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Surface Effects in Debye Screening*

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Abstract. A thermodynamic system of equally charged, plus and minus, classical particles constrained to move in a (spherical) ball is studied in a region of parameters in which Debye screening takes place. The activities of the two charge species are not taken as necessarily equal. We must deal with two physically interesting surface effects, the formation of a surface charge layer, and long range forces reaching around the outside of the spherical volume. This is an example in as much as 1) general charge species are not considered, 2) the volume is taken as a ball, 3) a simple choice for the short range forces (necessary for stability) is taken. We feel the present system is general enough to exhibit all the interesting physical phenomena, and that the methods used are capable of extension to much more general systems. The techniques herein involve use of the sine-Gordon transformation to get a continuum field problem which in turn is studied via a multi-phase cluster expansion. This route follows other recent rigorous treatments of Debye screening.

0. Introduction

The rigorous study of Debye screening was initiated in [4] by Brydges, with the treatment of a charge symmetric lattice Coulomb gas. This work was greatly generalized by Brydges and Federbush [7]. Their proof applies to continuum Coulomb systems with essentially arbitrary short range forces, and charge symmetry is not required. Imbrie [12] improved the convergence estimates of [7] and removed a restriction on the relative sizes of the activities. He also proved Debye screening in Jellium.

All of these treatments of Debye screening impose two important constraints on the system. First, there is a constraint on the activities z_i and charges e_i which is usually referred to as a "neutrality" condition. This condition may be viewed as essentially saying that $\sum_i z_i e_i = 0$. Second, Dirichlet boundary conditions for the

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Coulomb interaction are used. Physically, this means that the walls of the container are conducting. The significance of these constraints is that they minimize surface effects at the boundary of the container.

This paper is devoted to the study of a Coulomb system in which these two constraints are not imposed. Our system has two species with charges $\pm e$ and activities z_+ and z_-. The "neutrality" condition would require $z_+ = z_-$; we do not enforce this. Free boundary conditions are used for the Coulomb interaction. This corresponds to a container with insulating walls. We modify the Coulomb potential at short distances in a way which passes easily through the Sine-Gordon transformation [see Eq. (1.2)]. To facilitate computing covariances we take our finite volumes (containers) to be spherical with radius R.

The most interesting results we obtain for this system are for a finite volume. The charge density $J(x)$ is non-zero near the surface of the volume (see Theorems 1.1 and 1.2). This surface charge is a result of the unequal activities z_+ and z_-. Near the surface of the volume Debye screening breaks down. The correlation function between two charges near the surface decays as $1/r^3$ rather than exponentially (see Theorem 1.6). In the infinite volume limit these surface effects disappear, and there is Debye screening. Moreover, the infinite volume correlation functions of this system are equal to those of the system whose activities are both equal to $(z_+z_-)^{1/2}$ (see Theorem 1.5).

We will give two explanations of these surface effects. The first explanation uses the mean field treatment of Debye and Hückel. The second will use the Sine-Gordon transformation and serve as an introduction to our proofs. The Debye-Hückel equation for the mean field potential $\psi(x)$ is

$$\Delta\psi = (-z_+ e^{-\beta\psi} + z_- e^{\beta\psi})\chi, \tag{0.1}$$

where χ is the characteristic function of the volume (which we call Λ). In the infinite volume limit ($\chi = 1$), the solution is the constant potential $\psi_0 = (2\beta)^{-1}\ln(z_+/z_-)$.

The solution of this equation for a finite volume, the "instanton," is studied in Appendix A. Contributions of J. Rauch to this study are gratefully acknowledged. It is shown that $\psi(x)$ approaches the constant ψ_0 well inside the volume. Since the charge density is $-\Delta\psi$, the charge density is essentially zero away from the surface of the volume. Near the surface one finds that there is a charge per unit area of $(2R\beta)^{-1}\ln(z_+/z_-)$. It is easy to check that such a charge distribution yields the potential ψ_0 inside the ball.

With free boundary conditions the screening breaks down near the surface even if $z_+ = z_-$. So we will restrict our explanation of the $1/r^3$ decay to the simpler case of equal activities. The simplest explanation is that the wall of the container interferes with a test charge's attempt to surround itself with a screening cloud. The clouds of two test charges near the wall will have non-zero dipole moments. This dipole-dipole interaction produces the $1/r^3$ decay.

To see this $1/r^3$ decay in the Debye-Hückel theory, we start with the Debye-Hückel equation

$$-\Delta\psi(x) + 2z\chi(x)\sinh\beta\psi(x) = \delta(x-y).$$

We have included a test charge at y. If we linearize this equation by replacing $\sinh\beta\psi(x)$ by $\beta\psi(x)$, then $\psi(x) = (-\Delta + l_D^{-2}\chi)^{-1}(x,y)$. With Dirichlet boundary conditions this covariance has exponential decay even if x and y are near the surface. This covariance with free boundary conditions is studied in Appendix B. It only decays as $1/r^3$ along the surface. (For technical reasons we only prove a $1/r^{3-\varepsilon}$ decay.) Jancovici [13–16] studied a system with an infinite insulating plane wall using Debye-Hückel theory. In this approximation he found that the correlations along the wall decay as $1/r^3$.

The Sine-Gordon transformation expresses the partition function as

$$Z = \int d\mu(\phi) \exp\left\{\int_{\Lambda} dx[z_+ e^{+i\sqrt{\beta}\phi(x)} + z_- e^{-i\sqrt{\beta}\phi(x)}]\right\}.$$

$\phi(x)$ is a free (Gaussian) field whose covariance is essentially $(-\Delta)^{-1}(x,y)$. The correlation functions are given by expectations of products of the observables $z_\pm e^{\pm i\sqrt{\beta}\phi(x)}$. One is tempted to argue as follows. Translate $\phi(x)$ by the purely imaginary constant $i\beta^{-1/2}c$. Since $\Delta i\beta^{-1/2}c=0$, $d\mu(\phi)$ would be unchanged, while $z_\pm$ would change into $z_\pm e^{\mp c}$. So there would be a one-parameter family of activities which yield the same physical system. In particular, taking $c=\frac{1}{2}\ln(z_+/z_-)$, the two activities would both be equal to $(z_+z_-)^{1/2}$. For technical reasons one cannot translate $\phi(x)$ by a constant. However, in the infinite volume limit the above conclusions are correct. Lieb and Lebowitz proved the invariance of thermodynamic quantities like the pressure and densities under the transformation $z_i \to z_i e^{e_i c}$ [17]. For our model we show that the infinite volume correlation functions are invariant as well.

The objection to translating $\phi(x)$ by a constant is not merely a technical point. In a finite volume the system is not invariant under $z_\pm \to z_\pm e^{\mp c}$. The correct approach is to translate $\phi(x)$ to the stationary point of the functional integral. An easy computation reveals that this stationary point is $i\sqrt{\beta}\psi(x)$, where $\psi(x)$ is the instanton defined in Eq. (0.1). Since $\psi(x) \approx \psi_0$ well inside the ball, this translation essentially replaces z_+ and z_- by $(z_+z_-)^{1/2}$ well inside the ball. The translation introduces the term $\exp[i\sqrt{\beta}\int dx\,\phi(x)\Delta\psi(x)]$. This term shows the presence of a surface charge distribution. (Recall that under the Sine-Gordon transformation an external charge distribution $q(x)$ becomes a factor of $\exp[i\sqrt{\beta}\int dx\,\phi(x)q(x)]$.)

The explanation in the language of the Sine-Gordon transformation of the breakdown of screening near the boundary is similar to the first explanation. Debye screening occurs because the quadratic part of the $\cos\sqrt{\beta}\phi(x)$ terms acts like a mass for the field $\phi(x)$. Thus the inverse covariance $-\Delta$ becomes $-\Delta + l_D^{-2}\chi$. With Dirichlet boundary conditions the absence of this mass outside of the volume is irrelevant. With free boundary conditions the covariance $(-\Delta + l_D^{-2}\chi)^{-1}$ feels the absence of this mass, and so doesn't have exponential decay everywhere.

As in the previous rigorous studies of Debye screening we analyze the functional integrals by a multi-phase Glimm-Jaffe-Spencer cluster expansion [10]. This expansion is complicated by the fact that the cosine interaction has infinitely many minima. The standard approach, which we follow, is to introduce a function $h(x)$ which is constant on cubes and only takes on the values $2\pi\beta^{-1/2}n$, where n is an integer. $h(x)$ labels which minima $\phi(x)$ lies near, $h(x)$ is only defined inside of Λ; one should think of $h(x)$ as being zero outside of Λ.

The sum over the h's is controlled by a small factor e^{-E}. For each face between two cubes inside of Λ, there is a contribution to E of the order of $(\delta h)^2$, where δh is the change in h across the face. So e^{-E} would control the sum over h except for the translations $h(x) \rightarrow h(x) + 2\pi\beta^{-1/2}n$. With Dirichlet boundary conditions E also contains contributions of order $(\delta h)^2$ for the faces on the boundary of Λ. So these translations present no problem.

With free boundary conditions the situation is drastically different. Let h_0^n be the function which equals $2\pi\beta^{-1/2}n$ everywhere. Then the energy E of h_0^n is only of order $\beta^{-1}n^2R$ instead of $\beta^{-1}n^2R^2$. As a result of this the approach of [7] will not work.

We use the notion of a "sector" to handle the problem. To each h we assign an integer n, and say that h belongs to the n^{th} sector. For intuitive purposes one may define the sector of h to be the closest integer to the average of $\beta^{1/2}h/2\pi$ over the boundary of Λ. The partition function is a sum over all the h's. We split this sum up into a sum over each sector. Thus $Z = \sum_{n=-\infty}^{\infty} Z^{(n)}$.

Rather than doing a cluster expansion for Z, we do an expansion for each $Z^{(n)}$. Then we show that $Z^{(n)}/Z^{(0)}$ is bounded by $e^{-\varepsilon\beta^{-1}n^2R}$. So only the zero sector survives in the infinite volume limit. In a finite volume the contribution of the nonzero sectors to the correlation functions will be of order $e^{-\varepsilon\beta^{-1}n^2R}$. For small β this is much smaller than the zero sector contribution.

The correspondence $h \leftrightarrow h + h_0^n$ gives a one-to-one correspondence between the terms in $Z^{(0)}$ and $Z^{(n)}$. The corresponding terms differ in two important ways. First, the energy $E(h)$ is different from the energy $E(h+h_0^n)$. This difference is helpful since $E(h+h_0^n)$ is greater than $E(h)$ by an amount of order $\beta^{-1}n^2R$. So this provides the small factor of $e^{-\varepsilon\beta^{-1}n^2R}$. The second difference is that the integrand in the functional integrals for corresponding terms in $Z^{(n)}$ and $Z^{(0)}$ differ near the boundary. One would expect a contribution to $Z^{(n)}/Z^{(0)}$ of order e^{cR^2} from this. However, because the surface charge is only of order $1/R$, the difference in the functional integrands is only of order $1/R$. So the contribution to $Z^{(n)}/Z^{(0)}$ can be bounded by e^{cR}. $Z^{(n)}$ (and $Z^{(n)}/Z^{(0)}$) are studied by means of a polymer-type cluster expansion.

The use of free boundary conditions introduces another technical problem that must be handled differently from the treatment in [7]. In the cluster expansion, factors of $N!$ at each cube arise for various reasons. In [7] the exponential decay of the covariance is used to beat these factorials. We cannot use this "exponential pinning" since our covariance is only slightly better than integrable. The work of Battle and Federbush [2] provides an extra factor of $1/N!$ at each cube (see also [1, 3, 5, 8, 20]). Thus we can tolerate an $N!$ at each cube. This improvement is essential for our expansion. In [7] the convergence estimates actually contained $(N!)^p$ at each cube with p fairly large. By doing these estimates more carefully we obtain $p = 1 + \varepsilon$ with ε small. The $(N!)^\varepsilon$ can be overcome by a "power law pinning" since our covariance is slightly better than integrable.

Some familiarity with [7] is assumed. In particular we recommend reading Sects. 1 through 8 of [7], excluding details of the infinite volume limit (in Sect. 1), and the Mayer Series (Sect. 3). References to a few other sections of [7] are made, but these may be treated as isolated references to any other source. The present cluster expansion is rather different from that in [7], so one may well restrict ones attention to the sections mentioned above.

In the future one may want to treat the most general problem in classical Debye screening involving melding the techniques of [7, 12], and this paper. At present this seems possibly doable, but complicated much beyond its value.

There has been interesting work studying screening in an axiomatic setting (see [11] for example). Recent work in the physics literature studies surface charge and effective potentials near a plane surface [13–16, 19].

In addition to organizational details mentioned above, we wish to outline the paper's development as follows. The basic results are stated in Sect. 1. Section 2 displays the Peierls expansion. Section 3, and Appendix E are concerned with organizing regions of space that for a given term in the Peierls expansions are treated as units in the cluster expansion. Section 5 describes the interpolation procedure, and Sect. 6 the polymer expansion. Sections 9 and 10 present energy estimates, interesting geometrical analyses of the division of Coulomb-like energy over units in the cluster expansion. Sections 7, 8, 11, 12 handle the combinatoric aspects of the cluster expansion, as well as certain estimates of functional integrals. Appendix D studies some theorems of use to us, that fall in the domain of geometric measure theory.

1. Basics

We have two charges, $+1$ and -1, with (bare) activities z_+ and z_-. These need not be equal; this will mean we are not imposing a neutrality condition (for our system the imposed neutrality condition (3.8) of [7] would be $z_+ = z_-$). We let the charge density J be

$$J = \sum_{i=\pm} e_i \sigma_i , \tag{1.1}$$

where $e_i = \pm 1$ and σ_i is a sum of delta functions at the position of particles of species i. (The notation of [7] is a basic guide for us.) We let

$$u = \left(\frac{1}{-\Delta} - \frac{1}{-\Delta + \frac{1}{\lambda^2 l_D^2}} \right) . \tag{1.2}$$

λ will be a fixed small parameter, l_D will be specified, and the natural infinite volume Green's functions are always understood. We set

$$\tilde{z}_\pm = z_\pm e^{1/2\beta u(x,x)} , \tag{1.3}$$

$$z^2 = z_+ z_- , \tag{1.4}$$

$$\tilde{z}^2 = \tilde{z}_+ \tilde{z}_- , \tag{1.5}$$

$$l_D^2 = (2z\beta)^{-1} , \tag{1.6}$$

$$\tilde{l}_D^2 = (2\tilde{z}\beta)^{-1} , \tag{1.7}$$

$$U = \tfrac{1}{2} \int JuJ . \tag{1.8}$$

The particles are constrained to move inside a ball of radius R. We let Λ denote this volume

$$\Lambda = \{x \in \mathbb{R}^3, |x| \leqq R\} . \tag{1.9}$$

We let

$$I(A) = \sum \frac{\tilde{z}_+^{N_+} \tilde{z}_-^{N_-}}{N_+ ! N_- !} \int_{\Lambda^{(N_+ + N_-)}} e^{-\beta U} A, \tag{1.10}$$

$$\langle A \rangle = I(A)/I(1). \tag{1.11}$$

Z_0 is $I(1)$ with $U \equiv 0$, and $\tilde{z}_+$ and $\tilde{z}_-$ replaced by $\tilde{z}$.

$$Z = I(1)/Z_0. \tag{1.12}$$

The dependence of quantities on R is here suppressed in the notation. We construct a Gaussian measure $d\mu_0(\phi)$ on a measure space of continuous functions, $\phi(x)$, $x \in \mathbb{R}^3$, with covariance $u(x, y)$. One then has

$$\int d\mu_0(\phi) e^{i \int f \phi} = e^{-1/2 \int f u f}, \tag{1.13}$$

$$Z = \int d\mu_0 Z(\phi), \tag{1.14}$$

where

$$Z(\phi) = e^{\int \chi (\tilde{z}_+ \varepsilon_+ + \tilde{z}_- \varepsilon_-)}, \tag{1.15}$$

where χ is the characteristic function of Λ and

$$\varepsilon_\pm = \left(e^{\pm i \beta^{1/2} \phi} - \frac{\tilde{z}}{\tilde{z}_\pm} \right). \tag{1.16}$$

We write (1.14) formally in the familiar form

$$Z \sim \int d\phi e^{-1/2 \int (\phi u^{-1} \phi) + \int \chi (\tilde{z}_+ \varepsilon_+ + \tilde{z}_- \varepsilon_-)}, \tag{1.17}$$

$$= \int d\phi e^{-S}. \tag{1.18}$$

We find as a stationary point of the action S a solution of

$$u^{-1} \phi - (i \beta^{1/2} \tilde{z}_+ e^{i \beta^{1/2} \phi} - i \beta^{1/2} \tilde{z}_- e^{-i \beta^{1/2} \phi}) \chi = 0. \tag{1.19}$$

We rewrite this as

$$\left(D^4 + \frac{1}{\lambda^2 l_D^2} D^2 \right) \beta^{1/2} \psi + \frac{1}{\lambda^2 l_D^2} (\beta \tilde{z}_+ e^{\beta^{1/2} \psi} - \beta \tilde{z}_- e^{-\beta^{1/2} \psi}) \chi = 0, \tag{1.20}$$

with $D^2 = -\Delta$, and $\psi = i\phi$. We note that we will find a stationary point of purely imaginary ϕ! An important ingredient of the present procedure will be a complex translation of the integration in (1.17), so that the integration contour will pass through this stationary point.

Before we discuss (1.20) it will be convenient to detail our *restrictions on parameters*.

Parameter Manifesto. λ is a fixed small parameter. There are fixed positive constants $c_\pm$ that constrain z_+ and z_- as functions of β,

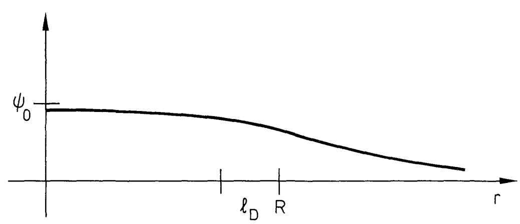
$$\begin{aligned} \beta z_+ &= c_+, \\ \beta z_- &= c_-. \end{aligned} \tag{1.21}$$

thus the only physical parameters we vary are R and β. All of our results will be proven under the condition that β is small enough. For convenience we require

$$l_D \sim 1\,. \tag{1.22}$$

We will also have mathematical parameters L and L', that must be chosen small enough and large enough respectively. Our estimates must be understood with the quantifiers: If L' is fixed large enough, if L is fixed small enough, and if λ is fixed sufficiently small, then for β sufficiently small depending on $L, L', \lambda, \ldots$. We will write this as "if PM, $\beta < \beta_0$." Notice with our conditions we may assume $\tilde{l}_D$ and l_D are arbitrarily close in value.

We will seek a solution of (1.20) that goes to zero at ∞ (it will be unique). In the interior of the sphere ψ will approach $\psi_0 = \frac{1}{2\beta^{1/2}} \ln(c_-/c_+)$ with distance from the boundary. This solution will be studied in Appendix A. Our choice of a spherical volume is mainly to simplify the study of (1.20) and not for any deep physical property of spheres, such as was used in [17]. $\psi(r)$ may be qualitatively viewed as follows



For $r > R$, ψ satisfies

$$D^2\left(D^2 + \frac{1}{\lambda^2 l_D^2}\right)\psi = 0\,, \tag{1.23}$$

and thus

$$\psi = c_1 \frac{1}{r} + c_2 \frac{1}{r} e^{-r/(\lambda l_D)}\,, \qquad r > R\,. \tag{1.24}$$

For $r < R$ we will prove in Appendix A:

Theorem 1.1. Surface Charge Estimate

$$R\beta^{1/2}|\psi(r) - \psi_0| < c e^{-\frac{(R-r)}{l_D}\cdot(1-\varepsilon)}\,, \qquad r < R\,. \tag{1.25}$$

We notice ψ approaches ψ_0 with distance from the boundary – of course for any given sphere this distance is bounded by R. $\psi - \psi_0$ "lives" within a distance from the boundary of order of magnitude l_D. This is a measure of the surface charge density that we find falls off with R, as $1/R$. The larger the sphere the less the surface charge density. Physically, the non-equality of activities z_+ and z_- leads to the development of a surface charge. The system in the deep interior, under the influence of the potential due to the surface charge together with the unequal activities, acts as a system in zero potential with equal activities z. (The potential due to the surface charge will be $-\frac{1}{\beta^{1/2}}\psi_0$.) Notice an infinitesimally thin layer of surface charge living on the boundary and with density (charge/area) of

$$\sigma_0 = -\frac{1}{\beta^{1/2}}\frac{1}{R}\psi_0 = -\frac{1}{2R\beta}\ln(c_-/c_+) \tag{1.26}$$

would yield the correct potential.

It is convenient to define

$$\tilde{\psi} = \beta^{1/2}\psi, \qquad \tilde{\psi}_0 = \beta^{1/2}\psi_0. \tag{1.27}$$

We then set

$$J_0(x) = J_0(|x|) = \chi 2\tilde{z}\sinh(\tilde{\psi} - \tilde{\psi}_0). \tag{1.28}$$

Theorem 1.2.

$$|\langle J(x)\rangle - J_0(x)| < c. \tag{1.29}$$

One should note $J(x) \sim 1/\beta$, so (1.29) has nontrivial content. Consistency between (1.29) and (1.26) leads us to expect our next result (shown in Appendix A).

Theorem 1.3.

a) *The following limit exists for each* $d > 0$.

$$f(d) = \lim_{R\to\infty} RJ_0(R-d), \tag{1.30}$$

b)

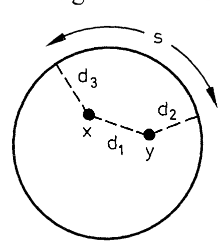
$$\int_0^\infty f(d)dd = -\frac{1}{2\beta}\ln(c_-/c_+). \tag{1.31}$$

To study Z and $I(A)$ we make the translation $\phi \to \phi - i\psi$. This is followed by a Peierls expansion, and subsequent further translations of ϕ as in [7]. These steps are pursued in Sect. 2. After the second set of translations one is naturally led to study fields with a covariance $C(x, y)$

$$C = \left(u^{-1} + \chi\frac{1}{l_D^2}\right)^{-1} \tag{1.32}$$

$$= \left(D^4 + \frac{1}{\lambda^2 l_D^2}D^2 + \chi\frac{1}{l_D^2}\frac{1}{\lambda^2 l_D^2}\right)^{-1}\frac{1}{\lambda^2 l_D^2}. \tag{1.33}$$

This is exactly what we would expect from Eq. (7.5) of [7]. We will need estimates for $C(x, y)$ with x, y in Λ. Consider a great circle cross section containing x and y.



Here $d_1 = d(x, y)$, $d_3 = R - |x|$, $d_2 = R - |y|$, and s is distance along the circle of radius R. Appendix B is devoted to proving the following estimate, with $l_D^+ = l_D/(1-\varepsilon)$.

Theorem 1.4. Covariance Estimate

$$0 < C(x,y) < c\,\mathrm{Max}\left\{e^{-d_1/l_D^+},\ e^{-(d_2+d_3)/l_D^+}\cdot\frac{1}{s^{3-\varepsilon}+1}\right\}. \tag{1.34}$$

We believe $s^{3-\varepsilon}$ can be replaced by s^3. To find intuition for the $1/s^3$ behavior, it is an easy exercise to show that $\dfrac{1}{-\Delta+\chi_{z>0}}$ has an inverse distance cube behavior in the $x-y$ plane. $\chi_{z>0}$ is the characteristic function for $\{z>0\}$. This estimate suggests the form of our main results which we now present.

