Long Range Order in the Anisotropic Quantum Ferromagnetic Heisenberg Model

Tom Kennedy*

Department of Physics, Princeton University, Princeton, NJ 08544, USA

Abstract. We study the anisotropic quantum mechanical ferromagnetic Heisenberg model. By anisotropic we mean that the x and y exchange constants are equal but smaller than the z exchange constant. We show that for any amount of anisotropy there is long range order in two or more dimensions at low enough temperature. We also develop a convergent low temperature expansion and use it to prove exponential decay of the truncated correlation functions.

1. Introduction

The Hamiltonian of the quantum mechanical ferromagnetic Heisenberg model is

$$- \sum_{\langle rs \rangle} (\alpha_x \sigma_r^x \sigma_s^x + \alpha_y \sigma_r^y \sigma_s^y + \alpha_z \sigma_r^z \sigma_s^z).$$

The isotropic Heisenberg model is obtained by taking $\alpha_x = \alpha_y = \alpha_z$. We will study the anisotropic case $\alpha_x = \alpha_y = \alpha$, $\alpha_z = 1$ with $\alpha < 1$. Our main result, Theorem 2.1, is that for two or more dimensions and any $\alpha < 1$, there is long range order (LRO) at sufficiently low temperature. By LRO we mean that $\langle \sigma_i^z \sigma_j^z \rangle$ is bounded away from zero uniformly in i and j. We use free boundary conditions to define the Gibbs state $\langle \ \rangle$. We will also show that a polymer expansion for the model converges for sufficiently low temperature. This yields more detailed information than LRO, e.g., the truncated correlation functions decay exponentially (see Theorem 2.2).

Ginibre [8] and Robinson [12] proved the existence of LRO at low temperature for sufficiently small α. Kirkwood [9] and Thomas and Yin [14, 15] proved LRO at low temperature in similar models. Fröhlich and Lieb [6] showed that if reflection positivity is true for the model then there is LRO for any $\alpha < 1$. However, Speer [18] has shown that reflection positivity is not always true for the model. For the classical Heisenberg model Malyshev [10] proved the occurrence of LRO for any $\alpha < 1$. This result was improved by Bricmont and Fontaine [17].

Our proof may be thought of as a Peierls argument combined with a resummation which takes care of much of the contour counting that must be done. We begin by following [8] and using the Trotter product formula to express our system in terms of Peierls contours that evolve in time. We can write the Hamiltonian as

$$H = (1 - \delta)H^{\text{iso}} + \delta H^z,$$

where H^{iso} is the isotropic Heisenberg Hamiltonian and H^z is the Ising model Hamiltonian ($\alpha_x = \alpha_y = 0$, $\alpha_z = 1$). So $\alpha = 1 - \delta$. The weight of a time-dependent contour Γ is a product of two factors, one from $(1 - \delta)H^{\text{iso}}$ and one from δH^z. The latter looks like $\exp(-\beta\delta|\Gamma|_{av})$, where $|\Gamma|_{av}$ is the time average of the number of bonds in Γ. To simplify the explanation we assume that $|\Gamma|_{av}$ is at least half as large as $|\Gamma(0)|$, the initial number of bonds in Γ. So we can bound $\exp(-\beta\delta|\Gamma|_{av})$ by $\exp(-\frac{1}{2}\beta\delta|\Gamma(0)|)$.

We do the sum over all time-dependent contours Γ in two stages. First, we sum over $\Gamma(0)$, the initial configuration of Γ. Then we sum over the time evolution of Γ. Given an initial configuration $\Gamma(0)$, the sum over the time evolution of Γ of the weight from the $(1 - \delta)H^{\text{iso}}$ term may be explicitly evaluated using the Trotter product formula. This resummation yields $\text{tr}(Pe^{-\beta(1-\delta)H^{\text{iso}}})$, where P is a two-dimensional projection which depends on $\Gamma(0)$. This trace is easily bounded. This leaves the sum

$$\sum_{\Gamma(0)} \exp(-\tfrac{1}{2}\beta\delta|\Gamma(0)|),$$

which can be bounded as in the classical Peierls argument. Thus the resummation reduces the estimate needed for the quantum mechanical model to a classical estimate.

The anisotropic model is quite different from the isotropic model, even if δ is small so that α_x and α_y are close to α_z. By the Mermin-Wagner theorem [11] the isotropic model, unlike the anisotropic model, does not have LRO in two dimensions at any temperature. (Our lower bound on the critical temperature goes to zero as δ goes to zero, so there is no contradiction between our result and the Mermin-Wagner theorem.) The ground state of the isotropic model is infinitely degenerate while the anisotropic model has only two ground states – all spins up in the z-direction and all spins down in the z-direction. A related observation is that in the anisotropic model it is natural to introduce Peierls contours with respect to the z–component of the spins. There is no natural way to introduce Peierls contours in the isotropic model.

We only consider the spin $1/2$ case in this paper. For higher spin S one can still define Peierls contours with respect to the spin in the z-direction. In the spin $1/2$ case there is only one type of bond in the Peierls contours. For higher spin the contours consist of different types of bonds since two spins can differ in more than one way. Our results and proofs extend to these more exotic contours. Unfortunately, the resulting bound on the critical temperature is very dependent on the spin. If we normalize the spin operators by dividing by S, then as $S \to \infty$ the quantum mechanical model looks like the classical model. One would expect that the critical temperature of the quantum mechanical model converges to that of the

classical model. However, our bound on the critical temperature goes to zero like $1/S$.

The reason for this poor bound on the critical temperature is that for large S our Peierls contours separate spins which may only differ by a small amount $(1/S)$. For large S one should probably introduce Peierls contours as in [6]. However, it is not clear how to use our resummation technique when Peierls contours are defined in this way.

This paper is organized as follows. In Sect. 2 we state our main results. Then we use the Trotter product formula to express our system in terms of time-dependent Peierls contours. This representation is standard, but our decomposition of these contours into connected components is not. Section 3 begins with the statement of the key estimate of this paper, Theorem 3.2. This estimate is used to prove LRO. Then the estimate itself is proven. A polymer expansion for the model is discussed in Sect. 4. The convergence of this expansion also follows from Theorem 3.2.

2. Statement of Results and the Contour Expansion

For a finite subset Λ of $\mathbb{Z}^\nu$ with $\nu \geq 2$ we let

$$H^{xy} = - \sum_{\langle rs \rangle} \tfrac{1}{2}(\sigma_r^x \sigma_s^x + \sigma_r^y \sigma_s^y),$$

$$H^z = \sum_{\langle rs \rangle} \tfrac{1}{2}(1 - \sigma_r^z \sigma_s^z). \tag{2.1}$$

The sum is over nearest neighbor pairs $\langle rs \rangle$ in Λ. Each such pair is counted once. σ_r^x, σ_r^y, and σ_r^z are the usual Pauli spin matrices. The Hamiltonian is

$$H = (1-\delta)H^{xy} + H^z$$

with $0 < \delta \leq 1$. Note that H is defined so that the ground state energy is 0. $\langle \ \rangle_\Lambda$ will denote the usual Gibbs state.

