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Non-linear σ-Models on Compact Riemann Surfaces

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Abstract. The classical $O(3)$ non-linear σ-model is generalised to a theory of fields defined on a compact Riemann surface M with values in a compact Kähler manifold V. The dimension of the space of self-dual fields from M to the complex projective space $\mathbb{P}^N$ is calculated and the classifying space for the inequivalent quantisations of the theory is also calculated.

1. Introduction

The main reason for studying the classical $O(3)$ non-linear σ-model in two dimensions is its similarities with pure Yang–Mills theory in four dimensions. The $O(3)$ model [1] is a theory of a smooth three component real field $\underline{\phi} = (\phi^a)$ $(a = 1, 2, 3)$ defined on $\mathbb{R}^2$, i.e. $\underline{\phi}: \mathbb{R}^2 \to \mathbb{R}^3$ is a smooth map. The action of the theory is

$$S[\underline{\phi}] = \frac{1}{2} \int_{\mathbb{R}^2} \partial_\mu \underline{\phi} \cdot \partial^\mu \underline{\phi} d^2x = \frac{1}{2} \int_{\mathbb{R}^2} \delta^{\mu\nu} \partial_\mu \phi^a \partial_\nu \phi^a d^2x, \tag{1.1}$$

where $\delta^{\mu\nu}$ is the Euclidean metric on $\mathbb{R}^2$. The field $\underline{\phi}$ is subject to the constraint

$$\underline{\phi}^2 \equiv \phi^a \phi^a = 1. \tag{1.2}$$

The action (1.1) is invariant under a conformal change in the metric

$$g_{\mu\nu} = \Omega^2 \delta_{\mu\nu} \tag{1.3}$$

for Ω a smooth real-valued function on $\mathbb{R}^2$. Taking

$$\Omega = 2/(1 + x^2) \tag{1.4}$$

for $x = (x_1, x_2) \in \mathbb{R}^2$, and assuming that the field $\underline{\phi}$ obeys the boundary condition

$$\underline{\phi}(x) \to \phi_\infty \quad \text{as} \quad |x| \to \infty, \tag{1.5}$$

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where ϕ_∞ is a constant, shows that the field defines a smooth map $\phi: S^2 \to S^2$, from the conformally compactified Euclidean 2-space to the unit 2-sphere in $\mathbb{R}^3$. The maps from S^2 to S^2 are partitioned into homotopy classes which form a group $\pi_2(S^2) \simeq \mathbb{Z}$; this isomorphism is given by the degree of the map. Associated with each homotopy class of maps is a topological charge

$$Q[\underline{\phi}] = \frac{1}{8\pi} \int_{\mathbb{R}^2} \varepsilon_{ab} \underline{\phi} \cdot (\partial_a \underline{\phi} \times \partial_b \underline{\phi}) d^2x, \tag{1.6}$$

and it follows from the inequality

$$\int_{\mathbb{R}^2} (\partial_a \underline{\phi} \pm \varepsilon_{ab} \underline{\phi} \times \partial^b \underline{\phi}) \cdot (\partial^a \underline{\phi} \pm \varepsilon^{ac} \underline{\phi} \times \partial_c \underline{\phi}) d^2x \geqq 0 \tag{1.7}$$

that

$$S \geqq 4\pi |Q|. \tag{1.8}$$

The equality in (1.8) will hold if and only if

$$\partial_a \underline{\phi} \pm \varepsilon_{ab} \underline{\phi} \times \partial^b \underline{\phi} = 0, \tag{1.9}$$

and such a field is said to be (anti-) self-dual. In discussing the solutions of (1.9) it is important to remember that the 2-sphere S^2 has a unique complex structure. This arises when it is regarded as the complex projective line $\mathbb{P}^1$. Under this identification the (anti-) self-dual fields correspond to (anti-) holomorphic maps from $\mathbb{P}^1$ to $\mathbb{P}^1$.

In this paper a generalisation of the $O(3)$ model is considered in which the field ϕ is a smooth map from a compact Riemann surface M into a compact Kähler manifold V. Using techniques from the theory of harmonic maps it is shown in Sect. 2 that the action of this theory is bounded below by a topological charge and that the fields which realise this absolute lower bound are the (anti-) holomorphic maps from M to V. For suitable choices of M and V this model coincides with the classical $O(3)$, $\mathbb{C}P^N$ and complex Grassmannian models (see, for example, [1, 2, 3 and 4]). In Sect. 3 the case when $V = \mathbb{P}^N$ (the N dimensional complex projective space) is discussed. In particular, the dimension of the space of self-dual fields from M to $\mathbb{P}^N$ of degree n is calculated in terms of N, n and the genus g of M. This result gives, for example, the number of independent instanton solutions (of a given degree) of the $O(3)$ or CP^N model. The existence of holomorphic maps from a compact Riemann surface to the complex Grassmannian $G_k(\mathbb{C}^n)$ is also briefly discussed. The topology of the configuration space $\mathcal{Q}$ of maps from M to V is considered in Sect. 4. The homotopy groups of the configuration space are calculated in terms of the homotopy groups of V and the genus g of M. The first homotopy group of $\mathcal{Q}$ is related to the existence of inequivalent quantisations of the theory and the classifying space for these quantisations is calculated. Finally, the relationship between the topology of the space of self-dual fields and the topology of the space of all fields is considered. It is shown, for example, that the space of self-dual fields, of degree greater than one, in the $O(3)$ model is not simply connected.