We first for two unit cubes Δ_1 and Δ_2 define $\mathscr{V}_\varepsilon(\Delta_1,\Delta_2)$

$$\mathscr{V}_\varepsilon(\Delta_1,\Delta_2) = \underset{\substack{x\in\Delta_1\\ y\in\Delta_2}}{\mathrm{Max}}\ \mathrm{Max}\left\{e^{-d_1/l_D^+},\ e^{-(d_2+d_3)/l_D^+}\cdot\frac{1}{s^{3-\varepsilon}+1}\right\}, \tag{1.35}$$

where s, d_1, d_2, d_3 are as in (1.34).

Theorem 1.5. *There is a $\beta_0>0$ such that if PM, $\beta<\beta_0$ then*

$$\lim_{R\to\infty}\langle A\rangle_R = \langle A\rangle_\infty, \tag{1.36}$$

where A is an operator as in (4.4) *of* [7], *$\langle\ \rangle_R$ is the expectation as a function of R, and $\langle\ \rangle_\infty$ is the infinite volume limit as calculated in* [7] *(with $z_+=z_-=z$).*

Theorem 1.6. *Given $\varepsilon>0$, there is a $\beta_0(\varepsilon)>0$, such that if PM, $\beta<\beta_0$, and if A and B live in Δ_1 and Δ_2 respectively, then*

$$|\langle AB\rangle_R - \langle A\rangle_R\langle B\rangle_R| < c(\varepsilon)\mathscr{V}_\varepsilon(\Delta_1,\Delta_2). \tag{1.37}$$

There is one basic conjecture we did not attempt to prove in this paper. Equation (1.26) must be the total surface charge density as $R\to\infty$, not just the density to order β^{-1}. We have not decided the most precise form of statement of this conjecture; we do not know how difficult a proof would be.

2. The Peierls Expansion, Two Translations

We will use the notation

$$[A] = \frac{1}{Z_0}I(A), \tag{2.1}$$

so that

$$Z = [1], \tag{2.2}$$

and if $\mathscr{A}(\phi)$ is a functional of ϕ

$$\langle\mathscr{A}(\phi)\rangle = \frac{1}{Z}[\mathscr{A}(\phi)], \tag{2.3}$$

$$[\mathscr{A}(\phi)] = \int d\mu_0 Z(\phi)\mathscr{A}(\phi). \tag{2.4}$$

We assume $\mathscr{A}(\phi)$ is analytic and sufficiently bounded so we may translate the contour $\phi\to\phi-i\psi$. Let

$$\mathscr{A}(\phi-i\psi) = \mathscr{A}''(\phi), \tag{2.5}$$

$$[\mathscr{A}(\phi)] = \int d\mu_0(\phi)e^{+U_1}\mathscr{A}''(\phi), \tag{2.6}$$

with

$$U_1=+\tfrac{1}{2}\int \psi u^{-1}\psi+i\int \phi u^{-1}\psi+\int \chi(\tilde{z}_+e^{\tilde{\psi}}e^{i\beta^{1/2}\phi}+\tilde{z}_-e^{-\tilde{\psi}}e^{-i\beta^{1/2}\phi}-2\tilde{z}). \tag{2.7}$$

We write

$$U_1=\tfrac{1}{2}\int \psi u^{-1}\psi-\tfrac{1}{2}\int \frac{1}{\tilde{l}_D^2}\phi^2\chi+U_2. \tag{2.8}$$

Henceforth we will often (but not always) set

$$\tilde{l}_D^2=1. \tag{2.9}$$

As in Sect. 6 of [7] we now introduce functions h, constant on each cube in the lattice of side L ($\{\Omega_\alpha\}$ the set of such cubes), but defined only on cubes that have non-empty intersection with Λ. The values assumed by h are integral multiples of $\frac{2\pi}{\beta^{1/2}}$. The set of such h we call $\mathscr{H}$. As in [7] we replace (2.6) by

$$\begin{gathered}[\mathscr{A}(\phi)]=\sum_h \int d\mu_0(\phi)e^{-1/2\int\frac{1}{\tilde{l}_D^2}(\phi-h)^2\chi}e^Q e^G \mathscr{A}''(\phi), \\ Q=\tfrac{1}{2}\int \psi u^{-1}\psi+\int \chi[\tilde{z}_+e^{\tilde{\psi}}+\tilde{z}_-e^{-\tilde{\psi}}-2\tilde{z}].\end{gathered} \tag{2.10}$$

We let

$$C=\left(u^{-1}+\chi\frac{1}{\tilde{l}_D^2}\right)^{-1} \tag{2.11}$$

and define g by

$$g=C\left(\frac{1}{\tilde{l}_D^2}h\chi\right). \tag{2.12}$$

We also introduce $E(h,h')$ by

$$E(h,h')=\int_\Lambda (g-h)(g'-h')+\int_{R^3} gu^{-1}g' \tag{2.13}$$

where g' is defined as in (2.12) with h replaced by h'. We now translate ϕ, by a change of variables $\phi\to\phi+g$. We also write this as

$$\phi_0=\phi+g, \tag{2.14}$$

where ϕ_0 is the "old" field.

We write

$$\mathscr{A}''(\phi+g)=\mathscr{A}'(\phi), \tag{2.15}$$

and let N be defined so that $d\mu$ is a normalized Gaussian of covariance C, and

$$Nd\mu=d\mu_0 e^{-\frac{1}{2}\int\frac{1}{\tilde{l}_D^2}\phi^2\chi}. \tag{2.16}$$

Equation (2.10) may now be put in the form

$$[\mathscr{A}(\phi)]=\sum_h N\int d\mu(\phi)e^Q e^{-\frac{1}{2}E(h,h)}e^{G(\phi+g)}\mathscr{A}'(\phi). \tag{2.17}$$

For some purposes it might be more aesthetic and technically advantageous to have defined U_2, with $\frac{1}{2}\int \frac{1}{l_D^2}\phi^2\chi$ replaced by $\frac{1}{2}\int \frac{1}{l^2}\phi^2\chi$, l^2 a spatially dependent function chosen to ensure $\frac{\delta^2 U_2}{\delta\phi^2}=0$ at $\phi=0$. Then one would consider separately

$$C=\left(u^{-1}+\chi\frac{1}{l^2}\right)^{-1}$$

and

$$C_0=\left(u^{-1}+\chi\frac{1}{l_D^2}\right)^{-1}.$$

This is pursued only in Appendix B, and otherwise C is used to denote C_0.

A Technical Point. In fact at the edge of the sphere the cubes are truncated. We thus find it necessary, at the edge, to combine some groups of cubes of side L to form unions whose truncated volume is $\geqq L^3$, but with diameter less than cL. Each of these unions is assigned to a fixed cube at the unit (and L' scale). A distinguished L-cube in each union determines this assignment. When we refer to unit (L or L') cubes, we include without comment the cubes as distorted at the boundary. h is constant on each "union L-cube."

3. Hunk and Sector Definitions

3.1. Hunks

We begin considering a specified fixed h. We let, as in [7], $\sum$ denote the closed set along which h has a step discontinuity. *However the boundary of Λ does not give rise to discontinuities.* h is undefined outside Λ. We only consider cubes having non-zero intersection with Λ. $\sum^\wedge$ is the set of unit cubes in Λ whose distance from $\sum$ is less than L'. Each connected component of $\sum^\wedge$ is called a *hunk*. The unit cubes in $\Lambda\backslash\sum^\wedge$ are called *atoms*. A hunk is a *B-hunk* if it intersects $\partial\Lambda$, otherwise an *I-hunk*, (*I* and *B* abbreviate interior and boundary respectively).

We now enlarge (and coalesce) the hunks in certain cases where h is particularly nasty. Given a hunk $\tilde{M}$ let $\tilde{h}_{\tilde{M}}$ be the function which agrees with h in $\tilde{M}$ and is defined off of $\tilde{M}$ by the requirement that $\tilde{h}_{\tilde{M}}$ have no discontinuities outside $\tilde{M}$. A hunk $\tilde{M}$ is a *monster* if

$$\beta\sum(\delta\tilde{h}_{\tilde{M}})^2\geqq\varepsilon_1|\Lambda|. \tag{3.1}$$

ε_1 is a positive constant later specified. (The sum as in (9.117) of [7].) A B-hunk $\tilde{M}$ is a *Jumbo* if for every subset S of $\partial\Lambda$ with $\tilde{h}_{\tilde{M}}$ constant on S, the area of S is $\leqq\frac{1}{2}|\partial\Lambda|$. Otherwise a B-hunk is *normal*.

We proceed to detail the enlargement process.

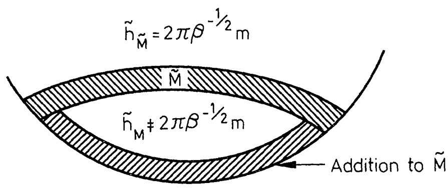
If $\tilde{M}$ is a B-hunk that is normal, then there is an integer m with

$$\tilde{h}_{\tilde{M}}=2\pi\beta^{-1/2}m$$

on more than one half of $\partial\Lambda$. We set

$$S^c = \{x \in \partial\Lambda : \tilde{h}_{\tilde{M}}(x) \neq 2\pi\beta^{-1/2}m\}$$

and add to $\tilde{M}$ the cubes in Λ within distance L' of S^c.



If $\tilde{M}$ is a jumbo, then we add to $\tilde{M}$ the cubes in Λ within distance L' of $\partial\Lambda$. Note that in this case there will be only one B-hunk at the end of the enlargement process.

If $\tilde{M}$ is a monster, then we add all of Λ to $\tilde{M}$.

In the above process of enlargement of hunks, two or more hunks may overlap, and then we coalesce them into a single new hunk. For example if there is a monster, at the end of the process there is only one hunk. In Subsect. 3.4 we will further enlarge the hunks. For now we work with the hunks after this first enlargement process. There is still a natural nomenclature of jumbo, monster, I-hunk, and B-hunk for those enlarged hunks.

3.2. Sectors

The assignment of a sector to h, is the assignment to h of an integer, n. We make this assignment distinguishing three cases.

I) We assume there is a monster among the hunks. We consider the average

$$\frac{\int_\Lambda \frac{\beta^{1/2}h}{2\pi}[\chi - C(\chi)]}{\int_\Lambda [\chi - C(\chi)]} \tag{3.2}$$

and let n be the integer closest to this average (the smallest in case of a tie).

II) We assume there is no monster among the hunks, but that there is a jumbo. We consider the average

$$\frac{1}{|\partial\Lambda|}\int_{\partial\Lambda} \frac{\beta^{1/2}}{2\pi}h \tag{3.3}$$

and let n be chosen as the closest integer to the average (the smallest in case of a tie).

III) We assume there is neither a jumbo nor a monster among the hunks.

Lemma 3.1. *Let M_1 and M_2 be normal B-hunks; so there exist S_{M_1} and S_{M_2} subsets of $\partial\Lambda$ such that $|S_{M_i}| > \frac{1}{2}|\partial\Lambda|$, and there are m_1 and m_2 with*

$$\tilde{h}_{M_i} = 2\pi\beta^{-1/2}m_i$$

on S_{M_i}. Then $m_1 = m_2$.

So in Case III) we can define the sector n as follows. If there are no B-hunks then h is constant on $\partial\Lambda$, say $h=2\pi\beta^{-1/2}n$. We let n be the sector of h. If M is any B-hunk, we let n be such that

$$\tilde{h}_M=2\pi\beta^{-1/2}n$$

on more than one half of $|\partial\Lambda|$. By Lemma 3.1 this is well defined.

3.3. Properties of the Sector Definitions

We now have a decomposition of $\mathscr{H}$ into sectors

$$\mathscr{H}=\bigcup_{n\in Z}\mathscr{H}^{(n)}. \tag{3.4}$$

h_0^n is the unique constant function in $\mathscr{H}^{(n)}$.

If M is a connected subset of unit cubes in Λ, we define the subset $\mathscr{H}_M^{(0)}$ of $\mathscr{H}^{(0)}$ as follows. h is contained in $\mathscr{H}_M^{(0)}$ if and only if h has exactly one hunk and it is M. If $M=\emptyset$ we set $\mathscr{H}_M^{(0)}$ to contain only the constant zero function.

Lemma 3.2. a) *If $h\in\mathscr{H}^{(n)}$, then $h+h_0^m\in\mathscr{H}^{(n+m)}$ and h and $h+h_0^m$ have the same hunks. They also fall into the same cases* (I, II, III) *in Sect. 3.2.*

b) *There is a one-to-one correspondence between elements h of $\mathscr{H}^{(0)}$ and a specification of*

1) *an integer $k\geqq 0$,*
2) *disjoint connected sets $M_1, M_2, \ldots, M_k$,*
3) $h_i\in\mathscr{H}_{M_i}^{(0)}$, $i=1,\ldots,k$.

Given 1), 2), *and* 3), *h is given by*

$$h=h_1+\ldots+h_k \quad (=0 \text{ if } k=0). \tag{3.5}$$

Given h, k is the number of hunks of h and $M_1,\ldots,M_k$ are the hunks. In 2) *we do not distinguish the order, a permutation of elements is not considered a new choice. We also implicitly assume $\mathscr{H}_{M_i}^{(0)}\neq\emptyset$ and $M_i\neq\emptyset$.*

From now on we will denote elements of $\mathscr{H}^{(n)}$ by h^n. Any h^n can be written uniquely as

$$h^n=h_0^n+h \tag{3.6}$$

with $h\in\mathscr{H}^{(0)}$. It follows from (3.5) that

$$h^n=h_0^n+\sum_M h_M. \tag{3.7}$$

This way of writing h^n is central to our decoupling procedure. We let

$$g_0^n=C(\chi h_0^n), \tag{3.8}$$

$$g_M=C(\chi h_M). \tag{3.9}$$

3.4. The Second Enlargement of the Hunks

The hunks we have developed so far will now be further enlarged and coalesced. This will be done in such a way that all the properties of hunks and sectors as presented in Subsect. 3.3 remain true. This is guaranteed by the way the enlargement is performed: The enlargement is done in the zero sector in Appendix E; the enlargement in other sectors is then uniquely determined by the requirement that Lemma 3.2 remains true.

The purpose of this process is to enable the estimation of functional integrals in Sect. 12 to be performed. The details are quite technical, using special mathematical results from Appendix D. The basic enlargement may be qualitatively understood by considering the following simple geometry. Suppose a hunk is approximately the shape of a hollow spherical shell. Suppose $h=0$ outside the shell, and $h=h_1 \neq 0$ inside the shell. We then desire the thickness of the shell to be proportional to $|h_1|$. In general hunks depend not only on the discontinuity set, but on the magnitude of the discontinuity.

From now on (unless otherwise noted) we work with hunks after the two enlargement processes. Likewise the spaces $\mathscr{H}_M^{(0)}$ are spaces of h's in $\mathscr{H}^{(0)}$ that after the two enlargements have the single hunk M.

4. Initial Analysis of Sector Contributions

4.1. Sector Decompositions and Transmutation

Corresponding to the decomposition of $\mathscr{H}$ into sectors, as given in (3.4)

$$\mathscr{H} = \bigcup_{n\in Z} \mathscr{H}^{(n)}. \tag{4.1}$$

One may decompose the sum in (2.17)

$$[\mathscr{A}(\phi)] = \sum_n [\mathscr{A}(\phi)]^{(n)}. \tag{4.2}$$

with $\mathscr{A}(\phi)=1$ one gets

$$Z = \sum_n Z^{(n)}, \tag{4.3}$$

which we explicitly write from (2.17) as

$$Z^{(n)} = Ne^Q \sum_{h^n \in \mathscr{H}^{(n)}} e^{-1/2E(h^n,h^n)} \int d\mu(\phi) e^{G(\phi+g^n)}. \tag{4.4}$$

It is suggestive of some further developments to write $Z^{(n)}$ (if $n \neq 0$) as

$$Z^{(n)} = Ne^Q \sum_{h\in\mathscr{H}^{(0)}} e^{-1/2E(h,h)} \int d\mu(\phi) e^T e^{G(\phi+g)} \tag{4.5}$$

with

$$T = -\tfrac{1}{2}E(h+h_0^n, h+h_0^n) + \tfrac{1}{2}E(h,h) + G(\phi+g^n) - G(\phi+g). \tag{4.6}$$

We may view the n-sector as having *transmuted* into a modified zero sector with e^T the modification. The modification is easily seen, from (4.6), to naturally divide itself into a numerical factor, the change in the energy, and a nontrivial distortion of the integrand. This division was qualitatively discussed in the introduction.

4.2. Extraction of Some (Easy) Z Ratios

In this subsection and the following two sections a polymer representation of $Z^{(n)}$ will be developed. We first specialize to the cluster expansion for $Z^{(n)}$, since $Z^{(n)}$ is our most basic object, and since we will pattern other developments after this. We will associate a partial partition function ϱ_i to each unit cube Δ_i, and multiply and divide by these factors in a natural way. We will also multiply and divide by certain other ratios of partition functions in a way much as done by Imbrie in [12] corresponding to changing boundary conditions – but our situation herein is much simpler than that in [12].

We first define ϱ_A (A for "atom"), and as in Eq. (8.5) of [7] we associate to Δ_A the function $G(\cdot, A)$. We then set

$$\varrho_A = \int d\mu(\phi) e^{G(\phi, A)} . \tag{4.7}$$

We proceed to define "normalized" $Z^{(n)}$, by dividing out some simple factors. First for the case $n=0$, we set

$$\bar{Z}^{(0)} = \frac{1}{N} e^{-Q} \prod_A (1/\varrho_A) Z^{(0)} . \tag{4.8}$$

We now define $\bar{Z}^{(n)}$ for $n \neq 0$ (although the expression also is correct for $n=0$) using a small parameter $\alpha_2 > 0$

$$\bar{Z}^{(n)} = \frac{1}{N} e^{-Q} e^{1/2(1-\alpha_2)E(h_0^n, h_0^n)} \prod_A (1/\varrho_A) e^{-i\int h_0^n u^{-1}\psi} Z^{(n)} . \tag{4.9}$$

Basically we are using part of the energy $E(h_0^n, h_0^n)$ to suppress the whole sector $Z^{(n)}$, and part to control our estimates for $\bar{Z}^{(n)}$.

We now present expressions for $\bar{Z}^{(n)}$ and $\bar{Z}^{(0)}$ as derived straightforwardly from (4.8), (4.9), (2.17) [with $\mathscr{A}(\phi)=1$], and the basic definitions of $E(h, h')$ and G. (It is important to note that $G(\phi)$ is constructed to be invariant under constant changes in its argument by multiples of $\frac{2\pi}{\beta^{1/2}}$ on cubes in $\{\Omega_\alpha\}$, except for the term $i\int \phi u^{-1}\psi$.)

$$\bar{Z}^{(0)} = \sum_{h \in \mathscr{H}^{(0)}} e^{-\frac{1}{2}E(h,h)} \int d\mu(\phi) e^{G(\phi+g-h)} \prod_M e^{i\int h_M u^{-1}\psi} \prod_A \frac{1}{\varrho_A} , \tag{4.10}$$

$$\begin{aligned}\bar{Z}^{(n)} = &\sum_{h \in \mathscr{H}^{(0)}} e^{-\frac{1}{2}\alpha_2 E(h_0^n, h_0^n) - E(h_0^n, h) - \frac{1}{2}E(h,h)} \\ &\cdot \int d\mu(\phi) e^{G(\phi + g_0^n - h_0^n + g - h)} \prod_M e^{i\int h_M u^{-1}\psi} \prod_A \frac{1}{\varrho_A} .\end{aligned} \tag{4.11}$$

In (4.10) and (4.11) the product over A is over all atoms in Λ; and the product over M is over the hunks of h, as expressed in the representation of (3.7).

5. Interpolation

Our polymer, P, will specify a sequence of disjoint sets $Y_1, Y_2, \ldots, Y_m$, each Y_i a hunk or an atom (see Sect. 8 of [7]). There will also be specified an (ordered) tree graph,

η_p (see Definition 8.2 of [2]). As in [7] and [2], and standard in cluster expansions, the sequence of Y_i is determined by differentiating interpolated expressions with respect to interpolation parameters $s_1, \ldots, s_{m-1}$. In this section we define the interpolation procedure, which in effect, determines the cluster expansion.

5.1. *Interpolation of Covariance*

The covariances are interpolated as in [7], explicitly detailed in Sect. 3.3 of [4].

5.2. *Terms that Factorize*

To uniquely determine the polymers developed by the cluster expansion procedure, it is sufficient to specify the distribution of the factors in (4.11) to the Y_i, for those factors that factorize, and the interpolation of those factors that do not. We first specify the handling of factors in (4.11) that completely factorize:

a) $\prod_A 1/\varrho_A$.

The region Y_i is assigned the factor

$$\prod_{A \in Y_i} 1/\varrho_A, \tag{5.1}$$

where $A \in Y_i$ states the corresponding cube is geometrically contained in the region of Y_i.

b) $\prod_M e^{i\int h_M u^{-1} \psi}$.

Y_i is associated a factor 1 if Y_i is an atom, and if Y_i is a hunk M it is associated the factor

$$e^{i\int h_M u^{-1} \psi}. \tag{5.2}$$

c) We now view the term $E(h_0^n, h_0^n)$. We note

$$E(h_0^n, h_0^n) = \int h_0^n(h_0^n - g_0^n) = \sum_Y \int_Y h_0^n(h_0^n - g_0^n). \tag{5.3}$$

We associate to Y_i the factor

$$e^{-\alpha_2/2 \int_{Y_i} h_0^n(h_0^n - g_0^n)}. \tag{5.4}$$

d) Similarly we observe for $E(h_0^n, h)$,

$$E(h_0^n, h) = \sum_M E(h_0^n, h_M). \tag{5.5}$$

We associate Y_i the factor of 1 if Y_i is an atom, and to Y_i if Y_i is the hunk M

$$e^{-E(h_0^n, h_M)}. \tag{5.6}$$

5.3. *Interpolation of the Energy Term*, $E(h, h)$

We view the identity

$$E(h, h) = \sum_M \sum_{M'} E(h_M, h_{M'}). \tag{5.7}$$

The diagonal terms factorize, assigning 1 to Y_i an atom, and if Y_i is the hunk M the factor

$$e^{-1/2E(h_M, h_M)}. \tag{5.8}$$

The expression $\sum_{M \neq M'} E(h_M, h_{M'})$ is treated as a two body potential between hunks. Thus at the end of the interpolation procedure our polymer will have associated to it the factor

$$e^{-1/2 \sum_{i,j}^{m} E(h_i, h_j)\sigma(i,j)}, \tag{5.9}$$

$\sigma(i,j)$ symmetric in i and j

$$\sigma(i,j) = \begin{cases} 1 & i=j \\ s_i s_{i+1} \dots s_{i-1} & i<j \end{cases}. \tag{5.10}$$

(Here we have set $h_i = 0$ if i is an atom.)

5.4. *Interpolations of $g-h$*

As in (8.5) of [7] we associate to region Y_i, $G(\cdot, Y_i)$, and we have associated to Y_i the factor

$$e^{G(\,, Y_i)}. \tag{5.11}$$

However the argument of G in (4.11) contains the term $g-h$, which unlike in [7] we must decouple from effects of other hunks. We write

$$g-h = \sum_M (g_M - h_M). \tag{5.12}$$

In the polymer, we interpolate the $g-h$ in $G(\,, Y_i)$ so that at the end of the interpolation procedure the $g-h$ has become

$$\sum_{j=1}^{m} \sigma(i,j)(g_j - h_j). \tag{5.13}$$

(Again $g_i = h_i = 0$ if i is an atom.)