Our main results are the following two theorems.

Theorem 2.1. *For $0 < \delta \leq 1$ there exists $\beta_c(\delta)$ such that for $\beta > \beta_c(\delta)$ there exists $M^2(\beta, \delta) > 0$ with*

$$\langle \sigma_i^z \sigma_j^z \rangle_\Lambda \geq M^2(\beta, \delta) \qquad \forall i, j \text{ and } \Lambda.$$

Moreover,

$$\lim_{\beta \to \infty} M^2(\beta, \delta) = 1$$

for fixed δ.

Theorem 2.2. *For $0 < \delta \leq 1$ there exists $\beta_0(\delta)$ such that for $\beta \geq \beta_0(\delta)$ there exists m_1, m_2, c_1, $c_2 > 0$, depending on β and δ, such that if we let*

$$\langle \sigma_i^\alpha \sigma_j^\alpha \rangle_\infty = \lim_{\Lambda \to \mathbb{Z}^\nu} \langle \sigma_i^\alpha \sigma_j^\alpha \rangle_\Lambda, \qquad \alpha = x, y, z,$$

and

$$M_\infty^2 = \lim_{|i-j| \to \infty} \langle \sigma_i^z \sigma_j^z \rangle_\infty,$$

then

$$|\langle \sigma_i^z \sigma_j^z \rangle_\infty - M_\infty^2| \leqq c_1 e^{-m_1|i-j|},$$

$$|\langle \sigma_i^x \sigma_j^x \rangle_\infty| = |\langle \sigma_i^y \sigma_j^y \rangle_\infty| \leqq c_2 e^{-m_2|i-j|}.$$

Remarks. 1. In both theorems we implicitly assume that the volume Λ is large enough that the distance from the line segment between i and j to $\partial\Lambda$ is at least as large as the distance from i to j.

2. We have used free boundary conditions in our definition of H. As with the Peierls argument for the Ising model, our argument can handle other boundary conditions. Of course, different boundary conditions can yield different infinite volume Gibbs states.

3. We prove Theorem 2.2 by showing that a polymer expansion for the system converges. Such an expansion gives results about all the correlation functions, not just the two-point function. The expansion can also be used to prove the existence of the infinite volume limit and the $|i-j| \to \infty$ limit in Theorem 2.2.

4. Theorem 2.2 says that the correlation length of the system is finite. For small δ and large β our upper bound on the correlation length is proportional to $1/\delta$ and independent of β. An interesting question is the true behavior of the correlation length for small δ and large β. In the classical case it is approximately $1/\sqrt{\delta}$ for small δ and large β [16].

5. Our proofs are valid for any dimension greater than or equal to two. However, we will use the terminology of two dimensions. In particular, our Peierls contours will consist of bonds in the dual lattice. In three dimensions they would consist of plaquettes. Our bound on the critical β and various constants depend on the number of dimensions.

We begin by introducing the usual Peierls contours. An orthonormal basis for the state space is given by the collection of vectors which are eigenvectors of all the σ_r^z, $r \in \Lambda$. There is a two-to-one correspondence between these vectors and Peierls contours. Given such a vector, the corresponding Peierls contour is the set of bonds b in the dual lattice such that the z-components of the spins on opposite sides of b are opposite. The correspondence is two-to-one since we can flip all the spins and still have the same Peierls contour. We will use G to denote a Peierls contour. Note that in our terminology a contour need not be connected. Figure 1 contains six examples of Peierls contours.

As in [8, 12] we use the following variant of the Trotter product formula,

$$e^{-\beta H} = \lim_{N \to \infty} \left[e^{-\frac{\beta}{N} H^z} \left(1 - \frac{\beta(1-\delta)}{N} H^{xy} \right) \right]^N. \tag{2.2}$$

Letting

$$\sigma_r^\pm = \tfrac{1}{2}(\sigma_r^x \pm i\sigma_r^y),$$

we have

$$-H^{xy} = \sum_{\langle rs \rangle} (\sigma_r^+ \sigma_s^- + \sigma_r^- \sigma_s^+). \tag{2.3}$$

a. $\Gamma(0), \Gamma(1/10), \Gamma(2/10)$

b. $\Gamma(3/10), \Gamma(4/10)$

c. $\Gamma(5/10), \Gamma(6/10)$

d. $\Gamma(7/10)$
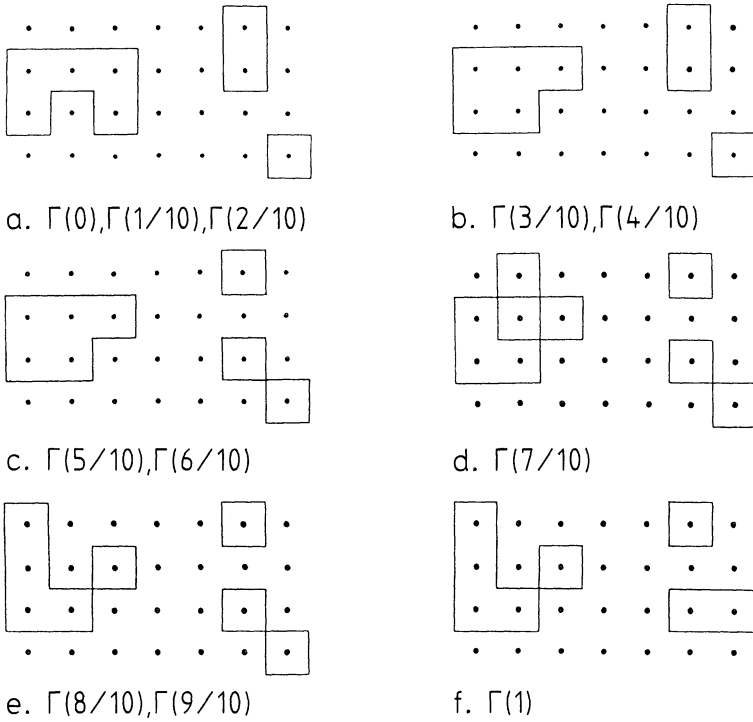
e. $\Gamma(8/10), \Gamma(9/10)$

f. $\Gamma(1)$

Fig. 1

Given a Peierls contour $G, (\sigma_r^+ \sigma_s^- + \sigma_r^- \sigma_s^+)$ annihilates the two corresponding basis vectors unless G contains the bond separating r and s. In this case this operator flips the spin at r and flips the spin at s. The Peierls contour changes accordingly. We denote the new contour by

$$(\sigma_r^+ \sigma_s^- + \sigma_r^- \sigma_s^+)G.$$

Figure 1 provides several examples of how a Peierls contour can change. Label the contours in $a, b, \ldots, f$ by $G_1, G_2, \ldots, G_6$. Then

$$G_i = (\sigma_{r_i}^+ \sigma_{s_i}^- + \sigma_{r_i}^- \sigma_{s_i}^+)G_{i-1},$$

where the bonds $(r_2, s_2), \ldots, (r_6, s_6)$ are $((2,2),(3,2))$, $((6,2),(6,3))$, $((2,3),(2,4))$, $((1,4),(2,4))$, and $((7,1),(7,2))$. We have specified "$x - y$ coordinates."

