Finite Temperature SU(2) Lattice Gauge Theory

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Abstract. We discuss SU(2) lattice gauge theories at non-zero temperature and prove several rigorous results including i) the absence of confinement for sufficiently high temperature in the pure gauge theory, and ii) the absence of spontaneous chiral symmetry breaking for sufficiently high temperature in the theory with massless fundamental representation fermions.

I. Introduction

Non-abelian gauge theories are the central building blocks of modern particle theories and have been widely studied in recent years. Despite this effort, very few physical results have been rigorously derived.

In this paper we consider SU(2) lattice gauge theories and derive several results concerning the finite temperature behavior of these theories.[1] These include the following:

A. In two or more space dimensions, there is a temperature $T_c < \infty$ (depending on the bare coupling and dimension) such that for temperatures $T > T_c$ static quarks cannot be confined.[2]

B. In any dimension, in the theory with dynamical massless fermions in the fundamental representation, there is a temperature $T_{ch} < \infty$ (depending on the dimension and number of fermions) above which chiral symmetry cannot be spontaneously broken.

The plan of this paper is as follows. In Sect. II we introduce our notation and review a variety of background information. This includes the definition of our confinement criteria and the equivalence between the lack of confinement and the spontaneous breakdown of a global $Z(2)$ symmetry. Section III contains our proof of the absence of confinement at high temperature. We use a Peierls argument to demonstrate the spontaneous breakdown of the global $Z(2)$ symmetry. However,

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1 Preliminary portions of this work have appeared in [1]
2 Recall that in one space dimension static quarks are confined for all temperatures $T < \infty$

the energetic estimates needed for our Peierls argument are far more delicate than those needed in conventional applications of this technique [2]. This is because the probability of a domain boundary is only suppressed by a power of $1/T$ instead of the usual exponential suppression. In order to deal with this problem we develop several renormalization group transformations which are simple enough to be rigorously controlled and are sufficiently accurate to preserve the overall power of T. In Sect. IV we consider the theory with dynamical massless quarks. We prove the convergence of a fermionic cluster expansion (i.e., an expansion in spacelike hops of the fermions) which converges for sufficiently high temperature for any value of the bare coupling. The existence of this convergent cluster expansion implies that all global fermionic symmetries are unbroken. Section V contains some brief concluding remarks. A number of technical lemmas are relegated to appendices. In addition, Appendix I describes some inequalities relating the various standard confinement criteria (Wilson loops, 't Hooft loops, the static quark potential, and the electric and magnetic flux free energies).

The major difficulty in proving our results is ultimately due to the paucity of useful analytical tools which are applicable to non-abelian gauge theories. (There are no non-trivial correlation inequalities [3], no random walk representations [4], no useful duality transformations [5], etc.) However in pure gauge theories the measure satisfies both ordinary positivity and reflection positivity [6, 7]. Ultimately, this is sufficient to derive the renormalization group bounds that we need to prove non-confinement.

In the presence of dynamical quarks, the measure contains Grassmann algebra valued terms and therefore one loses ordinary positivity (the integrand is no longer a simple number). However the measure is still reflection positive and using this alone, we are able to prove the convergence of our cluster expansion. (This may seem somewhat surprising; in most conventional proofs of cluster expansions one of the very first steps is a resummation yielding a single connected cluster during which one uses ordinary positivity to justify a small amount of over counting [6, 8, 9]. We are able to perform this resummation exactly and thus avoid any need for ordinary positivity.)

The absence of confinement at high temperatures has been previously demonstrated for any $SU(N)$ gauge group in three or more dimensions by Borgs and Seiler using an entirely different approach [10].

II. Background

The lattice Λ will be taken to be a $(d+1)$-dimensional periodic hypercubic lattice of size $L_t \times L_s^d$. For technical convenience we will assume that the lattice lengths in "time" (L_t) and "space" (L_s) are integer powers of two. Λ will be regarded as an anisotropic lattice with timelike and spacelike lattice spacings a_t and a_s, respectively. The physical temperature is defined as $T = (L_t a_t)^{-1}$. The "spatial" lattice obtained as a particular equal time slice of Λ will be denoted Λ_s.

On rare occasions the direction and coordinates of links (l) or plaquettes (p) will be indicated explicitly as $l_\mu(\bar{n})$ or $p_{\mu\nu}(\bar{n})$, where $0 \leq \mu, \nu \leq d$ (with $\mu = 0$ denoting the "time" direction) and where the coordinates $\{n_\mu\}$ are integers defined modulo the length of the lattice.

A. Pure Gauge Theory

An SU(2) valued link variable $U[l]$, is located on every link of the lattice. The pure gauge action is given by [11]

$$S_g = \beta_t \sum_{p_t \in \Lambda} \operatorname{tr}(U[\partial p_t] - 1) + \beta_s \sum_{p_s \in \Lambda} \operatorname{tr}(U[\partial p_s] - 1). \tag{2.1}$$

Here, $p_t(p_s)$ indicates a timelike (spacelike) plaquette and $U[\partial p]$ denotes the ordered product of link matrices around the boundary of the plaquette p. The timelike and spacelike gauge couplings are given by

$$\begin{aligned}
\beta_t &= \frac{2}{g^2} a_s^{d-3}(a_s/a_t) = \frac{2}{g^2} a_s^{d-3}(L_t a_s T), \\[4pt]
\beta_s &= \frac{2}{g^2} a_s^{d-3}(a_t/a_s) = \frac{2}{g^2} a_s^{d-3}(L_t a_s T)^{-1},
\end{aligned} \tag{2.2}$$

where g^2 is the conventional bare coupling constant. Note that increasing temperature corresponds to increasing timelike coupling (β_t) and decreasing spacelike coupling (β_s).[3]

Expectations of observables are computed in the measure

$$d\mu_\Lambda \equiv Z_\Lambda^{-1} \prod_{l \in \Lambda} dU[l] \exp S_g, \tag{2.3}$$

where the partition function Z_Λ is defined by $\int d\mu_\Lambda \equiv 1$, and $dU[l]$ denotes normalized Haar measure on the group SU(2). $d\mu_\Lambda$ is a reflection positive measure for reflections in d-dimensional planes parallel to two lattice axes which either contain lattice sites or bisect lattice bonds [6, 7]. In addition $d\mu_\Lambda$ is reflection positive with respect to timelike dihedral planes (i.e., planes parallel to the time axis and at 45° to a spatial axis). [Recall that reflection positivity for a reflection Θ which cuts the lattice into two equal pieces, Λ_+ and Λ_- [with $\Theta(\Lambda_+) = \Lambda_-$],[4] is the statement that

$$\langle F\Theta(F)\rangle \geqq 0 \tag{2.4}$$

for any observable F whose support is contained in Λ_+. This condition guarantees the existence of a positive definite transfer matrix [6]. Combined with the Schwarz inequality, (2.4) implies that

$$|\langle F\Theta(G)\rangle| \leqq \langle F\Theta(F)\rangle^{1/2} \langle G\Theta(G)\rangle^{1/2} \tag{2.5}$$

for any observables F and G with supports in Λ_+.]

For each spatial site $x \in \Lambda_s$, let S_x represent the set of L_t timelike link variables with the same spatial coordinates as the site x, and let $f_x(S_x)$ be an arbitrary (real) function of the links in S_x. By repeatedly using reflection positivity for timelike planes in between lattice sites, one may derive the chessboard estimate [12]

$$\left\langle \prod_{x \in \Lambda_s} f_x(S_x) \right\rangle \leqq \prod_{y \in \Lambda_s} \left\langle \prod_{x \in \Lambda_s} f_y(S_x) \right\rangle^{1/|\Lambda_s|}. \tag{2.6}$$

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3   The use of anisotropic couplings allows us to vary the temperature T, bare coupling g^2, and lattice length L_t independently

4   Due to the periodicity of the lattice, any reflection actually leaves invariant a pair of parallel planes which cut the lattice into two equal pieces

Similarly, for each spatial bond $\langle xy \rangle \in A_s$, (oriented such that x is a site in the even sublattice of A_s), let $f_{\langle xy \rangle}(S_x, S_y)$ be an arbitrary real function of the links in $S_x \cup S_y$. Using reflections in timelike planes through lattice sites plus timelike dihedral planes, one may derive the chessboard estimate for "bonds,"

$$\left\langle \prod_{\langle xy \rangle \in A_s} f_{\langle xy \rangle}(S_x, S_y) \right\rangle \leqq \prod_{\langle uv \rangle \in A_s} \left\langle \prod_{\langle xy \rangle \in A_s} f_{\langle uv \rangle}(S_x, S_y) \right\rangle^{1/(d|A_s|)}. \tag{2.7}$$

These estimates will be extensively used in Sect. III.

