, and from this it follows that $\mathscr{B}(K)$ commutes weakly on D_1 with $\mathscr{P}_0(K^c)$.

3) We consider the assertion in d). Let $\{\mathscr{A}_1(W), \mathscr{B}_1(K), \mathscr{A}_1(K^c)\}$ be the AB-system generated by the intrinsically local operator X_r, as in Theorem 4.6. Let $\mathscr{P}_{0r}(W) \subset \mathscr{P}_0(W)$ be defined in terms of X_r in analogy with the definition of $\mathscr{P}_{0s}(W)$ in terms of X_s. By part c) of the present theorem, $\mathscr{P}_{0r}(W)$ commutes weakly on $D_{0r} \subset D_1$ with $\mathscr{A}(\bar{W}^c)$. It follows, by part d) of Theorem 4.6, that $\mathscr{A}(\bar{W}^c) \subset \mathscr{A}_1(\bar{W}^c)$, and since both AB-systems satisfy the special condition of duality we conclude that they are equal. □

The significant improvements in the present theorem over Theorem 4.6 (and in particular the uniqueness property expressed in part d) all derive from the domain relation in (4.9). Hence conclusions similar to those in Theorem 4.8 also apply to intrinsically local operators which are *not* linear in the field, *provided* that they are such that the relation (4.9) holds.

As we said before, the notion of an intrinsically local operator could naturally be generalized to the notion of an intrinsically local *set* of operators, with implications analogous to Theorems 4.6 and 4.8. We felt it was of particular interest to consider the possibility of just *one* such operator defining the local AB-system.

5. On Fields Which Satisfy a Certain Regularity Condition

We shall now consider a condition on the field $\varphi(x)$ which goes beyond the usual minimum assumptions. It can be described as a condition which "regularizes" the high-energy behavior of the field. We state it as follows.

Definition 5.1. For any $\alpha > 0$ we employ the notation $\omega_\alpha(s)$ for the function

$$\omega_\alpha(s) = (1+s^2)^{\alpha/2} \tag{5.1}$$

of the real variable s, and we write ω_α for the selfadjoint operator $\omega_\alpha \equiv \omega_\alpha(H)$, where H is the Hamiltonian operator.

The field $\varphi(x)$ will be said to satisfy a *generalized H-bound* if and only if there exists a constant α, with $1 > \alpha > 0$, such that the following conditions hold:

a) For any test function f the domain $D(\overline{\varphi[f]})$ of the closure of $\varphi[f]$ (relative to D_1) contains $\exp(-\omega_\alpha)\mathscr{H}$.

b) For any test function f the operator $\overline{\varphi[f]}\exp(-\omega_\alpha)$ is a *bounded* operator.

Let α_0 be the infimum of all α for which the above conditions hold. We shall then say that the generalized H-bound is of *order* α_0.

Regularity conditions of this general character have been considered before, in studies of the connection between field operators and bounded local operators [15, 9, 13], although with stronger conditions on the field. Instead of the above condition it was assumed that $D(\overline{\varphi[f]})$ contains the intersection of the domains of all powers of H, and furthermore it was assumed that $\overline{\varphi[f]}(I+H)^{-r}$ is a bounded operator, for some $r > 0$. Such an H-bound is thus, with our terminology, a generalized H-bound of order 0. Since the considerations which follow are not more difficult in the case of a generalized H-bound than in the case of a power H-bound, we felt it worthwhile to discuss the situation under the weaker assumptions. For other applications of generalized H-bounds, see [10, 29].

The reason for the restriction $1 > \alpha$ will become clear in the following. The essential point is that we shall depend on the existence of test functions $f(t) \in \mathscr{S}(\mathbb{R}^1)$ of (arbitrarily prescribed) compact support whose Fourier transforms $\hat{f}(s)$ satisfy conditions of the form $|\hat{f}(s)| < b\exp(-\omega_\alpha(s))$. It is well-known (and easily shown) that such functions exist if and only if $\alpha < 1$.

The assumption of a generalized H-bound has rather drastic consequences for the field, which we shall now explore. For this paper the ultimate goals are Theorems 5.5 and 5.6. We shall proceed through a sequence of lemmas, some of which are also of interest in other contexts.

Lemma 5.2. *Suppose that $\varphi(x)$ satisfies a generalized H-bound of order α, with $1 > \alpha \geqq 0$. Let $D_H(\alpha)$ be the linear manifold defined by*

$$D_H(\alpha) = \operatorname{span}\{\exp(-\omega_\beta)\mathscr{H} \mid \beta > \alpha\}.$$

Let $f(x) \in \mathscr{S}(\mathbb{R}^4)$. We denote $X = \varphi[f]$. Then:

a) *The manifold $D_H(\alpha)$ is Poincaré-invariant. It is dense in $\mathscr{H}$, and it is contained in the domain $D(\bar{X})$ of the closure of X (on D_1).*

If $\beta > \alpha$, and if D is any dense linear manifold in $\mathscr{H}$, then $\exp(-\omega_\beta)D$ *is a core for $\bar{X}$. In particular $D_H(\alpha)$ is a core for $\bar{X}$.*

b) *Every dense, translation-invariant, linear sub-domain D_s of D_1 is a core for $\bar{X}$.*

Proof. 1) That $D_H(\alpha)$ is Poincaré-invariant follows readily from the fact that if $\beta' > \beta > 0$, and if e and e' are any two (fixed) forward timelike unit vectors in $\mathscr{M}$, then there exists a constant c such that $\omega_\beta(e \cdot p) < c + \omega_{\beta'}(e' \cdot p)$ for all p in the closed forward lightcone. It is obvious that $D_H(\alpha)$ is dense, and that $D_H(\alpha)$ is contained in $D(\bar{X})$. The remaining assertions in part a) follow readily from part b), which we shall now consider.