2. Generalised Non-Linear σ-Model

Let M be a compact Riemann surface with metric g and V a compact simply connected n-dimensional Riemannian manifold with metric h. Given the Riemannian metric $g \in \Gamma(TM \otimes TM)^*$, we write $\langle u, v \rangle$ for $g_x(u, v)$, $x \in M$, $u, v \in T_x M$, $\|u\|^2 = \langle u, v \rangle$, and similarly for $h \in \Gamma(TV \otimes TV)^*$. If $\phi: M \to V$ is a smooth map, then the differential of ϕ at $x \in M$ is a linear map

$$d\phi(x): T_x M \to T_{\phi(x)} V, \tag{2.1}$$

and hence $d\phi(x) \in T_x^* M \otimes T_{\phi(x)} V$. The norm $\|d\phi(x)\|$ is defined using the metric induced on $T_x^* M \otimes T_{\phi(x)} V$ from the Riemannian structures on M and V. The generalisation of the $O(3)$ model is a theory of smooth fields $\phi: M \to V$ with the action given by the "energy" of the field. The Lagrangian density $\mathscr{L}(\phi): M \to \mathbb{R}^{\geqq 0}$ is defined (see [5]) to be

$$\mathscr{L}(\phi)(x) = \tfrac{1}{2} \|d\phi(x)\|^2, \tag{2.2}$$

and the action is

$$S[\phi] = \tfrac{1}{2} \int_M \|d\phi(x)\|^2 \, d\mu(g), \tag{2.3}$$

where $d\mu(g)$ is the canonical volume measure associated with g. In local coordinates

$$\mathscr{L}(\phi) = \tfrac{1}{2} g^{\mu\nu} \frac{\partial \phi^a}{\partial x^\mu} \frac{\partial \phi^b}{\partial x^\nu} h_{ab}, \tag{2.4}$$

and the correspondence between (2.3) and (1.1) is clearly seen.

An important feature of the $O(3)$ model is that the range S^2 has a complex structure, $S^2 \simeq \mathbb{P}^1$. To incorporate this aspect of the $O(3)$ model into this generalisation it will be assumed that V has a complex structure.

An (almost) complex structure on the manifold V is a section $J_V \in \Gamma$ End TV such that $J_V^2 = -id$, similarly $J_M \in \Gamma$ End TM such that $J_M^2 = -id$ is an (almost) complex structure on M (see [6] for further details). It will be assumed here that these almost complex structures are integrable and hence define complex structures. A map $\phi: M \to V$ is holomorphic if its differential $d\phi$ commutes with the complex structures on M and V, i.e.

$$d\phi \cdot J_M = J_V \cdot d\phi. \tag{2.5}$$

A Hermitian metric on V is a Riemannian metric h such that

$$\langle u, v \rangle = \langle J_V u, J_V v \rangle \tag{2.6}$$

for all $u, v \in T_p V$, $p \in V$. The Kähler from $\omega \in \Gamma(\Lambda^2 T^* V)$ is defined by

$$\omega(u, v) = \langle u, J_V v \rangle. \tag{2.7}$$

If ω is closed then V is a Kähler manifold. The complexification of TM is $T^{\mathbb{C}} M = TM \otimes_{\mathbb{R}} \mathbb{C}$, and J_M may be extended by complex linearity to $J_M^{\mathbb{C}} \in \Gamma$ End $T^{\mathbb{C}} M$. Since $(J_M^{\mathbb{C}})^2 = -id$, there is a direct sum decomposition $T^{\mathbb{C}} M = T^{1,0} M \otimes T^{0,1} M$, where $T^{1,0} M$ and $T^{0,1} M$ are the eigenbundles corresponding to the eigenvalues $+i$ and $-i$ of $J_M^{\mathbb{C}}$, respectively. The differential of any map $\phi: M \to V$ can be extended by complex linearity to $d^{\mathbb{C}}\phi: T^{\mathbb{C}}M \to T^{\mathbb{C}}V$, with the canonical decomposition $d^{\mathbb{C}}\phi = \partial\phi + \bar{\partial}\phi$, where

$$\begin{aligned} \partial\phi &: T^{1,0}M \to T^{1,0}V, \\ \bar{\partial}\phi &: T^{1,0}M \to T^{0,1}V, \end{aligned} \tag{2.8}$$

are defined to be the composition of $d^{\mathbb{C}}$ followed by projection in $T^{\mathbb{C}}V$. A map $\phi: M \to V$ is (anti-) holomorphic if and only if $(\partial\phi = 0)\bar{\partial}\phi = 0$.

Using the derivatives given in (2.8) we can define the (1, 0) and (0, 1) Lagrangian densities by

$$\begin{aligned} \mathscr{L}^{(1,0)}(\phi)(x) &= \|\partial\phi(x)\|^2, \\ \mathscr{L}^{(0,1)}(\phi)(x) &= \|\bar{\partial}\phi(x)\|^2, \end{aligned} \tag{2.9}$$

with the corresponding actions

$$\begin{aligned} S^{(1,0)}[\phi] &= \int_M \|\partial\phi(x)\|^2 d\mu(g), \\ S^{(0,1)}[\phi] &= \int_M \|\bar{\partial}\phi(x)\|^2 d\mu(g) \end{aligned} \tag{2.10}$$

The natural decomposition of the Lagrangian density

$$\mathscr{L}(\phi) = \mathscr{L}^{(1,0)}(\phi) + \mathscr{L}^{(0,1)}(\phi) \tag{2.11}$$

induces the decomposition

$$S[\phi] = S^{(1,0)}[\phi] + S^{(0,1)}[\phi] \tag{2.12}$$

of the action.