We will use two different observables which may be interpreted as confinement criteria. The first of these is based on the "twist," $\Omega[x]$, defined on sites of the spatial lattice A_s as the (ordered) product of link variables along the straight timelike path beginning at the site $x \in A_s$ and running once around the (periodic) lattice. (This is a topologically non-trivial closed loop.) The static potential between an external quark and antiquark, $F^{q\bar{q}}$, may be expressed in terms of the two point correlation function of the trace of the twist [13],

$$\exp(-F^{q\bar{q}}(x, x')/T) = \langle \tfrac{1}{2} \operatorname{tr} \Omega[x] \tfrac{1}{2} \operatorname{tr} \Omega[x'] \rangle_A \equiv G_A(x, x'). \tag{2.6}$$

We will generally restrict our attention to the case where the separation $x - x'$ is directed along a lattice axis.[5]

If the "magnetization,"

$$m \equiv \lim_{|x - x'| \to \infty} \lim_{L_s \to \infty} G_A(x, x')^{1/2} \tag{2.7}$$

vanishes, then this implies that the quark-antiquark potential increases to infinity as the separation $|x - x'|$ grows, which is interpreted as confinement of static quarks. Conversely, a non-zero magnetization implies non-confinement of static quarks.

In addition to the local SU(2) gauge invariance [11], the action (2.1) is invariant under a global Z(2) transformation which changes the sign of all time-like link variables which emerge from the equal time slice A_s [14]. Under this transformation, products of link variables around topologically trivial loops (i.e., topologically trivial Wilson loops) remain invariant, however the twist, $\Omega[x]$, changes sign. In other words, $\operatorname{tr} \Omega[x]$ is an order parameter for this Z(2) symmetry. Consequently, just as in the d-dimensional Ising model, if the magnetization (2.7) is non-zero, then the measure (2.3) does not describe an ergodic state and multiple pure phases must occur [15]. In the pure phases $\left(\text{which could be selected by}\right.$

adding an infinitesimal "magnetic field," $h \sum_{x \in A_s} \operatorname{tr} \Omega[x]\left.\right)$, the order parameter has a non-zero expectation,

$$\lim_{h \to 0} \lim_{L_s \to \infty} \langle \operatorname{tr} \Omega[x] \rangle_A = \pm m$$

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5   The off-axis quark-antiquark potential is always bounded below by the on-axis potential,

$$F_A^{q\bar{q}}(x, x') \geqq F_A^{q\bar{q}}(x, x''),$$

where $x'' = x - |x - x'| e_1$ and $|x - x'| \equiv \max_i |x_i - x_i'|$. This is an immediate consequence of reflection positivity

and the $Z(2)$ global symmetry is spontaneously broken. Conversely, if the $Z(2)$ symmetry is not spontaneously broken, then the magnetization must vanish and static quarks are confined.

An alternative confinement criterion is based on the behavior of the electric or magnetic flux free energy [16, 5]. The magnetic flux free energy is defined by changing the sign of the gauge coupling on a topologically non-trivial stack of plaquettes. Explicitly,

$$\exp(-F_\Lambda^{\mathrm{mag}}/T) \equiv \langle \tau[S_{01}] \rangle_\Lambda \equiv \left\langle \prod_{p \in S_{01}} \exp - 2\beta_t \operatorname{tr}(U[\partial p]) \right\rangle_\Lambda, \qquad (2.8)$$

where S_{01} is any coclosed set of plaquettes which winds once through each [01] plane of the lattice. (For example, $S_{01} = \{p_{01}(\check{n}) | n_0 = n_1 = 0\}$.) [The magnetic free energy of a gauge theory has a simple analog in a spin system; it corresponds to the difference in free energy produced by changing periodic boundary conditions to antiperiodic in one direction, or equivalently the expectation of an operator which flips the sign of the coupling on a topologically non-trivial coclosed set of links [17].]

The electric flux free energy is simply related to the magnetic free energy [16],

$$\exp(-F_\Lambda^{\mathrm{elec}}/T) \equiv \tfrac{1}{2}(1 - \exp(-F_\Lambda^{\mathrm{mag}}/T)) = \langle \tfrac{1}{2}(1 - \tau[S_{01}]) \rangle_\Lambda. \qquad (2.9)$$

By rewriting the theory in terms of a transfer matrix acting on a suitable Hilbert space, one may show that $\exp(-F^{\mathrm{elec}}/T)$ is the expectation of a projection operator onto states with global electric flux flowing through the periodic lattice in the 1-direction [16, 18]. If static quarks are confined by an asymptotically linear potential, $F^{q\bar{q}}(x, x') \sim \sigma|x - x'|$, then the string tension σ may be interpreted as the energy per unit length of electric flux, and consequently one expects the electric flux free energy to grow with the size of the lattice, $F_\Lambda^{\mathrm{elec}} \sim \sigma L_s$. Conversely, if the electric free energy remains finite as $L_s \to \infty$, then this is interpreted as the absence of (linear) confinement.

The two confinement criteria,

$$\lim_{|x-x'| \to \infty} \lim_{L_s \to \infty} F_\Lambda^{q\bar{q}}(x, x') = \infty \quad \text{and} \quad \lim_{L_s \to \infty} F_\Lambda^{\mathrm{elec}} = \infty$$

are not strictly equivalent [18]. However, we prove in Appendix I that linearly confining behavior for the electric free energy implies the presence of an asymptotically linear static quark-antiquark potential. Specifically, we show that

$$F_\Lambda^{q\bar{q}}(x, x')/|x - x'| \geq (F^{\mathrm{elec}} - 2T \ln 2)/L_s \qquad (2.10)$$

for $|x - x'| \leq L_s/2$.

B. Dynamical Fermions

For part of our discussion, we will add to the theory massless fermion fields [11]. Therefore, to each site i of the lattice Λ we will associate ν independent generators of a Grassmann algebra, denoted by $\chi[i]_\alpha$, $\alpha = 1, \ldots, \nu$. The fermions at each site, $\chi[i]$, transform under some (reducible) representation of the SU(2) gauge group. We will assume that ν is a multiple of four, and that the fermion representation is equivalent to $(\nu/2)$ copies of the fundamental representation. The fermion

contribution to the action may be written as

$$S_f = \sum_{l=\langle ij \rangle} \chi[i]^T M_{\langle ij \rangle}(U[l])\chi[j], \tag{2.11a}$$

where for each link $l \in \Lambda$, M_l is a v-dimensional matrix depending on the link variable $U[l]$. The explicit form of the matrices $\{M_l\}$ depends on the particular fermion latticization scheme adopted. We will assume that M_l has the form

$$M_l = \frac{1}{2}(a_t/a_s)\begin{bmatrix} 0 & \gamma[l]^* \times U[l]^* \\ \gamma[l] \times U[l] & 0 \end{bmatrix} \tag{2.12a}$$

for spacelike links, and

$$M_l = \frac{1}{2}\begin{bmatrix} 0 & \gamma[l]^* \times U[l]^* \\ \gamma[l] \times U[l] & 0 \end{bmatrix} \tag{2.12b}$$

for timelike links (* denote complex conjugation).[6] Here, for each link $l \in \Lambda$, $\gamma[l]$ is a fixed $v/4$ dimensional unitary and hermitian matrix. The set of matrices $\{\gamma[l]\}$ must satisfy

$$\gamma[\partial p] = -1 \tag{2.13}$$

for every plaquette $p \in \Lambda$ in order to recover the Dirac equation in the continuum limit. Splitting the fermions into two parts, $\chi[i] \equiv \begin{pmatrix} \psi[i] \\ \bar\psi[i]^T \end{pmatrix}$, the action may be rewritten as[7]

$$S_f = \tfrac{1}{2}(a_t/a_s)\sum_{l_s = \langle ij \rangle}(\bar\psi[i]\gamma[l]U[l]\psi[j] - \bar\psi[j]\gamma[l]U[l]^\dagger\psi[i])$$
$$+ \tfrac{1}{2}\sum_{l_t = \langle ij \rangle}(\bar\psi[i]\gamma[l]U[l]\psi[j] - \bar\psi[j]\gamma[l]U[l]^\dagger\psi[i]). \tag{2.11b}$$