2) Let D_s be a dense, translation-invariant, linear sub-manifold of D_1. Let $u(t)$ be a test function of compact support, and such that $u(0) = 1$. We define the linear manifold $D_{s,c}$ in terms of u and D_s by $D_{s,c} = \operatorname{span}\{u(H/n)D_s | n = 1, 2, \ldots,\}$. Then $D_{s,c}$ is a dense, linear sub-manifold of D_1. For any $\phi \in D_s$ the sequence $\{\phi_n = u(H/n)\phi | n = 1, 2, \ldots\}$ is contained in $D_{s,c}$, and converges strongly to ϕ. Since the field is an operator-valued tempered distribution, the sequence $\{X\phi_n | n = 1, 2, \ldots,\}$ converges strongly to $X\phi$, and since D_s was assumed translation-invariant, it follows that $D_{s,c}$ is contained in the domain of $(X \restriction D_s)^{**}$ and that $(X \restriction D_s)^{**} = (X \restriction D_{s,c})^{**}$.

3) Let $\beta > \alpha$. By Definition 5.1 the operator $B_\beta = \bar{X} \exp(-\omega_\beta)$ is bounded, and we have $\bar{X} \supset B_\beta \exp(\omega_\beta)$. The operator $\exp(\omega_\beta)$ is defined and selfadjoint on the domain $\exp(-\omega_\beta)\mathscr{H}$. Every core for $\exp(\omega_\beta)$ is also a core for $(B_\beta \exp(\omega_\beta))^{**}$ [with $\exp(-\omega_\beta)\mathscr{H}$ regarded as the domain of $B_\beta \exp(\omega_\beta)$]. In view of the construction of $D_{s,c}$ in step 2 above, it is easily seen that $D_{s,c}$ is a core for $\exp(\omega_\beta)$. By the result in step 2 we can then conclude that $\bar{X} \supset (B_\beta \exp(\omega_\beta))^{**} = (\bar{X} \restriction D_{s,c})^{**} = (X \restriction D_s)^{**}$. If we select $D_s = D_1$, the first and fourth members are equal, from which it follows that $\bar{X} = (B_\beta \exp(\omega_\beta))^{**}$. Hence $\bar{X} = (X \restriction D_s)^{**}$ for *any* D_s which satisfies the stated premises. It is now trivial that $\exp(-\omega_\beta)\mathscr{H}$ is a core for $\bar{X}$ for any $\beta > \alpha$, and from this it follows that $D_H(\alpha)$ is also a core for $\bar{X}$. □

We remark here that D_0 and D_1 are not contained in the domain $D_H(\alpha)$, nor is the latter domain in general mapped into itself by $\bar{X}$, unless the test function $f(x)$ has special properties. We next state and prove a technical lemma which has to do with this issue, concerning a class of bounded operators which map $D_H(\alpha)$ into itself.

Lemma 5.3. *Let $D_H(\alpha)$ be defined as in Lemma 5.2. Let B_0 be a bounded operator. Let $g(t) \in \mathscr{S}(\mathbb{R}^1)$ be such that its Fourier transform*

$$\hat{g}(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt g(t) e^{-its}$$

satisfies the condition

$$|\hat{g}(s)| < \exp(-\omega_\beta(s)) \tag{5.2}$$

for some $\beta > \alpha$. Let $u(t) \in \mathscr{S}(\mathbb{R}^1)$ be such that $\operatorname{supp}(\hat{u}) \subset [-1, 1]$, *and let the bounded operator B be defined by*

$$B = \int_{-\infty}^{\infty} dt g(t) u(t) e^{itH} B_0 e^{-itH} . \tag{5.3}$$

Then $(U(\lambda) B U(\lambda)^{-1}) D_H(\alpha) \subset D_H(\alpha)$ and $(U(\lambda) B^* U(\lambda)^{-1}) D_H(\alpha) \subset D_H(\alpha)$ for all elements λ of the Poincaré group P.

Proof. 1) Let μ be the spectral measure in the spectral resolution of the Hamiltonian operator H. For any real q we write $F(q) = \mu((-\infty, q])$ [and we then have $F(q) = 0$ for $q < 0$, in view of the spectrum condition]. Let $\phi', \phi'' \in \mathscr{H}$. We define the function $h(t; q)$ by

$$h(t; q) = \langle \phi' | (I - F(2q)) e^{itH} B_0 e^{-itH} F(q) \phi'' \rangle . \tag{5.4}$$

As a function of t (with q fixed) $h(t; q)$ is continuous and bounded, and we trivially have

$$\int_{-\infty}^{\infty} dt |u(t) h(t; q)|^2 \leqq k_0^2 \cdot \|\phi'\|^2 \cdot \|\phi''\|^2 \tag{5.5}$$

for some constant k_0, independent of q, ϕ', and ϕ''.