To obtain a lower bound on the action we introduce the topological charge $Q[\phi]$ of the field $\phi: M \to V$ given by

$$Q[\phi] = \tfrac{1}{2} \int_M \phi^*\omega, \tag{2.13}$$

where ω is the Kähler form of V. Then a direct calculation (see [7]) shows that

$$\int_M \phi^*\omega = \int_M [\|\partial\phi(x)\|^2 - \|\bar{\partial}\phi(x)\|^2] d\mu(g) = S^{(1,0)}[\phi] - S^{(0,1)}[\phi]. \tag{2.14}$$

Thus the inequality

$$S^{(1,0)}[\phi] + S^{(0,1)}[\phi] \geqq |S^{(1,0)}[\phi] - S^{(0,1)}[\phi]| \tag{2.15}$$

is the equivalent to the inequality

$$S \geqq 2|Q|, \tag{2.16}$$

and we see that the action is bounded below by a multiple of the absolute value of the topological charge, just as in the $O(3)$ model. In general, the topological charge defined by (2.13) is not invariant under continuous deformations of the field ϕ, and thus does not define an absolute lower bound on the action in each homotopy class of maps from M to V. This defect can be remedied by requiring V to be a Kähler manifold. Let ϕ_1, $\phi_2 : M \to V$ be homotopic (denoted by $\phi_1 \sim \phi_2$) and let $\Phi : M \times [0, 1] \to V$ be a homotopy of ϕ_1 and ϕ_2. Then

$$\int_M \phi_1^* \omega - \int_M \phi_2^* \omega = \int_{M \times \{0\}} \Phi^* \omega - \int_{M \times \{1\}} \Phi^* \omega = \int_{\partial(M \times [0,1])} \Phi^* \omega = \int_{M \times [0,1]} \Phi^*(d\omega). \tag{2.17}$$

and thus

$$Q[\phi_1] - Q[\phi_2] = \int_{M \times [0,1]} \Phi^*(d\omega). \tag{2.18}$$

If V is a Kähler manifold, then $d\omega = 0$ and the topological charge Q defined by (2.13) is a homotopy invariant. Henceforth it will be assumed that V has a Kähler structure. Note that the topological lower bound on the action of the theory is exactly analogous to the topological lower bound on the Yang–Mills action which leads to instanton phenomena.

The space of maps $\phi : M \to V$ (which will be assumed to be basepoint preserving) are partitioned into homotopy classes, the set of which is denoted by $[M; V]_*$. The manifold V is simply connected, and thus by the Hopf classification theorem [8],

$$[M; V]_* \simeq H^2(M; \pi_2(V)) \simeq \pi_2(V). \tag{2.19}$$

Thus non-trivial topological classes of maps will exist for those spaces V which have a non-trivial second homotopy group. In each of these homotopy classes the action of the model will be bounded below by twice the absolute value of the topological charge Q. Those fields which realise this absolute lower bound are called instanton solutions of the model. It is clear from (2.15) that an instanton field satisfies either

$$\bar{\partial}\phi = 0 \quad \text{or} \quad \partial\phi = 0, \tag{2.20}$$

and hence is either holomorphic (self-dual) or anti-holomorphic (anti-self-dual). For certain choices of V such maps exist. The case when $V = \mathbb{P}^N$, the N-dimensional complex projective space, is discussed in the next section.

3. The Space of Self-Dual Maps from M to $\mathbb{P}^N$

The complex projective space $\mathbb{P}^N$ with the Fubini-Study metric is a compact simply connected Kähler manifold with $\pi_2(\mathbb{P}^N) \simeq \mathbb{Z}$, for all $N \geqq 1$. Thus, by (2.19) there exist non-trivial topological classes of maps from any compact Riemann surface M to $\mathbb{P}^N$. For $V = \mathbb{P}^N$ it is possible to write the topological charge (2.13) in terms of $\deg \phi$, the degree of the map $\phi : M \to \mathbb{P}^N$, and Q is given by [9]

$$Q[\phi] = 2\pi \deg \phi. \tag{3.1}$$

There is a bijective correspondence between $\deg \phi$ and the elements of $\pi_2(\mathbb{P}^N) \simeq \mathbb{Z}$, and thus within each homotopy class of maps of a given degree the action is minimised by the (anti-) holomorphic maps. If we denote the space of all maps from M to V by $\mathrm{Map}(M; V)$ and the space of all holomorphic maps by $\mathrm{Hol}(M; V)$, then $\mathrm{Map}(M; \mathbb{P}^N)_n$ and $\mathrm{Hol}(M; \mathbb{P}^N)_n$ denote the component of $\mathrm{Map}(M; \mathbb{P}^N)$ and

$\mathrm{Hol}(M;\mathbb{P}^N)$ of degree n, respectively. In this section we calculate the dimension of $\mathrm{Hol}(M;\mathbb{P}^N)_n$ which is the number of independent self-dual fields from M to $\mathbb{P}^N$ of degree n.

To calculate the dimension of $\mathrm{Hol}(M;\mathbb{P}^N)_n$ it is necessary to introduce a correspondence between holomorphic maps from M to $\mathbb{P}^N$ and holomorphic line bundles over M. Before explaining this correspondence we will first recall some notions from algebraic geometry (see [10]).