Conventional choices for the matrices $\{\gamma[l]\}$ are either

$$\gamma[n_\mu] = (-1)^{\sum_{v < \mu} n_v} \tag{2.14a}$$

(which gives "staggered" fermions [19, 20]), or

$$\gamma[n_\mu] = \gamma_\mu \tag{2.14b}$$

with $\{\gamma_\mu, \gamma_v\} = 2\delta_{\mu v}$ (which yields "naive" fermion [11]). (Since the irreducible representation of this Clifford algebra is of dimension $2^{\lfloor d+1/2 \rfloor}$, for naive fermions v must be a multiple of $4 \times 2^{\lfloor d+1/2 \rfloor}$. Naive fermions are unitarily equivalent to $2^{\lfloor d+1/2 \rfloor}$ copies of staggered fermions [20].[8] For later notational convenience, we

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6   In order to satisfy reflection positivity, fermions must obey antiperiodic boundary conditions [7]. In other words, the sign of the coupling matrices $\{M_l\}$ must be flipped on a coclosed set of bonds which intersects once with every closed loop which winds once around the lattice. In most of our discussion this will not be indicated explicitly. Note that we have rescaled the fermion fields so as to remove a conventional factor of $(a_s)^d$ from the coupling matrices

7   The fermions may be given a bare mass by adding the term [11] $m\sum_i \bar\psi[i]\psi[i]$. We will only consider the massless theory

8   The continuum limit of this theory actually describes $2^{d-\lfloor d+1/2 \rfloor}(v/4)$ species (or "flavors") of physical fermions [20]

will assume that for all timelike links $\gamma[l]$ equals the same matrix (which we will call) γ_0.

Let G be the subgroup of $U(v/4)$ which commutes with all the matrices $\{\gamma[l]\}$. [G contains at least a $U(1)$ subgroup and, for staggered fermions, equals $U(v/4)$.] The fermion action (2.11) is invariant under the global $G \times G$ symmetry corresponding to independent rotation of fermions on even and odd sublattices. (Explicitly, for any element $(u, v) \in G \times G$, $\psi[i] \to u\psi[i]$, and $\bar{\psi}[i] \to \bar{\psi}[i]v^\dagger$ for even sites, while $\psi[i] \to v\psi[i]$ and $\bar{\psi}[i] \to \bar{\psi}[i]u^\dagger$ for odd sites.) This symmetry is referred to as chiral symmetry.

The full measure is given by

$$d\mu_A = Z_A^{-1} \prod_{l \in A} dU[l] \prod_{i \in A} d\chi[i] \exp(S_g + S_f), \qquad (2.15)$$

where $d\chi[i] \equiv \prod_{\alpha=1}^{v} d\chi[i]_\alpha$ is the standard measure on a Grassman algebra [21]. The measure (2.15) is invariant under the global chiral symmetry [as well as the local SU(2) gauge symmetry].

In order to discuss reflection positivity in the presence of fermions, it is convenient to split each link variable, $U[l]$, running from site i to site j, into two "half-bonds" [7],

$$U[l] \equiv w[i,j]w[j,i]^\dagger. \qquad (2.16)$$

Here, for each nearest neighbor pair of sites (i,j), $w[i,j] \in SU(2)$ may be regarded as starting at the site i and running halfway toward the site j. In the measure $\prod_{l \in A} dU[l]$ is replaced by $\prod_{\langle ij \rangle} dw[i,j]dw[j,i]$. Since the action and every physical observable only depends on the half-bonds through the combination (2.16), this form of the theory is strictly equivalent to the original form. However, introducing redundant variables in this fashion simplifies the definition of reflections through planes which bisect links.[9]

In the presence of fermions, it is not simple to define reflections through planes containing lattice sites in such a way that the bare measure $\left(\prod_i d\chi[i] \right)$ is reflection positive. Reflections through planes between lattice sites may however be usefully defined as follows,

$$\Theta(zA) \equiv z^* \Theta(A),$$

$$\Theta(AB) \equiv \Theta(B)\Theta(A),$$

$$\Theta(w[i,j]) \equiv w[\Theta(i), \Theta(j)]^*, \qquad (2.17)$$

$$\Theta(\psi[i]) \equiv (\bar{\psi}[\Theta(i)]\gamma[\Theta(i), i])^T,$$

$$\Theta(\bar{\psi}[i]) \equiv (\gamma[i, \Theta(i)]\psi[\Theta(i)])^T.$$

Here Θ denotes a particular reflection which takes site i into site $\Theta(i)$, z is an arbitrary complex number, and A and B are arbitrary observables (polynomials in the basic fields) whose support is contained in A_+. $\gamma[i, \Theta(i)]$ denotes the product of the matrices $\gamma[l]$ along the straight line from site i to site $\Theta(i)$.

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9   Alternatively, the introduction of half bonds may be avoided by using a partial gauge fixing which sets certain link variables to one (see [6])

This definition of reflection is chosen so that i) the bare measure

$$\prod_{\langle ij \rangle} dw[i,j]dw[j,i] \prod_i d\psi[i]d\bar\psi[i]$$

is reflection positive, and ii) the action may be expressed in the form

$$S_g + S_f = A + \Theta(A) + \sum_i B_i \Theta(B_i),\tag{2.18}$$

where A and B_i are observables whose support is contained in Λ_+. This implies that the full measure satisfies reflection positivity [12]. Verifying (2.18) requires using the fact that

$$\bar\psi[i]\gamma[l]U[l]\psi[j] - \bar\psi[j]\gamma[l]U[l]^\dagger\psi[i]$$

$$= \sum_{\alpha=1}^{v/2} (\bar\psi[i]w[i,j])_\alpha \Theta(\bar\psi[i]w[i,j])_\alpha + (w[i,j]^\dagger\psi[i])_\alpha \Theta(w[i,j]^\dagger\psi[i])_\alpha,\tag{2.19}$$

where Θ is the reflection which bisects the link $l = \langle ij \rangle$. This decomposition of the fermion action will be used in Sect. IV.

In order to test for possible spontaneous breaking of the global chiral symmetry, we will study correlation functions of observables such as $\bar\psi[i]\psi[i]$. This is an order parameter for the chiral symmetry, and its expectation in the measure (2.15) vanishes identically due to the symmetry. However, if the large distance limit of the two-point function

$$\lim_{|i-j|\to\infty} \lim_{L_s\to\infty} \langle \bar\psi[i]\psi[i]\bar\psi[j]\psi[j]\rangle_\Lambda$$

does not vanish, then this demonstrates the spontaneous breakdown of chiral symmetry, and the existence of multiple pure states in which $\bar\psi\psi$ will have a non-zero expectation [15]. Conversely, if all correlation functions of chirally non-invariant, local fermion operators decay for large separations, then this implies that the chiral symmetry is unbroken.

III. Non-Confinement at High Temperature

In this section the following theorems demonstrating the absence of confinement for sufficiently high temperature will be proven.

Theorem I. *For every coupling $g^2 > 0$ and spatial dimension $d \geq 2$ there is a temperature $T^* < \infty$ and a function $\mu(T)$ such that for all temperatures $T < T^*$ and all sites $x, y \in \Lambda_s$,*

$$F_\Lambda^{q\bar q}(x,y)/T \leq \mu(T) < \infty,$$

with $\mu(T)$ bounded uniformly as the spatial lattice size $L_s \to \infty$. Furthermore $\mu(T) \to 0$ as $T \to \infty$.

Theorem II. *For every coupling $g^2 > 0$ and dimension $d \geq 2$ there is a temperature $T^{**} < \infty$ and a function $\varrho(T)$ such that for all $T > T^{**}$,*

$$F_\Lambda^{\mathrm{elec}}/T \leq \varrho(T) < \infty,$$

uniformly in L_s.

Using the definitions (2.6) and (2.7), Theorem I has the trivial corollary that for all $T > T^*$ the two point function of the twist is bounded from below,

$$G_A(x, y) \geq \exp(-\mu(T)) > 0 \quad \forall x, y \in \Lambda_s$$

as is the magnetization,

$$m \geq \exp(-\mu(T)/2).$$

Similarly, Theorem II is equivalent to the upper bound

$$\exp(-F_A^{\text{mag}}/T) \leq 1 - 2\exp(-\varrho(T)) < 1$$

for all $T > T^{**}$.