Furthermore we have

$$\langle \phi' | (I - F(2q)) B F(q) \phi'' \rangle = \int_{-\infty}^{\infty} dt g(t) u(t) h(t; q) . \tag{5.6}$$

2) Let $q \geqq 0$ be fixed. The function $h(t; q)$ of t can be regarded as a tempered distribution. Its Fourier transform $\hat{h}(s; q)$ is then well-defined as a tempered distribution, and by inspection of (5.4) we see that its support is contained in $[q, +\infty)$. The convolution of $\hat{h}(s; q)$ with $\hat{u}(s)$ is the Fourier transform of the function $h_1(t; q) = u(t) h(t; q) \in L_1(\mathbb{R}^1) \cap L_2(\mathbb{R}^1)$. In view of the assumed support properties of $\hat{u}(s)$ we conclude that the support of $\hat{h}_1(s; q)$ is contained in $[q-1, +\infty)$.

We can regard the integral at right in (5.6) as the scalar product of the elements $g^*(t)$ and $h_1(t; q)$ in the Hilbert space $L_2(\mathbb{R}^1)$. In view of the support properties of $\hat{h}_1(s; q)$, and in view of the inequalities (5.2) and (5.5), it follows by an application of Schwarz' inequality that

$$|\langle \phi' | (I - F(2q)) B F(q) \phi'' \rangle| \leqq k_0 \|\phi'\| \cdot \|\phi''\| I(q) , \tag{5.7}$$

where the positive function $I(q)$ is given by

$$I(q)^2 = \int_{q-1}^{\infty} ds \exp(-2\omega_\beta(s)) < k_1^2 \exp(-\omega_\beta(q)) \tag{5.8}$$

for some constant $k_1 > 0$ (independent of q). It follows from (5.7) and (5.8) that

$$\|(I - F(2q)) B F(q)\| < k_0 k_1 \exp(-\omega_\beta(q)/2) . \tag{5.9}$$

3) Let $\gamma > \alpha$ and let $q \geqq 0$. Since the function $\omega_\gamma(s)$ is monotonically increasing for positive s, we have $\|(I - F(q)) \exp(-\omega_\gamma)\| \leqq \exp(-\omega_\gamma(q))$. From this, and from

(5.9), it follows that

$$\begin{aligned}\|(I-F(2q))B\exp(-\omega_\gamma)\| &\leqq \|(I-F(2q))BF(q)\exp(-\omega_\gamma)\| \\ &\quad + \|(I-F(2q))B(I-F(q))\exp(-\omega_\gamma)\| \\ &\leqq k_0k_1\exp(-\omega_\beta(q)/2) + \|B\|\exp(-\omega_\gamma(q)) . \end{aligned} \tag{5.10}$$

4) We now select $\alpha' > \alpha$ such that $\beta > \alpha'$, $\gamma > \alpha'$. There then exists a constant k such that for all $s \geqq 0$,

$$k_0k_1\exp(-\omega_\beta(s/2)/2) + \|B\|\exp(-\omega_\gamma(s/2)) < k\exp(-2\omega_{\alpha'}(s)),$$

and hence, in view of (5.10),

$$\|(I-F(s))B\exp(-\omega_\gamma)\| < k\exp(-2\omega_{\alpha'}(s)).$$

This implies that $B\exp(-\omega_\gamma)\phi$ is in the domain of $\exp(\omega_{\alpha'})$ for all ϕ, and hence in $D_H(\alpha)$. Since γ is arbitrary, except for the condition $\gamma > \alpha$, it follows that $BD_H(\alpha) \subset D_H(\alpha)$. Similarly we conclude that $B^*D_H(\alpha) \subset D_H(\alpha)$, since B^* is given by an integral as in (5.3), with $g(t)u(t)$ replaced by its complex conjugate.

We have thus proved the assertion in the lemma for the operators $U(\lambda)BU(\lambda)^{-1}$ and $U(\lambda)B^*U(\lambda)^{-1}$ in the special case $\lambda = I$. The general case then follows from the Poincaré-invariance of $D_H(\alpha)$. □

We note here that Lemmas 5.2 and 5.3 actually hold for *all* $\alpha \geqq 0$. In the next lemma the restriction $1 > \alpha \geqq 0$ is, however, essential, which is why this restriction appears in Definition 5.1.