A *divisor* D on a compact Riemann surface is a finite sum $D=\Sigma n_i x_i$ of points $x_i \in M$ with multiplicities n_i. The set of divisors on M forms an additive group, denoted $\mathrm{Div}\,M$. If $n_i \geqq 0$, for all i, then D is called *effective*. In terms of sheaves, a divisor D on M is a global section of the quotient sheaf $\mathfrak{M}^*/\mathcal{O}^*$, where $\mathfrak{M}^*$ denotes the multiplicative sheaf of non-zero meromorphic functions on M and $\mathcal{O}^*$ the subsheaf of non-zero holomorphic functions on M. Thus we have the identification

$$\mathrm{Div}\,M = H^0(M;\mathfrak{M}^*/\mathcal{O}^*). \tag{3.3}$$

Let $\pi: L \to M$ be a holomorphic line bundle over M. For an open cover $\{U_\alpha\}$ of M there are trivialisations

$$\psi_\alpha : L|_{U_\alpha} \to U_\alpha \times \mathbb{C}$$

of $L|_{U_\alpha} = \pi^{-1}(U_\alpha)$ and transition functions $g_{\alpha\beta}: U_\alpha \cap U_\beta \to \mathbb{C}^*$ for L given by

$$g_{\alpha\beta}(x) = (\psi_\alpha \cdot \psi_\beta^{-1})|_{L_x} \in \mathbb{C}^*$$

The transition functions $g_{\alpha\beta}$ are holomorphic, non-vanishing and satisfy the standard cocycle condition. Given a holomorphic line bundle $L \to M$ with trivialisation $\{\psi_\alpha\}$ and transition functions $\{g_{\alpha\beta}\}$, then for any collection of non-zero holomorphic functions on U_α, $f_\alpha \in \mathcal{O}^*(U_\alpha)$, we can define a new trivialisation over $\{U_\alpha\}$ by

$$\psi'_\alpha = f_\alpha \cdot \psi_\alpha,$$

and new transition functions

$$g'_{\alpha\beta} = \frac{f_\alpha}{f_\beta} \cdot g_{\alpha\beta}. \tag{3.4}$$

As any trivialisation of L over $\{U_\alpha\}$ can be obtained in this way, the collections $\{g_{\alpha\beta}\}$ and $\{g'_{\alpha\beta}\}$ of transition functions define the same holomorphic line bundle if and only if there exist functions $f_\alpha \in \mathcal{O}^*(U_\alpha)$ satisfying (3.4). In terms of sheaves the transition functions $\{g_{\alpha\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)\}$ represent a Cech cocycle and two cocycles $\{g_{\alpha\beta}\}$ and $\{g'_{\alpha\beta}\}$ define the same line bundle if and only if their difference $\{g_{\alpha\beta} \cdot g'^{-1}_{\alpha\beta}\}$ is a Cech coboundary. Thus, the set of all line bundles L over M is $H^1(M;\mathcal{O}^*)$. The set of all line bundles over M has a group structure with multiplication given by tensor product and inverses given by dual bundles. This group structure coincides with the group structure of $H^1(M;\mathcal{O}^*)$ and is called the *Picard group* of M, denoted by $\mathrm{Pic}\,M$.

The exact exponential sequence of sheaves

$$O \to \mathbb{Z} \to \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \to 0 \tag{3.5}$$

induces in cohomology the boundary map

$$H^1(M;\mathcal{O}^*)\xrightarrow{\delta} H^2(M;\mathbb{Z}). \tag{3.6}$$

For a line bundle $L\in\operatorname{Pic} M = H^1(M;\mathcal{O}^*)$ the *first* Chern class $c_1(L)$ is defined to be $\delta(L)\in H^2(M;\mathbb{Z})$. The degree deg L of the line bundle L is defined to be $c_1(L)$. The set of all holomorphic line bundles $L\in\operatorname{Pic} M$ of degree n is denoted by $\operatorname{Pic}^n M$.

Let $L\to M$ be a holomorphic line bundle with trivialisation $\psi_\alpha\colon L|_{U_\alpha}\to U_\alpha\times\mathbb{C}$ over $\{U_\alpha\}$ and with transition functions $\{g_{\alpha\beta}\}$ relative to $\{\psi_\alpha\}$. The trivialisations ψ_α induce isomorphisms

$$\psi_\alpha^*\colon\mathcal{O}(L)(U_\alpha)\to\mathcal{O}(U_\alpha),$$

and from the correspondence

$$s\in\mathcal{O}(L)(U)\to\{s_\alpha=\psi_\alpha^*(s)\in\mathcal{O}(U\cap U_\alpha)\}$$

it is clear that a holomorphic section s of L over $U\subset M$ is equivalent to a collection of functions $s_\alpha\in\mathcal{O}(U\cap U_\alpha)$ satisfying

$$s_\alpha=g_{\alpha\beta}\cdot s_\beta$$

in $U\cap U_\alpha\cap U_\beta$. Similarly, a meromorphic section s of L over U is given by a collection of meromorphic functions $s_\alpha\in\mathfrak{M}(U\cap U_\alpha)$ which satisfy $s_\alpha=g_{\alpha\beta}\cdot s_\beta$ in $U\cap U_\alpha\cap U_\beta$. If s is a global meromorphic section of L then the order of s is independent of $\{\psi_\alpha\}$ and we may define the divisor (s) of s to be

$$(s)=\Sigma\operatorname{ord}_{x_i}(s)\cdot x_i.$$

The section s is holomorphic if and only if (s) is effective and the space of holomorphic sections of L over M is $\Gamma(L)=H^0(M;\mathcal{O}(L))$.

We now describe the correspondence between holomorphic maps from M to $\mathbb{P}^N$ and holomorphic line bundles L over M. Associated to any subspace E of the vector space $\Gamma(L)$ is the *linear system* $|E|$ of effective divisors corresponding to the sections in E, i.e.

$$|E| = \{(s)\}_{s\in E}\subset\operatorname{Div} M.$$

As M is compact $(s)=(s')$ only if $s=\lambda s'$, for $\lambda\in\mathbb{C}^*$, thus $|E|$ is parametrised by $\mathbb{P}(E)$, the projectivisation of E. The linear system $|E|$ is said to have no base points if not all the sections $s\in E$ vanish at any $x\in M$. In this case the set of sections $s\in E$ which vanish at $x\in M$ define a hyperplane $\bar{H}_x\subset E$. Equivalently, the set of divisors $D\in|E|$ which contain x forms a hyperplane $H_x\subset\mathbb{P}(E)$. Thus, one can define a map from M to the dual projective space $\mathbb{P}(E)^*$ ($\mathbb{P}(E)^*$ is the set of hyperplanes in $\mathbb{P}(E)$) as

$$f_E\colon M\to\mathbb{P}(E)^*$$

by sending a point $x\in M$ to the hyperplane $H_x\in\mathbb{P}(E)^*$.