A. Projections

We begin by introducing projection operators for the upper and lower hemispheres of SU(2),

$$P_x^\pm \equiv \begin{cases} 1 & \text{if} \quad \pm\frac{1}{2}\operatorname{tr}\Omega[x] \geq 0 \\ 0 & \text{otherwise}. \end{cases} \tag{3.1}$$

We may bound our confinement criteria in terms of expectations of these projections,

$$\begin{aligned} G_A(0, x) &= \langle \tfrac{1}{2}\operatorname{tr}\Omega[0]\tfrac{1}{2}\operatorname{tr}\Omega[x](P_0^+ + P_0^-)(P_x^+ + P_x^-)\rangle_A \\ &\geq 2\langle \tfrac{1}{2}\operatorname{tr}\Omega[0]\tfrac{1}{2}\operatorname{tr}\Omega[x]P_0^+ P_x^+\rangle - 2\langle P_0^+ P_x^-\rangle \\ &\geq 2\langle P_0^+ P_x^+\rangle - 2\langle P_0^+ P_x^-\rangle - 4\langle P_0^+(1 - \tfrac{1}{2}\operatorname{tr}\Omega[0])\rangle \\ &= 1 - 4\langle P_0^+ P_x^-\rangle_A - 2\langle 1 - |\tfrac{1}{2}\operatorname{tr}\Omega[0]|\rangle_A. \end{aligned}$$

Here we have used translation invariance and the global Z(2) symmetry to combine equivalent terms.

To bound the magnetic flux free energy, we introduce projection operators on two spatial sites halfway around the lattice from each other,

$$\begin{aligned} \exp(-F^{\text{mag}}/T) &= \langle (P_0^+ + P_0^-)\tau[S_{01}](P_x^+ + P_x^-)\rangle_A \\ &\leq 2\langle P_0^- P_x^- \tau[S_{01}]\rangle_A + 2\langle P_0^+ P_x^- \tau[S_{01}]\rangle_A \\ &= 2\langle P_0^+ P_x^- \tau[S_{01}']\rangle_A + 2\langle P_0^+ P_x^- \tau[S_{01}]\rangle_A. \end{aligned}$$

Here $x \equiv \frac{1}{2}L_s\hat{e}_1$, $S_{01} \equiv \{p_{01}(\bar{n})|n_0 = n_1 = 0\}$, and $S_{01}' = \{p_{01}(\bar{n})|n_0 = 0, n_1 = -1\}$. In the last step, we have simply made a change of variables in the first expectation which flipped the signs of the set of timelike links $\{l_0(\bar{n})|n_0 = n_1 = 0\}$. This moves the coclosed set of plaquettes with negative couplings from S_{01} to S_{01}' and changes the projection P_0^- to P_0^+. Next we use reflection positivity (2.4) for the reflection which leaves invariant planes perpendicular to the 1-axis containing sites 0 and x. Therefore

$$\begin{aligned} \exp(-F^{\text{mag}}/T) &\leq 4\langle P_0^+ P_x^- \tau[S_{01}]\rangle_A \\ &\leq 4\langle P_0^+ P_x^-\rangle^{1/2}\langle P_0^+ P_x^- \tau[S_{01}]\tau[S_{01}']\rangle^{1/2} \\ &\leq 4\langle P_0^+ P_x^-\rangle^{1/2}\langle P_0^- P_x^-\rangle^{1/2} \\ &\leq [8\langle P_0^+ P_x^-\rangle]^{1/2}. \end{aligned} \tag{3.3}$$

(Here, the first step is reflection positivity, the next step repeats the change of variables used above in the second expectation, and the last step uses the fact that $\langle P_0^- P_x^- \rangle \leqq \langle P_0^- \rangle = 1/2$.)

This result shows that the magnetic flux free energy will have non-confining behavior ($F^{\text{mag}} > 0$) if the probability of oppositely directed twists is sufficiently small. To demonstrate spontaneous magnetization, we must in addition show that the probability for the twist to deviate from ± 1 is small.

B. Peierls Argument

In order to control the dependence of $\langle P_0^+ P_x^- \rangle$ on the separation x, a modern version of the Peierls argument may be used to bound this expectation in terms of expectations of nearest neighbor projections [2]. Inserting $1 = P_y^+ + P_y^-$ on every site of the spatial lattice yields.

$$\langle P_0^+ P_x^- \rangle_A = \sum_{Q \subset A_s} \left\langle \prod_{y \in Q} P_y^+ \prod_{y \notin Q} P_y^- \right\rangle_A, \tag{3.4}$$

where Q represents an arbitrary set of spatial sites such that $0 \in Q$, $x \notin Q$. Now let C be a minimal length path (i.e. set of links) in A_s running from 0 to x, and define a contour γ to be any connected, coclosed set of links (in A_s) such that $\gamma[C] = 1$.[10] (Links in γ are defined to be connected if they are contained in the boundary of a common plaquette; in other words their coboundaries intersect. $\gamma[C]$ denotes the number of (oriented) links of γ contained in C.) Therefore, resumming all sets Q whose coboundaries contain a given contour, one finds

$$\langle P_0^+ P_x^- \rangle \leqq \sum_{\gamma \subset A_s} \left\langle P_0^+ P_x^- \prod_{\langle yy' \rangle \in \gamma} P_y^+ P_{y'}^- \right\rangle \leqq \sum_{\gamma} \left\langle \prod_{\langle yy' \rangle \in \gamma} P_y^+ P_{y'}^- \right\rangle. \tag{3.5}$$

($\langle yy' \rangle \in A_s$ denotes the link running from spatial site y to the nearest neighbor site y'.) We will show (in the next subsection) that this expectation of nearest neighbor projections around a contour obeys a bound

$$\left\langle \prod_{\langle yy' \rangle \in \gamma} P_y^+ P_{y'}^- \right\rangle \leqq \kappa^{|\gamma|} \tag{3.6}$$

for some (temperature and coupling dependent) constant κ. Assuming this for the moment, (3.6) implies that

$$\langle P_0^+ P_x^- \rangle \leqq \sum_{\substack{|\gamma| \geqq 2d \\ |\gamma| \text{ even}}} N(|\gamma|) \kappa^{|\gamma|},$$

where $N(|\gamma|)$ is the number of contours of length $|\gamma|$, and the sum starts from $2d$ since this is the size of the minimal coclosed contour. In Appendix II we give a simple counting argument which shows that

$$N(|\gamma|) \leqq \frac{(|\gamma| - 2)}{d - 1} 3^{|\gamma| - d} \tag{3.7}$$

---

10 Unlike previous applications of the Peierls argument [2], we define contours in such a way that topologically non-trivial contours do not cut the lattice into two disjoint pieces. Our definition simplifies the counting of contours of a given size

(provided $L_s \geq 4$). Therefore, if $3\kappa < 1$, we find that

$$\langle P_0^+ P_x^- \rangle \lesseqgtr \frac{2(3\kappa^2)^d}{d-1} \left[\frac{(d-1)-(d-2)(3\kappa)^2}{(1-(3\kappa)^2)^2} \right]. \tag{3.8}$$

C. Disorder Probabilities

The crux of the proof is now to bound the expectations of the contour projections $\left\langle \prod_{\langle xy \rangle \in \gamma} P_x^+ P_y^- \right\rangle$ as well as $\langle 1 - |\frac{1}{2} \operatorname{tr} \Omega[0]| \rangle$. These expectations essentially measure the probability of formation of domain boundaries or point defects, respectively; in other words they measure the disorder in the twist $\Omega[x]$.

We begin by using the chessboard estimates (2.6) and (2.7) to bound these local expectations by thermodynamic quantities,

$$\langle 1 - |\tfrac{1}{2} \operatorname{tr} \Omega[0]| \rangle \leq \left\langle \prod_{x \in \Lambda_s} (1 - |\tfrac{1}{2} \operatorname{tr} \Omega[x]|) \right\rangle^{1/|\Lambda_s|}, \tag{3.9}$$

$$\left\langle \prod_{\langle xy \rangle \in \gamma} P_x^+ P_y^- \right\rangle \leq \left\langle \prod_{x \in \Lambda_s^e} P_x^+ \prod_{y \in \Lambda_s^o} P_y^- \right\rangle^{|\gamma|/(d|\Lambda_s|)}. \tag{3.10}$$

Here $\Lambda_s^e(\Lambda_s^o)$ represents the even (odd) sublattice of Λ_s.