Lemma 5.4. *Let the field $\varphi(x)$ satisfy a generalized H-bound of order α, with $1 > \alpha \geqq 0$. Let $f(x)$ be a real test function. We write $X \equiv \varphi[f] = X^\dagger$. Let D be a core for $\bar{X}$. Let A be a bounded operator. We write $A(t) = \exp(itH)A\exp(-itH)$ for all real t. Suppose that for all $t \in (-\delta, \delta)$, for some $\delta > 0$, the operators $A(t)$ and $\bar{X}$ satisfy the weak commutation relation*

$$\langle A(t)^*\phi' | \bar{X}\phi''\rangle = \langle \bar{X}\phi' | A(t)\phi''\rangle, \quad \textit{all} \quad \phi', \phi'' \in D. \tag{5.11}$$

Then $A(t)$ commutes strongly with $\bar{X}$ for $t \in (-\delta, \delta)$, and in particular this holds for $A(0) = A$. Equivalently stated: $A(t) \in a(\bar{X})'$, where $a(\bar{X})$ is the von Neumann algebra generated by $\bar{X}$.

Proof. 1) Let $u(t) \in \mathscr{S}(\mathbb{R}^1)$ satisfy the conditions $u(0) = 1$ and $\operatorname{supp}(\hat{u}) \subset [-1, 1]$, where $\hat{u}(s)$ is the Fourier transform of $u(t)$. Let $g(t) \in \mathscr{S}(\mathbb{R}^1)$ be a function which satisfies the conditions: a) $\operatorname{supp}(g) \subset (-\delta, \delta)$; b) $|\hat{g}(s)| < k\exp(-\omega_{\beta'}(s))$ for some β' such that $1 > \beta' > \alpha$, and some constant k; c) $\int dt g(t) = 1$. As we remarked before, such functions exist. For any $\lambda \geqq 1$ we define $g_\lambda(t) = \lambda g(\lambda t)$. Hence $\int dt g_\lambda(t) = 1$, and $\operatorname{supp}(g_\lambda) \subset (-\delta, \delta)$. Let β be such that $\beta' > \beta > \alpha$. Since $\hat{g}_\lambda(s) = \hat{g}(s/\lambda)$, it follows that for all $\lambda \geqq 1$ there exists a constant k_λ such that $|\hat{g}_\lambda(s)| < k_\lambda \exp(-\omega_\beta(s))$ for all s.

We define the operator A_λ by

$$A_\lambda = \int_{-\delta}^{\delta} dt g_\lambda(t)u(t)A(t) \tag{5.12}$$

for all $\lambda \geqq 1$. It then follows, by Lemma 5.3, that $A_\lambda D_H(\alpha) \subset D_H(\alpha)$ and $A_\lambda^* D_H(\alpha) \subset D_H(\alpha)$.

2) In view of the construction in (5.12) it follows from (5.11) that

$$\langle A_\lambda^* \phi' | \bar{X} \phi'' \rangle = \langle \bar{X} \phi' | A_\lambda \phi'' \rangle, \quad \text{all} \quad \phi', \phi'' \in D. \tag{5.13}$$

Since D was assumed to be a core for $\bar{X}$, this relation also holds for all ϕ' and ϕ'' in the domain $D(\bar{X})$ of $\bar{X}$, and since $D_H(\alpha) \subset D(\bar{X})$, the relation (5.13) in particular holds for all $\phi', \phi'' \in D_H(\alpha)$. By Lemma 5.2, $D_H(\alpha)$ is a core for $\bar{X}$, and since $A_\lambda D_H(\alpha) \subset D_H(\alpha)$ and $A_\lambda^* D_H(\alpha) \subset D_H(\alpha)$ by step 1) above, it follows that $\bar{X} A_\lambda \supset A_\lambda \bar{X}$ and $\bar{X} A_\lambda^* \supset A_\lambda^* \bar{X}$, i.e., $A_\lambda \in a(\bar{X})'$. From the construction of A_λ in (5.12) it follows trivially that A_λ tends strongly to $A(0) = A$ as λ tends to $+\infty$, and hence $A(0) \in a(\bar{X})'$. We have thus proved the conclusion in the lemma for the case $t = 0$. The general case follows by the same reasoning applied to the operator $A(t_0)$ instead of A, for any particular $t_0 \in (-\delta, \delta)$. □

We are now prepared for the main results of this section, which we present in the form of two theorems; we prove these together.

Theorem 5.5. *Let $\varphi(x)$ be a local, irreducible, hermitian scalar field, which satisfies a generalized H-bound of order α, with $1 > \alpha \geqq 0$. Let K_s be a double cone, and let $f_s(x)$ be a real test function, with* $\mathrm{supp}(f_s) \subset K_s$, *and with a Fourier transform which vanishes nowhere. We write $X_s \equiv \varphi[f_s] = X_s^\dagger$. Let D be a core for $\bar{X}_s$, and let $a(\bar{X}_s)$ be the von Neumann algebra generated by $\bar{X}_s$.*

Let $\mathscr{C}$ be a selfadjoint set of bounded operators, such that the set $\mathscr{G} = \cup \{U(\lambda) \mathscr{C} U(\lambda)^{-1} | \lambda \in P\}$ is irreducible. Suppose, furthermore, that the elements of $\mathscr{C}$ satisfy the following weak condition of relative locality with respect to X_s: For each $A \in \mathscr{C}$ there exists a double cone $K(A)$ such that $U(\lambda) A U(\lambda)^{-1}$ commutes weakly with $\bar{X}_s$ on D, for all $\lambda \in P$ such that $K(A)_\lambda$ is spacelike relative to K_s. Then:

a) *The AB-system $\{\mathscr{A}(W), \mathscr{B}(K), \mathscr{A}(K^c)\}$ generated by $a(\bar{X}_s)$, through the definition*

$$\mathscr{A}(W) = \{U(\lambda) a(\bar{X}_s) U(\lambda)^{-1} | \lambda \in P,\ K_{s,\lambda} \subset W\}'', \tag{5.14}$$

is local and satisfies the special condition of duality, and Scenario G obtains for this AB-system and the field.