5.5. *Induction Step in Cluster Expansion*

At the n^{th} step in the formation of our polymer, $1 \leqq n \leqq m-1$, Y_{n+1} is introduced by differentiation with respect to s_n. Similarly to in Eq. (8.9) of [7] we introduce operators $\kappa(n+1, \eta(n+1))$,

$$\kappa(n+1, \eta(n+1)) = \sum_{t=1}^{4} \kappa^t(n+1, \eta(n+1)), \tag{5.14}$$

that describe terms differentiated down by $\frac{d}{ds_n}$ in the polymer. The decomposition in (5.14) is according to different types of terms that may be differentiated.

$t=1$) This term arises from differentiating the expression in (5.9) and thus

$$\kappa^1(\,,) = -E(h_{n+1}, h_{\eta(n+1)}) \cdot \sigma'(n+1, \eta(n+1)), \tag{5.15}$$

where in σ' the prime indicates differentiation with respect to s_n.

$t=2,3$) These terms arise from differentiating the dependencies of $g-h$ on s parameters, as indicated in Subsect. 5.4.

$$\kappa^2(\,,\,)=\int_{Y_{n+1}}(g_{\eta(n+1)}-h_{\eta(n+1)})\frac{\delta}{\delta\phi}\cdot\sigma'(\,,\,)\,,\tag{5.16}$$

$$\kappa^3(\,,\,)=\int_{Y_{\eta(n+1)}}(g_{n+1}-h_{n+1})\frac{\delta}{\delta\phi}\cdot\sigma'(\,,\,)\,.\tag{5.17}$$

$t=4$) These terms arise from differentiating the covariance, which are as in [7]

$$\kappa^4(\,,\,)=\int_{Y_{n+1}}\int_{Y_{\eta(n+1)}}C(x,y)\frac{\delta}{\delta\phi(x)}\frac{\delta}{\delta\phi(y)}\cdot\sigma'(\,,\,)\,.\tag{5.18}$$

6. The Polymer Expansion

The use of a polymer representation is modeled after [18, pp. 31–38] and [9], but with modifications.

6.1. Polymers

A polymer P is specified uniquely by

1) an integer $m\geqq 1$,

2) disjoint hunks or atoms $Y_1, \ldots, Y_m$,

3) a function $h_i\in\mathscr{H}^0_{Y_i}$ for each Y_i a hunk, (see the definition before Lemma 3.3. If Y_i is an atom h_i is zero.)

4) an ordered tree graph, η.

Each polymer can occur in any sector. But its activity is sector dependent.

6.2. Polymer Activity in the Zero Sector

To the polymer P we associate an activity $z_P=z_P^0$ in the zero sector.

$$z_P=\frac{1}{m}\frac{1}{F(P)}\int d\sigma\, e^{-DE}\int d\mu(\phi)\prod_2^n k(i,\eta(i))f(\eta,\sigma)e^G\,,\tag{6.1}$$

where

$$F(P)=\prod_{A\in P}\varrho_A\prod_{i=1}^m e^{-i\int h_i u^{-1}\psi}\,.\tag{6.2}$$

The inclusion $A\in P$ is geometrical inclusion in $\cup Y_i$.

$$DE=DE(P,s_1,\ldots,s_{m-1})=\tfrac{1}{2}\sum_{i,j=1}^m\sigma(i,j)E(h_i,h_j)\,,\tag{6.3}$$

$$\begin{aligned}G&=G(\phi+g_P-h_P)\,,\\ g_P-h_P&=\sum_{i=1}^m(g_i-h_i)\,.\end{aligned}\tag{6.4}$$

$\int d\sigma$ is the integral over parameters s_i. $f(\eta,\sigma)$ is the usual monomial in the s's associated to η. We have k defined by

$$\left(\prod_{i=2}^m k(i,\eta(i))\right)f(\eta,\sigma)=\prod_{i=2}^m\kappa(i,\eta(i))\,.\tag{6.5}$$

6.3. Polymer Activity in General Sector

The polymer P is assigned activity z_P^n in the n^{th} sector,

$$z_P^n = \frac{1}{m}\frac{1}{F(P)}\int d\sigma e^{-DE-CT-\alpha_2 SE}\int d\mu(\phi)\prod_2^m \kappa(i,\eta(i))f(\eta,\sigma)e^G. \tag{6.6}$$

$F(P)$ is as in (6.2). And

$$CT = CT(P) = \sum_{i=1}^m E(h_i, h_0^n), \tag{6.7}$$

$$SE = SE(P) = \tfrac{1}{2}\sum_{i=1}^m \int_{Y_i} h_0^n(h_0^n - g_0^n), \tag{6.8}$$

$$G = G(\phi + g_0^n - h_0^n + g_P - h_P). \tag{6.9}$$

A polymer is said to be *trivial* if $m=1$ and Y_1 is an atom.

6.4. The Polymer Expansions for $Z^{(0)}$ and $Z^{(n)}$

$$\bar{Z}^{(n)} = \sum_{s=0}^\infty \frac{1}{s!}\sum_{\substack{P_1,\dots,P_s\\ \cup P_i = \Lambda}} z_{P_1}^n \dots z_{P_s}^n \prod_{i<j\leq s}(1-\chi_{ij}). \tag{6.10}$$

The sum over polymers in (6.10) includes trivial polymers. $\chi_{ij}=1$ if the polymers P_i and P_j have non-zero geometrical intersection, and is otherwise zero. In the zero sector z_P^0 is 1 for trivial polymers. Thus we may remove the restriction that $\cup P_i = \Lambda$ in the sum (6.10), *provided* that we restrict the sum to non-trivial polymers. Thus we have

$$\bar{Z}^{(0)} = \sum_{s=0}^\infty \frac{1}{s!}\sum_{P_1,\dots,P_{s(nt)}} z_{P_1}\dots z_{P_s}\prod_{i<j\leq s}(1-\chi_{ij}), \tag{6.11}$$

where the (nt) indicates sums are taken over non-trivial polymers. One views (6.11) as a partition function for a gas of polymers with activities z_{P_i} and Boltzmann factor $(1-\chi_{ij})$. This view leads to the expressions

$$\bar{Z}^{(0)} = e^{S^0}, \tag{6.12}$$

$$S^0 = \sum_{s=1}^\infty \sum_{|\eta|=s}\frac{1}{s}\sum_{P_1,\dots,P_{s(nt)}}\prod_{r=2}^s(-\chi_{r,\eta(r)})z_{P_1}\dots z_{P_s}g(\eta). \tag{6.13}$$

This is formally a Mayer Series for a hard sphere gas.

$$0 < g(\eta).$$

We suppress some dependences of $g(\eta)$ that are unnecessary to our estimates. We write $\eta_1 \sim \eta_2$ if η_1 are topologically isomorphic ($\eta_1 = P\eta_2P^{-1}$ for some permutation of the integers) and then find

$$\sum_{\eta\sim\eta_0} g(\eta) \leq 1 \tag{6.14}$$

for each η_0. This estimation of $g(\eta)$ is modeled after results in Sect. 8 in [2], and the Appendix of [6] (see also Sect. 3 of [5]).

6.5. Good and Bad Trivial Polymers

We are now going to redefine the activity of trivial polymers in the n^{th} sector, for $n \neq 0$. We change from

$$z_P^n = e^{-\alpha_2 1/2 \int_P h_0^n(h_0^n - g_0^n)} \frac{\varrho_P^n}{\varrho_P} \tag{6.15}$$

to

$$z_P^n = e^{-\alpha_2 1/2 \int_P h_0^n(h_0^n - g_0^n)} \frac{\varrho_P^n}{\varrho_P} - 1 . \tag{6.16}$$

ϱ_P^n is defined as ϱ_P is in Eq. (4.7), with ϕ replaced by $\phi + g_0^n - h_0^n$. It is then easy to see we have the expansion, in terms of these new activities,

$$\bar{Z}^{(n)} = \sum_{s=0}^{\infty} \frac{1}{s!} \sum_{P_1, \ldots, P_s} z_{P_1}^n \ldots z_{P_s}^n \prod_{i<j\leqq s} (1-\chi_{ij}) . \tag{6.17}$$

A trivial polymer P is said to be *good* if

$$\int_P h_0^n(h_0^n - g_0^n) \leqq 1 , \tag{6.18}$$

otherwise it is said to be *bad*. The activity of a good trivial polymer will be small since α_2 is small. All non-trivial polymers are good.

We may write (6.17) as follows

$$\bar{Z}^{(n)} = \sum_{s=0}^{\infty} \sum_{t=0}^{\infty} \frac{1}{s!} \frac{1}{t!} \sum_{P_1, \ldots, P_s(G)} \sum_{A_1, \ldots, A_t(B)} z_{P_1}^n \ldots z_{A_t}^n \prod_{i<j\leqq s+t} (1-\chi_{ij}) , \tag{6.19}$$

P_i and A_i label polymers. The (B) and (G) indicate the sums are restricted to bad and good polymers respectively. We write (6.19), with some introduction of notation, in the form

$$\bar{Z}^{(n)} = \sum_{t=0}^{\infty} \frac{1}{t!} \sum_{A_1, \ldots, A_t(B)} z_{A_1}^n \ldots z_{A_t}^n \prod_{i<j\leqq t} (1-\chi_{ij}) \bar{Z}^{(n)}(A_1, \ldots, A_t) \tag{6.20}$$

with

$$\bar{Z}^{(n)}(A_1, \ldots, A_t) = \sum_{s=0}^{\infty} \frac{1}{s!} \sum_{P_1, \ldots, P_s(GCA)} z_{P_1}^n \ldots z_{P_s}^n \prod_{i<j\leqq s} (1-\chi_{ij}) . \tag{6.21}$$

(GCA) indicates the sum is over good polymers in the complement of the union of the A_i. We may follow the development from (6.11) to (6.12), (6.13) and write

$$\bar{Z}^{(n)}(A_1, \ldots, A_t) = e^{S^n(A_1, \ldots, A_t)} , \tag{6.22}$$

$$S^n(A_1, \ldots, A_t) = \sum_{s=1}^{\infty} \sum_{|\eta|=s} \frac{1}{S} \sum_{P_1, \ldots, P_s(GCA)} \prod_{r=2}^{s} (-\chi_{r,\eta(r)}) z_{P_1}^n \ldots z_{P_s}^n g(\eta) . \tag{6.23}$$

6.6. Polymer Expansion for $[\mathscr{A}(\phi)]^{(n)}$

The polymer expansion for $Z^{(n)}$ extends to an expansion for $[\mathscr{A}(\phi)]^{(n)}$ in a natural way. We sketch the details. $\mathscr{A}'(\phi)$ is local so

$$\mathscr{A}'(\phi) = \prod_{\Delta} \mathscr{A}'(\phi; \Delta) ,$$

where the product is over unit cubes which intersect the support of $\mathscr{A}(\phi)$. We write this as

$$\prod_{\Delta}\{1+[\mathscr{A}'(\phi;\Delta)-1]\}=\sum_{S}\prod_{\Delta\subset S}[\mathscr{A}'(\phi;\Delta)-1],$$

where the sum is over all subsets of the set of unit cubes intersecting the support of $\mathscr{A}(\phi)$.

We are assuming that $\mathscr{A}'(\phi;\Delta)-1$ and its derivatives are small. The most important example of $\mathscr{A}'(\phi)$ arises when one considers correlation functions. Then

$$\mathscr{A}(\phi)=\prod_{i} e^{i\beta^{1/2}\phi(x_i)}.$$

In the polymer expansion for $[\mathscr{A}(\phi)]^{(n)}$ a polymer is a specification of 1) through 4) as in Subsect. 6.1 and

5) a subset S of the set of unit cubes which intersect the support of $\mathscr{A}(\phi)$.

We denote these polymers by $\tilde{P}$. The activity of $\tilde{P}$ is given by Eq. (6.6) with $\prod_{\Delta\subset S}[\mathscr{A}'(\phi;\Delta)-1]$ included in the integrand. A trivial polymer is a polymer with $m=1$, Y_1 an atom, and $S=\phi$. The activities of trivial polymers are redefined as before.

7. Proofs of Main Results

In this section we reduce the proofs of the main results stated in Sect. 1 to various estimates on the cluster expansion. These estimates are

Theorem 7.1. *If PM, $\beta<\beta_0$ and $R>R_0$, then*

(A) *There is a positive constant ε such that*

$$\frac{Z^{(n)}}{Z^{(0)}}\leqq e^{-\varepsilon\beta^{-1}n^2R}, \tag{7.1}$$

and

$$\frac{[\mathscr{A}(\phi)]^{(n)}}{Z^{(0)}}\leqq c_{\mathscr{A}}e^{-\varepsilon\beta^{-1}n^2R},$$

where $c_{\mathscr{A}}$ is a constant depending on $\mathscr{A}(\phi)$.

(B) *For any unit cube Δ and any integer n,*

$$\sum_{s=1}^{\infty}\sum_{\eta}\frac{1}{S}\sum_{\substack{P_1\dots P_s(G)\\ \Delta\subset\bigcup_i P_i}}\prod_{r=2}^{s}(-\chi_{r,\eta(r)})\,|z_n(P_1)\dots z_n(P_s)|g(\eta)\leqq\delta'. \tag{7.2}$$

$P_1,\dots,P_s$ are summed over good polymers with the constraint that at least one P_i must contain Δ. δ' can be made arbitrarily small by taking β_0 sufficiently small and R_0 sufficiently large. Moreover, $P_1,\dots,P_s$ can be replaced by $\tilde{P}_1,\dots,\tilde{P}_s$ if we replace δ' by $c_{\mathscr{A}}\delta'$. $c_{\mathscr{A}}$ is a constant depending on $\mathscr{A}(\phi)$.

(C) *There is a constant c so that for any unit cubes Δ_1 and Δ_2, the quantity in part* (B) *is*

$$\leqq c\mathscr{V}_{\varepsilon}(\Delta_1,\Delta_2),$$

if we replace the constraint that $\bigcup_i P_i$ *contains* Δ *by the constraint that the union contains* Δ_1 *and* Δ_2. *Again* $P_1, \ldots, P_s$ *may be replaced by* $\tilde{P}_1, \ldots, \tilde{P}_s$.

Remarks. 1. Part (A) says that only the zero sector contributes to the infinite volume limit.

2. Part (B) implies that the sums in $S^n(\Delta_1, \ldots, \Delta_t)$ and S^0 converge absolutely. This justifies equations like

$$\bar{Z}^{(n)}(\Delta_1, \ldots, \Delta_t) = e^{S^n(\Delta_1, \ldots, \Delta_t)}, \qquad \bar{Z}^{(0)} = e^{S^0}.$$

We can now reduce the proofs of the main results to the estimates stated above.

Proof of Theorem 1.5.

$$\begin{aligned}\langle \mathscr{A}(\phi)\rangle_R = \frac{1}{Z}[\mathscr{A}(\phi)] &= \left\{\frac{1}{Z^{(0)}}[\mathscr{A}(\phi)]^{(0)} + \sum_{n \neq 0} \frac{1}{Z^{(0)}}[\mathscr{A}(\phi)]^{(n)}\right\} \\ &\quad \cdot \left\{1 + \sum_{n \neq 0} \frac{Z^{(n)}}{Z^{(0)}}\right\}^{-1} = \frac{1}{Z^{(0)}}[\mathscr{A}(\phi)]^{(0)} + E,\end{aligned} \tag{7.3}$$

where by part (A) of the previous theorem

$$|E| \leqq c_{\mathscr{A}} e^{-\varepsilon\beta^{-1}R}.$$

So $E \to 0$ as $R \to \infty$. Now

$$\frac{1}{Z^{(0)}}[\mathscr{A}(\phi)]^{(0)} = e^{S^0_{\mathscr{A}} - S^0}. \tag{7.4}$$

Using standard arguments and Theorem 1.1, $\lim_{R \to \infty} (S^0_{\mathscr{A}} - S^0)$ exists and equals the corresponding limit for the charge symmetric system with $z_+ = z_- = z$. □

Proof of Theorem 1.6. As in the previous proof

$$\langle \mathscr{A}\mathscr{B}\rangle_R - \langle \mathscr{A}\rangle_R \langle \mathscr{B}\rangle_R = \frac{1}{Z^{(0)}}[\mathscr{A}\mathscr{B}]^{(0)} - \frac{1}{Z^{(0)}}[\mathscr{A}]^{(0)} \frac{1}{Z^{(0)}}[\mathscr{B}]^{(0)} + E, \tag{7.5}$$

with

$$|E| \leqq c_{\mathscr{A}\mathscr{B}} e^{-\varepsilon\beta^{-1}R}.$$

If β is sufficiently small then

$$\varepsilon\beta^{-1} \geqq \frac{2}{l_D^+}.$$

So

$$|E| \leqq c_{\mathscr{A}\mathscr{B}} e^{-\frac{2R}{l_D^+}} \leqq \mathscr{V}_\varepsilon(\Delta_1, \Delta_2). \tag{7.6}$$

Part (C) of Theorem 7.1 and the usual argument show that the rest of Eq. (7.5) is $\leqq c\mathscr{V}_\varepsilon(\Delta_1, \Delta_2)$. □

We leave the proof of Theorem 1.2 to the reader. The idea is to use the cluster expansion to show that $\langle J(x)\rangle$ equals $J_0(x)$ plus terms that are of order 1 in β by the estimates in Theorem 7.1.

To state the next theorem we introduce some notation. The "jump energy" of a function h_i is

$$JE(h_i) = \sum_f [\delta h_i(f)]^2 .$$

The sum is over faces f, and $\delta h_i(f)$ is the jump of h_i across f. For a polymer P let

$$JE(P) = \sum_{i=1}^{m} JE(i) ,$$

with $JE(i) = JE(h_i)$. By a standard argument there is a $c > 0$ with $DE(P) \geqq cJE(P)$. For example, see Lemma 5.2 of [4].

Theorem 7.2. *There exist positive numbers $\bar{z}(P)$ for each good polymer P such that if PM, $\beta < \beta_0$ and $R > R_0$, then*

(1) $|z_n(P)| \leqq \bar{z}(P) \quad \forall n$.

(2) *Given $\delta > 0$, if β_0 is sufficiently small and R_0 sufficiently large, then*

$$\sum_{P : \Delta \subset P} \bar{z}(P) e^{|P|} \leqq \delta .$$

Δ is any unit cube.

(3) *Let $\mathscr{B} \subset \Lambda$ be the union of all the bad polymers. If P is a good polymer with $P \cap \mathscr{B} = \phi$, then*

$$|z_n(P) - z_0(P)| \leqq c(h_0^n)^2 \sup_{\Delta \subset P} \int_{\Delta} [\chi - C(\chi)] \bar{z}(P) .$$

The sup *is over the unit cubes Δ in P.*

(4) *In* (2) *we can replace δ by $\delta \mathscr{V}_\varepsilon(\Delta_1, \Delta_2)$ if we replace the constraint $\Delta \subset P$ by $(\Delta_1 \cup \Delta_2) \subset P$.*

Moreover, these estimates all hold with P replaced by $\tilde{P}$. In this case the estimates will contain constants that depend on $\mathscr{A}(\phi)$.

We end this section by using Theorem 7.2 to prove Theorem 7.1.

Proof of Theorem 7.1.

Part (B). The proof of (B) using parts (1) and (2) of Theorem 7.2 is standard. For example, see Theorem 3.4 of [5].

Part (A). By the definition of $\bar{Z}^{(n)}$,

$$\left| \frac{Z^{(n)}}{Z^{(0)}} \right| = e^{-1/2(1-\alpha_2)E(h_0^n, h_0^n)} \left| \frac{\bar{Z}^{(n)}}{\bar{Z}^{(0)}} \right| . \tag{7.7}$$

By Eqs. (6.12), (6.20), and (6.22),

$$\frac{\bar{Z}^{(n)}}{\bar{Z}^{(0)}} = \sum_{t=0}^{\infty} \frac{1}{t!} \sum_{A_1, \ldots, A_t(B)} z_n(A_1) \ldots z_n(A_t) \cdot \prod_{i<j \leqq t} (1 - \chi_{ij}) e^{S^n(A_1, \ldots, A_t) - S^0} . \tag{7.8}$$

We complete the proof by establishing the following two bounds:

$$\sum_{t=0}^{\infty} \frac{1}{t!} \sum_{A_1, \ldots, A_t(B)} |z_n(A_1) \ldots z_n(A_t)| \prod_{i<j} (1 - \chi_{ij}) \leqq e^{1/2(\delta_1 + \alpha_2)E(h_0^n, h_0^n)} , \tag{7.9}$$

and

$$\sup_{A_1,\ldots,A_t(B)} |e^{S^n(A_1,\ldots,A_t)-S^0}| \leqq e^{1/2\delta_2 E(h_0^n,h_0^n)}. \tag{7.10}$$

This will prove part (A) provided α_2, δ_1 and δ_2 are small enough that $1-2\alpha_2-\delta_1-\delta_2>0$.

To prove (7.9) we begin by noting that the left-hand side of (7.9) is less than

$$\prod_{A:(B)} [1+|z_n(A)|].$$

So it suffices to show

$$1+|z_n(A)| \leqq e^{1/2(\delta_1+\alpha_2)SE(A)}, \tag{7.11}$$

where

$$SE(A)=\int_A h_0^n(h_0^n-g_0^n).$$

By Eq. (6.16),

$$\begin{aligned}|z_n(A)| &\leqq \left|\frac{\varrho_A^n}{\varrho_A}-1\right| e^{-\alpha_2 1/2SE(A)}+|e^{-\alpha_2 1/2SE(A)}-1| \\ &\leqq \frac{1}{|\varrho_A|}|\varrho_A^n-\varrho_A|+\alpha_2\tfrac{1}{2}SE(A).\end{aligned} \tag{7.12}$$

It is routine to show

$$|\varrho_A| \geqq c. \tag{7.13}$$

We write $\varrho_A^n-\varrho_A$ as

$$\int d\mu \int_0^1 dt \frac{d}{dt} e^{G(\phi+t(g_0^n-h_0^n);A)} = \int d\mu \int_0^1 dt \int_A dx (g_0^n-h_0^n)(x) \frac{\delta}{\delta\phi(x)} e^{G(\phi+t(g_0^n-h_0^n);A)}.$$

Using Lemmas 11.2 and 11.3 this shows

$$|\varrho_A^n-\varrho_A| \leqq c\int_A |g_0^n-h_0^n| e^{\varepsilon SE(A)}, \tag{7.14}$$

where ε can be made as small as desired by suitable choices of β_0 and R_0. Since

$$\int_A |g_0^n-h_0^n| = \frac{1}{|h_0^n|} SE(A) \leqq c\beta^{1/2} SE(A),$$

(7.12), (7.13), and (7.14) show

$$1+|z_n(A)| \leqq 1+c\beta^{1/2}SE(A)e^{\varepsilon SE(A)}+\alpha_2\tfrac{1}{2}SE(A),$$

which proves (7.11) with $\frac{1}{2}\delta_1=c\beta^{1/2}+\varepsilon$.