These resulting thermodynamic expectations each have the form $\left\langle \prod_{y \in \Lambda_s} f_y(\Omega[y]) \right\rangle$ for some set of non-negative class functions $\{f_y\}$. It will be convenient to express these expectations in terms of the transfer matrix T,

$$
\begin{aligned}
\left\langle \prod_{y \in \Lambda_s} f_y(\Omega[y]) \right\rangle &= Z^{-1} \int \prod_{l \in \Lambda} dU[l] \prod_{y \in \Lambda_s} f_y(\Omega[y]) \\
&\quad \cdot \exp\left[\beta_t \sum_{p_t \in \Lambda} \operatorname{tr}(U[\partial p]-1) + \beta_s \sum_{p_s \in \Lambda} \operatorname{tr}(U[\partial p]-1) \right] \\
&= Z^{-1} \int \prod_{x \in \Lambda_s} d\Omega[x] f_x(\Omega[x]) \prod_{l \in \Lambda_s} dU[l] T^{L_t}(\{U^\Omega[l]\}, \{U[l]\}; \beta_t, \beta_s).
\end{aligned} \tag{3.11}
$$

Here $U^\Omega[l] \equiv \Omega[i] U[l] \Omega[j]^\dagger$ for $l = \langle ij \rangle$, and we have introduced (the L_t^{th} power of) the transfer matrix T whose matrix elements are given by

$$
\begin{aligned}
&T^L(\{U'[l]\}, \{U[l]\}; \beta_s, \beta_t) \\
&\equiv \int \prod_{t=1}^{L-1} \prod_{l \in \Lambda_s} dU_t[l] \exp \beta_t \sum_{t=1}^{L} \sum_{l \in \Lambda_s} \operatorname{tr}(U_t[l] U_{t-1}^\dagger[l]-1) \\
&\quad \cdot \exp \beta_s \sum_{t=1}^{L} \sum_{p \in \Lambda_s} \operatorname{tr}(U_t[\partial p]-1).
\end{aligned} \tag{3.12}
$$

(In this last expression, the subscript on $U_t[l]$ denotes the time of the link matrix, and $U_0[l] \equiv U[l]$, $U_L[l] \equiv U'[l]$.) To derive (3.11) one makes a change of variables (gauge transfomration) which effectively sets to one all timelike link variables except for those that end in the spacelike slice Λ_s which are transformed into the twist $\Omega[x]$.

In Appendix III, we derive a simple recursion relation for the transfer matrix which yields the bound,

$$T^{L_t}(\{U'[l]\}, \{U[l]\}; \beta_t, \beta_s) \leq \exp\left(\beta_> \sum_{l \in \Lambda} \operatorname{tr}(U'[l] U[l]^\dagger - 1) + |\Lambda_s| F_> \right). \tag{3.13}$$

Here $F_>$ is an explicit constant (depending on β_t and L_t) and, if $L_t = 2^k$, then the "renormalized" coupling $\beta_>$ is the image of β_t under the kth iterate of the mapping

$$\beta \to \tfrac{1}{8} \ln(I_1(4\beta)/2\beta). \tag{3.14}$$

[$I_1(z)$ is a modified Bessel function.] As $T \to \infty$, $\beta_> \sim \beta_t/L_t + O(\ln\beta_t)$, and

$$F_> \sim -(L_t - 1)\frac{d}{2}\ln(4\pi\beta_t^3) - k\frac{d}{2}\ln 8.$$

[For the precise definition of $F_>$ see (A3.5).]

Inserting this bound on the transfer matrix yields the result

$$\left\langle \prod_{y \in \Lambda_s} f_y(\Omega[y]) \right\rangle \leq Z^{-1} \int \prod_{x \in \Lambda} d\Omega[x] f_x(\Omega[x]) \prod_{l \in \Lambda_s} dU[l]$$
$$\cdot \exp\left(\beta_> \sum_{l \in \Lambda_s} \text{tr}(U^\Omega[l]U[l]^\dagger - 1) + |\Lambda_s|F_>\right). \tag{3.15}$$

This reduces the original theory with L_t time slices to an effective theory defined on a single time slice.

At this point, one can exactly integrate over the remaining spacelike link variables and find

$$\left\langle \prod_{y \in \Lambda_s} f_y(\Omega[y]) \right\rangle \leq Z^{-1} e^{|\Lambda_s|F_>} \int \prod_{x \in \Lambda_s} \frac{d\omega[x]}{\pi/2} \sin^2 \omega[x] f_x(e^{i\tau_3 \omega[x]})$$
$$\cdot \prod_{\langle xy \rangle \in \Lambda_s} J(\omega[x], \omega[y]; \beta_>), \tag{3.16}$$

where

$$J(\omega, \omega'; \beta) \equiv e^{-2\beta(1 - \cos\omega \cos\omega')}\left(\frac{\sinh(2\beta \sin\omega \sin\omega')}{2\beta \sin\omega \sin\omega'}\right), \tag{3.17}$$

and $e^{i\omega[x]}$ is the eigenvalue of the twist $\Omega[x]$ in the upper half plane (i.e., $0 \leq \omega[x] \leq \pi$).

We must now perform the final integral over the eigenvalues $\{\omega[x]\}$. Consider first the right-hand side of (3.9) and note that

$$\text{and} \qquad\qquad J(\omega, \omega'; \beta) \leq \exp 2\beta(\cos(\omega - \omega') - 1),$$

$$\sin\omega \sin\omega' J(\omega, \omega'; \beta) \leq (4\beta)^{-1} \exp 2\beta(\cos(\omega - \omega') - 1).$$

Furthermore, $1 - |\tfrac{1}{2}\text{tr}\,\Omega[x]| = 1 - |\cos\omega[x]| \leq \sin^2 \omega[x]$. Hence,

$$\left\langle \prod_{x \in \Lambda_s}(1 - |\tfrac{1}{2}\text{tr}\,\Omega[x]|) \right\rangle \leq (e^{F_>}/(2\beta_>)^2)^{|\Lambda_s|} Z_{xy}(2\beta_>)/Z, \tag{3.18}$$

where $Z_{xy}(\beta)$ is the partition function of a d-dimensional xy model,

$$Z_{xy}(\beta) = \int \prod_{x \in \Lambda_s} \frac{d\theta[x]}{2\pi} \exp \sum_{\langle xy \rangle \in \Lambda_s} \beta(\cos(\theta[x] - \theta[y]) - 1). \tag{3.19}$$

To evaluate the contour expectation (3.10) we will use the fact that

$$P_x^+ P_y^- J(\omega[x], \omega[y]) \leq P_x^+ P_y^- [J(\omega[x], \pi/2) J(\pi/2, \omega[y])]^{1/2}.$$

Inserting this bound yields $|\Lambda_s|$ decoupled integrals, so that

$$\left\langle \prod_{x \in \Lambda_s^e} P_x^+ \prod_{y \in \Lambda_s^o} P_y^- \right\rangle \leq (e^{F_>}j(\beta_>))^{|\Lambda_s|}/Z, \tag{3.20}$$

where

$$j(\beta) \equiv \int_0^{\pi/2} \frac{d\omega}{\pi/2} \sin^2 \omega \, J(\omega, \pi/2)^d \tag{3.21}$$

and $j(\beta) \sim (4\beta)^{-d}/\sqrt{\pi d\beta}$ as $\beta \to \infty$.