This map can be described more explicitly by letting $E\subset\Gamma(L)$ be $N+1$ dimensional with a basis $s_0,\dots,s_N$. For any trivialisation $\{\psi_\alpha\}$of L over $U\subset M$ let

$s_{i,\alpha} = \psi_\alpha^*(s_i) \in \mathcal{O}(U)$, then the point $[s_{0,\alpha}(x), \dots, s_{N,\alpha}(x)] \in \mathbb{P}^N$ is independent of the trivialisation $\{\psi_\alpha\}$ and can be written as $[s_0(x), \dots, s_N(x)]$. The map $f_E: M \to \mathbb{P}(E)^* = \mathbb{P}^N$ is then defined by

$$f_E(x) = [s_0(x), \dots, s_N(x)]$$

for $x \in M$, and f_E is seen to be holomorphic. Thus a subspace E of the space of holomorphic sections of a line bundle $L \to M$ determines a holomorphic map to $\mathbb{P}^N$. Conversely, let $f_E: M \to \mathbb{P}^N$ be a holomorphic map and let H be the hyperplane bundle on $\mathbb{P}^N$, then $L = f_E^* H$ and any section $s \in E$ is the pull-back of a section of H on $\mathbb{P}^N$, i.e.,

$$E = f_E^* H^0(\mathbb{P}^N; \mathcal{O}(H)) \subset H^0 M; \mathcal{O}(L))$$

Thus, the map f_E: $M \to \mathbb{P}^N$ determines both the line bundle L and the subspace $E \subset \Gamma(L)$. This results in the following:

Correspondence Holomorphic maps f: $M \to \mathbb{P}^N$, modulo projective automorphisms $\leftrightarrow$ holomorphic line hundles $L \to M$ with $E \subset \Gamma(L)$ such that $|E|$ has no base points.

Note that the maps f are only determined up to automorphisms of $\mathbb{P}^N$ because a different choice of basis $s_0, \dots, s_N$ for E gives different homogeneous coordinates on $\mathbb{P}^N$. Also note that maps $f: M \to \mathbb{P}^N$ of degree n correspond to $E \subset \Gamma(L)$ for line bundles L of degree n.

To obtain the dimension of Hol $(M; \mathbb{P}^N)_n$, we need the following result:

Lemma 3.1. *Let L be a holomorphic line bundle of degree n over a compact Riemann surface of genus g. Then for $n \geqq 2g$ the complete linear system $|\Gamma(L)|$ has no base points.*

Proof. For any $x \in M$, we have the short exact sequence of sheaves

$$0 \to \mathcal{O}(L-x) \to \mathcal{O}(L) \to L_x \to 0 \tag{3.7}$$

which gives rise in cohomology to the sequence

$$\dots \to H^0(M; \mathcal{O}(L)) \xrightarrow{r_x} H^0(M; L_x) \to H^1(M; \mathcal{O}(L-x)) \to \dots, \tag{3.8}$$

where r_x is evaluation at x. Let K_M be the canonical bundle of M and L_1 any line bundle over M. Then it follows from the Kodaira vanishing theorem that if $\deg L > \deg K_M$, then $H^1(M; \mathcal{O}(L)) = 0$. On a Riemann surface of genus g the degree of K_M is given by the Riemann–Hurwitz formula to be $\deg K_M = 2g-2$. Applying this to the line bundle $L-x$ we obtain that if $\deg(L-x) = \deg L - 1 > 2g-2$ then $H^1(M; \mathcal{O}(L-x)) = 0$. Thus, for deg $L \geqq 2g$ the exact sequence (3.8) reduces to

$$\dots \to H^0(M; \mathcal{O}(L)) \xrightarrow{r_x} L_x \to 0.$$

Hence, the evaluation map r_x is surjective and not all the section $s \in \Gamma(L)$ can vanish at x.

We now calculate the dimension of $\mathrm{Hol}(M; \mathbb{P}^N)_n$.

Theorem 3.2. *Let M be a compact Riemann surface of genus g, then for $n \geqq 2g$ the dimension of* $\mathrm{Hol}(M;\mathbb{P}^N)_n$ *is given by*

$$\dim \mathrm{Hol}(M;\mathbb{P}^N)_n = (N+1)n - N(g-1).$$

Remarks. (i) If L is a line bundle of negative degree over M then $H^0(M;\mathcal{O}(L)) = 0$. Thus, by the correspondence introduced above there are no holomorphic maps from M to $\mathbb{P}^N$ of negative degree.

(ii) If L is a line bundle of degree n over a compact Riemann surface M of genus g, then for $n \geqq 2g-1$ the dimension of $\Gamma(L) = H^0(M;\mathcal{O}(L))$ is given by the Riemann-Roch theorem to be $n-g+1$. Thus, the dimension of $|\Gamma(L)| = \dim \mathbb{P}(\Gamma(L)) = n-g$.