We now fix $A_1,\ldots,A_t(B)$ and bound

$$|S^n(A_1,\ldots,A_t)-S^0|.$$

Let $\mathscr{B}$ denote the union of the atoms associated with the bad polymers. Recall that $S^n(A_1,\ldots,A_t)$ is a sum over various choices of polymers $P_1,\ldots,P_s$. Each P_i must be good and contained in the complement of $\bigcup_{i=1}^t A_i$. We split up the sum into three parts

$$S^n(A_1,\ldots,A_t)=S_1^n+S_2^n+S_3^n,$$

by putting additional constraints on the P_i. In S_1^n each P_i must be nontrivial and contained in the complement of $\mathscr{B}$. In S_2^n each P_i must be contained in the complement of $\mathscr{B}$, and at least one P_i must be trivial. In S_3^n at least one P_i must intersect $\mathscr{B}$. Similarly

$$S^0 = S_1^0 + S_2^0 + S_3^0 .$$

However, $S_2^0 = 0$, since there are no trivial P_i in S^0.

There is a one-to-one correspondence between the terms in S_1^n and those in S_1^0. The only difference between corresponding terms is that the S_1^n terms contain $z_n(P_1)\dots z_n(P_s)$, while the S_1^0 terms contain $z_0(P_1)\dots z_0(P_s)$. Using parts of (1) and (3) of Theorem 7.2,

$$\begin{aligned}&|z_n(P_1)\dots z_n(P_s) - z_0(P_1)\dots z_0(P_s)|\\ &\quad \leqq sc(h_0^n)^2 \sup_{\Delta \subset (\bigcup_i P_i)} \int_\Delta [\chi - C(\chi)] \bar{z}(P_1)\dots\bar{z}(P_s) .\end{aligned} \tag{7.15}$$

For each choice of $P_1, \dots, P_s$ the sup is attained by at least one $\Delta \subset \bigcup_i P_i$. So a crude bound on $|S_1^n - S_1^0|$ is

$$\begin{aligned}&\leqq \sum_\Delta c(h_0^n)^2 \int_\Delta [\chi - C(\chi)] \sum_{s=1}^\infty s \sum_\eta \sum_{\substack{P_1\dots P_s(G)\\ \Delta \subset (\bigcup_i P_i)}} \prod_{r=2}^s (-\chi_{r,\eta(r)}) \bar{z}(P_1)\dots\bar{z}(P_s) g(\eta)\\ &\leqq \sum_\Delta c(h_0^n)^2 \int_\Delta [\chi - C(\chi)] \delta\end{aligned}$$

by the techniques used to prove part (B). Now

$$\sum_\Delta (h_0^n)^2 \int_\Delta [\chi - C(\chi)] = E(h_0^n, h_0^n) ,$$

so our bound is

$$\leqq c\delta E(h_0^n, h_0^n) ,$$

which contributes $c\delta$ to $\frac{1}{2}\delta_2$.

In S_2^n at least one P_i is trivial. Using the proof of part (B) this implies

$$|S_2^n| \leqq \delta \sum_A |z_n(A)| ,$$

where the sum is over trivial polymers A. By estimates in the proof of (7.11), this is

$$\leqq \delta(c\beta^{1/2} + \alpha_2)\tfrac{1}{2} E(h_0^n, h_0^n) ,$$

which contributes $\delta(c\beta^{1/2} + \alpha_2)$ to δ_2.

Finally, in S_3^n at least one P_i intersects $\mathscr{B}$. So the techniques used to prove part (B) show

$$|S_3^n| \leqq |\mathscr{B}| \delta .$$

Recall that for a bad unit cube Δ

$$\int_\Delta h_0^n (h_0^n - g_0^n) \geqq 1 .$$

So

$$E(h_0^n, h_0^n) \geqq \int_B h_0^n(h_0^n - g_0^n) \geqq |\mathscr{B}|,$$

and

$$|S_3^n| \leqq \delta E(h_0^n, h_0^n).$$

The same bound holds for S_3^0. This completes the proof of (7.10).

Part (C). The term $\prod_r (-\chi_{r,\eta(r)})$ forces the P_i to overlap so that $\bigcup_i P_i$ is connected. Thus we can find unit cubes $A_1, A_2, \ldots, A_m$ such that Δ_1 and A_1 are in the same P_i, A_1 and A_2 are in the same $P_i, \ldots, A_{m-1}$ and A_m are in the same P_i, and A_m and Δ_2 are in the same P_i. Using part (4) of Theorem 7.2 with ε replaced by $\varepsilon/2$ and the usual techniques, one can show (7.2) is bounded by

$$\sum_{m=0}^{\infty} \delta^{m+1} \sum_{A_1,\ldots,A_m} \mathscr{V}_{\varepsilon/2}(\Delta_1, A_1)\mathscr{V}_{\varepsilon/2}(A_2, A_3)\ldots\mathscr{V}_{\varepsilon/2}(A_{m-1}, A_m)\mathscr{V}_{\varepsilon/2}(A_m, \Delta_2). \tag{7.16}$$

It is easy to show there is a constant c so that

$$\sum_B \mathscr{V}_\varepsilon(A, B)\mathscr{V}_{\varepsilon/2}(B, C) \leqq c\mathscr{V}_\varepsilon(A, C),$$

so (7.16) is $\leqq c'\delta\mathscr{V}_\varepsilon(\Delta_1, \Delta_2)$ provided δ is small enough. □

8. Combinatorics

We begin the proof of Theorem 7.2. In this section we define $\bar{z}(P)$ and do the combinatorial parts of the proof.

Definition of $\bar{z}(P)$. Recall that $k^{t_i}(i, \eta(i))$ contains an integral over Y_i for $t_i = 2$ or 4 and an integral over $Y_{\eta(i)}$ for $t_i = 3$ or 4. We break each occurrence of these integrals up into integrals over unit cubes:

$$\begin{aligned} \int_{Y_i} dx &= \sum_{C_i \subset Y_i} \int_{C_i} dx, \\ \int_{Y_{\eta(i)}} dy &= \sum_{B_i \subset Y_{\eta(i)}} \int_{B_i} dy. \end{aligned} \tag{8.1}$$

So at each vertex i in η we have a sum over unit cubes C_i in Y_i if $t_i = 2$ or 4. In addition for each vertex j with $\eta(j) = i$ and $t_j = 3$ or 4 we get another sum over unit cubes B_j in Y_i. We summarize this by defining

$$\sum{}^{(i)} = \sum_{\substack{C_i \subset Y_i \\ \text{if } t_i = 2 \text{ or } 4}} \prod_{\substack{j:\eta(j)=i \\ \text{and } t_j = 3 \text{ or } 4}} \left[\sum_{B_j \subset Y_i}\right], \tag{8.2}$$

where the sum over C_i appears only when $t_i = 2$ or 4.

We now have

$$\prod_{i=2}^{m} k^{t_i}(i, \eta(i)) = \prod_{k=1}^{m} \sum{}^{(k)} \prod_{i=2}^{m} \tilde{k}^{t_i}(i, \eta(i)),$$

where $\tilde{k}^{t_i}(i, \eta(i))$ is $k(i, \eta(i))$ with the integrations over Y_i and $Y_{\eta(i)}$ replaced by integrations over C_i and B_i respectively, when they occur. The point of these definitions is to restrict the functional derivatives to unit cubes. For each unit cube B, let n_B be the number of functional derivatives in $\prod_{i=2}^{m} \tilde{k}^{t_i}(i,\eta(i))$ which are with respect to $\phi(x)$ with $x \in B$. So n_B is just the number of cubes C_i or B_i equal to B. We will always use C_i to denote a cube in Y_i and B_i to denote a cube in $Y_{\eta(i)}$. So each Y_i contains at most one C-cube, but it can contain many B-cubes if η has many bonds which hit vertex i.

Finally, we can define $\bar{z}(P)$.

$$\bar{z}(P) = \prod_{k=2}^{m} \left(\sum_{t_k=1}^{4} \right) \prod_{k=1}^{m} (\Sigma^{(k)}) \prod_{B \subset P} (n_B!)^{3/2-\varepsilon_0}$$

$$\cdot \prod_{i=2}^{m} \bar{k}^{t_i}(i,\eta(i)) \delta^m e^{-\alpha JE(P)} \int d\sigma f(\eta,\sigma). \tag{8.3}$$

α and ε_0 are positive constants which will be specified later. $\delta = \delta(\beta_0, R_0)$ can be made arbitrarily small by taking β_0 sufficiently small and R_0 sufficiently large.

$$\begin{aligned} \bar{k}^1(i,j) &= E(h_i, h_j), \\ \bar{k}^2(i,j) &= \int_{C_i} dx |g_j - h_j|(x), \\ \bar{k}^3(i,j) &= \int_{B_i} dy |g_i - h_i|(y), \\ \bar{k}^4(i,j) &= \int_{C_i} dx \int_{B_i} dy C(x,y). \end{aligned} \tag{8.4}$$

Proof of Theorem 7.2

Part (1). We reduce the proof to theorems in Sects. 9–12 as follows. Choose α_1 small enough for Theorem 11.1 to hold. We split up DE as $\alpha_1 DE + (1-\alpha_1)DE$. By Theorem 9.1

$$\exp[-\tfrac{1}{2}\alpha_1 DE - \tfrac{1}{2}\alpha_2 SE - CT] \leqq 1. \tag{8.5}$$

Next we apply Theorem 11.1 with $\varepsilon = \frac{1}{2}\alpha_2$ to bound the functional derivatives and functional integrals. This reduces the proof to showing

$$\frac{1}{|F(P)|} e^{-1/2\alpha_1 DE} \delta_0^m \leqq \delta^m e^{-\alpha JE(P)}. \tag{8.6}$$

We have

$$|F(P)| = \prod_{A \in P} \varrho_A \geqq c^{-|P|}.$$

Now $|P| = |P|_H + |P|_A$, where $|P|_H$ is the sum of the sizes of the hunks in P and $|P|_A$ is the number of atoms in P. By (E.6), $|P|_H \leqq c\beta^{1/2} DE$. So

$$c^{|P|_H} e^{-1/4\alpha_1 DE} \leqq 1 \tag{8.7}$$

for small enough β. The remaining $e^{-1/4\alpha_1 DE}$ provides the factor of $e^{-\alpha JE(P)}$. By choosing δ_0 small enough

$$c^{|P|_A} \delta_0^m \leqq \delta^m.$$

Part (2). Let

$$\mathscr{S}=\prod_{B\subset P}(n_B!)^{3/2-\varepsilon_0}\prod_{i=2}^{m}\bar{k}^{t_i}(i,\eta(i))\delta^m e^{-\alpha JE(P)}f(\eta,\sigma)e^{|P|}. \tag{8.8}$$

Then

$$\sum_{P:\Delta\subset P}\bar{z}(P)e^{|P|}=S_{(1)}\dots S_{(5)}\mathscr{S},$$

where $S_{(1)}\dots S_{(5)}$ stand for the sum or integral over the following:

(1) the integer m,

(2) the tree graph η and the interpolation parameters $\sigma=(s_1,\dots,s_{m-1})$,

(3) the integers $t_1,\dots,t_m$,

(4) the hunks or atoms $Y_1,\dots,Y_m$ such that $\Delta\subset\bigcup_i Y_i$ and the cubes B_1, $C_1,\dots,B_m$, C_m, which occur in $\prod_k\sum^{(k)}$,

(5) the functions $h_1,\dots,h_m$.

Note that each sum may depend on the sums preceding it. We will bound these sums in the opposite order from the order in which they are listed. For $i=2,\dots,5$ we will convert the sums to sups by introducing quantities W_i which depend on the objects in (1) through (5). We will show

$$S_{(i)}W_i^{-1}\leqq 1\quad\text{for}\quad i=2,\dots,5 \tag{8.9}$$

and

$$\sup W_2\dots W_5\mathscr{S}\leqq c^m\delta^m, \tag{8.10}$$

where the sup is over all allowed choices in (1) through (5). Thus

$$\sum_P\bar{z}(P)e^{|P|}\leqq\sum_{m=1}^{\infty}c^m\delta^m.$$

This will complete the proof since the right-hand side can be made arbitrarily small by choosing β_0 sufficiently small and R_0 sufficiently large.

To define the W_i we introduce some notation. Let

$$d_k=|\{i:\eta(i)=k\}|,$$
$$d_k^1=|\{i:\eta(i)=k\text{ and }t^i=1\text{ or }2\}|,$$
$$d_k^2=|\{i:\eta(i)=k\text{ and }t^i=3\text{ or }4\}|.$$

So $d_k=d_k^1+d_k^2$. Note also that $\sum_k d_k=m-1$.

We are going to take advantage of the fact that $C(x,y)$ is slightly better than integrable to get rid of $(n_B!)^{1/2-\varepsilon_0}$. Let

$$P(x,y)=\max\left\{e^{-\varepsilon d_1/l_D},\, e^{-\varepsilon(d_2+d_3)/l_D}\frac{1}{1+s^{1-2\varepsilon}}\right\} \tag{8.11}$$

with notation, including ε, as in Theorem 1.4. Let

$$\begin{gathered}P^3(i,\eta(i))=P^4(i,\eta(i))=\sup_{x\in Y_i,\,y\in B_i}P(x,y),\\ P^1(i,\eta(i))=P^2(i,\eta(i))=1.\end{gathered} \tag{8.12}$$

Define $\hat{k}^{t_i}$ by the equation

$$\bar{k}^{t_i}(i,\eta(i)) = P^{t_i}(i,\eta(i))\hat{k}^{t_i}(i,\eta(i)). \tag{8.13}$$

We now list the W_i:

$$W_5 = e^{\alpha/4JE(P)},$$

$$W_4 = \prod_{B\subset P}(n_B!)^{-1}\prod_k [d_k^1!(d_k^2+1)!\,2^{d_k^2}]$$

$$\cdot 2^{|P|}\left[\sup_{(5)} e^{-\alpha/2JE(P)}\prod_{i=2}^{m}\hat{k}^{t_i}(i,\eta(i))\right]^{-1}.$$

$\sup_{(5)}$ denotes the sup over the objects in (5).

$$W_3 = 4^m,$$

$$W_2 = \prod_k (d_k!)^{-1}\frac{1}{f(\eta,\sigma)}4^m. \tag{8.14}$$

We verify (8.10). By (E.6) for any $\varepsilon > 0$ we can choose β_0 sufficiently small so that

$$\sum_j |Y_j| \leqq \varepsilon JE(P), \tag{8.15}$$

where the sum is only over Y_j which are hunks. The sum over atoms is trivially bounded by m. These observations and some easy cancellations and bounds reduce (8.10) to showing

$$\prod_i P^{t_i}(i,\eta(i))\prod_B (n_B!)^{1/2-\varepsilon_0} \leqq c^m. \tag{8.16}$$

This is the analog of "exponential pinning," Lemma 9.10 of [7], for a covariance whose decay is better than integrable but not exponential.

To each $B\subset P$ we associate the factors of $P^{t_i}(i,\eta(i))$ with $B_i = B$. There are at least $n_B - 1$ such factors. The proof of (8.16) is thus reduced to showing that given B, $y_1, \ldots, y_n \in B$ and points $x_1, \ldots, x_n$ in different unit cubes,

$$\prod_{i=1}^{n} P(x_i, y_i) \leqq c^n[(n+1)!]^{-(1/2-\varepsilon_0)}. \tag{8.17}$$

This inequality is proven in the same way as exponential pinning.

We now verify (8.9). For $i=3$ this is trivial. For $i=5$ it reduces to showing

$$\sum_{h\in\mathscr{H}_Y^{(0)}} e^{-\alpha/4JE(h)} \leqq 1. \tag{8.18}$$

Fix $x_0 \in Y$. For $h \in \mathscr{H}_Y^{(0)}$, let $\tilde{h}$ be the translate

$$\tilde{h} = h - h(x_0). \tag{8.19}$$

The map $h \to \tilde{h}$ is injective. So the above sum is

$$\leqq \sum_{\tilde{h}:\,\tilde{h}(x_0)=0} e^{-\alpha/4JE(\tilde{h})}, \tag{8.20}$$

where $\tilde{h}$ has discontinuities only in Y. This sum is bounded by the usual argument, e.g., see p. 216 of [7].

For $i=2$ we need

$$\sum_{\eta}\prod_{k} d_k! \int d\sigma f(\eta,\sigma) \leqq 4^m . \tag{8.21}$$

This is an example of the "extra $N!$" of [3]. The above inequality is Theorem 1.4 of [1].

We are left with the $i=4$ case. The factors in W_4^{-1} can each be associated with a vertex k in η or a bond $(i,\eta(i))$ in η. Among other things, vertex k has a factor of

$$\frac{1}{d_k^1!} e^{-\alpha/4JE(k)}$$

associated with it. Using the inequality $d!\,e^x \geqq x^d$, for $x \geqq 0$, this gives a factor of $JE(k)^{-d_k^1}$ at vertex k. $S_{(4)}$ is a sum over the various possibilities for each vertex in η. The constraint $\Delta \subset \bigcup_i Y_i$ ties down the tree η. The usual argument shows

$$S_{(4)} W_4^{-1} \leqq mc\varrho^{m-1} \tag{8.22}$$

with

$$\varrho = \sup_t \sup_{Y_j} \sup{}^{(j)} \sum_{Y_i} \sum{}^{(i)} [K^t(i,j) c(i)] ,$$
$$c(i) = \prod_{B \subset Y_i} n_B! 2^{-d_i^2 - |Y_i|} [(d_i^2+1)!]^{-1} , \tag{8.23}$$

$$K^t(i,j) = \begin{cases} e^{-\alpha/4JE(i)} \hat{k}^t(i,j) JE(j)^{-1} & \text{for} \quad t=1,2 \\ e^{-\alpha/4JE(i)} \hat{k}^t(i,j) & \text{for} \quad t=3,4 . \end{cases} \tag{8.24}$$

The $\sup^{(j)}$ is over the terms in $\sum^{(j)}$. Consider

$$\sum{}^{(i)} \prod_{B \subset Y_i} n_B! . \tag{8.25}$$

From the definition of n_B we see that

$$\sum_{B \subset Y_i} n_B = \begin{cases} d_i^2 & \text{if} \quad t_i = 1 \quad \text{or} \quad 3 \\ d_i^2+1 & \text{if} \quad t_i = 2 \quad \text{or} \quad 4 . \end{cases}$$

Denote this integer by d. Then an easy combinatoric argument shows that (8.25) equals $(d+s-1)!/(s-1)!$, where $s=|Y_i|$ is the number of unit cubes in Y_i. Thus (8.25) is bounded by $d!\,2^{d+s-1}$.

So we have shown

$$\sum{}^{(i)} c(i) \leqq 1 . \tag{8.26}$$

Using (8.26), (8.23) is

$$\leqq \sup_t \sup_{Y_j} \sup{}^{(j)} \sum_{Y_i} \sup{}^{(i)} K^t(i,j) .$$

It will be crucial to our proof that $\sup^{(j)}$ is taken *after* the sum over Y_i. If we had converted all of the $\sum^{(k)}$ to $\sup^{(k)}$ and then converted the sum over the $Y_1, \ldots, Y_m$ to a sup, then this would not be the case.

Now $\sup^{(i)}$ is over a finite number of terms. So we can choose a term which attains the sup and drop the $\sup^{(i)}$. Fix a term in $\sup_{Y_j} \sup^{(j)}$ and consider

$$\sum_{Y_i} K^t(i,j) . \tag{8.27}$$

We will find a bound independent of the term in $\sup_{Y_j} \sup^{(j)}$. Note that the cubes B_k, C_i in Y_i depend on Y_i, but those in Y_j do not. We reduce the four cases $t=1,2,3,4$ to energy estimates.

$t=1$. By (8.12) and (8.4), $\hat{k}^1(i,j)=E(h_i,h_j)$. So this case is Theorem 10.2.

$t=2$. $\hat{k}^2(i,j)=\int_{C_i}|g_j-h_j|$, where $C_i \subset Y_i$. So a crude bound on (8.27) is

$$JE(j)^{-1}\sum_C \int_C |g_j-h_j| \sum_{Y_i:Y_i\supset C} e^{-\alpha/4JE(i)}.$$

We use

$$\sum_{Y_i:Y_i\supset C} e^{-\alpha/4JE(i)} \leqq c \tag{8.28}$$

and

$$\sum_C \int_C |g_j-h_j| \leqq \int_\Delta |g_j-h_j| \leqq cJE(j) \tag{8.29}$$

by part (a) of Theorem 10.3.

$t=3$. $\hat{t}^3(i,j)=P^3(i,j)^{-1}\int_{B_i}|g_i-h_i|$ with $B_i \subset Y_j$. So B_i is fixed in the sum over Y_i. Let $\operatorname{supp}(Y_i)$ denote the union of Y_i and any regions enclosed by Y_i. Then by Theorem 10.3(b), Lemma 10.1, Lemma 9.2(c) and (E.6),

$$\int_{B_i} |g_i-h_i| e^{-\alpha/8JE(i)} \leqq c\mathscr{V}_\varepsilon(B_i, \operatorname{supp}(h_i)), \tag{8.30}$$

where $\mathscr{V}_\varepsilon$ is given by (1.35) and $\operatorname{supp}(h_i)=\{x:h_i(x)\neq 0\}$. Let $\operatorname{supp}(Y_i)$ denote the union of Y_i and the regions enclosed by Y_i. Then the construction of the hunks insure that $\operatorname{supp}(h_i)\subset\operatorname{supp}(Y_i)$. So (8.27) is bounded by

$$c\sum_{Y_i} e^{-\alpha/8JE(i)}\mathscr{V}_\varepsilon(B_i,\operatorname{supp}(Y_i))P^3(i,j)^{-1}. \tag{8.31}$$

For sets S and R define

$$D(S,R)=\sup_{x\in S,\, y\in R} \frac{\mathscr{V}_\varepsilon(x,y)}{P(x,y)}. \tag{8.32}$$

Then

$$\frac{\mathscr{V}_\varepsilon(B_i,\operatorname{supp}(Y_i))}{P^3(i,j)} \leqq D(\Delta, B_i)$$

for some unit cube $\Delta\subset\operatorname{supp}(Y_i)$.

A crude bound on (8.31) is thus

$$c\sum_\Delta D(\Delta,B_i) \sum_{Y_i:\Delta\subset\operatorname{supp}(Y_i)} e^{-\alpha/4JE(i)} \leqq c'\sum_\Delta D(\Delta,B_i)\leqq c'',$$

since $D(\,,\,)$ is integrable.

$t=4$. The methods of the $t=3$ case easily handle this case.

Part (3).

$$z_n(P)-z_0(P)=\frac{1}{m}\frac{1}{F(p)}\int d\sigma\, e^{-DE}\int d\mu \prod_2^m k(i,\eta(i))f(\eta,\sigma)\,[D_1+D_2] \tag{8.33}$$

with

$$D_1 = e^{-CT-\alpha_2 SE}\left[e^{G(\phi+g_0^n-h_0^n+g_P-h_P)} - e^{G(\phi+g_P-h_P)}\right],$$
$$D_2 = e^{G(\phi+g_P-h_P)}\left[e^{-CT-\alpha_2 SE} - 1\right]. \tag{8.34}$$

To bound the D_2 term we use Theorem 9.1 as follows.

$$|e^{-CT-\alpha_2 SE} - 1| = |CT+\alpha_2 SE| \int_0^1 dt\, e^{-t(CT+\alpha_2 SE)} \leqq (|CT|+\alpha_2 SE)e^{1/2\alpha_1 DE}. \tag{8.35}$$

For the D_1 term we use

$$e^{G(\phi+g_0^n-h_0^n+g_P-h_P)} - e^{G(\phi+g_P-h_P)} = \int_0^1 dt \frac{d}{dt} e^{G(\phi+t(g_0^n-h_0^n)+g_P-h_P)}$$
$$= \int_0^1 dt \int_P dx (g_0^n - h_0^n)(x) \frac{\delta}{\delta\phi(x)} e^{G(\phi+t(g_0^n-h_0^n)+g_P-h_P)}. \tag{8.36}$$

We substitute this expression in Eq. (8.33) and then use Theorem 11.1 to bound the functional derivatives and functional integral as we did in the proof of part (1). The only difference is the extra functional derivative $\frac{\delta}{\delta\phi(x)}$. Its effects are harmless. For example, one n_B will be increased by 1. We leave it to the reader to check that a bound like Eq. (8.6) holds with some of the constants modified.