To complete our estimates we must derive an upper bound on the xy model partition function, and a lower bound on the full gauge theory partition function. This is done in Appendix IV where we find explicit functions $z_{xy}(\beta)$ and $z(\beta_t, \beta_s)$ such that

$$Z_{xy}(\beta) \leq z_{xy}(\beta)^{|A_s|},$$
$$Z_A \geq z(\beta_t, \beta_s)^{|A_s|}. \tag{3.22}$$

Asymptotically,

$$\ln z_{xy}(\beta) \sim -\tfrac{1}{2} \ln 8\pi\beta + (2^d - 1)^{-1} \ln 2,$$

as $\beta, L_s \to \infty$ and

$$z(\beta_t, \beta_s) e^{-F_>} \sim c\beta_>^{-3/2},$$

as $T \to \infty$. The positive constant c depends on the bare coupling g^2 and L_t. Combining these results yields

$$\left\langle \prod_{x \in A_s} (1 - |\tfrac{1}{2} \operatorname{tr} \Omega[x]|) \right\rangle \leq [r/(4\beta_>)]^{|A_s|} \tag{3.23}$$

and

$$\left\langle \prod_{x \in A_s^e} P_x^+ \prod_{y \in A_s^o} P_y^- \right\rangle \leq [s/(4\beta_>)^{(d-1)}]^{|A_s|}, \tag{3.24}$$

where $r \equiv \dfrac{z_{xy}(2\beta_>) e^{F_>}}{z(\beta_t, \beta_s)\beta_>}$ and $s \equiv \dfrac{(4\beta_>)^{d-1} j(\beta_>) e^{F_>}}{z(\beta_t, \beta_s)}$. Note that the ratios r and s remain bounded as $T \to \infty$ (uniformly in L_s). Returning finally to (3.9) and (3.10), we have

$$\langle 1 - |\tfrac{1}{2} \operatorname{tr} \Omega[0]| \rangle \leq r/(4\beta_>) \sim O(T)^{-1} \quad \text{as} \quad T \to \infty,$$

and

$$\left\langle \prod_{\langle xy \rangle \in \gamma} P_x^+ P_y^- \right\rangle \leq \kappa^{|\gamma|},$$

where

$$\kappa \equiv [s/(4\beta_>)^{(d-1)}]^{1/d} \sim O(T)^{-(d-1)/d} \quad \text{as} \quad T \to \infty.$$

Thus, we have established the basic bound (3.26) needed for the Peierls argument. Inserting (3.8), (3.25), and (3.26) into the initial bounds on confinement criteria, (3.2) and (3.3), yields the theorems stated at the beginning of this section.

IV. Chiral Symmetry at High Temperature

In this section we derive the following theorem:

Theorem III. *For every coupling $g^2 > 0$ and spatial dimension $d \geq 1$ there is a temperature $T_{ch} > \infty$ such that*

$$\lim_{|i-j| \to \infty} \lim_{|A_s| \to \infty} \langle \bar{\psi}[i]\psi[i]\bar{\psi}[j]\psi[j] \rangle_A$$

vanishes for all $T > T_{ch}$. *More generally, for* $T > T_{ch}$ *all correlation functions of chirally non-invariant local fermion operators exhibit exponential clustering. Our result for* T_{ch} *will actually be independent of the bare coupling.*

The starting point of our proof is the observation that spacelike terms in fermion action (2.12) are suppressed relative to timelike terms by a factor of (a_t/a_s) $= 1/(L_t a_s T)$. Therefore an expansion in powers of the number of spacelike hops should converge for sufficiently large temperature.

Consider the expectation of some product of local fermion operators,

$$\mathcal{O} \equiv \prod_{i \in O} (\chi[i]^T \Gamma_i \chi[i]) \equiv \prod_{i \in O} \mathcal{O}_i, \tag{4.1}$$

where O is some finite set of sites and each Γ_i is an arbitrary unitary matrix. Define $\mathcal{O}_i$ to equal one for sites not contained in the set O. The expectation of $\mathcal{O}$ is given by

$$\langle \mathcal{O} \rangle = Z^{-1} \int \prod_{l \in \Lambda} dU[l] \prod_{i \in \Lambda} d\chi[i] \left(\prod_{i \in O} \mathcal{O}_i \right) \exp(S_g + S_{f,t})$$
$$\cdot \prod_{l = \langle ij \rangle} \exp(\chi[i]^T M_{\langle ij \rangle}(U[l]) \chi[j]), \tag{4.2}$$

where S_g is the pure gauge action (2.1), $S_{f,t}$ is the timelike part of the fermion action (2.11), and $l_s = \langle ij \rangle$ denotes an arbitrary spacelike link running from site i to site j.

Our cluster expansion is generated by the representation,

$$\exp(\chi[i]^T M_{\langle ij \rangle} \chi[j]) = \prod_{\alpha=1}^{v} (1 + f^\alpha_{\langle ij \rangle}), \tag{4.3}$$

where

$$f^\alpha_{\langle ij \rangle} \equiv (\chi[i]^T w[i,j])_\alpha (w[i,j]^\dagger M_{\langle ij \rangle} \chi[j])_\alpha. \tag{4.4}$$

(The half-bond $w[i,j]$ is to be taken in the representation under which $\chi[i]$ transforms. Note that $\exp f^\alpha_{\langle ij \rangle} = 1 + f^\alpha_{\langle ij \rangle}$, since $(f^\alpha_{\langle ij \rangle})^2 = 0$. This decomposition is equivalent to (2.19).) Inserting this representation in (4.2) yields

$$\langle \mathcal{O} \rangle = \sum_{C \subset Z_v \times L} \left\langle \mathcal{O} \prod_{(\alpha,l) \in C} f^\alpha_l \exp - S_{f,s} \right\rangle. $$

Here $L \equiv \{l_s \in \Lambda\}$ is the set of all spacelike links and C is an arbitrary subset of the set consisting of v copies of L. Each set C may be decomposed into a number of connected components, where connectivity is defined by first projecting $Z_v \times L$ onto L, then projecting L onto the set of links in Λ_s, and finally using the ordinary definition of connectivity in the spacelike slice Λ_s. (In other words, two spacelike links are connected if they may be joined by a sequence of timelike links.) Next, we may resum all components of C which do not intersect the set O. This yields the cluster expansion,

$$\langle \mathcal{O} \rangle = \sum_Q \left\langle \mathcal{O} \prod_{(\alpha,l) \in Q} f^\alpha_l \prod_{(\alpha,l) \in \bar{Q}} (1 - f^\alpha_l) \right\rangle \equiv \sum_Q I_Q, \tag{4.5}$$

where the sum is over all sets $Q \subset Z_v \times L$ which are composed of clusters which are i) connected, and ii) connected to some site of the set O. $\bar{Q}$ is the "closure" of Q, defined as the set of all elements of $Z_v \times L$ which are connected to (any element of) $Q \cup O$.

Our main result for this section is

Theorem IV. *The cluster expansion* (4.5) *converges absolutely and uniformly in* $|A_s|$ *if*

$$\gamma \equiv 2d^2 v 2^{\nu}/(Ta_s 2^{(\nu-1)/L_t}) < 1. \tag{4.6}$$

To prove Theorem IV, we need a bound on the generic term I_Q in the expansion (4.5). To obtain this, we will use the fact that the measure (2.13) satisfies reflection positivity about planes without sites. Therefore, I_Q may be bounded by a chessboard estimate,

$$|I_Q|^A \leq \prod_{i \in Q \cup O} \left\langle \left(\prod_{j \in A} T_i^j(\mathcal{O}_i) \right) \left(\prod_{\substack{(\alpha, l) \in Q \\ l \in \delta i}} \prod_{l' \sim l} f_{l'}^{\alpha} \right) e^{-S_{f,s}} \right\rangle$$

$$\cdot \prod_{i \in \bar{Q} \backslash Q} \left\langle \left(\prod_{\substack{(\alpha, l) \notin \bar{Q} \\ l \in \delta i}} \prod_{l' \sim l} (1 + f_{l'}^{\alpha}) \right) e^{-S_{f,s}} \right\rangle. \tag{4.7}$$

Here T_i^j represents a sequence of reflections between lattice planes which moves the site i to the site j. $l' \sim l$ indicates that l' is one of the $|A|/2$ links which may be moved by a sequence of reflections to the link l. The notation $i \in Q$ means that there exists an element $(\alpha, l) \in Q$ such that $l \in \delta i$ (δi is the coboundary of the site i). To derive (4.7) we have used the fact that

$$f_{\langle ij \rangle}^{\alpha} = \tfrac{1}{2} (a_t/a_s) (\chi^T[i] w[i,j])_{\alpha} \Theta(\chi^T[i] w[i,j])_{\alpha},$$

where Θ is the reflection through the plane which bisects the link $\langle ij \rangle$. To simplify the expectations in (4.7), note that due to reflection positivity

$$Z \equiv \int \prod_l dU[l] \prod_i d\chi[i] e^{S_g} \prod_{l \in A} \prod_{\alpha} (1 + f_l^{\alpha}) \geq \int \prod_l dU[l] \prod_i d\chi[i] e^{S_g} \prod_{l \in T} \prod_{\alpha} (1 + f_l^{\alpha}), \tag{4.8}$$

where $T \subset A$ is any set of links which is invariant under all reflections through planes without sites. Since each of the numerators in the second set of expectations in (4.7) is equal to the right-hand side of (4.8) for some set T, these expectations are all less than one. Consequently, we find

$$|I_Q|^A \leq \prod_{i \in Q \cup O} (N_i/D), \tag{4.9}$$

where

$$N_i = \left\langle \left(\prod_{j \in A} T_i^j(\mathcal{O}_i) \right) \left(\prod_{\substack{(\alpha, l) \in Q \\ l \in \delta i}} \prod_{l' \sim l} f_{l'}^{\alpha} \right) e^{-S_{f,s}} \right\rangle (Z/Z_g)$$

$$= \left\langle \int \prod_{j \in A} d\chi[j] T_i^j(\mathcal{O}_i) \left(\prod_{\substack{(\alpha, l) \in Q \\ l \in \delta i}} \prod_{l' \sim l} f_{l'}^{\alpha} \right) \exp \sum_{l_t = \langle jk \rangle} \chi[j]^T M_{\langle jk \rangle} \chi[k] \right\rangle_g, \tag{4.10}$$

and

$$D = \langle e^{-S_{f,s}} \rangle (Z/Z_g) = \left\langle \int \prod_{j \in A} d\chi[j] \exp \sum_{l_t = \langle jk \rangle} \chi[j]^T M_{\langle jk \rangle} \chi[k] \right\rangle_g. \tag{4.11}$$