b) *Let $K \in \mathscr{K}$, and let f be a test function with* $\mathrm{supp}(f) \subset K$. *Let $X = \varphi[f]$. Then $\bar{X}$ is affiliated with $\mathscr{B}(K)$. If f is also real, and has a Fourier transform which vanishes nowhere, then X is intrinsically local, in the sense of Definition 4.1.*

c) *Every intrinsically local hermitian operator in $\mathscr{P}_0(\mathscr{M})$ generates the AB-system defined in* a) *above, in the sense of Theorem 4.6.*

d) *For each $A \in \mathscr{C}$, $A \in \mathscr{B}(K(A))$.*

Theorem 5.6. *Let $\varphi(x)$ be local, irreducible, hermitian scalar field, which satisfies a generalized H-bound of order α, with $1 > \alpha \geqq 0$.*

a) *Suppose that Scenario A in Definition 2.4 obtains for an AB-system $\{\mathscr{A}(W), \mathscr{B}(K), \mathscr{A}(K^c)\}$, and a sub-algebra $\mathscr{P}_{0r}(\mathscr{M})$ of $\mathscr{P}_0(\mathscr{M})$, and suppose furthermore that $\mathscr{P}_{0r}(K_s)$, for some $K_s \in \mathscr{K}$, contains an operator $X_s \equiv \varphi[f_s]$, where f_s is a real test function with* $\mathrm{supp}(f_s) \subset K_s$, *and with a Fourier transform which vanishes* *nowhere. Then X_s is intrinsically local, and Scenario G obtains, and all the conclusions in* a)–c) *in Theorem 5.5 apply for this AB-system.*

b) *Suppose, instead, that for some $K_r \in \mathscr{K}$ there exists a hermitian intrinsically local operator $X_r \in \mathscr{P}_0(K_r)$, not necessarily linear in the field. Then Scenario G obtains for the AB-system generated by X_r in the sense of Theorem 4.6, and all the conclusions in* b) *and* c) *of Theorem 5.5 apply to this AB-system.*

Proof. 1) We first prove Theorem 5.5. Let $\mathscr{G}_w$ be defined as the set $\{U(\lambda)AU(\lambda)^{-1}|\lambda \in P,\ A \in \mathscr{C},\ K(A)_\lambda \subset K_s^c\}$. Since $K(A)_\lambda$ is closed and K_s^c is open, it follows that for any $B \in \mathscr{G}_w$ there exists a $\delta > 0$ such that $\exp(itH)B\exp(-itH) \in \mathscr{G}_w$ for all $t \in (-\delta, \delta)$, and hence these latter operators commute weakly with $\bar{X}_s$ on D, by the premises of the theorem. Since D is a core for $\bar{X}_s$ we conclude, by Lemma 5.4, that $\mathscr{G}_w \subset a(\bar{X}_s)'$, and hence $\mathscr{G}_w'' \subset a(\bar{X}_s)'$.

2) Let the AB-system $\{\mathscr{A}_1(W), \mathscr{B}_1(K), \mathscr{A}_1(K^c)\}$ be defined through

$$\mathscr{A}_1(W) = \{U(\lambda)AU(\lambda)^{-1}|A \in \mathscr{C},\ \lambda \in P,\ K(A)_\lambda \subset W\}''. \tag{5.15}$$

By the premises of the theorem, the set $\cup\{\mathscr{B}_1(K)|K \in \mathscr{K}\}$ is irreducible. With reference to Theorem 2.8 we define $\mathscr{F}(K_s) = \{X_s\}$, and $\mathscr{F}(K) = \emptyset$ for $K \neq K_s$, and we define $\mathscr{P}_{0s}(R)$ as in that theorem. By part a) of Lemma 4.7 the linear manifold $D_{0s} = \mathscr{P}_{0s}(\mathscr{M})\Omega$ is then dense in $\mathscr{H}$. From this, and from the result in step 1 above, it follows that $\mathscr{P}_{0s}(\mathscr{M})$ and the AB-system defined through (5.14) satisfy the premises in part a) of Theorem 2.8, and we conclude that Scenario A obtains. By part b) of Lemma 4.7, and by the result in step 1 above, we have $a(\bar{X}_s) = a((X_s \upharpoonright D_{0s})^{**}) \subset \mathscr{G}_w' \subset \mathscr{A}_1(K_s^c)' = \mathscr{B}_1(K_s)$. This implies that X_s is intrinsically local, and hence the conclusions in Theorem 4.8 apply to the AB-system defined through (5.14). Since $a(\bar{X}_s) \subset \mathscr{B}_1(K_s)$ it follows that $\mathscr{A}(W) \subset \mathscr{A}_1(W)$, and from this we conclude that the AB-systems defined through (5.14) and through (5.15) are identical. This implies the assertion in d), and the assertion in c) follows from part d) of Theorem 4.8.