We now follow the proof of part (1) and conclude

$$|z_n(P) - z_0(P)| \leqq c\left(|CT| + \alpha_2 SE + \int_P |g_0^n - h_0^n|\right)\bar{z}(P). \tag{8.37}$$

By an easy modification of this proof we can add a factor of $e^{-|P|-\varepsilon JE(P)}$ with $\varepsilon > 0$ to the right-hand side of (8.37). So the proof is reduced to showing

$$\left[|CT| + \alpha_2 SE + \int_P |g_0^n - h_0^n|\right] e^{-|P|-\varepsilon JE(P)} \leqq c(h_0^n)^2 \sup_{\Delta\subset P} \int_\Delta [\chi - C(\chi)]. \tag{8.38}$$

Using Lemmas 10.1 and 9.2,

$$|CT| = \left|\sum_i \int h_i(h_0^n - g_0^n)\right| \leqq \sup_i \|h_i\|_\infty |h_0^n| \int_{\mathrm{supp}(P)} [\chi - C(\chi)]$$
$$\leqq cJE(P)\,|P|^{3/2}\,|h_0^n| \sup_{\Delta\subset P} \int_\Delta [\chi - C(\chi)]$$
$$\leqq c' e^{\varepsilon JE(P)+|P|} |h_0^n| \sup_{\Delta\subset P} \int_\Delta [\chi - C(\chi)].$$

For the other two terms in (8.38) we use

$$SE = \int_P h_0^n(h_0^n - g_0^n) \leqq |P|\,(h_0^n)^2 \sup_{\Delta\subset P} \int_\Delta [\chi - C(\chi)],$$

and

$$\int_P |g_0^n - h_0^n| \leqq SE.$$

Part (4). We leave the proof of part (4) to the reader. The basic idea is to combine the proof of part (2) with the idea in the proof of part (C) of Theorem 7.1 □

9. Energy Estimates – I

All the inequalities in this section are homogeneous in β. So in the proofs we take $\beta = 1$. The constant ε_1 is the constant that appears in the definition of monster; see Eq. (3.1). Some of the constants in this section and the following one depend on L. The main result of this section is the following theorem.

Theorem 9.1. *For R sufficiently large, depending on ε_1,*

$$|CT(P)| \leqq c\varepsilon_1^{1/6}[DE(P; s_1, \ldots, s_{m-1}) + SE(P)]. \tag{9.1}$$

We begin with an easy lemma.

Lemma 9.2. *There exist constants c_1, c_2, and c_3 such that for any hunk M and $h \in \mathscr{H}_M^{(0)}$, if we denote $\{x : h(x) \neq 0\}$ by* supp(h), *then*

(a)
$$\int_{\mathrm{supp}(h)} [\chi - C(\chi)] \leqq c_1 \int_M [\chi - C(\chi)]. \tag{9.2}$$

(b)
$$\sup_{A \subset \mathrm{supp}(h)} \int_A [\chi - C(\chi)] \leqq c_2 \sup_{A \subset M} \int_A [\chi - C(\chi)]. \tag{9.3}$$

The sup's *are over L-cubes A in* supp(h) *and M,*

(c)
$$|\mathrm{supp}(h)| \leqq c_3 |M|^{3/2}. \tag{9.4}$$

Proof. Thanks to the enlargements of hunks carried out in Sect. 3, $\{x \notin M : h(x) \neq 0\}$ is surrounded by M. In particular, along any ray from the center of Λ to a point p on $\partial\Lambda$,

$$d(p, \{x \notin M : h(x) \neq 0\}) \geqq d(p, M) + L'.$$

Parts (a) and (b) are easy consequences of this fact and the bounds

$$\frac{a_1}{R} e^{-(R-|x|)/l_D^+} \leqq \chi - C(\chi) \leqq \frac{a_2}{R} e^{-(R-|x|)/l_D^+}. \tag{9.5}$$

Part (c) is also immediate since the volume of a region can grow no faster than its surface area raised to the 3/2 power. □

The heart of the proof of Theorem 9.1 is the following technical lemma.

Lemma 9.3. *If R is sufficiently large, depending on ε_1, then for any hunk M and function $h \in \mathscr{H}_M^{(0)}$, which is not a monster,*

$$\int_\Lambda h^2[\chi - C(\chi)] \leqq c\varepsilon_1^{1/3} JE(h). \tag{9.6}$$

We postpone the proof of this lemma until after the proof of Theorem 9.1

Proof of Theorem 9.1. We consider two cases.

Case 1. There is a monster hunk. So $m = 1$ and

$$\sum_f \delta h_1(f)^2 \geqq \varepsilon_1 |\Lambda|. \tag{9.7}$$

This is case I of the sector definitions in Sect. 3.2.

Case 2. There is no monster. This is cases II and III of the sector definitions.

In case 1,

$$CT(P)=E(h,h_0^n)=\int_\Lambda h(h_0^n-g_0^n)=h_0^n\int_\Lambda h[\chi-C(\chi)],$$

where $h=h_1$. The sector n was defined so that

$$\left|\frac{\int_\Lambda \frac{1}{2\pi}h[\chi-C(\chi)]}{\int_\Lambda [\chi-C(\chi)]}\right|\leqq\tfrac{1}{2}. \tag{9.8}$$

Hence

$$|CT(P)|\leqq c|h_0^n|\int_\Lambda [\chi-C(\chi)]=c\left\{(h_0^n)^2\int_\Lambda [\chi-C(\chi)]\right\}^{1/2}\left\{\int_\Lambda [\chi-C(\chi)]\right\}^{1/2}. \tag{9.9}$$

Now M_1 is all of Λ, so

$$SE(P)=\int_\Lambda h_0^n(h_0^n-g_0^n)=(h_0^n)^2\int_\Lambda [\chi-C(\chi)].$$

By (9.5) and (9.7),

$$\int[\chi-C(\chi)]\leqq cR\leqq\frac{c'}{\varepsilon_1R^2}DE(P).$$

So (9.9) implies

$$|CT(P)|\leqq\frac{c}{\varepsilon_1^{1/2}R}SE(P)^{1/2}DE(P)^{1/2}\leqq\frac{c}{\varepsilon_1^{1/2}R}[SE(P)+DE(P)].$$

If R is large enough then $\frac{1}{\varepsilon_1^{1/2}R}\leqq\varepsilon_1^{1/6}$. So this proves Case 1.

Since $DE(P;s_1,\ldots,s_{m-1})\geqq c\sum_{i=1}^m JE(h_i)$, Case 2 reduces to showing that

$$\left|\int_\Lambda h_i(h_0^n-g_0^n)\right|\leqq c\varepsilon_1^{1/6}\left[\int_{M_i} h_0^n(h_0^n-g_0^n)+JE(h_i)\right] \tag{9.10}$$

for $i=1,2,\ldots,m$. By the Cauchy-Schwarz inequality,

$$\begin{aligned}\left|\int_\Lambda h_i(h_0^n-g_0^n)\right|&=\left|h_0^n\int_{\mathrm{supp}(h_i)} h_i[\chi-C(\chi)]\right|\\&\leqq|h_0^n|\left\{\int_{\mathrm{supp}(M_i)}[\chi-C(\chi)]\right\}^{1/2}\left\{\int h_i^2[\chi-C(\chi)]\right\}^{1/2}.\end{aligned} \tag{9.11}$$

By part (a) of Lemma 9.2 and Lemma 9.3 this is

$$\begin{aligned}&\leqq c|h_0^n|\left\{\int_{M_i}[\chi-C(\chi)]\right\}^{1/2}\{\varepsilon_1^{1/3}JE(h_i)\}^{1/2}\\&=c\varepsilon_1^{1/6}\left\{\int_{M_i}h_0^n(h_0^n-g_0^n)\right\}^{1/2}\{JE(h_i)\}^{1/2},\end{aligned} \tag{9.12}$$

which proves (9.10) and thus Case 2. □

Proof of Lemma 9.3. For any two points x, y we can express $h(x)-h(y)$ as the sum of the jumps that h makes at each face f which intersects the line segment from x to y. We write this as

$$h(x)-h(y)=\sum_{f\in[x,y]}\delta h(f),$$

and refer to the right-hand side as a "line integral."

We use $1(\,)$ to denote characteristic functions. For example, $1(f\in[x,y])$ is 1 if $f\in[x,y]$, 0 if $f\notin[x,y]$. This notation yields equations like

$$\int dy \sum_{f\in[x,y]}\delta h(y)=\sum_f \delta h(f)\int dy\, 1(f\in[x,y]).$$

All sums over faces f will only be over f with $\delta h(f)\neq 0$. Since h is not a monster, $\sum_f \delta h(f)^2 \leqq \varepsilon_1|\Lambda|$. So the total number of f with $\delta h(f)\neq 0$ is $\leqq c\varepsilon_1 R^3$.

We must replace the $h^2(x)$ in (9.6) by something like $[h(x)-h(y)]^2$ to use these line integrals. y will be a point in $\partial\Lambda$. Since $h\in\mathscr{H}^{(0)}$, the average of h over a large subset of $\partial\Lambda$ is near zero.

For $S\subset\partial\Lambda$ let

$$A=\frac{1}{|S|}\int_S dy h(y). \tag{9.13}$$

We use the sector definitions of Sect. 3.2 to choose S and to bound A. If M is an I-hunk, we let $S=\partial\Lambda$. Then $A=0$. If M is a normal B-hunk we let $S=\{x\in\partial\Lambda : h(x)=0\}$. Then $A=0$ and $|S|\geqq\frac{1}{2}|\partial\Lambda|$. If M is a jumbo B-hunk we let $S=\partial\Lambda$. Then by the sector definition for jumbos, $|A|\leqq\pi$. So

$$\int_\Lambda A^2[\chi-C(\chi)]\leqq cR\leqq \frac{c'}{R}JE(h), \tag{9.14}$$

since $JE(h)\geqq cR^2$ for a jumbo hunk. Note that in all three cases $|S|\geqq cR^2$.

For large R, $\frac{1}{R}\leqq\varepsilon_1^{1/3}$. So the preceding paragraph reduces the proof to bounding

$$\int_\Lambda |h-A|^2[\chi-C(\chi)].$$

For $x\in\Lambda$ let p_x be the projection with respect to the origin of x onto $\partial\Lambda$. Then it suffices to bound

$$\int_\Lambda dx[h(x)-h(p_x)]^2\,[\chi-C(\chi)] \tag{9.15}$$

and

$$\int_\Lambda dx[h(p_x)-A]^2\,[\chi-C(\chi)]. \tag{9.16}$$

In both expressions we will bound $[\chi-C(\chi)]$ by $\frac{c}{R}e^{-(R-|x|)/l_D^+}$.

To bound (9.15) we use a line integral from x to p_x. The number of $f\in[x,p_x]$ is $\leqq c|x-p_x|=c(R-|x|)$. So by Cauchy-Schwartz,

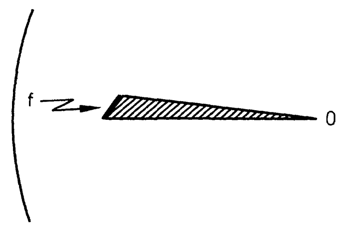
$$|h(x)-h(p_x)|^2=\left|\sum_{f\in[x,p_x]}\delta h(f)\right|^2\leqq c(R-|x|)\sum_{f\in[x,p_x]}\delta h(f)^2. \tag{9.17}$$

Thus (9.15) is

$$\leqq \frac{c}{R} \int_{\Lambda} dx(R-|x|)e^{-(R-|x|)/l_D^+} \sum_{f \in [x, p_x]} \delta h(f)^2$$

$$= \sum_f \delta h(f)^2 \frac{c}{R} \int_{\Lambda} dx(R-|x|)e^{-(R-|x|)/l_D^+} 1(f \in [x, p_x]) . \tag{9.18}$$

The function $1(f \in [x, p_x])$ is the characteristic function of the shaded region in the figure.



It easily follows that

$$\int_{\Lambda} dx(R-|x|)e^{-(R-|x|)/l_D^+} 1(f \in [x, p_x]) \leqq c .$$

The factor of $\frac{1}{R}$ in (9.18) is $\leqq \varepsilon_1^{1/3}$ for large R, so this takes care of (9.15).

Bounding the volume integral in (9.16) is easily reduced to bounding

$$\frac{1}{R} \int_{\partial\Lambda} dx[h(x)-A]^2 . \tag{9.19}$$

For the rest of the proof all integrals will be over $\partial\Lambda$ or a subset of $\partial\Lambda$. By (9.13),

$$|h(x)-A| = \frac{1}{|S|} \left| \int_S dy[h(x)-h(y)] \right|$$

$$\leqq \frac{c}{R^2} \left[\int_{|y-x| \geqq 1/2R} dy|h(x)-h(y)| + \int_{|y-x| < 1/2R} dy|h(x)-h(y)| \right],$$

where we have used $|S| \geqq \frac{1}{2}|\partial\Lambda| \geqq cR^2$ and bounded the integral over S by the integral over all of $\partial\Lambda$. So bounding (9.19) reduces to bounding

$$\frac{1}{R^5} \int dx \left(\int_{|y-x| \geqq 1/2R} dy|h(x)-h(y)| \right)^2 \tag{9.20}$$

and

$$\frac{1}{R^5} \int dx \left(\int_{|y-x| < 1/2R} dy|h(x)-h(y)| \right)^2 . \tag{9.21}$$

In (9.20) we will write $h(x)-h(y)$ as a line integral from x to y. If we did this in (9.21) the line could lie close to $\partial\Lambda$. This causes technical problems, so we will bound (9.21) in terms of (9.20). Let $z \in \partial\Lambda$ with $|z-x| \geqq \frac{3}{2}R$. Using the triangle inequality and then averaging over these z, we obtain

$$|h(x)-h(y)| \leqq \frac{a}{R^2} \int_{|z-x| \geqq 3/2R} dz(|h(x)-h(z)|+|h(z)-h(y)|), \tag{9.22}$$

where a is such that the area of $\{z \in \partial\Lambda : |z-x| \geqq \frac{3}{2}R\}$ is R^2/a.

Inequality (9.22) reduces bounding (9.21) to bounding

$$\frac{1}{R^9} \int dx \Bigg(\int_{|z-x| \geqq 3/2R} dz \int_{|y-x| \leqq 1/2R} dy |h(x)-h(z)| \Bigg)^2 \tag{9.23}$$

and

$$\frac{1}{R^9} \int dx \Bigg(\int_{|z-x| \geqq 3/2R} dz \int_{|y-x| \leqq 1/2R} dy |h(z)-h(y)| \Bigg)^2 . \tag{9.24}$$

In (9.23) we drop the constraint $|y-x| \leqq \frac{1}{2}R$, and do the integral over y. The result is (9.20). In (9.24) we note that $|z-x| \geqq \frac{3}{2}R$ and $|x-y| \leqq \frac{1}{2}R$ imply $|z-y| \geqq \frac{1}{2}R$. Using this observation and then doing the integral over x, (9.24) is bounded by

$$\frac{1}{R^7} \Bigg(\int dz \int_{|y-z| \geqq \frac{1}{2}R} dy |h(z)-h(y)| \Bigg)^2 .$$

Applying Cauchy-Schwartz to the integral over z, this is

$$\leqq \frac{1}{R^7} \int dz \Bigg(\int_{|y-z| \geqq 1/2R} |h(z)-h(y)| \Bigg)^2 \int dz\, 1 ,$$

which is (9.20).

It remains to bound (9.20).

$$\begin{aligned} \int_{|y-x| \geqq 1/2R} dy |h(x)-h(y)| &\leqq \int_{|y-x| \geqq 1/2R} dy \sum_{f \in [x,y]} |\delta h(f)| \\ &\leqq \Bigg(\int_{|y-x| \geqq 1/2R} dy \sum_{f \in [x,y]} \delta h(f)^2 \Bigg)^{1/2} \\ &\quad \cdot \Bigg(\int_{|y-x| \geqq 1/2R} dy \sum_{f \in [x,y]} 1 \Bigg)^{1/2} \end{aligned} \tag{9.25}$$

by Cauchy-Schwartz. We will show that

$$\int_{|y-x| \geqq 1/2R} dy \sum_{f \in [x,y]} 1 \leqq c\varepsilon_1^{1/3} R^3 . \tag{9.26}$$

Using (9.25) and (9.26), (9.20) is

$$\begin{aligned} &\leqq c \frac{\varepsilon_1^{1/3}}{R^2} \int dx \int_{|y-x| \geqq 1/2R} dy \sum_{f \in [x,y]} \delta h(f)^2 \\ &= c \frac{\varepsilon_1^{1/3}}{R^2} \sum_f \delta h(f)^2 \int_{|y-x| \geqq 1/2R} dx \int dy\, 1(f \in [x,y]) \\ &\leqq c\varepsilon_1^{1/3} \sum_f \delta h(f)^2 , \end{aligned} \tag{9.27}$$

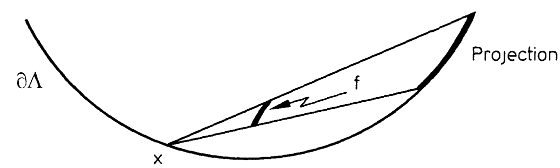
provided

$$\int_{|y-x|\geqq 1/2R} dx \int dy\, 1(f\in[x,y]) \leqq cR^2 . \qquad (9.28)$$

To prove (9.26) and (9.28) it is useful to first show that

$$\int_{|x-y|\geqq 1/2R} dy\, 1(f\in[x,y]) \leqq c\frac{R^2}{d(x,f)^2} . \qquad (9.29)$$

Without the constraint $|x-y|\geqq \frac{1}{2}R$, this integral would be the area of the projection with respect to x of f onto $\partial\Lambda$. See figure.



The constraint $|x-y|\geqq\frac{1}{2}R$ implies that for f such that (9.29) is not zero, the angles between the lines in the figure and $\partial\Lambda$ are bounded away from zero. Inequality (9.29) now follows.

To prove (9.26) we begin with

$$\sum_{f\in[x,y]} 1 \leqq c\varepsilon_1^{1/3}R + \sum_{\substack{f\in[x,y] \\ d(f,x)\geqq \varepsilon_1^{1/3}R}} 1 . \qquad (9.30)$$

The $c\varepsilon_1^{1/3}R$ contributes $c\varepsilon_1^{1/3}R^3$ to (9.26). The second term in (9.30) contributes

$$\begin{aligned} \int_{|y-x|\geqq 1/2R} dy \sum_{\substack{f\in[x,y] \\ d(f,x)\geqq \varepsilon_1^{1/3}R}} 1 &= \sum_{f:d(x,f)\geqq \varepsilon_1^{1/3}R} \int_{|y-x|\geqq 1/2R} dy\, 1(f\in[x,y]) \\ &\leqq \sum_{f:d(x,f)\geqq \varepsilon_1^{1/3}R} c\frac{R^2}{d(x,f)^2} \\ &\leqq \frac{c}{\varepsilon_1^{2/3}} \sum_f \leqq c'\varepsilon_1^{1/3}R^3 , \end{aligned}$$

using (9.29) and the fact that the total number of f with $\delta h(f)\neq 0$ is $\leqq c\varepsilon_1 R^3$.

Finally, we prove (9.28). It is symmetric in x and y, so at the expense of a factor of 2 we can assume $d(x,f)\geqq d(y,f)$. Since $|x-y|\geqq\frac{1}{2}R$, this implies $d(f,x)\geqq\frac{1}{4}R$. Using (9.29),

$$\begin{aligned} &\int_{d(x,f)\geqq 1/4R} dx \int_{|y-x|\geqq 1/2R} dy\, 1(f\in[x,y]) \\ &\leqq \int_{d(x,f)\geqq 1/4R} dx\, c\frac{R^2}{d(x,f)^2} \leqq 16c'R^2 , \end{aligned}$$

which proves (9.28). □

Remark. In this section, and the next, we ignore the second enlargement of hunks, as given in Subsect. 3.4. Its inclusion would not change the results or proofs, only complicate the notation.

10. Energy Estimates – II

We begin by stating a very crude estimate. We leave the easy proof to the reader.

Lemma 10.1. *For any* $h \in \mathscr{H}^{(0)}$,

$$\|h\|_\infty \leqq c\beta^{1/2} JE(h). \tag{10.1}$$

The following theorem was used in Sect. 8 to control $k^1(i, \eta(i))$.

Theorem 10.2. *Let* $\varepsilon > 0$. *Then there exists* $\beta_0 > 0$ *such that for* $\beta \leqq \beta_0$ *and any hunk* M' *and* $h' \in \mathscr{H}_{M'}^{(0)}$,

$$\sum_{M : M \cap M' = \phi} \sup_{h \in \mathscr{H}_M^{(0)}} e^{-\varepsilon JE(h)} |E(h, h')| \leqq cJE(h'), \tag{10.2}$$

where M *is summed over all hunks disjoint from* M'.

We will reduce the proof of Theorem 10.2 to part (a) of the following theorem. Part (a) was also used in Sect. 8 to control $k^2(i, \eta(i))$. Part (b) was used to control $k^3(i, \eta(i))$ and $k^4(i, \eta(i))$.

Theorem 10.3. *Let* M *be any hunk,* $h \in \mathscr{H}_M^{(0)}$ *and* $g = C(\chi h)$. *Then*

(a)
$$\int_\Lambda |g - h| \leqq c\beta^{1/2} JE(h), \tag{10.3}$$

(b)
$$\int_B |g - h| \leqq c \|h\|_\infty |\mathrm{supp}(h)| \mathscr{V}_\varepsilon(B, \mathrm{supp}(h)), \tag{10.4}$$

where $\mathscr{V}_\varepsilon$ *is defined as in Eq.* (1.35).

The bound in part (b) is very crude. If M is an annulus-like region and $h \neq 0$ in the region surrounded by M, then $\int_B |g - h|$ will fall off as B moves away from M whether B is inside the region surrounded by M or outside it. But $\mathscr{V}_\varepsilon(B, \mathrm{supp}(h))$ only falls off for B outside this region.