Definition 2.3. A quantum contour is a function $\Gamma = \Gamma(t)$ from $\left\{0, \frac{1}{N}, \frac{2}{N}, \ldots, \frac{N-1}{N}, 1\right\}$ into the set of all possible Peierls contours such that

(1) $\Gamma(1) = \Gamma(0)$.   (2.4)

(2) For each $m = 1, 2, \ldots, N$ either

(a) $\Gamma\left(\dfrac{m}{N}\right) = \Gamma\left(\dfrac{m-1}{N}\right),$ (2.5)

or

(b) $$\Gamma\left(\frac{m}{N}\right) = (\sigma_r^+ \sigma_s^- + \sigma_r^- \sigma_s^+)\Gamma\left(\frac{m-1}{N}\right) \tag{2.6}$$

for some $\langle rs \rangle$.

Figure 1 provides an example of a $\Gamma(t)$ with $N=10$. However, this example does not satisfy the $\Gamma(0) = \Gamma(1)$ condition. Nonetheless, Fig. 1 will still serve to illustrate our definitions.

For each Γ we let $n(\Gamma)$ denote the number of flips in Γ, i.e., the number of m such that condition (b) above holds. The average of the z-component of the Hamiltonian is

$$\bar{H}^z(\Gamma) = \frac{1}{N} \sum_{m=1}^{N} H^z\left(\Gamma\left(\frac{m}{N}\right)\right). \tag{2.7}$$

$H^z\left(\Gamma\left(\frac{m}{N}\right)\right)$ is just $\left|\Gamma\left(\frac{m}{N}\right)\right|$, the number of bonds in $\Gamma\left(\frac{m}{N}\right)$. The weight of Γ is

$$W(\Gamma) = \left[\frac{\beta(1-\delta)}{N}\right]^{n(\Gamma)} e^{-\beta \bar{H}^z(\Gamma)}. \tag{2.8}$$

Finally, we define $\mathrm{sgn}(\Gamma) = \pm 1$ by

$$\sigma_i^z \sigma_j^z(\Gamma(0)) = \mathrm{sgn}(\Gamma)\Gamma(0). \tag{2.9}$$

Graphically, $\mathrm{sgn}(\Gamma)$ is $(-1)^n$, where n is the number of times a path from i to j crosses $\Gamma(0)$.

The point of these definitions is the following expansion.

$$\mathrm{tr}(e^{-\beta H}) = \lim_{N \to \infty} 2 \sum_{\Gamma} W(\Gamma). \tag{2.10}$$

The trace in the left-hand side corresponds to the sum over $\Gamma(0)$ and the condition $\Gamma(0) = \Gamma(1)$. The sum over $\Gamma\left(\frac{1}{N}\right)$, $\Gamma\left(\frac{2}{N}\right)$, ..., $\Gamma\left(\frac{N-1}{N}\right)$ comes from expanding out the factors

$$\left(1 - \frac{\beta(1-\delta)}{N} H^{xy}\right).$$

Similarly,

$$\mathrm{tr}((1 - \sigma_i^z \sigma_j^z)e^{-\beta H}) = \lim_{N \to \infty} 4 \sum_{\Gamma : \mathrm{sgn}(\Gamma) = -1} W(\Gamma). \tag{2.11}$$

Next we define the support $S(\Gamma)$ of a quantum contour Γ. We think of the dual lattice as a subset of $\mathbb{R}^2$. Each site in Λ is at the center of a square in the dual lattice. We take these squares to be closed. Λ^* is the union of the closed squares corresponding to the sites in Λ. The support $S(\Gamma)$ will be a subset of Λ^* made up of closed squares and bonds in the dual lattice.
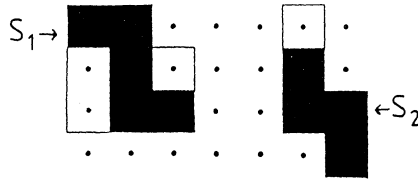
Fig. 2

Definition 2.4. $F(\Gamma)$ is the union of the closed squares such that the spin at the corresponding site flips at least once in Γ. So the square corresponding to site k is in $F(\Gamma)$ if for some $m \in \{1, 2, ..., N\}$ condition (b) of Definition 2.3 holds for some $\langle rs \rangle$ with $k = r$ or $k = s$.

The support of Γ is

$$S(\Gamma) = F(\Gamma) \cup \left[\bigcup_t \Gamma(t) \right]. \tag{2.12}$$

Γ is said to be connected if $S(\Gamma)$ is connected. Γ_1 and Γ_2 are said to be disjoint if their supports are disjoint. Figure 2 shows $S(\Gamma)$ for the $\Gamma(t)$ in Fig. 1.

We will use γ rather than Γ to denote connected quantum contour configurations. We caution the reader that γ may be connected without $\gamma(t)$ being connected for any t.

If a bond b is contained in $S(\Gamma)$ but not in $F(\Gamma)$, then the two squares on opposite sides of b are not in $F(\Gamma)$. So the two spins on opposite sides of b never flip. Hence b is in $\Gamma(t)$ for all t. So for any t

$$S(\Gamma) = F(\Gamma) \cup \Gamma(t). \tag{2.13}$$

Given disjoint Γ_1 and Γ_2 we can combine Γ_1 and Γ_2 by defining $(\Gamma_1 \cup \Gamma_2)(t) = \Gamma_1(t) \cup \Gamma_2(t)$. The crucial property to note is that $W(\Gamma_1 \cup \Gamma_2) = W(\Gamma_1)W(\Gamma_2)$. More importantly, we can go the other way and decompose a given Γ into connected pieces. Write $S(\Gamma)$ as a union of disjoint connected components.

$$S(\Gamma) = \bigcup_{i=1}^{n} S_i.$$

Let $\gamma_i(t) = \Gamma(t) \cap S_i$. Then the γ_i's are connected and disjoint. And

$$\Gamma = \bigcup_{i=1}^{n} \gamma_i, \qquad W(\Gamma) = \prod_{i=1}^{n} W(\gamma_i).$$

In the example in Fig. 1, $S(\Gamma)$ has two components. They are labelled in Fig. 2. Thus $\Gamma = \gamma_1 \cup \gamma_2$. γ_1 and γ_2 provide examples of two extremes. $\gamma_1(t)$ is connected for all t, while $\gamma_2(t)$ is never connected.