3) Let $K \in \mathscr{K}$, and let $W \in \mathscr{W}$, $W \supset K$. Let f be any test function with $\operatorname{supp}(f) \subset K$. We write $X = \varphi[f]$. Since K is closed and W is open it follows that there exists a $\delta > 0$ such that for all $t \in (-\delta, \delta)$, $\exp(itH)X\exp(-itH) \in \mathscr{P}_0(W)$. By part c) of Theorem 4.8, $\exp(itH)X\exp(-itH)$ commutes weakly on D_1 with $\mathscr{A}(\bar{W}^c)$ for all $t \in (-\delta, \delta)$, or, what amounts to the same, if $A \in \mathscr{A}(\bar{W}^c)$, then $\exp(itH)A\exp(-itH)$ commutes weakly with X on D_1, for all $t \in (-\delta, \delta)$. For any $A \in \mathscr{A}(\bar{W}^c)$, and any $\lambda \geqq 1$, we define A_λ by the integral in (5.12), as in the proof of Lemma 5.4, and we then have $X^{\dagger *}A_\lambda \supset A_\lambda X$, and hence $X^{\dagger *}A_\lambda \supset A_\lambda \bar{X}$. By Lemma 5.2 we have $D_H(\alpha) \subset D(\bar{X})$, where $D(\bar{X})$ is the domain of $\bar{X}$, and by Lemma 5.3 we have $A_\lambda D_H(\alpha) \subset D_H(\alpha)$. Hence $\bar{X}A_\lambda = A_\lambda \bar{X}$ on $D_H(\alpha)$, and since $D_H(\alpha)$ is a core for $\bar{X}$, by Lemma 5.2, it follows that $\bar{X}A_\lambda \supset A_\lambda \bar{X}$. In a similar fashion we conclude that $\bar{X}A_\lambda^* \supset A_\lambda^* \bar{X}$, and hence $A_\lambda \in a(\bar{X})'$. As λ tends to $+\infty$ the operator A_λ converges strongly to A, and hence $A \in a(\bar{X})'$, which means that $\mathscr{A}(\bar{W}^c) \subset a(\bar{X})'$. Hence $a(\bar{X}) \subset \mathscr{A}(\bar{W}^c)' = \mathscr{A}(W)$, and since this holds for all $W \supset K$, it follows that $a(\bar{X}) \subset \mathscr{B}(K)$.

From the above result it readily follows that X is intrinsically local if f is real and such that its Fourier transform vanishes nowhere. Furthermore we conclude that Scenario G obtains for the AB-system and the field.

4) This concludes the proof of Theorem 5.5, and we now consider Theorem 5.6. We first assume the premises in part a). From the description of Scenario A in Definition 2.4 it follows that the operator X_s commutes weakly on D_{0r} with $\mathscr{A}(K_s^c)$. By part b) of Lemma 5.2 we have $\bar{X}_s=(X_s \upharpoonright D_{0r})^{**}$, and it follows that X_s commutes weakly with $\mathscr{A}(K_s^c)$ on D_1. We can then continue the reasoning as in step 3 above, and the conclusion follows readily.

5) We consider part b) of Theorem 5.6. By Theorem 4.6 the intrinsically local operator X_r generates an AB-system $\{\mathscr{A}(W), \mathscr{B}(K), \mathscr{A}(K^c)\}$, and a polynomial algebra $\mathscr{P}_{0r}(\mathscr{M})$, for which Scenario A obtains. Let $K \in \mathscr{K}$, and let f be a real test function with $\mathrm{supp}(f) \subset K$. Let $W \in \mathscr{W}$, $W \supset K$. By part f) of Theorem 4.6 the operator $X \equiv \varphi[f] = X^\dagger$ commutes weakly on D_{0r} with $\mathscr{A}(\bar{W}^c)$. By Lemma 5.2 we have $\bar{X}=(X \upharpoonright D_{0r})^{**}$, and hence X commutes weakly on D_1 with $\mathscr{A}(\bar{W}^c)$. The reasoning in step 3 above then applies, and the assertions in part b) of Theorem 5.6 follow. □

As we said in the Introduction, the assumption of an H-bound thus has remarkable implications for the "selfadjointness problem." By Theorem 5.6 the existence and uniqueness of a local AB-system with which the field is locally associated is assured if the field satisfies a generalized H-bound *and* if there exists at least *one* intrinsically local operator. It then follows that there also exists a multitude of intrinsically local operators which are *linear* in the field. What is perhaps more remarkable is that, according to part a) of Theorem 5.6, the requirement of intrinsic locality can be omitted from the premises if X_s is *linear* in the field. This provides potential "tests" for whether the field is locally associated with a net of local von Neumann algebras or not: we can select *any* single real test function f of compact support with a Fourier transform which vanishes nowhere, and then "check" whether $\varphi[f]$ is intrinsically local or not. We regard this as a very substantial reduction of the "selfadjointness problem."