Proof of Theorem 10.2. Letting $g' = C(\chi h')$,

$$E(h, h') = \int_\Lambda h(h' - g') = \int_{\mathrm{supp}(h)} h(h' - g'). \tag{10.5}$$

So

$$|E(h, h')| \leqq \|h\|_\infty \sum_{\Delta \subset \mathrm{supp}(h)} \int_\Delta |h' - g'|, \tag{10.6}$$

where Δ is summed over unit cubes in $\mathrm{supp}(h)$. Since $\mathrm{supp}(h)$ is surrounded by M, $\Delta \subset \mathrm{supp}(h)$ implies $d(\Delta, M) \leqq \mathrm{diam}(M)$. So we can bound the sum over $\Delta \subset \mathrm{supp}(h)$ by the sum over Δ with $d(\Delta, M) \leqq \mathrm{diam}(M)$. By Lemma 10.1,

$$\|h\|_\infty e^{-\varepsilon/2\, JE(h)} \leqq 1$$

for small enough β. So

$$|E(h, h')| e^{-\varepsilon/2\, JE(h)} \leqq \sum_{\Delta : d(\Delta, M) \leqq \mathrm{diam}(M)} \int_\Delta |h' - g'|. \tag{10.7}$$

In the other factor of $e^{-\varepsilon/2\, JE(h)}$, we use $JE(h) \geqq c\beta^{-1/2}|M|$. The sup over $h \in \mathscr{H}_M^{(0)}$ then becomes trivial, and the proof reduces to showing

$$\sum_M e^{-\varepsilon/2\, c\beta^{-1/2}|M|} \sum_{\Delta : d(\Delta, M) \leqq \mathrm{diam}(M)} \int_\Delta |h' - g'| \leqq cJE(h'). \tag{10.8}$$

The left-hand side equals

$$\sum_{\Delta} \int_{\Delta} |h'-g'| \sum_{M:d(\Delta,M)\leqq \operatorname{diam}(M)} e^{-\varepsilon/2c\beta^{-1/2}|M|}. \tag{10.9}$$

For small enough β the sum over M converges and is less than 1. So (10.9) is

$$\leqq \sum_{\Delta} \int_{\Delta} |h'-g'| = \int_{\Lambda} |g'-h'|,$$

which is bounded by part (a) of Theorem 10.3. □

Proof of Theorem 10.3. Both inequalities are homogeneous in β, so we set $\beta=1$.

Part (a). We begin with

$$|h-g| \leqq |h-hC(\chi)| + |hC(\chi)-g|. \tag{10.10}$$

The first term contributes

$$\int_{\Lambda} |h-hC(\chi)| = \int_{\Lambda} |h| [\chi - C(\chi)].$$

We can bound this using Lemma 9.3 since $|h| \geqq 1$.

The second term in (10.10) contributes

$$\begin{aligned} \int_{\Lambda} |hC(\chi)-g| &= \int_{\Lambda} dx \left| h(x) \int_{\Lambda} dy C(x,y) - \int_{\Lambda} dy C(x,y) h(y) \right| \\ &\leqq \int_{\Lambda} dx \int_{\Lambda} dy C(x,y) |h(x)-h(y)|. \end{aligned} \tag{10.11}$$

Theorem 1.4 says

$$C(x,y) \leqq c[C_1(x,y) + C_2(x,y)], \tag{10.12}$$

where

$$C_1(x,y) = e^{-|x-y|/l_D^+},$$

$$C_2(x,y) = e^{-(R-|x|+R-|y|)/l_D^+} \frac{1}{1+s^{3-\varepsilon}},$$

with s defined as in Theorem 1.4. So it suffices to bound (10.11) with $C(x,y)$ replaced by $C_1(x,y)$ and $C_2(x,y)$.

In the integral containing $C_1(x,y)$ we write $h(x)-h(y)$ as a line integral from x to y. Then the integral is

$$\leqq \sum_{f} |\delta h(f)| \int_{\Lambda} dx \int_{\Lambda} dy e^{-|x-y|/l_D^+} 1(f \in [x,y]).$$

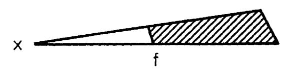
So it suffices to show

$$\int_{\Lambda} dx \int_{\Lambda} dy e^{-|x-y|/l_D^+} 1(f \in [x,y]) \leqq c. \tag{10.13}$$

At the expense of a factor of 2 we can add to (10.13) the constraint that $d(f,y) \leqq d(f,x)$. Consider

$$\int_{d(f,y)\leqq d(f,x)} dy e^{-|x-y|/l_D^+} 1(f \in [x,y]). \tag{10.14}$$

This is the integral of $e^{-|x-y|/l_D^+}$ over y in the shaded region



So (10.14) is $\leqq ce^{-d(x,f)/l_D^+}$, which proves (10.13).

In the integral containing $C_2(x, y)$, we use

$$|h(x)-h(y)| \leqq |h(x)-h(p_x)| + |h(p_x)-h(p_y)| + |h(p_y)-h(y)|,$$

where p_x is the projection (with respect to the origin) of x onto $\partial\Lambda$. The contribution of the $|h(x)-h(p_x)|$ and $|h(p_y)-h(y)|$ terms to the integral are bounded in essentially the same way that (9.15) was bounded in the proof of Lemma 9.3.

We are left with

$$\int_\Lambda dx \int_\Lambda dy\, e^{-(R-|x|+R-|y|)/l_D^+} \frac{1}{1+s^{3-\varepsilon}} |h(p_x)-h(p_y)|, \tag{10.15}$$

which reduces to the surface integral

$$\int_{\partial\Lambda} dx \int_{\partial\Lambda} dy \frac{1}{1+s^{3-\varepsilon}} |h(x)-h(y)|. \tag{10.16}$$

We will use inequality (9.28) from the proof of Lemma 9.3. Since our integral does not have the constraint $|x-y| \geqq \frac{1}{2}R$, we cannot simply write $h(x)-h(y)$ as a line integral from x to y.

Instead we use

$$|h(x)-h(y)| \leqq |h(x)-h(z)| + |h(z)-h(y)|, \tag{10.17}$$

where $z \in \partial\Lambda$. There is a $c>0$ such that for any $x, y \in \partial\Lambda$, the area of $\{z \in \partial\Lambda : |x-z| \geqq \frac{1}{2}R$ and $|y-z| \geqq \frac{1}{2}R\}$ is $\geqq R^2/c$. So we can average (10.17) over these z and conclude that (10.16) is

$$\leqq \frac{c}{R^2} \int dz \int_{|x-z| \geqq 1/2R} dx \int_{|y-z| \geqq 1/2R} dy \frac{1}{1+s^{3-\varepsilon}} [|h(x)-h(z)| + |h(z)-h(y)|]. \tag{10.18}$$

The two terms in (10.18) are identical. In the first we use

$$\int dy \frac{1}{1+s^{3-\varepsilon}} \leqq c.$$

Then we write $h(x)-h(z)$ as a line integral from x to z and conclude this term is

$$\leqq \sum_f |\delta h(f)| \frac{c}{R^2} \int_{|x-z| \geqq 1/2R} dz \int dx\, 1(f \in [x, y]) \leqq c \sum_f |\delta h(f)|$$

by (9.28).

Part (b). If $B \cap \mathrm{supp}(h) \neq \phi$, then $\mathscr{V}_\varepsilon(B, \mathrm{supp}(h)) = 1$, and the inequality is trivial. If $B \cap \mathrm{supp}(h) = \phi$, then

$$\int_B |g-h| = \int_B |g| = \int_B dx \left| \int_\Lambda dy\, C(x, y) h(y) \right|$$
$$\leqq \|h\|_\infty \int_B dx \int_{\mathrm{supp}(h)} dy\, C(x, y),$$

and the inequality follows. $\square$

11. Functional Derivatives

This section, the next section, and Appendices C, D, and E are all devoted to proving the following estimate.

Theorem 11.1. *Let ε, $\delta_0 > 0$. If α_1 is sufficiently small and PM $\beta < \beta_0$, $R > R_0$, then*

$$e^{-(1-\alpha_1/2)DE-\varepsilon SE}\left|\int d\mu \prod_{i=2}^{m} k(i,\eta(i))e^{G}\right|$$

$$\leqq \sum_{t_2 \ldots t_k} \sum^{(1)} \ldots \sum^{(m)} \prod_{i=2}^{m} \bar{k}^{t_i}(i,\eta(i)) \prod_{B\subset P} (n_B!)^{3/2-\varepsilon_0} \delta_0^{m}, \qquad (11.1)$$

In this section we bound the functional derivatives and do the easy part of the functional integral estimate. The techniques of [7] could be used to bound the functional derivatives. Unfortunately, the resulting bound would contain $(n_B!)^p$ with $p > 3/2$. Obtaining the above bound requires a few improvements in the usual techniques. We explain these improvements and leave most of the routine work to the reader.

Proof of Theorem 11.1. Recall that $d_k = |\{i : \eta(i) = k\}|$. So $\sum_k d_k = m-1$, and hence $\sum_k (2-d_k) = m+1$. So if we can associate a factor of $\delta_0^{2-d_k}$ with vertex k, then we will have the desired factor of δ_0^m. Note that for vertices with $d_k > 2$ this procedure allows derivatives to contribute large factors $(1/\delta_0)$ rather than small ones. We take advantage of this fact in Appendix C.

Depending on t, $k^t(i,\eta(i))$ need not contain a derivative in Y_i or $Y_{\eta(i)}$. So there can be vertices k without any functional derivatives in Y_k. A glance at the definitions of $k^t(i,\eta(i))$ reveals this can happen only when Y_k is a hunk. Since $DE \geqq c\beta^{-1}$ (number of hunks), we can write

$$e^{-(1-\alpha_1/2)DE} = e^{-(1-\alpha_1)DE}\, e^{-\alpha_1/2\, DE},$$

and use $e^{-\alpha_1/2\,DE}$ to provide the factors of $\delta_0^{2-d_k}$ for these hunks without derivatives.

Now consider the hunks or atoms Y_k that do contain derivatives. Recall that n_B is the number of derivatives in B. Our bound will contain a factor of $\delta_0^{3-n_B}$ for each B with $n_B \geqq 1$. Since

$$\sum_{B\subset Y_k} n_B \leqq d_k + 1, \qquad (11.2)$$

this gives the desired factor of $\delta_0^{2-d_k}$.

We have reduced the proof to the following two lemmas.

Lemma 11.2. *There are constants $\gamma < 1$ and $c > 0$ such that if $\delta_0 > 0$ and PM $\beta < \beta_0$, $R > R_0$, then the following is true. Let P be a polymer. Let $B_1, \ldots, B_n$ be L-cubes in P with repetitions allowed. Let n_B be the number of times B appears in $B_1, \ldots, B_n$. Let $f(x_1, \ldots, x_n)$ be a function on $B_1 \times \ldots \times B_n$. Then*

$$\left|\int_{B_1\times\ldots\times B_n} f(x_1,\ldots,x_n) \prod_{i=1}^{n} \frac{\delta}{\delta\phi(x_i)} e^{G}\right|$$

$$\leqq L^{3n}\left[\sup_{B_1\times\ldots\times B_n} |f|\right] e^{\gamma/2 \int_P A^2 + 2\int_P \delta^2} \sum_a C_a F_a, \qquad (11.3)$$

where a is summed over some index set.

$$F_a = \prod_{B\subset P} \left\{ |A_B|^{K_a(B)} \left[L^{-3} \int_B \delta^2(x) \right]^{1/2 l_a(B)} \right\}. \tag{11.4}$$

A_B and $\delta(x)$ are defined in the proof. $K_a(B)$ and $l_a(B)$ are nonnegative integers. For each a let

$$f_a = \prod_{B\subset P} [\tfrac{1}{2}K_a(B) + \tfrac{1}{2}l_a(B)]!\,. \tag{11.5}$$

Then the coefficients C_a are such that

$$\sum_a C_a f_a \leqq c^n \prod_{B\subset P:\, n_B \geqq 1} [(n_B!)^{3/2-\varepsilon_0} \delta_0^{3-n_B}]\,. \tag{11.6}$$

Lemma 11.3. *Let $\varepsilon>0$ and let F_a, f_a and γ be as in the previous lemma. Then if PM $\beta<\beta_0$, $R>R_0$, then*

$$\int d\mu e^{\gamma/2 \int_P A^2 + 2\int_P \delta^2} F_a \leqq f_a e^{(1-\alpha_1)DE + \varepsilon SE}\,. \tag{11.7}$$

Remark. When we use Lemma 11.1 in the proof of Theorem 11.2, the function $f(x_1, \ldots, x_n)$ will be a product of functions that appear in k^t, i.e., $(g_k - h_k)(x)$ and $C(x, y)$. The resulting bound does not contain $\bar{k}^t$ as it is defined in Eq. (8.4). Instead, expressions like $\int_C |g-h|$ should be replaced by $L^3 \sup_C |g-h|$ in (8.4). The proofs in Sects. 8–10 all work for this modified $\bar{k}^t$.

Proof of Lemma 11.2. We have

$$e^G = \prod_{B\subset P} e^{G_B}$$

with G_B defined in the obvious way. Thus it suffices to prove the lemma with P replaced by an L-cube B. For the remainder of the proof we denote G_B and n_B by G and n.

Recall that e^G is $e^{G(\phi+(g-h)(s))}$. Let

$$\phi_s = \phi + (g-h)(s)\,. \tag{11.8}$$

So $e^G = e^{G(\phi_s)}$. We introduce average and fluctuation fields in the usual way.

$$A = L^{-3} \int_B \phi_s(x)dx\,, \qquad \delta(x) = \phi_s(x) - A\,. \tag{11.9}$$

The function e^G is defined implicitly by Eq. (2.10). We find

$$\begin{aligned} e^{G(\phi_s)} = \exp\Big[& i \int_B \phi_s u^{-1} \psi - \int_B (\tilde{z}_+ e^{\tilde{\psi}} + \tilde{z}_- e^{-\tilde{\psi}} - 2\tilde{z}) \\ & + \int_B (\tilde{z}_+ e^{\tilde{\psi} + i\beta^{1/2}\phi_s} + \tilde{z}_- e^{-\tilde{\psi} - i\beta^{1/2}\phi_s} - 2\tilde{z}) \Big] \\ & \cdot \left\{ \sum_{n=-\infty}^{\infty} \exp\left[-\tfrac{1}{2} \int_B (\phi_s - n\tau)^2 \right] \right\}^{-1} \end{aligned} \tag{11.10}$$

with $\tau=\beta^{-1/2}2\pi$. Let

$$r(A)=\exp[L^3 2\tilde{z}(\cos\beta^{1/2}A-1)]\left\{\sum_{n=-\infty}^{\infty}\exp[-\tfrac{1}{2}L^3(A-n\tau)^2]\right\}^{-1},$$

$$G_1=2\tilde{z}\int_B(\cos\beta^{1/2}\delta-1+\tfrac{1}{2}\beta\delta^2)+2\tilde{z}\int_B\mathrm{Re}[(e^{i\beta^{1/2}A}-1)(e^{i\beta^{1/2}\delta}-i\beta^{1/2}\delta-1)],$$

$$G_2=\tilde{z}\int_B[(e^{\tilde{\psi}-\tilde{\psi}_0}-1)(e^{i\beta^{1/2}\phi_s}-i\beta^{1/2}\phi_s-1)$$
$$+(e^{-(\tilde{\psi}-\tilde{\psi}_0)}-1)(e^{-i\beta^{1/2}\phi_s}+i\beta\phi_s-1)]. \tag{11.11}$$

We leave it to the reader to check that $e^G=r(A)e^{G_1+G_2}$. We have used Eq. (1.20) to rewrite the $i\int_B\phi_s u^{-1}\psi$ term in G.

Derivatives of $r(A)$ are dealt with in Appendix C.

Derivatives of e^{G_1} are bounded by the usual techniques. Such derivatives provide small factors since $\beta^{1/2}$ is small. These derivatives can also introduce factors of fields $\delta(x)$ or A. We leave it to the reader to check that any order derivative of G_1 contributes at most two factors of fields. When n derivatives act on e^{G_1}, the number of terms grows like $n!$ Each term contains at most $2n$ factors of fields. So a crude bound on $\sum_a C_a f_a$ contains $(n!)^2$. To obtain a bound that only contains $n!$ we use Lemma 11.4 below.

In G_2 derivatives of $\tilde{z}(e^{\pm i\beta^{1/2}\phi_s}\mp i\beta^{1/2}\phi_s-1)$ do not necessarily give us a small factor. However, by Theorem 1.1,

$$|e^{\pm(\tilde{\psi}-\tilde{\psi}_0)}-1|\leqq\frac{c}{R},$$

so the $\frac{1}{R}$ provides the needed smallness. As with G_1 we also use Lemma 11.4.

Besides bounding derivatives of G_1 and G_2 we must also bound the factors e^{G_1} and e^{G_2}. Easy estimates give

$$|e^{G_1}|\leqq e^{3/2(2\beta\tilde{z})\int_B\delta^2}, \tag{11.12}$$

$$|e^{G_2}|\leqq e^{c/R\int_B\phi_s^2}\leqq e^{2c/R\int_B(\delta^2+A^2)}. \tag{11.13}$$

By Lemma C.1, $D^N r(A)$ contributes a factor of $e^{\gamma/2\int_B A^2}$. Since $\frac{c}{R}$ is small, we can bound the product of these three factors by

$$e^{\gamma/2\int_B A^2+2\int_B\delta^2},$$

if we increase γ slightly. □

Lemma 11.4. *Let $D_1,\ldots,D_n$ denote functional derivatives, and F a function of the fields, then*

$$D_1D_2\ldots D_n e^F=\sum_a F_a e^F, \tag{11.14}$$

where F_a is a product of derivatives of F. More precisely, for each a there is an *integer K_a and K_a disjoint subsets I_i of $\{1,2,\dots,n\}$ with $\bigcup_{i=1}^{K_a} I_i=\{1,2,\dots,n\}$ such that*

$$F_a=\prod_{i=1}^{K_a}\left[\left(\prod_{j\in I_i} D_j\right)F\right]$$

and

$$\sum_a K_a!\leqq n!2^n. \tag{11.15}$$

Proof. Easy induction. □

Remark. Equation (11.14) is trivial. The important part of the lemma is the bound (11.15). If some of the $D_1,\dots,D_n$ are equal then some of the F_a will be equal. Nonetheless, we do not combine these terms in (11.14). So F_a does not contain any "counting factors" of 2 or 3 or ....

Proof of Lemma 11.3. Choose a positive integer q which is large enough that $p\gamma<1$, where $\frac{1}{p}+\frac{1}{q}=1$ and γ is as in Lemma 11.2. By Holder's inequality it suffices to bound

$$\left(\int d\mu e^{p\gamma/2\int_P A^2+p2\int_P \delta^2}\right)^{1/p} \quad \text{and} \quad \left(\int d\mu F_a^q\right)^{1/q}.$$

The first expression is bounded in Theorem 12.1.

In the second expression we bound A's and δ's in terms of ϕ's and $(g-h)$'s. Then we use Lemma 11.5 below and bounds like

$$\left[\int_B (g-h)^2\right]^K\leqq K!\left(\frac{1}{\varepsilon}\right)^K e^{\varepsilon\int_B (g-h)^2}.$$

By results in Sect. 12, $\int_P (g-h)^2\leqq cDE$. The lemma follows by choosing α_1 and ε sufficiently small. □

Lemma 11.5. *Let $C(x,y)$ be a covariance which is integrable in the sense that*

$$\sum_B \sup_{y\in B}|C(x,y)|\leqq c_0 \quad \forall x, \tag{11.16}$$

where B is summed over L-cubes. Let $x_1,\dots,x_l$ be points in different L-cubes. Let $m_1,\dots,m_l$ be positive integers. Then

$$\int d\mu \prod_{i=1}^{l} \phi(x_i)^{2m_i}\leqq \prod_{i=1}^{l}\left[c(4c_0)^{m_i}m_i!\right], \tag{11.17}$$

where c is a universal constant.

Remarks. 1. If the power of $2m_i$ is replaced by m_i, then this result is standard.

2. Lemma 11.5 is not essential. By taking advantage of the fact that derivatives need only provide small factors for $n_B=1$ and 2, one can bound the derivatives in such a way that the standard result alluded to in Remark 1 is sufficient.

3. The constant c_0 will of course depend on L.

4. In cases where the covariance is not bounded for coincident arguments, one may derive similar results for smeared fields.

Proof. For any $y \geqq 0$, $y^m \leqq m!\ c^m e^{y/c}$. So

$$\phi(x_i)^{2m_i} \leqq m_i!(4c_0)^{m_i} e^{1/(4c_0)\phi^2(x_i)}.$$

Thus it suffices to prove

$$\int d\mu \exp\left[\frac{1}{4c_0}\sum_{i=1}^{l} \phi^2(x_i)\right] \leqq c^l. \tag{11.18}$$

We use the identity

$$\exp\left[\frac{1}{4c_0}\phi^2(x_i)\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk_i \exp\left[-\tfrac{1}{2}k_i^2 + \frac{1}{\sqrt{2c_0}} k_i \phi(x_i)\right]$$

in the left-hand side of (11.18). Then we can compute the integral with respect to $d\mu$. We find that the left-hand side of (11.18) is

$$\prod_{i=1}^{l}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk_i\right] \exp\left[-\tfrac{1}{2}\sum_{i=1}^{l} k_i^2 + \frac{1}{4c_0}\sum_{i,j=1}^{l} k_i k_j C(x_i, x_j)\right]. \tag{11.19}$$

A well-known consequence of the Cauchy Schwartz inequality is that

$$\sum_{i,j=1}^{l} k_i k_j C(x_i, x_j) \leqq \left(\sum_{i=1}^{l} k_i^2\right) \sup_j \sum_m |C(x_j, x_m)|.$$

So (11.19) is

$$\leqq \prod_{i=1}^{l}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk_i\right] \exp\left[-\tfrac{1}{4}\sum_{i=1}^{l} k_i^2\right] = c^l. \quad \square$$

12. The Vacuum Energy

This section studies the problem treated in Sect. 9.6 of [7] in the present context. This is the non-trivial aspect of the functional integration estimates. We follow many of the ideas of and some of the notation of [7]; but new and interesting problems also arise. In particular the mathematical machinery of Appendices D and E is necessary; in this connection one needs the second enlargement of hunks as carried out in Subsect. 3.4.

For a given polymer P we study

$$\int d\mu_s e^{p\gamma B}, \tag{12.1}$$

with

$$B = \tfrac{1}{2}\int (\phi + g - h)^2 + \frac{2}{\gamma}\int \delta^2, \tag{12.2}$$

the integrals over the region of the polymer.

Theorem 12.1. *Given $p\gamma < \frac{1}{\tilde{l}_D^2}$, if $L/\tilde{l}_D$ and β are sufficiently small, and $L'/\tilde{l}_D$ and R are sufficiently large, then*

$$\left(\int d\mu_s e^{p\gamma B}\right)^{1/p} \leqq e^{c|P|} e^{\delta DE} e^{\varepsilon SE}, \tag{12.3}$$

where $\delta < 1$ and ε may be picked arbitrarily small by picking R large.