Proof. Consider the short exact sequence (which follows from Lemma 3.1)

$$0 \to K_x \to \Gamma(L) \overset{r_x}{\to} L_x \to 0,$$

where $K_x = \ker r_x$ and thus $\dim K_x = n-g$. The point $x \in M$ is a base point of $|E|$ if and only if all the sections $s \in E$ vanish at x. Thus, if x is a base point, the map

$$r_x|_E : E \subset \Gamma(L) \to L_x$$

obtaining by restricting r_x to E, which takes a section $s \in E$ to $s(x) \in \mathbb{C}$, is zero, i.e. $E = \ker r_x|_E \subset \ker r_x$. Thus, x is a base point of $|E|$ if and only if $E \subset K_x$, and conversely, $|E|$ has no base points if any only if $E \not\subset K_x$, for all $x \in M$. For a given $x \in M$, $K_x = \ker r_x$ gives a hyperplane in the projective space $\mathbb{P}(\Gamma(L))$ parametrising $\Gamma(L)$, and thus $K_x \in \mathbb{P}(\Gamma(L))^*$, the dual projective space. For a fixed $K_x \in \mathbb{P}(\Gamma(L))^*$ we have the Grassmannian $G_{N+1}(K_x)$ of $N+1$ dimensional spaces E in the $n-g$ dimensional space K_x. This Grassmanian is the fibre over K_x of the fibre bundle

$$\begin{array}{ccc} G_{N+1}(K) & \longrightarrow & \mathcal{F} \\ & & \big\downarrow {\scriptstyle pr_1} \\ & & \mathbb{P}(\Gamma(L))^* \end{array}$$

where $\mathcal{F}$ is the flag manifold consisting of pairs (K, E) with $E \subset K \subset \Gamma(L)$ and $\dim E = N+1$, $\dim K = n-g$. The total space $\mathcal{F}$ has two canonical projections $pr_1(K,E) = K \in \mathbb{P}(\Gamma(L))^*$ and $pr_2(K,E) = E \in G_{N+1}(\Gamma(L))$. By Lemma 3.1, if $\deg L \geqq 2g$ then $|\Gamma(L)|$ has no base points and there is a well defined map $f: M \to \mathbb{P}(\Gamma(L))^*$ given by the correspondence introduced earlier. Thus we have the diagram

$$\begin{array}{ccc} G_{N+1}(K) & & G_{N+1}(K) \\ \downarrow & & \downarrow \\ f^*\mathcal{F} & \xrightarrow{\hat{f}} & \mathcal{F} \overset{pr_2}{\to} G_{N+1}(\Gamma(L)) \\ \downarrow & & \big\downarrow {\scriptstyle pr_1} \\ M & \xrightarrow[f]{} & \mathbb{P}(\Gamma(L))^* \end{array}$$

$E \subset K_x$, for some $x \in M$, if and only if $E \in \mathrm{im}\, pr_2 \circ \hat{f}$, thus there is no $x \in M$ such that $E \subset K_x$ if and only if $E \notin \mathrm{im}\, pr_2 \circ \hat{f}$. Hence, $\mathrm{im}\, pr_2 \circ \hat{f}$ consists of exactly those E for which $|E|$ has a base point. The dimension of $G_{N+1}(\Gamma(L))$ is $(N+1)[n-g+1-(N+1)] = (N+1)(n-g) + (N+1) - (N+1)^2$ and $\dim(\mathrm{im}\, pr_2 \circ \hat{f}) \leqq \dim f^*\mathcal{F} =$

$1+(N+1)(n-g)-(N+1)^2$. Thus, $\dim(\operatorname{im} pr\circ\hat{f}) < \dim G_{N+1}(\Gamma(L))$ if $N \geqq 1$ and hence $pr_2\circ\hat{f}$ is not surjective. $\operatorname{Im} pr_2\circ\hat{f}$ is a closed subvariety in $G_{N+1}(\Gamma(L))$. The complement $G_{N+1}(\Gamma(L))\backslash \operatorname{im} pr_2\circ\hat{f}$ is open and consists of those E's with no base points. The Grassmannian $G_{N+1}(\Gamma(L))$ can be considered as the fibre over $L\in \operatorname{Pic}^n(M)$ of the fibre bundle

$$\begin{array}{ccc} G_{N+1}(\Gamma(L)) & \longrightarrow & \mathscr{G}_{N+1}(M) \\ & & \Big\downarrow \not{p} \\ & & \operatorname{Pic}^n(M) \end{array}$$

where the total space $\mathscr{G}_{N+1}(M)$ consists of pairs (L, E) $E\subset\Gamma(L)$, and $\not{p}(L,E)=L\in\operatorname{Pic}^n(M)$. From the above argument those E's for which $|E|$ has no base points from a Zariski open set in $\mathscr{G}_{N+1}(M)$ which is the complement of a subvariety in $\mathscr{G}_{N+1}(M)$. Thus the dimension of the space of holomorphic maps from M to $\mathbb{P}^N$, modulo projective automorphism, is equal to $\dim\mathscr{G}_{N+1}(M)=g+(N+1)(n-g)-N(N+1)$. Finally, the dimension of $\operatorname{Hol}(M;\mathbb{P}^N)_n$ is equal to $\dim\mathscr{G}_{N+1}(M)$ plus the dimension of $PGL_{N+1}(\mathbb{C})$, the group of automorphisms of $\mathbb{P}^N$. Hence

$$\begin{aligned}\dim\operatorname{Hol}(M;\mathbb{P}^N)_n &= \dim\mathscr{G}_{N+1}(M)+\dim PGL_{N+1}(\mathbb{C})\\ &= g+(N+1)(n-g)-N(N+1)+(N+1)^2-1\\ &=(N+1)n-N(g-1).\end{aligned}$$

An application of this result is to calculate the number of independent self-dual solutions, of degree n, of the classical CP^N model. This corresponds to calculating the dimension of $\operatorname{Hol}(S^2;\mathbb{P}^N)_n$. Recall from the remark made earlier that there are no holomorphic maps from S^2 to $\mathbb{P}^N$ of negative degree and therefore there are no self-dual fields of negative topological charge. As S^2 has $g=0$ we have for all $n\geqq 0$ that

$$\dim\operatorname{Hol}(S^2;\mathbb{P}^N)_n=(N+1)n+N. \tag{3.9}$$

The classical $O(3)$ model corresponds to the CP^1 model and hence, for all $n\geqq 0$,

$$\dim\operatorname{Hol}(S^2;\mathbb{P}^N)_n=2n+1, \tag{3.10}$$

which agrees with the number of independent parameters in the general, explicitly known, self-dual solution of degree n.