We want to say a few words here about generalizations. First of all our regularity condition could be relaxed in various ways. We might thus assume that the conditions a) and b) in Definition 5.1 hold only for *some* test function $f(x)$. Our reasoning then applies to the corresponding operators $\varphi[f]$, and it is clear that we arrive at conclusions similar to (although possibly somewhat weaker than) the conclusions in Theorems 5.5 and 5.6.

Secondly we note that the "bounding operators" $\exp(-\omega_\alpha)$ could have been chosen differently, i.e., the functions $\omega_\alpha(s)$ in (5.1) can be replaced by functions in a somewhat larger class without upsetting the conclusions in the crucial Lemmas 5.2 and 5.4. For the latter lemma it is required that there exist functions of *compact* support with Fourier transforms which are asymptotically similar to the bounding functions $\exp(-\omega(s))$. A function $\omega(s)$ which increases linearly with s is not acceptable, but the arrangements leading to Theorem 5.6 *could* be carried through for functions with an asymptotic behavior such as $s/(\ln(s))^2$. The more general permissible bounding functions are to be found in the class of functions discussed by Jaffe [19] in his generalization of the notion of local fields. Since the handling of the more general functions is mildly complicated we felt that our results were best presented within the framework of the simple conditions in Definition 5.1.

6. About Local Nets Associated with Borchers Classes of Fields

The considerations in the preceding sections can readily be generalized to the case of a theory of an arbitrary number of finite-component Bose-fields. A particular case of this is a set of fields in the Borchers class [4] of a single irreducible field $\varphi(x)$. The following question then arises. Suppose that $\varphi(x)$ is related to a local AB-system in such a way that Scenario A (or G) obtains. Is every field $\psi(x)$ in the Borchers class of $\varphi(x)$ then also so related to the *same* AB-system? At this level of generality we have no answer to the question, but as we shall see, a rather satisfactory answer can be given in the presence of a generalized H-bound. We remark here that the question makes good sense only for local nets which are AB-systems. It is easy to construct examples [22] (involving generalized free fields) in which one field is locally associated with a local net, but such that some other field in its Borchers class is not locally associated with the same net.

We shall now consider the situation in which at least *one* field in a Borchers class satisfies a generalized H-bound, as in Definition 5.1. For reasons of simplicity we shall actually consider only the case of two fields, $\varphi(x)$ and $\psi(x)$, both defined on a common dense domain D_1, as described in [28]. The remarkable circumstances which we wish to discuss are already manifest in this simplest special case. We next state and prove a theorem on this issue.

Theorem 6.1. *Let $\varphi(x)$ and $\psi(x)$ be two irreducible hermitian scalar fields, local and relatively local. It is assumed that $\psi(x)$ satisfies a generalized H-bound of order α, with $1>\alpha\geqq 0$. For any $R\subset\mathscr{M}$, $\mathscr{P}_{0\varphi}(R)$ denotes the polynomial algebra generated by all $\varphi[f]$ with $\operatorname{supp}(f)\subset R$, and $D_{0\varphi}=\mathscr{P}_{0\varphi}(\mathscr{M})\Omega$ and $D_{1\varphi}$ denote the standard domains constructed from the field $\varphi(x)$ alone. Similarly the objects $\mathscr{P}_{0\psi}(R)$, $D_{0\psi}$ and $D_{1\psi}$ refer to the field $\psi(x)$ alone. Finally it is assumed that there exists an intrinsically local hermitian operator $X_s\in\mathscr{P}_{0\varphi}(K_s)$, for some $K_s\in\mathscr{K}$, which generates the AB-system $\{\mathscr{A}(W),\mathscr{B}(K),\mathscr{A}(K^c)\}$ (in the sense of Theorem 4.6), and hence Scenario A obtains for this AB-system and the polynomial algebra $\mathscr{P}_{0s}(\mathscr{M})$ generated by X_s. Then:*

a) *For any test function f,*

$$\overline{\psi[f]}\equiv(\psi[f]\restriction D_1)^{**}=(\overline{\psi[f]}\restriction D_H(\alpha))^{**}=(\psi[f]\restriction D_{0\psi})^{**}=(\psi[f]\restriction D_{0\varphi})^{**}, \tag{6.1}$$

where the domain $D_H(\alpha)$ is defined as in Lemma 5.2.

b) *Scenario G in Definition 2.4 obtains for the AB-system and the field $\psi(x)$, and hence any intrinsically local (hermitian) operator in $\mathscr{P}_{0\psi}(\mathscr{M})$ generates this AB-system. In particular, if $K\in\mathscr{K}$ and if f is any real test function with support in K and with a Fourier transform which vanishes nowhere, then the hermitian operator $\psi[f]$ is intrinsically local, and $(\psi[f]\restriction D_{0\psi})^{**}$ is affiliated with $\mathscr{B}(K)$.*

c) *The AB-system is unique in the sense that it is equal to any other AB-system which satisfies the same premises, but with respect to some other intrinsically local operator $X_r\in\mathscr{P}_{0\varphi}(\mathscr{M})$.*

Proof. 1) We first note that it makes no difference whether the condition that $\psi(x)$ satisfies a generalized H-bound refers to the closure $\overline{\psi[f]}$ (relative to D_1) or to

$(\psi[f] \upharpoonright D_{1\psi})^{**}$. In the first case it follows, by the reasoning in the proof of Lemma 5.2, that $\overline{\psi[f]} = (\psi[f] \upharpoonright D_{1\psi})^{**} = (\overline{\psi[f]} \upharpoonright D_H(\alpha))^{**}$, since $D_{1\psi}$ is a dense, translation-invariant sub-manifold of D_1. In the second case we note that since $\overline{\psi[f]} \supset (\psi[f] \upharpoonright D_{1\psi})^{**}$, the field in fact satisfies the generalized H-bound condition with reference to the domain D_1. The relations (6.1) follow trivially from Lemma 5.2.