This section is devoted to the proof of this theorem. We find it convenient to temporarily consider $g-h$ in the polymer in the case when all interpolation parameters equal 1,

$$g-h=g_0^n-h_0^n+\sum_{i\in P}(g_i-h_i), \tag{12.4}$$

where

$$g_i=C(h_i). \tag{12.5}$$

For each hunk M_i in P, let g_i^* be as in Subsect. E.2. We define

$$h_i^*=C^{-1}g_i^*. \tag{12.6}$$

We also define

$$\tilde{g}_i=g_i-g_i^*, \qquad \tilde{h}_i=h_i-h_i^*, \tag{12.7}$$

$$\tilde{g}-\tilde{h}=\sum_i(\tilde{g}_i-\tilde{h}_i). \tag{12.8}$$

It is important to note that $\tilde{h}_i$ and $(g_i^*-h_i^*)$ are supported in M_i. We also trivially have

$$\tilde{g}=C(\tilde{h}). \tag{12.9}$$

We may write Eq. (12.4) as

$$g-h=\tilde{g}-\tilde{h}+g_0^n-h_0^n+\sum_i(g_i^*-h_i^*)=\tilde{g}-\tilde{h}+e. \tag{12.10}$$

We now no longer restrict ourselves to the situation when all interpolation parameters equal 1. Equation (12.10) becomes

$$g-h=(g-h)(s)=(\tilde{g}-\tilde{h})(s)+e, \tag{12.11}$$

where

$$\tilde{h}(s)=\tilde{h}(0), \qquad \tilde{g}(s)=C(s)(\tilde{h}), \tag{12.12}$$

and e in unchanged. Our purpose in introducing the function g_i^* is to yield (12.12). (It is only hunks M_i, for which $h_{M_i}\neq 0$ outside M_i, that have thus caused us difficulties.)

In (12.1) we make the substitution $\phi\to\phi-\tilde{g}$ and arrive at

$$(12.1)=e^{\tilde{E}}\int d\mu_s e^{-1/2\int_\Lambda(\tilde{h}^2-2\phi\tilde{h})}e^{p\gamma B}, \tag{12.13}$$

where

$$\tilde{E}=\tfrac{1}{2}\int(\tilde{h}\tilde{h}-\tilde{g}\tilde{h}), \tag{12.14}$$

and in B the indicated substitution is understood. It is enough to estimate (12.13) with the measure $d\mu_s$ replaced by $d\mu_i$ by the argument of (9.67) of [7]. We define

$$\langle h,h\rangle=\tfrac{1}{2}\int[hh-hCh], \tag{12.15}$$

for any function h, where C is the covariance (associated to $d\mu_s$) including interpolation. So $\tilde{E}=\langle\tilde{h},\tilde{h}\rangle$.

Lemma 12.2. *For each $\varepsilon>0$, one has*

$$|E-\tilde{E}|\leqq \varepsilon E+\left(1+\frac{1}{\varepsilon}\right)\langle h^*, h^*\rangle\,, \tag{12.16}$$

$$\langle h^*, h^*\rangle=\sum_i \langle h_i^*, h_i^*\rangle\,. \tag{12.17}$$

Lemma 12.3. *For each $\varepsilon>0$, L', L, one has for β sufficiently small*

$$\langle h_i^*, h_i^*\rangle \leqq \varepsilon J E_i\,. \tag{12.18}$$

Proof of Lemmas 12.2 and 12.3. Equation (12.17) follows from the fact that $g_i^*-h_i^*$ is supported in M_i, and (12.6) and (12.15). Inequality (12.18) follows from (E.6), (E.9), and (12.6). To show (12.16), we realize the covariance (associated to $d\mu_s$) as the convex sum of (9.65), (9.66), of [7],

$$C=C(s)=\sum \lambda_i C_i\,, \qquad \lambda_i=\lambda_i(s)\,, \tag{12.19}$$

$$C_i=\sum_j \chi_{ij} C \chi_{ij}\,. \tag{12.20}$$

We write $k\in ij$ if M_k is contained in the support of χ_{ij}. We let

$$\hat{h}_{ij}=\sum_{k\in ij} h_k \tag{12.21}$$

and define $\langle h, h\rangle_0$ as $\langle h, h\rangle$ defined with C as the uninterpolated covariance. We have

$$E=\sum_i \lambda_i \sum_j \langle \hat{h}_{ij}, \hat{h}_{ij}\rangle_0\,, \tag{12.22}$$

$$\langle h^*, h^*\rangle=\sum_i \lambda_i \sum_j \langle \hat{h}_{ij}^*, \hat{h}_{ij}^*\rangle_0\,, \tag{12.23}$$

$$\tilde{E}=\langle \tilde{h}, \tilde{h}\rangle=\sum_i \lambda_i \sum_j \langle \hat{\tilde{h}}_{ij}, \hat{\tilde{h}}_{ij}\rangle_0\,. \tag{12.24}$$

Equation (12.16) follows from (12.7) and (12.22)–(12.24) using

$$|\langle a, b\rangle_0|\leqq \frac{1}{2\varepsilon}\langle a, a\rangle_0+\frac{\varepsilon}{2}\langle b, b\rangle_0\,, \tag{12.25}$$

the ultra-useful ultra-trivial form of Schwartz's inequality.

We now return to our study of (12.13). We need only consider the following objects

$$\int d\mu e^{-1/2\int \chi_{ij}[\tilde{h}^2-2\phi\tilde{h}]} e^{p\gamma B_{ij}}\,, \tag{12.26}$$

where B_{ij} is B with the integrals restricted to the support of χ_{ij}. We change the gaussian measure similar to in (9.629)–(9.631) of [7],

$$(12.26)=N_{0ij}^{-1}\int d\mu_0 e^{-1/2\int \chi_{ij}(\phi-\hat{h})^2} e^{p\gamma B_{ij}(\phi-\hat{h}+e)}\,, \tag{12.27}$$

where

$$\prod_j N_{0ij}^{-1}\leqq e^{c|P|}\,. \tag{12.28}$$

Using the idea of (12.25) again,

$$xy \leqq \frac{\varepsilon}{2}x^2 + \frac{1}{2\varepsilon}y^2,$$

one gets

$$(12.27) \leqq N_{0ij}^{-1} \int d\mu_0 e^{-1/2\int \chi_{ij}[(\phi-\hat{h})^2 - \gamma p'(\phi-\tilde{h})^2 - c(\delta\phi)^2]} \cdot e^{c\int \chi_{ij}[e^2 + (-\delta\tilde{h}+\delta e)^2]}, \tag{12.29}$$

$$\gamma p' < 1. \tag{12.30}$$

The functional integral in (12.29) is controlled exactly as in [7]. The following lemmas complete the proof of Theorem 12.1:

Lemma 12.4. *For each $\varepsilon > 0$, one has for R sufficiently large*

$$\int_{M_i} (g_0^n - h_0^n)^2 \leqq \varepsilon S E_i. \tag{12.31}$$

Lemma 12.5. *With the same conditions as Lemma 12.3*

$$\int (g_i^* - h_i^*)^2 \leqq \varepsilon (JE)_i. \tag{12.32}$$

Lemma 12.6. *With the same conditions as Lemma 12.3*

$$\int (\delta h_i^*)^2 \leqq \varepsilon (JE)_i. \tag{12.33}$$

Lemma 12.4 follows easily from properties of g_0^n. Lemmas 12.5 and 12.6 are proved as Lemma 12.3. In fact Lemma 12.5 is an immediate consequence of Lemma 12.3, from the identity that if

$$g = C(h),$$

then

$$\int (hh - gh) = \int g(C^{-1} - 1)g + \int (g-h)^2.$$

Appendix A. The Complex Translation ψ

The results in Subsects. A.1 and A.2 are for R sufficiently large.

A.1. The Linearized Solution

We write Eq. (1.20) for $\tilde{\psi} = \beta^{1/2}\psi$,

$$\left(D^4 + \frac{1}{\lambda^2 l_D^2} D^2\right)\tilde{\psi} + \frac{1}{\lambda^2 l_D^2}(\beta \tilde{z}_+ e^{\tilde{\psi}} - \beta \tilde{z}_- e^{-\tilde{\psi}})\chi = 0. \tag{A.1}$$

For $r > R$, $\tilde{\psi}$ will be of the form

$$(\tilde{\psi}_0 + A)\frac{R}{r} + B\frac{R}{r} e^{-(r-R)/(\lambda l_D)}, \tag{A.2}$$

for some constants A and B. For $r < R$ we linearize about $\tilde{\psi} = \tilde{\psi}_0$,

$$\tilde{\psi} = \tilde{\psi}_0 + \tilde{\psi}_1, \tag{A.3}$$

$$\left(D^4 + \frac{1}{\lambda^2 l_D^2} D^2\right)\tilde{\psi}_1 + \frac{1}{\lambda^2 l_D^2}\frac{1}{\tilde{l}_D^2}\tilde{\psi}_1 = 0, \tag{A.4}$$

Thus for $r<R$ the solution of the linearized equation (which we still now call $\tilde{\psi}$) is of the form

$$\tilde{\psi}_0 + C\frac{R}{r}\frac{\sinh(r/l_1)}{\sinh(R/l_1)} + D\frac{R}{r}\frac{\sinh(r/l_2)}{\sinh(R/l_2)}, \tag{A.5}$$

for constants C and D, l_1 and l_2 are solutions of

$$\frac{1}{l^4} - \frac{1}{\lambda^2 l_D^2}\frac{1}{l^2} + \frac{1}{\lambda^2 l_D^2}\frac{1}{\tilde{l}_D^2} = 0, \tag{A.6}$$

with

$$l_1 \sim l_D, \qquad l_2 \sim \lambda l_D, \tag{A.7}$$

for our choice of parameters. A, B, C, D are determined uniquely by matching value and first three derivatives ar $r=R$. It is easy to show that

$$|A|+|B|+|C|+|D| \leqq c/R. \tag{A.8}$$

Thus we expect the linearized solution to be a better approximation to the true $\tilde{\psi}$ the larger R is ($\tilde{\psi}-\tilde{\psi}_0$ goes to zero for $r<R$ as $R\to\infty$). This reflects the physical fact that the surface charge density goes to zero with increasing R.

If we set $\varrho(r)=r\tilde{\psi}(r)$ and write the linearized form of Eq. (A.1) for $\varrho(r)$, we get

$$\frac{d^4}{dr^4}\varrho - \frac{1}{\lambda^2 l_D^2}\frac{d^2}{dr^2}\varrho + \chi\cdot\left(\frac{1}{\lambda^2 l_D^2 \tilde{l}_D^2}\right)\cdot(\varrho-\varrho_0) = 0, \tag{A.9}$$

with $\varrho_0 = r\tilde{\psi}_0$. Integrating this equation from 0 to ∞ we get

$$-\left.\frac{d^3\varrho}{dr^3}\right|_{r=0} + \frac{1}{\lambda^2 l_D^2}\left.\frac{d}{dr}\right|_{r=0} + \frac{1}{\lambda^2 l_D^2 \tilde{l}_D^2}\int_0^R r(\tilde{\psi}-\tilde{\psi}_0)dr = 0, \tag{A.10}$$

or

$$\int_0^R r(\tilde{\psi}-\tilde{\psi}_0)dr \to -\tilde{l}_D^2\tilde{\psi}_0, \tag{A.11}$$

as R goes to ∞. Thus we find from (A.11) that the linearized solution $\tilde{\psi}$ satisfies the linearized form of Eq. (1.31). Outside this subsection we refer to the linearized solution we have just found as $\tilde{\psi}_L$.

A.2. The Full Equation, Large R Situation

We write the linearized solution we have found in Sect. A.1 as $\tilde{\psi}_L$

$$\tilde{\psi}_L = \begin{cases} \tilde{\psi}_{1L} & r>R \\ \tilde{\psi}_0 + \psi_{1L} & r<R. \end{cases} \tag{A.12}$$

We seek a solution for (A.1) of the form

$$\tilde{\psi} = \tilde{\psi}_L + \tilde{\phi}. \tag{A.13}$$

The equation for $\tilde{\phi}$ is

$$\left(D^4 + \frac{1}{\lambda^2 l_D^2}D^2 + \frac{1}{\lambda^2 l_D^2}\frac{1}{\tilde{l}_D^2}\chi\right)\tilde{\phi} + \frac{1}{\lambda^2 l_D^2}\frac{1}{\tilde{l}_D^2}(\sinh(\tilde{\psi}_{1L}+\tilde{\phi}) - (\tilde{\psi}_{1L}+\tilde{\phi}))\chi = 0, \tag{A.14}$$

or

$$\tilde{\phi} = -\frac{1}{\tilde{l}_D^2} C * [(\sinh(\tilde{\psi}_{1L} + \tilde{\phi}) - (\tilde{\psi}_{1L} + \tilde{\phi}))\chi] . \tag{A.15}$$

Noting that we may view this as an equation for $\tilde{\phi}$ on $r < R$ only, we put a norm on the functions $\tilde{\phi}$ on $[0, R]$,

$$|\tilde{\phi}|' = \sup_{0<r<R} |\tilde{\phi}(r) e^{(1-2\varepsilon)(R-r)/l_D}| . \tag{A.16}$$

Using Theorem 1.4 for estimates on C, and the results of the last subsection, it is straightforward to show a (unique) solution of (A.15) exists, of finite $|\,|'$ norm (bounded uniformly in R) by the contraction mapping principle. The result

$$R|\tilde{\phi}|' \xrightarrow[R\to\infty]{} 0 ,$$

is also easy in the contraction mapping setting. These facts establish Theorems 1.1 and 1.3.

A.3. General Existence and Uniqueness Results (This subsection is due to J. Rauch)

Very general existence and uniqueness results for instantons are herein derived. The volume Λ need not be a ball, but may be an arbitrary bounded set. In addition a more general form of p.d.e. is considered; that may be important to the extension of the present treatment to include other types of short range forces and charge species.

We are given $F \in C^\infty(\mathbb{R})$ satisfying

$$F'' \geqq \alpha > 0 , \tag{A.17}$$

and define

$$F' = f . \tag{A.18}$$

We also have a_0, $a_1 \in L^\infty(\mathbb{R}^3)$ with

$$a_i \geqq 0 , \qquad i = 0, 1 . \tag{A.19}$$

$a_0 > 0$ on an open set, and having compact support. For $\psi \in C_0^\infty(\mathbb{R}^3)$ (the present discussion applies to dimensions <3 as well) we define

$$\|\psi\|_H^2 = \tfrac{1}{2} \int [|\Delta\psi|^2 + a_1(x)|\nabla\psi|^2 + a_0(x)\psi^2] . \tag{A.20}$$

We let H be the completion of C_0^∞ in this norm. Notice we have

$$H^2 \subset H \subset H^2_{\mathrm{loc}} \subset C(\mathbb{R}^3) . \tag{A.21}$$

Since the ψ in H are all continuous, $J(\psi)$ defined as

$$J(\psi) = \tfrac{1}{2} \int [|\Delta\psi|^2 + a_1|\nabla\psi|^2 + 2a_0 F(\psi)] , \tag{A.22}$$

makes sense as a clearly continuous map from H to $\mathbb{R}$.

Theorem A.1. *J has a unique minimum in H. The minimum is a solution of the differential equation*

$$\Delta^2\psi - \nabla(a_1\nabla\psi) + a_0 f(\psi) = 0 . \tag{A.23}$$

We first prove that J is strictly convex which implies the uniqueness assertion. We define

$$J_2(\psi) = \int a_0 F(\psi), \tag{A.24}$$

$$J(\psi) = J_1 + J_2. \tag{A.25}$$

As both J_1 and J_2 are convex, if

$$J\left(\frac{\psi_1+\psi_2}{2}\right) = \tfrac{1}{2}(J(\psi_1)+J(\psi_2)), \tag{A.26}$$

then

$$J_i\left(\frac{\psi_1+\psi_2}{2}\right) = \tfrac{1}{2}(J_i(\psi_1)+J_i(\psi_2)), \qquad i=1,2. \tag{A.27}$$

Since F is strictly convex, (A.27) for $i=2$ implies

$$a_0(\psi_1-\psi_2)=0 \quad \text{a.e.}. \tag{A.28}$$

Equations (A.26) and (A.28) together imply

$$\int [|\Delta(\psi_1-\psi_2)|^2 + a_1|\nabla(\psi_1-\psi_2)|^2] = 0, \tag{A.29}$$

or

$$\|\psi_1-\psi_2\|_H = 0 \Rightarrow \psi_1 = \psi_2. \tag{A.30}$$

To prove existence we prove that J is coercive and lower semicontinuous. Note that from

$$\begin{aligned} F(s) &= F(0) + F'(0)s + s^2 \int_0^1 (1-\theta)F''(\theta s)d\theta \\ &\geqq \frac{\alpha s^2}{2} + F'(0)s + F(0), \end{aligned} \tag{A.31}$$

we see that

$$J(\psi) \geqq \min\{1,\alpha\} \cdot \|\psi\|_H^2 + F'(0)\int a_0\psi + F(0)\int a_0 \tag{A.32}$$

$$\geqq c_1\|\psi\|^2 - c_2\|\psi\|_H - c_3, \qquad c_1>0 \tag{A.33}$$

$$\geqq \frac{c_1}{2}\|\psi\|_H^2 - c_4. \tag{A.34}$$

This is the desired coerciveness.

If $\psi_n \to \psi$ weakly in H, then $\psi_n \to \psi$ uniformly on compacts, so

$$J_2(\psi_n) \to J_2(\psi). \tag{A.35}$$

The lower semicontinuity of the norm in H then yields

$$J_1(\psi) \leqq \lim J_1(\psi_n). \tag{A.36}$$

Thus $J = J_1 + J_2$ is weakly lower semicontinuous.

Let

$$i = \inf_{\psi \in H} J(\psi). \tag{A.37}$$

By (A.34)

$$i \geqq -c_4 > -\infty\,. \tag{A.38}$$

Choose $\psi_n \in H$ so that $J(\psi_n) \searrow i$. By (A.34) the ψ_n are bounded in H. Passing to a subsequence we may suppose $\psi_n \to \psi$ weakly in H. Weak lower semicontinuity and (A.37) yield $J(\psi) = i$, and the existence of a minimum is established.

The differential equation is the Euler equation expressing the fact that $\frac{d}{dt} J(\psi + t\phi) = 0$ for any ϕ.

We do not here explore the fall off properties at infinity of ψ.

Appendix B. Estimates for $C(x,y)$, $C_0(x,y)$

Lemma B.1. *Suppose $x^2 - ax + b = 0$ has two distinct positive roots, and suppose*

$$0 \leqq b_1(x) \leqq b_2(x) \leqq b\,, \qquad x \in \mathbb{R}^3\,, \tag{B.1}$$

then pointwise

$$0 \leqq (D^4 + aD^2 + b_2(x))^{-1} \leqq (D^4 + aD^2 + b_1(x))^{-1}\,. \tag{B.2}$$

Proof. Expand

$$\begin{aligned}(D^4 + aD^2 + b_i)^{-1} &= (D^4 + aD^2 + b - (b - b_i))^{-1} = (D^4 + aD^2 + b)^{-1} \\ &\quad + (D^4 + aD^2 + b)^{-1}(b - b_i)\,(D^4 + aD^2 + b)^{-1} + \dots\,.\end{aligned} \tag{B.3}$$

The lemma now follows since $(D^4 + aD^2 + b)^{-1}$ has a positive kernel. (It is easy to show the convergence of (B.3).)

Corollary B.2.

$$0 \leqq C(x,y) \leqq C_0(x,y)\,. \tag{B.4}$$

We assume parameters satisfying conditions of Lemma B.1. We proceed to study C_0. We expand $C_0(x,y)$ with $x, y \in \Lambda$. We write $\lambda^2 l_D^2 C_0 = (D^4 + aD^2 + b\chi)^{-1}$ with a, b satisfying conditions of Lemma B.1, and in addition let $\mathscr{L} = D^4 + aD^2$, and $\chi^c = (1 - \chi)$,

$$\lambda^2 l_D^2 C_0 = (\mathscr{L} + b)^{-1} + (\mathscr{L} + b)^{-1} b \chi^c (\mathscr{L} + b)^{-1} \dots\,. \tag{B.5}$$

The first two terms we have explicitly exhibited in (B.5) can clearly be absorbed into the first expression in the Max of (1.34). We must look at the remaining terms in (B.5). We write the sum of the remaining terms in (B.5) as R.

$$\begin{aligned}R = (\mathscr{L} + b)^{-1} \chi^c [\, & b(\mathscr{L} + b)^{-1} + b(\mathscr{L} + b)^{-1} \\ & \cdot \chi^c b(\mathscr{L} + b)^{-1} + \dots] \chi^c b (\mathscr{L} + b)^{-1}\,.\end{aligned} \tag{B.6}$$

Notice we need the integral kernel in brackets only for arguments in Λ^c. We let K be the expression in brackets, and $k = b(\mathscr{L} + b)^{-1}$. We define $*'$ as

$$(r *' s)(x,y) \equiv \int_{\Lambda^c} dz\, r(x - z) s(z - y)\,. \tag{B.7}$$

Then we have

$$K = k + k *' k + k *' k *' k + \dots . \tag{B.8}$$

We note the following properties for k:

P1) $$k(x, y) = k(|x - y|), \tag{B.9}$$

P2) $$k(x, y) \geqq 0, \tag{B.10}$$

P3) $$\int dx\, k(x, y) = 1, \tag{B.11}$$

P4) $$k(x, y) \leqq c e^{-\alpha|x-y|} \quad \text{some} \quad \alpha > 0. \tag{B.12}$$

We believe (B.8) and P1)→P4) are sufficient to derive the estimate for K we need (at the end of this section we state our conjecture), but we only know a procedure using detailed properties of k. We have by assumption

$$D^4 + aD^2 + b = (D^2 + c_1)(D^2 + c_2), \tag{B.13}$$

$$0 \leqq c_1 < c_2 . \tag{B.14}$$

(Usually we will want $c_1 > 0$.) We first state two results about "infinite barriers" outside a nice domain $\mathscr{S} \subset R^3$.

Lemma B.3.

$$(D^4 + aD^2 + b + c\chi_{\mathscr{S}^c})^{-1} \xrightarrow[c \to +\infty]{} (D^4 + aD^2 + b)_1^{-1}, \tag{B.15}$$

where the subscript 1 *indicates the closure of the differential operator restricted to functions vanishing with their normal derivatives on* $\partial\mathscr{S}$, i.e. $\{f \mid f = \partial_n f = 0$ *on* $\partial\mathscr{S}\}$. *The limit in* (B.15) *is strong.*

Lemma B.4. *Noting*

$$(D^4 + aD^2 + b)^{-1} = (c_2 - c_1)^{-1}[(D^2 + c_1)^{-1} - (D^2 + c_2)^{-1}], \tag{B.16}$$

$$\begin{aligned}&(c_2 - c_1)^{-1}[(D^2 + c_1 + c\chi_{\mathscr{S}^c})^{-1} - (D^2 + c_2 + c\chi_{\mathscr{S}^c})^{-1}] \xrightarrow[c \to +\infty]{} \\ &\quad \cdot (c_2 - c_1)^{-1}[(D^2 + c_1)_0^{-1} - (D^2 + c_2)_0^{-1}] = (D^4 + aD^2 + b)_0^{-1},\end{aligned} \tag{B.17}$$

where the subscript 0 *indicates the closure of the differential operator restricted to functions vanishing with* D^2 *times themselves on* $\partial\mathscr{S}$, i.e. $\{f \mid f = D^2 f = 0$ *on* $\partial\mathscr{S}\}$. *The limit in* (B.17) *is uniform.*

The two domains in these two lemmas determine two different self adjoint extensions of the formal differential operator. We are indebted to Rauch for information on these results. We will not give a proof here. Lemma B.3 will not be used. We let $\mathscr{S}^c$ be a ball of radius R', $R' < R$ centered about the origin; and χ' its characteristic function. We define

$$k_d = b(c_2 - c_1)^{-1}[(D^2 + c_1 + d\chi')^{-1} - (D^2 + c_2 + d\chi')^{-1}], \tag{B.18}$$

$$\Delta k = k - k_d . \tag{B.19}$$

We now write K as

$$K = K_d + \Delta k + K_d *' \Delta k + \Delta k *' K + K_d *' \Delta k *' K, \tag{B.20}$$

where K_d is K with k replaced by k_d.