The decomposition $\Gamma \to \gamma_1, ..., \gamma_n$ is unique except for the ordering of $\gamma_1, ..., \gamma_n$. We will sum over all orderings and then divide by $n!$. Note also that

$$\text{sgn}(\Gamma) = \prod_{i=1}^{n} \text{sgn}(\gamma_i).$$

So we have the following lemma.

Lemma 2.5.

$$\text{tr}(e^{-\beta H}) = \lim_{N \to \infty} 2 \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\gamma_1, \dots, \gamma_n} \prod_{i=1}^{n} W(\gamma_i),$$

$$\text{tr}(\sigma_i^z \sigma_j^z e^{-\beta H}) = \lim_{N \to \infty} 2 \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\gamma_1, \dots, \gamma_n} \prod_{i=1}^{n} [W(\gamma_i) \operatorname{sgn}(\gamma_i)],$$

$$\text{tr}((1 - \sigma_i^z \sigma_j^z) e^{-\beta H}) = \lim_{N \to \infty} 4 \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\substack{\gamma_1, \dots, \gamma_n: \\ \operatorname{sgn}(\gamma_1 \cup \dots \cup \gamma_n) = -1}} \prod_{i=1}^{n} W(\gamma_i).$$

In all three sums the γ_i must be connected and disjoint.

3. Proof of LRO

We must take the $N \to \infty$ limit before we take the infinite volume limit. So throughout this section we work in a finite volume Λ. So all quantum contours γ have $S(\gamma) \subset \Lambda^*$. The bounds we obtain on quantities like $\langle \sigma_i^z \sigma_j^z \rangle_\Lambda$ are independent of Λ and so carry over to the infinite volume limit.

The following theorem is the heart of our proof of LRO. It is also the estimate needed to prove the convergence of the polymer expansion of the next section. We need a definition to state the estimate.

Definition 3.1. $|S(\gamma)|$ is the number of bonds and closed squares in $S(\gamma)$. The bonds which are part of the closed squares in $S(\gamma)$ are not counted in $|S(\gamma)|$.

In the example in Figs. 1 and 2, $|S(\gamma_1)| = 10$ and $|S(\gamma_2)| = 7$.

Theorem 3.2. *Given δ, $0 < \delta \leq 1$, there exists $\beta_o(\delta) > 0$ and $\varepsilon = \varepsilon(\delta) > 0$ such that for $\beta \geq \beta_o(\delta)$ and any closed square $P \subset \Lambda^*$,*

$$\varlimsup_{N \to \infty} \sum_{\gamma: S(\gamma) \cap P \neq \emptyset} W(\gamma) e^{\varepsilon |S(\gamma)|} \leq r(\beta, \delta), \tag{3.1}$$

where $r(\beta, \delta) \to 0$ as $\beta \to \infty$ with δ fixed. The sum is over connected γ whose support intersects P. $r(\beta, \delta)$ is independent of Λ.

We use Theorem 3.2 and the usual Peierls argument to prove Theorem 2.1.

Proof of Theorem 2.1. We must show that $\langle 1 - \sigma_i^z \sigma_j^z \rangle_\Lambda$ is small uniformly in i, j, and Λ. We claim that

$$\langle 1 - \sigma_i^z \sigma_j^z \rangle_\Lambda \leq \varlimsup_{N \to \infty} 2 \sum_{\gamma: \operatorname{sgn}(\gamma) = -1} W(\gamma). \tag{3.2}$$

By Lemma 2.5 this inequality is implied by

$$\sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\substack{\gamma_1, \dots, \gamma_n: \\ \operatorname{sgn}(\gamma_1 \cup \dots \cup \gamma_n) = -1}} \prod_{i=1}^{n} W(\gamma_i)$$

$$\leq \left[\sum_{\gamma: \operatorname{sgn}(\gamma) = -1} W(\gamma) \right] \left[\sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\gamma_1, \dots, \gamma_n} \prod_{i=1}^{n} W(\gamma_i) \right]. \tag{3.3}$$

To prove inequality (3.3) we note that given γ with $\mathrm{sgn}(\gamma) = -1$ and disjoint $\gamma_1, \ldots, \gamma_n$ such that $\mathrm{sgn}(\gamma_1 \cup \ldots \cup \gamma_n) = +1$ and $\gamma_1 \cup \ldots \cup \gamma_n$ is disjoint from γ, then $\gamma, \gamma_1, \ldots, \gamma_n$ are disjoint and $\mathrm{sgn}(\gamma \cup \gamma_1 \cup \ldots \cup \gamma_n) = -1$. In this way every term in the left-hand side of (3.3) appears at least once in the right-hand side.

Thus the proof is reduced to showing that

$$\varlimsup_{N \to \infty} \sum_{\gamma : \mathrm{sgn}(\gamma) = -1} W(\gamma) \tag{3.4}$$

is small. We must use the constraint $\mathrm{sgn}(\gamma) = -1$ to tie down $S(\gamma)$ to a square. If $\mathrm{sgn}(\gamma) = -1$, then either $S(\gamma)$ encloses i or j, or $S(\gamma)$ intersects both $\partial\Lambda$ and $[i,j]$, the line segment from i to j. In both cases $\mathrm{dist}(\{i,j\}, S(\gamma)) \leq |S(\gamma)|$, provided Λ is large enough that $\mathrm{dist}([i,j], \partial\Lambda) \geq |i-j|$. Hence $S(\gamma)$ intersects a square P in Λ^* with $\mathrm{dist}(P, i) \leq |S(\gamma)|$ or $\mathrm{dist}(P, j) \leq |S(\gamma)|$. So

$$\sum_{\gamma : \mathrm{sgn}(\gamma) = -1} W(\gamma) = \sum_{m=1}^{M} \sum_{\substack{\gamma : \mathrm{sgn}(\gamma) = -1 \\ |S(\gamma)| = m}} W(\gamma)$$

$$= \sum_{m=1}^{M} e^{-\varepsilon m} \sum_{\substack{\gamma : \mathrm{sgn}(\gamma) = -1 \\ |S(\gamma)| = m}} W(\gamma) e^{\varepsilon |S(\gamma)|}$$

$$\leq \sum_{m=1}^{M} e^{-\varepsilon m} \sum_{\substack{P : d(P, i) \leq m \\ \text{or } d(P, j) \leq m}} \sum_{\gamma : S(\gamma) \cap P \neq \emptyset} W(\gamma) e^{\varepsilon |S(\gamma)|}.$$