2) Let $W \in \mathscr{W}$. Since $\varphi(x)$ commutes with $\psi(y)$ on D_1 for $x-y$ spacelike, we conclude, by reasoning similar to the reasoning in the proof of Proposition 4.4, that $\mathscr{P}_{0\psi}(W)$ commutes weakly on D_{0s} with $\mathscr{A}(\bar{W}^c)$. Let $K \in \mathscr{K}$, $K \subset W$, and let f be a real test function with $\mathrm{supp}(f) \subset K$ and with a Fourier transform which vanishes nowhere. We write $X \equiv \psi[f] = X^\dagger$. Since $X \in \mathscr{P}_{0\psi}(W)$, it commutes weakly on D_{0s} with $\mathscr{A}(\bar{W}^c)$. Since D_{0s} is dense and translation-invariant, it is, by Lemma 5.2, a core for $\bar{X}$. Hence $\bar{X}$ commutes weakly with $\mathscr{A}(\bar{W}^c)$ on $D_{1\psi} \subset D(\bar{X})$. The reasoning in step 3 in the proof of Theorem 5.5 now applies, and we conclude that X is intrinsically local, and that X generates the given AB-system. The remaining assertions in the theorem now follow readily from Theorem 5.5 applied to the field $\psi(x)$. □

The above result suggests that a field $\varphi(x)$ which is in the same Borchers class as a field $\psi(x)$ which satisfies a generalized H-bound is better behaved than a field in general. In this context the following (open) questions present themselves.

a) Could it be the case that *every* local field is in a Borchers class with *some* field which satisfies a generalized H-bound?

b) Could it happen that a Borchers class which contains a field which satisfies a generalized H-bound also contains a field which does not?

We have no basis for any conjectures concerning the above. An affirmative answer to question a) would, of course, be rather pleasing since the analysis could then be shifted, so to say, to the field which satisfies the H-bound. A negative answer to question b) would somewhat reduce the significance of Theorem 6.1, although we still have the result that both fields (which now both satisfy generalized H-bounds) do generate the same unique local AB-system if they generate any local net at all. If the answer to question b) is in the affirmative, one may hope that no field of interest in physics can be *so* bad that it is not in a Borchers class with *some* field which satisfies a generalized H-bound. Theorem 6.1 is then of obvious interest.

The theorem has an obvious application to the case when $\varphi(x)$ is a free scalar field for a particle of mass m, and $\psi(x)$ is any (irreducible) field in the Borchers class of $\varphi(x)$. In this case it is known [12] that the Borchers class is the set of all Wick polynomials in $\varphi(x)$ and its derivatives. It is also known [18] that the fields in this class satisfy generalized H-bounds of order 0. It is well-known that $\varphi(x)$ generates a unique local AB-system (via the Weyl group elements $\exp(i\overline{\varphi[f]})$, with f real). The field averaged with a real test function is, in fact, essentially selfadjoint on its domain $D_{0\varphi}$. It was shown by Langerholc and Schroer [23] that a Wick polynomial $\psi(x)$ is irreducible if it contains a term of odd order. Suppose that such a Wick polynomial $\psi(x)$ is a scalar field. It now follows from Theorem 6.1 that the closure $(\psi[f] \upharpoonright D_{0\psi})^{**}$, where the field $\psi(x)$ is a field in its own right, regarded as defined on *its* canonical domain $D_{1\psi}$ (or $D_{0\psi}$), is affiliated with $\mathscr{B}(K)$ if $\mathrm{supp}(f) \subset K$.

If the Fourier transform of f vanishes nowhere, and if f is real, then the operator $(\psi[f] \upharpoonright D_{0\psi})^{**}$ generates the unique AB-system associated with $\varphi(x)$. If f satisfies the further condition that $f(x)=f(-x)$, we have $\Theta_0\psi[f]\Theta_0^{-1}=\psi[f]$, and it then follows, as we remarked at the end of Sect. 3, that $\psi[f]$ has a selfadjoint extension affiliated with $\mathscr{B}(K_0)$, for any K_0 which contains K in its *interior*. We do not know whether such an extension is actually equal to $(\psi[f] \upharpoonright D_{0\psi})^{**}$. Irrespective of the answer to this question we conclude that a Wick polynomial (of odd order) of a free field, regarded as a field in its own right, does generate, all by itself, a unique local AB-system, and in a rather trivial fashion, as described above.

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