We let K_d' be K_d with $*'$ replaced by a similar integration over $\mathscr{S}$, instead of Λ^c. We have

$$0 < K_d < K_d' . \tag{B.21}$$

Taking the limit $d \to +\infty$

$$\lim_{d \to +\infty} K_d' = b(\mathscr{L})_0^{-1} = b \cdot \frac{1}{a} \cdot [(D^2)_0^{-1} - (D^2 + a)_0^{-1}], \tag{B.22}$$

$$\leqq b \cdot \frac{1}{a} (D^2)_0^{-1} . \tag{B.23}$$

So in the limit $d \to +\infty$,

$$K_d < c(D^2)_0^{-1} . \tag{B.24}$$

The remaining terms on the right-hand side of (B.20) are estimated using the smallness of Δk,

$$0 < \Delta k < c(D^2 + c_1)^{-1} d\chi'(D^2 + c_1 + d\chi')^{-1} . \tag{B.25}$$

In the limit $d \to \infty$ we may estimate Δk by random walk techniques, or by the maximum principle

$$\Delta k(x, y) < c e^{-\alpha|x-y|} e^{-\alpha|R-R'|} e^{-\alpha(|x|-R)} e^{-\alpha(|y|-R)} \quad \text{for some} \quad \alpha > 0 . \tag{B.26}$$

In (B.26) we assume $c_1 > 0$, and α is a fraction of c_1. We let $G_{R'} = (D^2)_0^{-1}$ (computed for a given value of R'). In fact we will estimate K simply as

$$0 \leqq K < cG_{R'} + ce^{-\alpha|R-R'|}(|x| - R' + 1)(|y| - R' + 1) . \tag{B.27}$$

We have used the easy estimates $|K|$, $|K_d| < c$. We have also used properties of $G_{R'}$ to derive (B.27), in particular to control the last term in (B.20). We need properties of $G_{R'}$ to use after inserting (B.27) back into (B.6). The following lemma serves our purposes:

Lemma B.5. *With $|x|$, $|y| > R$ and s as described after Eq.* (1.33),

$$0 \leqq G_{R'}(x, y) \leqq c \operatorname{Min} \left\{ \frac{1}{s}, (|x| - R')(|y| - R') \frac{1}{s^3} \right\} . \tag{B.28}$$

$G_{R'}$ may be constructed explicitly using a single image charge in a familiar manner. The term in $1/s$ is immediate, as the distance between y and x, and y and the image of x, are both greater than a constant times s. The term in $1/s^3$ may be derived in noticing that $G_{R'}$ is zero when either $|x| = R'$ or $|y| = R'$. One integrates along a ray from $|x| = R'$ to $|x| = |x|$ the radial derivative of $G_{R'}$ in the variable x, and similarly integrates the radial derivative of $G_{R'}$ in the variable y. The double derivatives of $|x - y|^{-1}$ and $|x' - y|^{-1} \left(\frac{R'}{|x|} \right)$ (x' the image charge position) are bounded by $\frac{c}{s^3}$.

Letting R' be a function of s,

$$e^{-\alpha(R - R')} \sim \frac{c}{s^3 + 1} , \tag{B.29}$$

and using (B.28), (B.27), and (B.6), derivation of Theorem 1.4 is straightforward. We are not very pleased with our proof of Theorem 1.4 on two accounts.

1) We feel a proof should use only (B.9)→(B.12) to derive properties of K.

2) We have derived a $1/s^{3-\varepsilon}$ law instead of $1/s^3$ law we feel is correct. This line of proof would yield a $1/s^{3-\varepsilon}$ law for a plane surface boundary also.

It is likely that viewing (B.20) as an integral equation for K, and solving this integral equation by iteration, would with a little care yield a proof of the $1/s^3$ law. We will not pursue this direction.

Conjecture. (B.8)→(B.12) imply that

$$0 \leqq K \leqq \bar{c} G_{R'},$$

where $R' = R - 1$, and $\bar{c}$ is independent of R. [$\bar{c}$ may depend only on the variables α and c in (B.12).]

This conjecture is for Λ a ball of radius R. Similar conjectures may be made for a large class of other shapes.

Appendix C. Derivatives of $r(A)$

This appendix gives a stronger form of some of the results in Sect. 9.5 of [7]. In particular we prove the following lemma:

Lemma C.1. *There exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$ there exists $\delta > 0$, $\gamma < 1/\tilde{l}_D^2$, and c, $c' > 0$ (all four depending on ε) such that*

$$|D^N r(A)| \beta^{\varepsilon(N-1)} e^{-L^3 \gamma^{1/2} A^2} \leqq c' c^N \beta^{2\varepsilon} e^{(3/2-\delta) N \log N}.$$

As in [7] we write (with $\tilde{l}_D = 1$)

$$r(x+iy) = \frac{\mathrm{I} \cdot \mathrm{II}}{\mathrm{III} \cdot \mathrm{IV}},$$

$$\mathrm{I} = \exp[L^3 2\tilde{z}(\cos \beta^{1/2} x - 1)],$$

$$\mathrm{II} = \exp\{L^3 \tilde{z}[e^{i\beta^{1/2}x}(e^{-\beta^{1/2}y} - 1) + e^{-i\beta^{1/2}x}(e^{\beta^{1/2}y} - 1)]\},$$

$$\mathrm{III} = \exp[\tfrac{1}{2} L^3 y^2],$$

$$\mathrm{IV} = \sum_n \exp\left[-\frac{L^3}{2}\{(x - n\tau)^2 + 2iy(x - n\tau)\}\right].$$

We split up II as

$$\mathrm{II} = (\mathrm{IIa}) + (\mathrm{IIb}),$$

$$\mathrm{IIa} = \exp[L^3 \tilde{z}\{(e^{i\beta^{1/2}x} - 1)(e^{-\beta^{1/2}y} - 1) + (e^{-i\beta^{1/2}x} - 1)(e^{\beta^{1/2}y} - 1)\}],$$

$$\mathrm{IIb} = \exp[L^3 2\tilde{z}(\cosh \beta^{1/2} y - 1)].$$

We note the following:

$$|\mathrm{IIa}| = \exp[L^3 2\tilde{z}(\cos \beta^{1/2} x - 1)(\cosh \beta^{1/2} y - 1)] \leqq 1,$$

$$\frac{\mathrm{IIb}}{\mathrm{III}} = \exp\left[L^3 2\tilde{z}\left(\cosh \beta^{1/2} y - 1 - \frac{\beta}{2} y^2\right)\right].$$

So

$$\left|\frac{\mathrm{IIb}}{\mathrm{III}}\right| \leqq \exp[L^3 2\tilde{z} c_1 \beta^2 y^4]$$

for some $c_1 > 0$. ($|\beta^{1/2} y|$ will never exceed 1.)

For $N = 1, 2$, the lemma follows from Lemma 9.7 of [7]. For $N \geqq 3$ and $|A| \leqq \beta^{-1/6}$ the lemma also follows from the proof of Lemma 9.7 of [7]. (See the statement after Eq. (9.57) of [7].) So we need only consider the situation for $N \geqq 3$ and $|A| \geqq \beta^{-1/6}$. We break this up into two cases.

Case 1. $\beta^{-1/6} \leqq |A| \leqq \beta^{-1/2+\alpha}$. α is a small positive constant which will be fixed later. Consider the region

$$|x| \leqq 2\beta^{-1/2+\alpha}, \qquad |y| \leqq 1 .$$

In this region $\frac{\mathrm{IIb}}{\mathrm{III}}$ is bounded, as

$$2\tilde{z}\beta^2 y^4 \leqq \beta .$$

The $n=0$ term dominates in IV so we have

$$\begin{aligned}\left|\frac{\mathrm{I}}{\mathrm{IV}}\right| &\leqq c_0 \exp\left[2L^3\tilde{z}\left(\cos\beta^{1/2}x - 1 + \frac{\beta}{2}x^2\right)\right] \\ &\leqq c_0 \exp\left[L^3 \tilde{z}\frac{\beta^{3/2}}{3}|x|^3\right] \\ &\leqq c_0 \exp\left[L^3 \frac{\beta^\alpha}{3} x^2\right].\end{aligned}$$

We use a contour centered at A with radius 1. Then $x^2 \leqq (2A)^2$, so the above is

$$\leqq c_0 \exp[L^2 \tfrac{4}{3}\beta^\alpha A^2] .$$

Thus

$$|D^N r(A)| \leqq c' \exp[L^3 \tfrac{4}{3}\beta^\alpha A^2] N! c^N .$$

For small β, $\frac{4}{3}\beta^\alpha < \frac{1}{2}$, so the lemma follows since $\beta^{\varepsilon(N-1)} \leqq \beta^{2\varepsilon}$ for $N \geqq 3$.

Case 2. $\beta^{-1/2+\alpha} \leqq |A| \leqq \beta^{-1/2}\pi$. Consider the region

$$\tfrac{1}{2}\beta^{-1/2+\alpha} \leqq |x| \leqq 2\beta^{-1/2}\pi, \qquad |y| \leqq \bar{\varepsilon}/|x| ,$$

where $\bar{\varepsilon}$ is picked small enough so that the two largest terms in the sum in IV are within $\pi/2$ in phase.

As before, $|\mathrm{IIa}|$ and $\left|\frac{\mathrm{IIb}}{\mathrm{III}}\right|$ are bounded and

$$\left|\frac{\mathrm{I}}{\mathrm{IV}}\right| \leqq c_0 \exp\left[2L^3\tilde{z}\left(\cos\beta^{1/2}x - 1 + \frac{\beta}{2}x^2\right)\right].$$

We now use the following easily proven estimate:

Sublemma. *There exists a $\gamma' < 1$ such that*

$$\cos x - 1 + \tfrac{1}{2}x^2 \leqq \gamma' \tfrac{1}{2}x^2 \quad \textit{for} \quad |x| \leqq \pi .$$

By this sublemma

$$\left|\frac{\mathrm{I}}{\mathrm{IV}}\right| \leqq c_0 \exp[\tfrac{1}{2}L^3\gamma' x^2] ,$$

so a contour of radius $\frac{\bar{\varepsilon}\beta^{1/2}}{2\pi}$ centered at A yields

$$|D^N r(A)| \leqq c' c^N \beta^{-1/2N} N! \exp\left[\tfrac{1}{2}L^3\gamma' \left(A + \frac{\bar{\varepsilon}\beta^{1/2}}{2\pi}\right)^2\right].$$

Pick γ such that $\gamma' < \gamma < 1$. Then we have

$$|D^N r(A)| \beta^{\varepsilon(N-1)} e^{-L^3\gamma^{1/2}A^2} \beta^{-2\varepsilon} \leqq c' N! \beta^{-1/2N} \beta^{\varepsilon N} \beta^{-3\varepsilon} e^{-L^3 a\beta^{-1+2\alpha}} c^N ,$$

with $a > 0$.

Maximizing this as a function of β we find it is

$$\leqq c' c^N N! \exp\left[(\tfrac{1}{2} - \varepsilon)\left(\frac{1}{1-2\alpha}\right) N \log N\right].$$

Given ε we may choose α small enough to yield the lemma.

Appendix D. Some Theorems (From Geometric Measure Theory)

To determine the choice of hunks, appropriate for yielding satisfactory estimation of the functional integration, we have used the mathematical techniques of this Appendix. We found and proved the theorems herein, only later discovering the results (and presumably also the flavor of proofs) are known in the general context of geometric measure theory. We are indebted to F. Almgren for a discussion of this material. Since we need more than the theorems, also details of their constructive proof, we shall give in addition to the theorems, constructions yielding their solution. Complete verification that the constructions work is not presented, but the interested reader may fill in the details.

Theorem D.1. *There is a c (depending only on the dimension d), such that for any $f(x)$ on $\mathbb{R}^d$ satisfying*

$$\int |\nabla f| < \infty , \tag{D.1}$$

and any $M > 0$, there is a $g(x) = g_{M,f}(x)$ such that

a) $$|\nabla g| \leqq M , \tag{D.2}$$

b) $$g = f \textit{ off a set of measure } \leqq \frac{c}{M} \int |\nabla f| . \tag{D.3}$$

Theorem D.2. *Let B be the ball of radius R centered at the origin in $\mathbb{R}^d$. There is a c (depending only on d) such that for any f on B satisfying*

$$\int_B |\nabla f| < \infty \tag{D.4}$$

and any $M>0$, *there is a* $g=g_{M,f}$ *such that*

a) $$|\nabla g| \leqq M, \tag{D.5}$$

b) $$g=f \text{ off a set of measure } \leqq \frac{c}{M}\left[\int_B |\nabla f| + \int_{\partial B} |f|\right] \tag{D.6}$$

c) $$g=0 \quad \text{on} \quad \partial B. \tag{D.7}$$

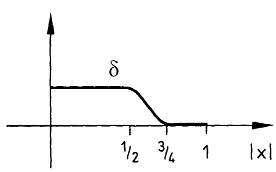
In Theorems D.1 and D.2 g is a function whose weak derivative is an ordinary function, pointwise bounded as in (D.2) and (D.5). f is a function whose weak derivative is a measure. In our application f will be h, a piecewise constant function; and thus ∇f will be a (δ-function) measure supported on the discontinuity set of h. If we set $f=0$ outside the ball B in Theorem D.2, we note that we may interpret the expression in brackets in (D.6) as

$$\int_B |\nabla f| + \int_{\partial B} |f| = \int_{R^d} |\nabla f|, \tag{D.8}$$

(the weak derivative being interpreted in $\mathbb{R}^d$, rather than in the interior of B). Another useful observation is that it is sufficient to prove the two theorems with $M=1$, a simple scaling argument yields the general result.

We introduce a number of functions useful in constructing the g's of the two theorems. (r and s appearing below are integers.)

$\delta(x)$:



$$\begin{aligned} &\text{a) } \delta(x)=\delta(|x|)\\ &\text{b) } \delta \in C^\infty\\ &\text{c) } \textstyle\int \delta = 1\\ &\text{d) } \delta \geqq 0\\ &\text{e) } \delta=0,\ |x| \geqq 3/4\\ &\text{f) } \delta=c>0,\ |x| \leqq 1/2, \end{aligned} \tag{D.9}$$

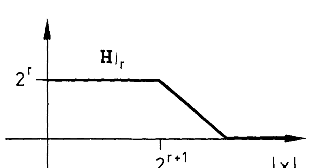
$\delta_r(x)$:

$$\delta_r(x)=\frac{1}{2^{rd}}\delta\left(\frac{1}{2^r}x\right), \tag{D.10}$$

$\delta^h(x)$:

$$\delta^h(x)=\frac{1}{h^d}\delta(x/h), \tag{D.11}$$

$H_r(x)$:



$$\begin{aligned} &\text{a) } H_r(x)=H_r(|x|)\\ &\text{b) } H_r(x)=2^r,\ |x| \leqq 2^{r+1}\\ &\text{c) } H_r(x)=0,\ |x| \geqq 3\cdot 2^r\\ &\text{d) } H_r(x)=3\cdot 2^r-|x|,\ 2^{r+1} \leqq |x| \leqq 3\cdot 2^r. \end{aligned} \tag{D.12}$$

We let x_{rs}, as s varies, be a cubical lattice of points in $\mathbb{R}^d$, with edge size $2^r \cdot \dfrac{1}{10d}$.

$\delta_{rs}(x)$: $$\delta_{rs}(x)=\delta_r(x-x_{rs}) \tag{D.13}$$

$H_{rs}(x)$: $$H_{rs}(x)=H_r(x-x_{rs}) \tag{D.14}$$

We let $\varepsilon > 0$ be a small number. We say rs is *rough* if

$$\int \delta_{rs}(x)\,|\nabla f(x)| > \varepsilon\,. \tag{D.15}$$

We define $H(x)$ by

$$H(x) = \sup_{rs\ \text{rough}} (H_{rs}(x))\,, \tag{D.16}$$

and

$$g(x) = (\delta^{H(x)} * f)(x) = \int dy\, \delta^{H(x)}(x-y) f(y) \tag{D.17}$$

[interpreting $\delta^0(x)$ as a delta function]. With $M=1$, picking $\varepsilon = \varepsilon_0$, ε_0 a small enough absolute constant, (D.17) provides a solution for $g(x)$ satisfying the conditions of Theorem D.1. Two lemmas are useful in showing (D.17) works

Lemma D.3. *The* H *of* (D.16) *satisfies*

$$|\nabla H| \leqq 1\,. \tag{D.18}$$

Lemma D.4. *The* δ^H *of* (D.11) *satisfies*

$$\left| \frac{d}{dH} \int \delta^H(y) f(y) \right| \leqq \int \delta^H(y)\,|\nabla f(y)|\,. \tag{D.19}$$

To construct a function $g(x)$ satisfying the conditions of Theorem D.2, we first define $f_1(x)$ by

$$f_1(x) = \begin{cases} f(x) & |x| \leqq R \\ 0 & |x| > R\,, \end{cases} \tag{D.20}$$

and construct $g_1(x)$ using (D.17) with f_1, and with $H(x)$ constructed for f_1, and ε picked equal $\frac{1}{10}\varepsilon_0$. g_1 certainly satisfies the conditions of Theorem D.1 (with $f = f_1$). $g(x)$ for Theorem D.2 is given as follows:

a) If $g_1(x)$ is not identically zero for $|x| \geqq \frac{3}{2}R$, we set

$$g(x) \equiv 0\,. \tag{D.21}$$

b) If $g_1(x)$ is identically zero for $|x| \geqq \frac{3}{2}R$, we set

$$g(x) = g_1(x) - g_1\left(\frac{(2R - |x|)}{|x|} x \right). \tag{D.22}$$

Appendix E. Enlargement of Hunks, Definition of g^*

In this appendix we make explicit the second enlargement of hunks, as referred to in Subsect. 3.4. We also will construct the functions g^* used in Sect. 12; the hunk enlargement herein detailed has as its sole purpose the construction of the g^*. We often enlarge the hunks more than necessary, by the process of this Appendix; we sought an enlargement procedure that is easy to describe, at the price of other considerations. (In fact only hunks M with h not constant on $\partial M - \partial \Lambda$ may possibly require enlargement.)

E.1. Hunks

We now detail the enlargement process for hunks associated with a given element

$\hat{h}(x)$ in $\mathscr{H}^{(0)}$ (as defined in Subsect. 3.3). We will later also construct the g^*'s for the associated enlarged hunks. The enlarged hunks are developed inductively. We start with

$$\hat{h} = \sum_i h_i, \tag{E.1}$$

corresponding to hunks M_i, $h_i \in \mathscr{H}^{(0)}_{M_i}$, the decomposition of Eq. (3.5). At the onset of the n^{th} stage we have

$$\hat{h} = \sum_i h_{ni}, \qquad h_{ni} \in \mathscr{H}^{(0)}_{M_{ni}}. \tag{E.2}$$

($h_{1i} = h_i$, $M_{1i} = M_i$, and $\mathscr{H}^{(0)}_{M_{ni}}$ is, naturally, functions as so far expanded associated to the hunk M_{ni}.) We develop the n^{th} enlargement stage via a number of steps.

1) We set for all time

$$1/M = L', \tag{E.3}$$

and identify B of Theorem D.2 with Λ.

2) We pick $h_{ni}(x)$, and carry out the construction of Theorem D.1 with

$$f(x) = \begin{cases} 0 & |x| > R \\ h_{ni}(x) & |x| < R \end{cases}, \tag{E.4}$$

arriving at an approximating function $g_{ni}(x)$. In the course of the construction we set ε of (D.15) to be $\left(\frac{\varepsilon_0}{L'}\right)$, and denote the corresponding $H(x)$ of (D.16) by $H^{ni}(x)$.

3) For x in Λ we define x_R by

$$x_R = \frac{2R - |x|}{|x|} x. \tag{E.5}$$

We let S_{ni} be the set of points in Λ where either $H^{ni}(x) \neq 0$ or $H^{ni}(x_R) \neq 0$. We let $\hat{S}_{ni}$ be the set of points in Λ within distance $2L'$ of S_{ni}.

4) We enlarge M_{ni} by adding a minimal number of unit cubes to M_{ni} so that the union of M_{ni} with these cubes cover $\hat{S}_{ni}$. If these enlarged $\{M_{ni}\}_i$ are disjoint the induction stops. If there are overlaps, we coalesce the corresponding hunks and begin the next induction step with the coalesced hunks (and the associated combined h's).

We note the enlarged hunks have the following properties.

a) They are connected.

b) They contain an L' neighborhood of the discontinuity set.

c) The volume of the hunk is

$$\begin{aligned} &\leqq cL' \sum |\delta h| = cL' \int |\nabla h| \\ &\leqq cL' \beta^{1/2} \sum |\delta h|^2 \leqq cL' \beta^{1/2} JE \end{aligned} \tag{E.6}$$

(where h is the h associated to the hunk). The last term involves the discontinuity energy of the hunk.

d) The hunk decomposition of $\hat{h}$ is consistent with the polymer representation. See Subsect. E.3 for some discussion of c).

E.2. Construction of g^*

We let M be a hunk (at the end of the two enlargement processes) and $h_M \in \mathscr{H}_M^{(0)}$. We want to define a $g^* = g_M^*$ for use in Sect. 12. g^* will be required to satisfy:

a) $$g^* = h_M \quad \text{outside} \quad M. \tag{E.7}$$

b) $$g^* = 0 \quad \text{in a neighborhood of} \quad \partial\Lambda. \tag{E.8}$$

c) $$|D^\alpha g^*| \leqq c_\alpha(L')^{-|\alpha|}, \qquad |\alpha| \geqq 1. \tag{E.9}$$

We apply the construction of Theorem D.2 with f picked as h_M. The associated g constructed we call $\hat{g}_M$. We let $s_B(x)$ be defined by

$$s_B = \begin{cases} 0 & R - |x| < L' \\ 1 & R - |x| > 2L' \\ \dfrac{R}{L'} - 1 - \dfrac{|x|}{L'} & L' \leqq R - |x| \leqq 2L' \end{cases}. \tag{E.10}$$

We then define

$$g'_M(x) = s_B(x) \cdot \hat{g}_M(x). \tag{E.11}$$

Finally we set

$$g_M^*(x) = (\delta^{(L'/2)} * g'_M)(x). \tag{E.12}$$

If $h_M = 0$ everywhere outside M, we may instead choose $g_M^* = 0$.

E.3. The Boundary Discontinuity

In the first line of (E.6) we understand the discontinuity of h at $\partial\Lambda$ to be included, as naturally arises in Theorem D.1. In the second line of (E.6) we do not, the discontinuity energy does not include contributions from the boundary of Λ. In the case that all the boundary hunks that have coalasced into M were normal B-hunks it is easy to see that the second line of (E.6) follows from the first with only a change of c. It is only the case of a jumbo B-hunk appearing in M, that must be further studied. In this case we need only observe the inequality for a function $f(x)$ defined on a ball B and its boundary ∂B

$$\int_{\partial B} |f - \bar{f}| dA \leqq c \int |\nabla f| dV, \tag{E.13}$$

where

$$\frac{1}{\mathrm{Area}(\partial B)} \int_{\partial B} f dA = \bar{f}. \tag{E.14}$$

(E.13) follows from the fundamental theorem of calculus and some simple geometry in the style of Sect. 9.

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