We have taken advantage of the fact that $S(\gamma) \subset \Lambda^*$ implies $|S(\gamma)| \leq M$, where M is an integer which depends on Λ. Since the lim sup of a finite sum is less than or equal to the sum of the lim sup's, we have shown that (3.4) is

$$\leq \sum_{m=1}^{M} e^{-\varepsilon m} \sum_{\substack{P : d(P, i) \leq m \\ \text{or } d(P, j) \leq m}} \varlimsup_{N \to \infty} \sum_{\gamma : S(\gamma) \cap P \neq \emptyset} W(\gamma) e^{\varepsilon |S(\gamma)|}$$

$$\leq r(\beta, \delta) \sum_{m=1}^{\infty} e^{-\varepsilon m} c m^2$$

by Theorem 3.2. The sum over m converges and is independent of β. Since $r(\beta, \delta) \to 0$ as $\beta \to \infty$ this completes the proof. $\quad\square$

We now turn to the proof of Theorem 3.2. We introduce some notation.

$$W_o(\Gamma) = \left[\frac{\beta\left(1 - \dfrac{\delta}{2}\right)}{N}\right]^{n(\Gamma)} e^{-\beta\left(1 - \frac{\delta}{2}\right)\bar{H}^z(\Gamma)}, \tag{3.5}$$

$$\varrho = \frac{1 - \delta}{1 - \dfrac{\delta}{2}}. \tag{3.6}$$

So $\varrho < 1$ and ϱ depends on δ,

$$P(\Gamma) = \varrho^{n(\Gamma)} e^{-\beta \frac{\delta}{2} \bar{H}^z(\Gamma)}. \tag{3.7}$$

So $W(\Gamma) = W_o(\Gamma)P(\Gamma)$. Finally,

$$H_o = \left(1 - \frac{\delta}{2}\right)[H^{xy} + H^z]. \tag{3.8}$$

Note that $W_o(\Gamma)$ is the weight associated with $e^{-\beta H_o}$.

We motivate our proof by sketching a simpler proof that does not quite work. Split up the sum over γ as

$$\sum_G \sum_{\gamma\,:\,\gamma(0)=G} .$$

The sum is over all Peierls contours G in Λ, not just over connected G. We can control the second sum by the resummation

$$\lim_{N\to\infty} 2 \sum_{\Gamma\,:\,\Gamma(0)=G} W_o(\Gamma) = \mathrm{tr}(e^{-\beta H_o}P_G), \tag{3.9}$$

where P_G is the projection onto the two dimensional subspace associated with G. Since $H_o \geq 0$, (3.9) is bounded by 2. We would like to use $P(\gamma)$ to control the sum over G as in the classical Peierls argument. There are two problems to be overcome. First, G need not be connected. This is compensated for by the factor of $\varrho^{n(\Gamma)}$ which will provide exponential decay between the components of G. Second, $H^z(\gamma(0))$ can be large without $\bar{H}^z(\gamma)$ being large since the number of bonds in $\gamma(t)$ can decrease. We overcome this second problem by replacing $\gamma(0) = G$ by $\gamma(t) = G$, where t is chosen so that $H^z(\gamma(t))$ is not too large relative to $\bar{H}^z(\gamma)$.

Proof of Theorem 3.2. Define a subset $T(\gamma)$ of $\left\{\dfrac{1}{N}, \dfrac{2}{N}, \ldots, 1\right\}$ by

$$T(\gamma) = \{t : H^z(\gamma(t)) \leq 2\bar{H}^z(\gamma)\}. \tag{3.10}$$

By the definition of $\bar{H}^z$, Eq. (2.7), and Chebyshev's inequality,

$$|T(\gamma)| \geq \frac{N}{2}. \tag{3.11}$$

For a closed square P in the dual lattice and a Peierls contour G, we define an integer $D_P(G)$ which measures how spread out the components of G are and how far G is from P. Let F be a union of closed squares in the dual lattice such that $P \cup G \cup F$ is connected. Then $D_P(G)$ is the minimum over such F of the number of squares in F.

We can now begin bounding the sum in (3.1). This sum over γ will have various constraints added to it. To simplify notation we use the following abbreviations:

$$P \quad \text{means} \quad P \cap S(\gamma) \neq \emptyset,$$

$$T \quad \text{means} \quad t \in T(\gamma), \tag{3.12}$$

$$G \quad \text{means} \quad \gamma(t) = G.$$

As always, the use of γ as opposed to Γ implies γ must be connected. The sum in Theorem 3.2 is

$$\sum_{\gamma\,:\,P} W(\gamma)e^{\varepsilon|S(\gamma)|} = \sum_t \sum_{\gamma\,:\,P,\,T} \frac{1}{|T(\gamma)|} W(\gamma)e^{\varepsilon|S(\gamma)|}. \tag{3.13}$$

Using (3.11) and bounding $\dfrac{1}{N}\sum\limits_{t}$ by $\sup\limits_{t}$, the above is

$$\leqq 2\sup_{t}\sum_{\gamma:P,T} W(\gamma)e^{\varepsilon|S(\gamma)|}.$$

Fix a time t for the remainder of the proof. All estimates will be uniform in t. If $t \in T(\gamma)$, then

$$e^{-\beta\frac{\delta}{2}\bar{H}^z(\Gamma)} \leqq e^{-\beta\frac{\delta}{4}H^z(\gamma(t))} = e^{-\beta\frac{\delta}{4}|\gamma(t)|}, \tag{3.14}$$

where $|\gamma(t)|$ is the number of bonds in $\gamma(t)$. We now drop the constraint T. So

$$\sum_{\gamma:P,T} W(\gamma)e^{\varepsilon|S(\gamma)|} \leqq \sum_{\gamma:P} W_o(\gamma)\varrho^{n(\gamma)}e^{-\beta\frac{\delta}{4}|\gamma(t)|}e^{\varepsilon|S(\gamma)|}. \tag{3.15}$$

By Eq. (2.13),

$$S(\gamma) = \gamma(t) \cup F(\gamma).$$

So $|S(\gamma)| \leqq |\gamma(t)| + |F(\gamma)|$. And $|F(\gamma)| \leqq 2n(\gamma)$ since each flip changes two sites. Thus

$$\varrho^{\frac{1}{2}n(\gamma)}e^{-\beta\frac{\delta}{8}|\gamma(t)|}e^{\varepsilon|S(\gamma)|} \leqq 1$$

if ε is sufficiently small, depending on δ. So (3.15) is

$$\leqq \sum_{\gamma:P} W_o(\gamma)\varrho^{\frac{1}{2}n(\gamma)}e^{-\beta\frac{\delta}{8}|\gamma(t)|}.$$