To conclude this section we note that a theory of maps from M to the complex Grassmannian $G_K(\mathbb{C}^m)$ generalises the complex Grassmannian model (see [4]). The Grassmannian $G_K(\mathbb{C}^m)$ is a simply connected Kähler manifold, and thus the self-dual fields from M to $G_K(\mathbb{C}^m)$ are given by the holomorphic maps $\operatorname{Hol}(M;G_K(\mathbb{C}^m))$. Although the analogoue of Theorem 3.2 for the dimension of $\operatorname{Hol}(M;G_K(\mathbb{C}^m))$ is not known, certain holomorphic maps from M to $G_K(\mathbb{C}^m)$ do exist. For example, if M is holomorphically immersed in $\mathbb{P}^N$, then the Gauss map (see [10])

$$\gamma: M\to G_2(\mathbb{C}^{N+1})$$

is holomorphic (see [5], for example).

4. Topology of the Configurations Space

An interesting feature of field theories with non-simply connected configuration spaces is that they can possess inequivalent quantisations. If $\mathscr{Q}$ is the configuration space of the theory in question then the the inequivalent quantisations are classified by (see [11 and 12])

$$\theta = \mathrm{Hom}\,(\pi_1(\mathscr{Q}), U(1)). \tag{4.1}$$

In fact, the arguments leading to this result are not quite complete as they ignore the possibility of the theory possessing a Wess–Zumino type term. This problem can be seen most clearly from the canonical viewpoint. Let $\mathscr{Q}$ be the configuration space of the theory and the cotangent bundle $\pi{:}T^*\mathscr{Q} \to \mathscr{Q}$ is the phase space; this carries a canonical non-degenerate symplectic 2-form Ω_0, defining the natural Hamiltonian structure. In canonical quantisation we choose a complex line bundle $\mathscr{L} \to \mathscr{Q}$; the Hilbert space $\mathscr{H}$ of states of the quantised theory is the space of sections of $\mathscr{L}$ and the equations of motion of he theory are implemented as operator equations on $\mathscr{H}$. If the canonical symplectic structure on $T^*\mathscr{Q}$ defined by Ω_0 can be changed by adding a curvature term pulled-back from $\mathscr{Q}$, then the equations of motion defined by this new symplectic structure will differ from those defined by Ω_0. An example of such a change in the symplectic structure occurs when one considers the motion of a charged particle in the field of a magnetic monopole. The quantisation of the magnetic charge of the monopole is a consequence of the modification in the symplectic structure. A second important example of such a modification in the equation of motion of a physical system is the addition of the Wess–Zumino term in the SU(3) non-linear σ-model. It is the presence of this term in the model that is responsible for the important consequences discovered by Witten [13]. The way in which the Wess–Zumino term arises in the SU(3) σ-model by changing the symplectic structure has been investigated by Ramadas [14]. If, however, we consider a theory which has no Wess–Zumino typer term then to eliminate the possibility of altering the canonical symplectic structure we can require that the complex line bundle $\mathscr{L} \to \mathscr{Q}$ must be flat. Then it is well known that the flat complex line bundles over $\mathscr{Q}$ are classified by $\mathrm{Hom}\,(\pi_1(\mathscr{Q}), U(1))$, which gives (4.1). As there are no Wess–Zumino type terms in the non-linear σ-models being considered here the classification (4.1) is valid.

For a non-linear σ-model in $1+1$ dimensions $\mathscr{Q} = \Omega(V)$. Thus, for $V = \mathbb{P}^N$,

$$\Theta = \mathrm{Hom}(\mathbb{Z}, U(1)) \simeq U(1)$$

For a non-linear σ-model in $2+1$ dimensions the spatial topology may be represented by a compact Riemann surface M. The configuration space is $\mathscr{Q} = \mathrm{Map}_*(M; V)$ and the homotopy groups $\pi_q(\mathscr{Q})$ are given by the following theorem (the space $\mathrm{Map}_*(M; V)$ is assumed to have the compact-open topology (see [18])).

Theorem 4.1. *Let M be a compact Riemann surface of genus g and V a compact topological space. The homotopy groups of* $\mathrm{Map}_*(M; V)$ *are given by*

$$\pi_q\,\mathrm{Map}_*(M; V) \simeq [\pi_{q+1}(V)]^{2g} \oplus \pi_{q+2}(V)$$

for $q \geqq 1$.