Here $\langle \ldots \rangle_g$ denotes expectations computed with the pure gauge measure (2.3), and Z_g is the pure gauge partition function. We have used (4.8) to justify deleting the spacelike part of the fermion action in the denominator of the first expectations in (4.7).

At this point, we may finally integrate out the fermions explicitly. Consider the denominator D first,, and note that it consists of $|A_s|$ independent fermion integrals,

each involving vL_t fermionic degrees of freedom coupled together in a linear chain. It is convenient to make a gauge transformation which, for each spatial site $x \in \Lambda_s$, sets every timelike link variable with the same spatial coordinates as x to the same value $(\Omega[x])^{1/L_t}$. Hence,

$$D = \left\langle \prod_{x \in \Lambda_s} d(\Omega[x]) \right\rangle_g , \qquad (4.12)$$

where

$$d(\Omega) = \int \prod_{t=1}^{L_t} d\psi_t d\bar\psi_t \exp\tfrac{1}{2} \sum_{t=1}^{L_t} (\bar\psi_{t-1}\gamma_0 \Omega^{1/L_t}\psi_t - \bar\psi_t\gamma_0 \Omega^{-1/L_t}\psi_{t-1}). \qquad (4.13)$$

This function is explicitly computed in Appendix V where we find

$$d(\Omega) = 2^{-vL_t}(\mathrm{tr}(1+\Omega))^v \geqq 2^{-v(L_t-1)}P^+(\Omega), \qquad (4.14)$$

where the projection $P^+(\Omega)$ is one if $\mathrm{tr}\,\Omega \geqq 0$, and zero otherwise. Thus

$$D \geqq 2^{-v|\Lambda|}2^{v|\Lambda_s|}\left\langle \prod_{x \in \Lambda_s} P^+(\Omega[x]) \right\rangle_g \geqq 2^{-v|\Lambda|}2^{v|\Lambda_s|}\langle P^+(\Omega[0])\rangle_g^{|\Lambda_s|} = 2^{-v|\Lambda|}2^{(v-1)|\Lambda_s|}. \qquad (4.15)$$

[The next to last step was an (inverse) chessboard estimate, and the last step follows from the global $Z(2)$ symmetry of the pure gauge theory.]

For each numerator N_i a very simple bound will suffice. First recall that each $f_{l'}^\alpha$ term contains a factor of $\tfrac{1}{2}(a_t/a_s)$. After extracting these overall factors, the remaining fermion integrations are of the form

$$\int d\tilde\chi \prod_q (\tilde\chi \cdot \tilde v_q) \exp\tfrac{1}{2} \tilde\chi \cdot m\tilde\chi, \qquad (4.16)$$

where $\tilde\chi$ is a vector containing all $v|\Lambda|$ fermion components, and each vector $\tilde v_q$ is a unit vector. (If $\mathcal{O}_i$ is not one, then N_i is a sum of $v^{|\Lambda|}$ terms of this form.) From Appendix III, one easily sees that the eigenvalues of the matrix m (which describes timelike fermion hopping) are all bounded by one. By the usual rules of Grassmann integration, this implies that the integral (4.16) is bounded by one. Consequently, we find

$$|N_i| \leqq \left(\prod_{\substack{(\alpha, l) \in Q \\ l \in \delta i}} (\tfrac{1}{2}(a_t/a_s))^{|\Lambda|/2} \right) (v^{|\Lambda|})^{|i \cap O|}. \qquad (4.17)$$

Inserting these bounds into (4.9) yields

$$|I_Q| \leqq (vk)^{|O|}(\tfrac{1}{2}k(a_t/a_s))^{|Q|}, \qquad (4.18)$$

where

$$k \equiv 2^v/2^{(v-1)/L_t}. \qquad (4.19)$$

Finally, we must count the number of sets, $N(Q)$, of a given size $|Q|$. Since vL_t elements of $Z_v \times L$ project onto a given spacelike link $l \in \Lambda_s$, we have

$$N(Q) \leqq (vL_t)^{|Q|}n(|Q|),$$

where $n(|Q|)$ is the number of sets of links in the spacelike slice Λ_s of size $|Q|$ (and where each connected component is connected to the set O). A standard counting argument (based on Euler's Konigsberg bridge problem) [22] implies that the number of connected sets of size q touching a given site is bounded by $(2d)^{2q}$. Since

there are at most $|O|$ connected clusters in the set Q, we obtain

$$|\langle \mathcal{O} \rangle| \leqq (vk)^{|O|} \sum_{|Q|=Q_{\min}}^{\infty} \sum_{q_1,\ldots,q_{|O|}=0}^{\infty} \delta\left(|Q| - \sum_i q_i\right) \prod_{i=1}^{|O|} (2d)^{2q_1}(\tfrac{1}{2}vk(L_t a_t/a_s))^{q_i}$$

$$= (vk)^{|O|} \int_C \frac{dz}{2\pi i} \frac{(\gamma/z)^{Q_{\min}}}{(z-\gamma)(1-z)^{|O|}}$$

$$= \gamma^{Q_{\min}} \frac{(vk)^{|O|}}{(|O|-1)!} (\partial/\partial z)^{|O|-1}(z^{-Q_{\min}}/z - \gamma)|_{z=1}, \tag{4.20}$$

provided

$$\gamma \equiv 2d^2 vk/(Ta_s) < 1. \tag{4.21}$$

Here $Q_{\min}$ is the minimal size of a set Q with a non-zero contribution, and the contour C circles the points $z=0$ and $z=\gamma$ counterclockwise. This proves Theorem IV.

If $\mathcal{O} = \bar{\psi}[i]\psi[i]\bar{\psi}[j]\psi[j]$, then since $\bar{\psi}\psi$ is chirally non-invariant the contribution from sets consisting of two disjoint clusters vanishes identically. Hence $Q_{\min}$ equals the minimal number of links connecting site i to j, and

$$\langle \bar{\psi}[i]\psi[i]\bar{\psi}[j]\psi[j]\rangle \sim \gamma^{|i-j|}$$

as $|i-j| \to \infty$. This establishes Theorem III with

$$T_{\mathrm{ch}} = 2d^2 v 2^v/(a_s 2^{(v-1)/L_t}). \tag{4.22}$$

Note that T_{ch} remains finite in the time continuum limit ($L_t \to \infty$).

V. Discussion

We have established that for sufficiently high temperature static quarks cannot be confined (in the pure gauge theory), and that chiral symmetry with massless fundamental representation fermions cannot be spontaneously broken. Our methods are applicable for all values of the bare coupling and in all possible dimensions. However, our bounds on the critical temperatures are obviously crude. In particular, they behave poorly in the weak coupling (or continuum) limit where the physical critical temperatures [in units of $(a_s)^{-1}$] are expected to vanish like $\exp(-c/g^2)$ with $c>0$ [23]. Proving a bound on any physical quantity which has the correct weak coupling behavior is a major unsolved problem in non-abelian gauge theories. This is the basic stumbling block which currently prevents the actual construction of the continuum limit.