We rewrite this as

$$\sum_{G}\sum_{\gamma:P,G} W_o(\gamma)\varrho^{\frac{1}{2}n(\gamma)}e^{-\beta\frac{\delta}{8}|G|}, \tag{3.16}$$

where the sum over G is over all Peierls contours, not just connected ones. Note that $\gamma(t) \neq \emptyset$. This is a consequence of the fact that $(\sigma_r^+\sigma_s^- + \sigma_r^-\sigma_s^+)G \neq \emptyset$ if $G \neq \emptyset$. So the sum over G in (3.16) does not include the $G = \emptyset$ term. The definition of $D_P(G)$, the connectedness of $F(\gamma) \cup \gamma(t)$, and the constraint $P \cap [F(\gamma) \cup \gamma(t)] \neq \emptyset$ imply

$$D_P(G) \leqq |F(\gamma)| \leqq 2n(\gamma). \tag{3.17}$$

So (3.16) is

$$\leqq \sum_{G}\varrho^{\frac{1}{4}D_P(G)}e^{-\beta\frac{\delta}{8}|G|}\sum_{\gamma:G} W_o(\gamma). \tag{3.18}$$

We have dropped the constraint P on γ.

Consider the sum $\sum\limits_{\gamma:G} W_o(\gamma)$. Recall that the constraint G means that $\gamma(t) = G$. Because $\gamma(0) = \gamma(1)$, this sum is independent of t. So take $t = 0$. Our bound (3.18) is thus independent of t. The only quantity in (3.18) which depends on N is $W_o(\gamma)$. So taking the lim sup as $N \to \infty$ we have

$$\varlimsup_{N\to\infty}\sum_{\gamma:P} W(\gamma)e^{\varepsilon|S(\gamma)|} \leqq \varlimsup_{N\to\infty}\sum_{G}\varrho^{\frac{1}{4}D_P(G)}e^{-\beta\frac{\delta}{8}|G|}\sum_{\gamma:G} W_o(\gamma)$$

$$\leqq \sum_{G}\varrho^{\frac{1}{4}D_P(G)}e^{-\beta\frac{\delta}{8}|G|}\varlimsup_{N\to\infty}\sum_{\gamma:G} W_o(\gamma).$$

We have used the fact that the sum over G is finite since we are in a finite volume. As in the introduction to our proof, see Eq. (3.9),

$$\overline{\lim_{N \to \infty}} \sum_{\gamma : G} W_o(\gamma) \leqq \overline{\lim_{N \to \infty}} \sum_{\Gamma : G} W_o(\Gamma) = \tfrac{1}{2} \mathrm{tr}(e^{-\beta H_o} P_G) \leqq 1, \tag{3.19}$$

since $H_o \geqq 0$ and P_G is two-dimensional. The change from γ to Γ indicates that we drop the constraint that γ be connected.

We have thus reduced the proof to bounding

$$\sum_G \varrho^{\frac{1}{4} D_P(G)} e^{-\beta \frac{\delta}{8} |G|}. \tag{3.20}$$

The idea behind bounding this sum is the following. The sum over G can be done by first summing over the choice of the shapes of the connected components of G, and then summing over the locations of those components. The first sum is controlled by the factor of $e^{-\beta \frac{\delta}{8} |G|}$ in the usual way. The second sum is controlled by the factor $\varrho^{\frac{1}{4} D_P(G)}$, since this factor decays exponentially as we move the components of G apart.

We implement the above idea by thinking of G as being generated by a random walk on the set of bonds in the dual lattice. We will show that (3.20) is

$$\leqq \sum_\omega e^{-\beta \delta_o |\omega|} \prod_{i=1}^{|\omega|} \varrho_o^{d(\omega(i-1), \omega(i))}. \tag{3.21}$$

The sum is over walks on the set of bonds in the dual lattice, i.e., ω is a sequence $\omega(0), \omega(1), \ldots, \omega(n)$ of bonds for some $n \geqq 1$. $|\omega|$ denotes n. The distance $d(\omega(i-1), \omega(i))$ is the minimal number of closed squares needed to connect $\omega(i-1)$ and $\omega(i)$. In particular, $d(\omega(i-1), \omega(i)) = 0$ if $\omega(i-1)$ and $\omega(i)$ share an endpoint. $\delta_o > 0$ and $\varrho_o < 1$. These two constants depend on the number of dimensions and on δ. Also, we require $\omega(0)$ to be one of the bonds contained in P. We will show that for each G there is an ω with $G = \bigcup_i \omega(i)$ and

$$\varrho^{\frac{1}{4} D_P(G)} e^{-\beta \frac{\delta}{8} |G|} \leqq e^{-\beta \delta_o |\omega|} \prod_{i=1}^{|\omega|} \varrho_o^{d(\omega(i-1), \omega(i))}. \tag{3.22}$$

Let F be a collection of closed squares in the dual lattice with $F \cup G \cup P$ connected and $D_P(G) = |F|$. The number of squares or bonds that are connected to a given square or bond is bounded by a constant which depends on the number of dimensions. By the Konigsberg bridge problem there is a walk on the bonds in G and squares in $F \cup P$ which visits each bond or square at least once but no more than c_o times, where c_o depends on the number of dimensions. Furthermore, this walk only jumps between connected squares and bonds, and we can start this walk at P. We take ω to be this walk with the visits to squares deleted. Note that when ω jumps between two bonds that are not connected, the original walk must visit at least $d(\omega(i-1), \omega(i))$ squares in between the two bonds. The left-hand side of (3.22) contains a factor of $\varrho^{1/4}$ for each square in F and a factor of $e^{-\beta \delta/8}$ for each bond in G. So we have factors of $\varrho^{1/(4c_o)}$ and $e^{-\beta \delta/(8c_o)}$ for each visit to a square and bond, respectively. Inequality (3.22) follows.

Finally, to bound (3.21) note that

$$\sup_{b'} \sum_b \varrho_o^{d(b,b')}$$

is finite and depends only on δ. Call this supremum K. Then (3.21) is bounded by

$$c \sum_{n=1}^{\infty} K^n e^{-\beta \delta_o n},$$

which converges for sufficiently large β, and $\to 0$ as $\beta \to \infty$. $\square$

4. The Polymer Expansion

We have expressed our system as a gas of time dependent Peierls contours with a hard core condition, specifically, the supports of the contours are disjoint. In this section we give the polymer expansion for this system and discuss how this expansion yields results like Theorem 2.2. These are standard techniques so we will provide references to the literature in lieu of detailed proofs. A relatively simple proof of the convergence of the polymer expansion, based on arguments in [4] and [1, 2], may be found in Sect. 3 of [3] (see also [7] and [13]).