Proof. Recall that $\pi_1(M) =$ free group on $a_1 b_1 a_2 b_2 \ldots a_g b_g$ subject to the relation $a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_g b_g a_g^{-1} b_g^{-1} = 1$. M can be obtained from the wedge product of $2g$ circles by attaching a cell in dimension two via the map $a = a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_g b_g a_g^{-1} b_g^{-1}$, i.e.

$$M \simeq \bigvee_{2g} S^1 \bigcup_a e^2.$$

Now

$$a \in \pi_1\big(\bigvee_{2g} S^1\big).$$

and its suspension

$$Sa \in \pi_2\big(\bigvee_{2g} S^2\big)$$

is null-homotopic ($Sa \simeq 0$) because π_2 is Abelian. Thus, the suspension of M is

$$SM \simeq \bigvee_{2g} S^2 \bigcup_{Sa} e^3 \simeq \bigvee_{2g} S^2 \vee S^3.$$

Suspending this $q-1$ times gives

$$S^q M \simeq \bigvee_{2g} S^{q+1} \vee S^{q+2}.$$

and the homotopy groups of $\mathrm{Map}_*(M; V)$ are given by

$$\begin{aligned} \pi_q \mathrm{Map}_*(M; V) &\simeq [S^q M; V]_* \\ &\simeq [S^{q+1}; V]_*^{2g} \oplus [S^{q+2}; V]_* \\ &\simeq [\pi_{q+1}(V)]^{2g} \oplus \pi_{q+2}(V). \end{aligned}$$

We also have the following consequences:

Corollary 4.2.

$$\pi_1 \mathrm{Map}_*(M; \mathbb{P}^N) \simeq \begin{cases} \mathbb{Z}^{2g} \oplus \mathbb{Z}, & \text{for } N = 1 \\ \mathbb{Z}^{2g} & \text{for } N \geqq 2 \end{cases}.$$

Proof. This follows from the homotopy groups of $\mathbb{P}^N$ which are obtained from the exact homotopy sequence of the Hopf fibration

$$U(1) \to S^{2N+1} \to \mathbb{P}^N.$$

Corollary 4.3. *For* $m \geqq k+2$,

$$\pi_1 \mathrm{Map}_*(M; G_k(\mathbb{C}^m) \simeq \mathbb{Z}^{2g}.$$

Proof. This follows from the homotopy result (see Appendix)

$$\pi_q(G_k(\mathbb{C}^m)) \simeq \pi_{q-1}(U(k)) \quad \text{for} \quad q < 2(m-k).$$

Thus, the classifying space for inequivalent quantisations for $V = \mathbb{P}^N$ is

$$\Theta = \begin{cases} \mathrm{Hom}(\mathbb{Z}^{2g} \oplus \mathbb{Z}, U(1)), & \text{for } N = 1 \\ \mathrm{Hom}(\mathbb{Z}^{2g}, U(1)), & \text{for } N \geqq 2 \end{cases}$$

and for $V = G_k(\mathbb{C}^m)$ is

$$\Theta = \mathrm{Hom}(\mathbb{Z}^{2g}, U(1)) \quad \text{for} \quad m > k + 1.$$

Note that for $M = S^2$ (i.e., $g = 0$) both the complex Grassmannian model and the $CP^N (N \geqq 2)$ model have a unique quantisation. Only the $O(3)$ model has a non-trivial Θ for $g = 0$.

To conclude, we briefly consider the relationship between the topology of the space of self-dual fields $\mathrm{Hol}_*(M; V)$ and the topology of the space of all fields $\mathrm{Map}_*(M; V)$. For $M = S^2$ and $V = \mathbb{P}^N$ this problem has been solved by a theorem of Segal's [15]. This theorem states that the inclusion $\mathrm{Hol}_*(S^2; \mathbb{P}^N)_n \subsetneq \mathrm{Map}_*(S^2; \mathbb{P}^N)_n$ is a homotopy equivalence up to dimension $n(2N - 1)$. For example, when $N = 1$,

$$\pi_q \mathrm{Hol}_*(S^2; S^2)_n \simeq \pi_q \mathrm{Map}_*(S^2; S^2)_n \simeq \pi_{q+2}(S^2)$$

for $q < n$. For $q = 1$, we obtain

$$\pi_1 \mathrm{Hol}_*(S^2; S^2)_n \simeq \pi_3(S^2) \simeq \mathbb{Z}$$

for $n > 1$, and hence the space of self-dual fields of degree greater than 1 in the $O(3)$ model is not simply connected.

Appendix

We prove here the formula for the stable homotopy of $G_k(\mathbb{C}^m)$ used in Corollary 4.3, namely

$$\pi_q(G_k(\mathbb{C}^m)) \simeq \pi_{q-1}(U(k)). \tag{A1}$$

for $q < 2(m - k)$.

First recall that as a homogeneous space

$$G_k(\mathbb{C}^m) = \frac{U(m)}{U(k) \times U(m-k)}. \tag{A2}$$

We know that $U(m+1)/U(m) = S^{2m+1}$ and from the homotopy exact sequence of the fibration

$$\begin{array}{ccc} U(m) \rightarrow & U(m+1) \\ & \downarrow \\ & S^{2m+1} \end{array}$$

we see that the inclusion $U(m) \subsetneq U(m+1)$ is a homotopy equivalence up to dimension $2m$, i.e., $\pi_q(U(m)) \simeq \pi_q(U(m+1))$, for $q < 2m$. Applying this result to the inclusion $U(m-k) \subsetneq U(m)$ gives

$$\pi_q(U(m-k)) \simeq \pi_q(U(m)), \tag{A3}$$

for $q < 2(m-k)$. The homotopy exact sequence of the fibration

$$\begin{array}{ccc} U(m-k) & \rightarrow & U(m) \\ & & \downarrow \\ & & U(m)/U(m-k) \end{array}$$

together with (A3) result in

$$\pi_q(U(m)/U(m-k)) = 0, \tag{A4}$$

for $q < 2(m-k)$. Finally, from the expression (A2) for $G_k(\mathbb{C}^m)$ as a homogeneous space it is clear that we have a fibration

$$\begin{array}{ccc} U(k) & \rightarrow & U(m)/U(m-k) \\ & & \downarrow \\ & & G_k(\mathbb{C}^m) \end{array}$$

and the homotopy exact sequence of this together with (A4) results in the desired formula (A1).

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