Definition 4.1. For a connected set $S \subset \Lambda^*$ which is made up of bonds and squares, let

$$K(S) = \lim_{N \to \infty} \sum_{\gamma: S(\gamma) = S} W(\gamma), \tag{4.1}$$

$$\bar{K}(S) = \lim_{N \to \infty} \sum_{\gamma: S(\gamma) = S} \text{sgn}(\gamma) W(\gamma). \tag{4.2}$$

We discuss the existence of these limits below.

We write each sum over connected quantum contours γ as the following double sum

$$\sum_{\gamma} = \sum_{S \subset \Lambda^*} \sum_{\gamma: S(\gamma) = S}.$$

The sum over S is only over connected S. Then Lemma 2.5 yields the following expansions:

$$\text{tr}(e^{-\beta H}) = 2 \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\substack{S_1, \ldots, S_n \\ \subset \Lambda^*, \text{disjoint}}} \prod_{i=1}^{n} K(S_i), \tag{4.3}$$

$$\text{tr}(\sigma_i^z \sigma_j^z e^{-\beta H}) = 2 \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\substack{S_1, \ldots, S_n \\ \subset \Lambda^*, \text{disjoint}}} \prod_{i=1}^{n} \bar{K}(S_i). \tag{4.4}$$

In both sums the $S_1, \ldots, S_n$ must be connected, disjoint subsets of Λ^*.

Definition 4.1 assumes that the limits as $N \to \infty$ in (4.1) and (4.2) exist. If we had been using the formalism of stochastic processes, then $K(S)$ would be given by the integral of a characteristic function. Even without this formalism we can give an easy proof of the existence of these limits. Fix an S and consider

$$2 \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\substack{\gamma_1, \ldots, \gamma_n: \\ \bigcup_i S(\gamma_i) \subset S}} \prod_{i=1}^{n} W(\gamma_i). \tag{4.5}$$

Using Eq. (2.2), the limit as $N \to \infty$ of the above expression exists and equals $\mathrm{tr}(P_S e^{-\beta H}s)$, where P_S and H_S are operators depending on S. Assume by induction that $K(S_i)$ exists for $S_i \subsetneqq S$. Since every γ_i that appears in Eq. (4.5) either has $S(\gamma_i) \subsetneqq S$ or $S(\gamma_i) = S$, the existence of the limit as $N \to \infty$ of (4.5) implies the existence of $K(S)$.

We now use the polymer expansion to take the logarithm of the sums in (4.3) and (4.4). We use the notation and convergence criteria of the review article by Brydges [3],

$$\log(\tfrac{1}{2}\mathrm{tr}(e^{-\beta H})) = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\substack{S_1,\dots,S_n \\ \subset \Lambda^*}} \prod_{i=1}^{n} K(S_i)\psi_c(S_1,\dots,S_n). \tag{4.6}$$

$\psi_c(S_1,\dots,S_n)$ is the usual connected part of the hard core interaction. See Theorem 3.1 of [3] or Eq. (1.17) and Theorem 2 of [4] for its definition. (In [4] ψ_c is denoted by ϕ^T.) In the above sum the $S_1,\dots,S_n$ need not be disjoint. In fact, $\psi_c(S_1,\dots,S_n)$ will be 0 unless $\bigcup_k S_k$ is connected. The same expansion holds for $\log(\tfrac{1}{2}\mathrm{tr}(\sigma_i^z \sigma_j^z e^{-\beta H}))$ with $K(S_i)$ replaced by $\bar{K}(S_i)$.

Expansion (4.6) converges if

$$\sup_P \sum_{S: S \cap P \neq \emptyset} K(S)e^{\varepsilon|S|} < \varepsilon \tag{4.7}$$

for some $\varepsilon > 0$. The sup is over closed squares P in Λ^*. This criterion for convergence is a slight modification of Theorem 3.4 of [3]. Inequality (4.7) holds with $\varepsilon = \varepsilon(\delta)$ and β sufficiently large by Theorem 3.2. Since $|\bar{K}(S)| \leq K(S)$, the expansion for $\log(\tfrac{1}{2}\mathrm{tr}(\sigma_i^z \sigma_j^z e^{-\beta H}))$ also converges. Thus we have the following expansion for the two point function.

Lemma 4.2. *For β sufficiently large, depending on δ but not on Λ,*

$$\log\langle \sigma_i^z \sigma_j^z \rangle_\Lambda = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\substack{S_1,\dots,S_n \\ \subset \Lambda^*}} \left[\prod_{i=1}^{n} \bar{K}(S_i) - \prod_{i=1}^{n} K(S_i) \right] \psi_c(S_1,\dots,S_n). \tag{4.8}$$

In expansion (4.8) we need only sum over $S_1,\dots,S_n$ with $\prod_{i=1}^{n} \bar{K}(S_i) \neq \prod_{i=1}^{n} K(S_i)$. From the definition of $\bar{K}(S)$ we see that $\bar{K}(S) \neq K(S)$ implies there are γ with $S(\gamma) = S$ and $\mathrm{sgn}(\gamma) = -1$. This implies that $[i,j]$, the line from i to j, intersects $\gamma(0)$ and thus intersects S. So the terms $S_1,\dots,S_n$ that contribute to the sum in (4.8) must have $\left(\bigcup_k S_k \right) \cap [i,j] \neq \emptyset$ and must fall in at least one of the following classes:

(1) $\bigcup_k S_k$ intersects $\partial \Lambda^*$.

(2) $\bigcup_k S_k$ encloses i or j but not both.

(3) $\bigcup_k S_k$ encloses both i and j.

A set S encloses i if every path from i to outside Λ^* intersects S.

The proof of Theorem 2.2 is now routine. See [13] for similar proofs. We sketch the key points.

The existence of the infinite volume limit of $\langle \sigma_i^z \sigma_j^z \rangle_\Lambda$ follows from the proof of the convergence of the polymer expansion and the observation that

$\left(\bigcup_k S_k\right) \cap [i,j] \neq \emptyset$. For terms in (1), $\sum_k |S_k| \geq \text{dist}([i,j], \partial \Lambda^*)$, so these terms do not contribute to the infinite volume limit.

For terms in (3), $\sum_k |S_k| \geq |i-j|$. So only terms in (2) contribute to the limit

$$M_\infty^2 = \lim_{|i-j| \to \infty} \langle \sigma_i^z \sigma_j^z \rangle_\infty.$$

To prove the exponential decay,

$$|\langle \sigma_i^z \sigma_j^z \rangle_\infty - M_\infty^2| \leq c e^{-m|i-j|}.$$

We note that only terms in (3) contribute to $\langle \sigma_i^z \sigma_j^z \rangle_\infty - M_\infty^2$. We split up $e^{\varepsilon|S|}$ as $e^{\varepsilon|S|/2} e^{\varepsilon|S|/2}$ and replace ε by $\dfrac{\varepsilon}{2}$ in condition (4.7). (We may have to increase β to do this.) The proof of convergence of the polymer expansion is then easily modified to show

$$\left| \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{S_1,\ldots,S_n:(3)} \left[\prod_i \bar{K}(S_i) - \prod_i K(S_i) \right] \psi_c(S_1,\ldots,S_n) \right|$$
$$\leq c \exp\left[-m \inf_{S_1,\ldots,S_n:(3)} \sum_k |S_k| \right],$$

where $m < \varepsilon/2$ and c depends on m. The sum over $S_1,\ldots,S_n$ and the inf over $S_1,\ldots,S_n$ are only over terms in (3). So the above is $\leq c e^{-m|i-j|}$, where $|i-j|$ is the minimal number of squares in a connected set which contains i and j.

Finally, we briefly describe how to prove the exponential decay of $\langle \sigma_i^x \sigma_j^x \rangle$ and $\langle \sigma_i^y \sigma_j^y \rangle$. It is not necessary to use the polymer expansion for this. By symmetry these two correlation functions are equal, and so $\langle \sigma_i^x \sigma_j^x \rangle = \langle \sigma_i^+ \sigma_j^- + \sigma_i^- \sigma_j^+ \rangle$. As in Sect. 2 we can expand $\text{tr}[(\sigma_i^+ \sigma_j^- + \sigma_i^- \sigma_j^+)e^{-\beta H}]$ in terms of time dependent contours. There is one important difference. The condition $\Gamma(1) = \Gamma(0)$ is replaced by $\Gamma(1) = (\sigma_i^+ \sigma_j^- + \sigma_i^- \sigma_j^+)\Gamma(0)$. This implies that the support of Γ has a connected component which contains the squares in Λ^* corresponding to the sites i and j.

As before we decompose Γ into connected quantum contours $\gamma_1,\ldots,\gamma_n$. Letting γ_k be the quantum contour whose support contains i and j, we have $\gamma_k(1) = (\sigma_i^+ \sigma_j^- + \sigma_i^- \sigma_j^+)\gamma_k(0)$ and $\gamma_m(1) = \gamma_m(0)$ for $m \neq k$. Thus the Peierls argument reduces bounding $\langle \sigma_i^x \sigma_j^x \rangle$ to bounding

$$\sum_\gamma{}' W(\gamma), \tag{4.8}$$

where the $'$ indicates that the sum is over connected γ with $\gamma(1) = (\sigma_i^+ \sigma_j^- + \sigma_i^- \sigma_j^+) \cdot \gamma(0)$.

The proof of Theorem 3.2 can be applied to the sum (4.8) with one modification. The proof used the condition $\gamma(1) = \gamma(0)$ in the resummation in Eq. (3.19). The resummation now becomes

$$\lim_{N \to \infty} \sum_{\Gamma:\Gamma(t)=G}{}' W_0(\Gamma) = \tfrac{1}{2}\text{tr}[(\sigma_i^+ \sigma_j^- + \sigma_i^- \sigma_j^+)e^{-t\beta H_0} P_G e^{-(1-t)\beta H_0}] \leq 1,$$

since $\sigma_i^+ \sigma_j^- + \sigma_i^- \sigma_j^+$ has norm equal to 1. In the sum (4.8), $S(\gamma)$ is connected and contains i and j. Thus the proof of Theorem 3.2 shows that (4.8) is bounded by $c e^{-m|i-j|}$.

References

1. Battle, G., Federbush, P.: A phase cell cluster expansion for Euclidean field theories. Ann. Phys. **142**, 95–139 (1982)
2. Battle, G.: A new combinatoric estimate for cluster expansions. Commun. Math. Phys. **94**, 133–139 (1984)
3. Brydges, D.: A short course on cluster expansions. In: Proceeding of 1984 Les Houches Summer School. Osterwalder, K. (ed.) (to be published)
4. Cammarota, C.: Decay of correlations for infinite range interactions in unbounded spin systems. Commun. Math. Phys. **85**, 517–528 (1982)
5. Dyson, F., Lieb, E., Simon, B.: Phase transitions in quantum spin systems with isotropic and non-isotropic interactions. J. Stat. Phys. **18**, 335–383 (1978)
6. Fröhlich, J., Lieb, E.: Phase transitions in anisotropic lattice spin systems. Commun. Math. Phys. **60**, 233–267 (1978)
7. Gallavotti, G., Martin-Löf, A., Miracle-Sole, S.: Some problems connected with the description of coexisting phases at low temperatures in the Ising model. Lecture Notes in Physics, Vol. 20. Berlin, Heidelberg, New York: Springer 1971
8. Ginibre, J.: Existence of phase transitions for quantum lattice systems. Commun. Math. Phys. **14**, 205–234 (1969)
9. Kirkwood, J.: Phase transitions in the Ising model with transverse field. J. Stat. Phys. **37**, 407–417 (1984)
10. Malyshev, V.: Phase transitions in classical Heisenberg ferromagnets with arbitrary parameter of anisotropy. Commun. Math. Phys. **40**, 75–82 (1975)
11. Mermin, N., Wagner, H.: Absence of ferromagnetism or antiferromagnetism in one- or two-dimensional isotropic Heisenberg models. Phys. Rev. Lett. **17**, 1133–1136 (1966)
12. Robinson, D.: A new proof of the existence of phase transitions in the anisotropic Heisenberg model. Commun. Math. Phys. **14**, 195–204 (1969)
13. Seiler, E.: Gauge theories as a problem of constructive quantum field theory and statistical mechanics. Lecture Notes in Physics, Vol. 159. Berlin, Heidelberg, New York: Springer 1982
14. Thomas, L., Yin, Z.: Low temperature expansions for the Gibbs states of quantum Ising lattice systems. J. Math. Phys. **25**, 3128–3134 (1984)
15. Thomas, L., Yin, Z.: Low temperature expansions for the Gibbs states of weakly interacting quantum Ising lattice systems. Commun. Math. Phys. **91**, 405–417 (1983)
16. Bricmont, J., Lebowitz, J.L., Pfister, C.E.: Some inequalities for anisotropic rotators (preprint)
17. Bricmont, J., Fontaine, J.-R.: Infrared bounds and the Peierls argument in two dimensions. Commun. Math. Phys. **87**, 417–427 (1982)
18. Speer, E.R.: Failure of reflection positivity in the quantum Heisenberg ferromagnet. To appear in Lett. Math. Phys.