Extending our methods to gauge groups other than SU(2) should be possible but will be technically more involved. For example, to generalize the Peierls argument of Sect. III it appears necessary to introduce a set of projection operators which controls the individual eigenvalues of the twist (instead of merely controlling the trace of the twist). Furthermore, although our proof of the absence of chiral symmetry breaking is essentially independent of the pure gauge dynamics (and thus yields a bound independent of the bare coupling) this will no longer be possible for groups larger than SU(2) or for fermions in representations other than the fundamental. (The difficulty arises in controlling the probability of

fermion zero modes in the partition function.) Finally, it should be possible to prove the convergence of the fermionic cluster expansion for sufficiently large chemical potential at any temperature. However, the fact that a non-zero chemical potential destroys reflection positivity (with respect to timelike planes) frustrates the obvious extension of our proof.[11]

Appendix I. Confinement Criteria Inequalities

In this appendix we discuss the inequalities relating Wilson loops, 't Hooft loops, the static quark potential, and the electric and magnetic flux free energies which may be derived using reflection positivity. (Similar inequalities have been derived using other methods in [10].)

A. Wilson Loops

The Wilson loop $W[C]$ for any closed loop C is defined by the expectation (in the pure gauge measure),

$$W[C] = \left\langle \tfrac{1}{2} \mathrm{tr} \left(\prod_{l \in C} U[l] \right) \right\rangle \tag{A1.1}$$

(where the product is understood to be ordered around the loop). For simplicity we will restrict our attention to rectangular loops whose side lengths do not exceed the length of the lattice. At zero temperature, the static quark potential may be extracted from Wilson loops [11],

$$V^{q\bar{q}}(r) = \lim_{t \to \infty} -\frac{1}{t} \ln W_{t,r}, \tag{A1.2}$$

where $W_{t,r}$ is the expectation of a loop of length t in the time direction and r is space. (This relation is false at non-zero temperature.)

Let $W_{I,J}$ denote the expectation of a rectangular loop of size $I \times J$ in a particular plane of the lattice. First note that reflection positivity implies that $W_{I,J}$ is non-negative. This follows since any rectangular loop may be written in the form $\langle A\Theta(A) \rangle$, where Θ is a reflection bisecting the loop. Next, applying reflection positivity to the 1×1 loop with the plane of reflection perpendicular to the plane of the loop and containing one of its sides yields

$$W_{1,1} \leqq W_{1,2}{}^{1/2}.$$

[To derive this, one considers one of the links of the 1×1 loop to be on the opposite side of the plane of reflection from the other three links, applies (2.5), and then applies the Schwarz inequality to the sum over color indices linking the two pieces of the loop together.] Generalizing this argument, one may easily prove

$$W_{1,1} \leqq W_{1,J}{}^{1/J} \tag{A1.3}$$

for J less than or equal to the length of the lattice in the chosen direction. [If J is not a multiple of 2, then after reaching the largest power of two smaller than J, one

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11   This fact was overlooked in [1b] where it was incorrectly claimed that the extension to non-zero chemical potential was straightforward

must choose a reflection which yields the $1 \times J$ loop times a smaller loop, both to fractional powers. Repeatedly applying this procedure to the leftover pieces eventually yields (A1.3).]

Reapplying this technique using perpendicular directions produces the well known area law bound

$$W_{1,1} \leqq W_{I,J}^{1/IJ} . \tag{A1.4}$$

This implies that the zero temperature quark potential cannot rise faster than linearly. (This result was first derived in [24].)

This argument based on reflection positivity actually yields the stronger result that $(W_{I,J})^{1/IJ}$ is monotonically increasing in I and J.

B. Static Quark Potential

The static quark potential is determined by the two point function of the trace of the twist (also called a Wilson line).

$$e^{-F^{q\bar{q}}(x,x')/T} = G(x,x') \equiv \langle \tfrac{1}{2} \operatorname{tr} \Omega[x] \tfrac{1}{2} \operatorname{tr} \Omega[x'] \rangle_A . \tag{A1.5}$$

We will restrict our consideration to the case where the separation $x - x'$ is directed along a lattice axis, and use G_J to denote the two point function at a separation J.

Exactly the same procedure of successively applying various reflections that was used for Wilson loops may be applied to the two point function (A1.5) to show that $(G_J)^{1/J}$ is monotonically increasing in J (for $1 \leqq J \leqq L_s/2$). In other words $F^{q\bar{q}}(|x - x'|)/|x - x'|$ is monotonically decreasing.

In addition, the expectations of timelike Wilson loops may be related to the two point function of Wilson lines. Consider a rectangular Wilson loop which extends a length $L_t/2$ in time and an arbitrary distance $J (\leqq L_s)$ in space. This loop may be written in the form

$$W_{L_t/2,J} = \langle \tfrac{1}{2} \operatorname{tr}(UVU'^{\dagger}V'^{\dagger}) \rangle ,$$

where U and U' are the timelike legs of the loop, and V and V' are the spacelike legs. We wish to get rid of the spacelike legs. To do so consider the spacelike reflection Θ which leaves invariant a pair of planes containing the spacelike legs V and V' (see footnote 3). We may consider the spacelike legs of our loop to be on the opposite side of the planes of reflection from the timelike legs and apply reflection positivity. This yields

$$W_{L_t/2,J} \leqq \tfrac{1}{2} \sum_{i,j,k,l} \langle U_{ij} \Theta(U_{ij}) U_{kl}'^{\dagger} \Theta(U_{kl}'^{\dagger}) \rangle^{1/2} \langle V_{jk} V_{jk}^* V_{il}'^* V_{il}' \rangle^{1/2} .$$

Applying the Schwarz inequality for sums gives

$$W_{L_t/2,J} \leqq \left\langle \sum_{i,j} U_{ij} \Theta(U_{ij}) \sum_{k,l} U_{kl}'^{\dagger} \Theta(U_{kl}'^{\dagger}) \right\rangle^{1/2} = 2 G_J^{1/2} . \tag{A1.6}$$

Combining this result with the monotonicity of Wilson loops yields

$$W_{I,J}^{1/I} \leqq (4 G_J)^{1/L_t} \tag{A1.7}$$

for timelike Wilson loops with time extent $I \leqq L_{t/2}$.

C. Electric Flux Free Energy

The electric flux free energy is defined by

$$\exp(-F_A^{\mathrm{elec}}/T) = \langle \tfrac{1}{2}(1 - \tau[S_{01}]) \rangle_A ,$$

where $\tau[S_{01}]$ [defined in (2.8)] is an operator which flips the coupling on a coclosed stack of plaquettes which we may choose to be

$$S_{01} = \{ p_{01}(\vec{n}) | n_0 = n_1 = 0 \} .$$

In order to relate the static quark potential to the electric free energy, one may begin with the two point function at a separation of half the lattice length, $G_{L_s/2}$, and insert one in the form,

Hence $1 = \tfrac{1}{2}(1 - \tau[S_{01}]) + \tfrac{1}{2}(1 + \tau[S_{01}]) .$

$$G_{L_s/2} = \langle \tfrac{1}{2}\operatorname{tr}\Omega[0]\tfrac{1}{2}(1 - \tau[S_{01}])\tfrac{1}{2}\operatorname{tr}\Omega[x] \rangle_A + \langle \tfrac{1}{2}\operatorname{tr}\Omega[0]\tfrac{1}{2}(1 + \tau[S_{01}])\tfrac{1}{2}\operatorname{tr}\Omega[x] \rangle_A ,$$

where $x = (L_s/2)e_1$. In the part of the second expectation involving $\tau[S_{01}]$, we may make a change of variables which flips the signs of the set of timelike links, $L_0 \equiv \{ l_0(\vec{n}) | n_0 = n_1 = 0 \}$. This yields

$$G_{L_s/2} = \langle \tfrac{1}{2}\operatorname{tr}\Omega[0]\tfrac{1}{2}(1 - \tau[S_{01}])\tfrac{1}{2}\operatorname{tr}\Omega[x] \rangle + \langle \tfrac{1}{2}\operatorname{tr}\Omega[0]\tfrac{1}{2}(1 - \tau[S_{01}'])\tfrac{1}{2}\operatorname{tr}\Omega[x] \rangle$$
$$= 2\langle \tfrac{1}{2}\operatorname{tr}\Omega[0]\tfrac{1}{2}(1 - \tau[S_{01}])\tfrac{1}{2}\operatorname{tr}\Omega[x] \rangle_A ,$$

where $S_{01}' = \{ p_{01}(\vec{n}) | n_0 = 0, n_1 = -1 \}$. We may now apply reflection positivity using a reflection which leaves invariant timelike planes containing $\Omega[0]$ and $\Omega[x]$. Considering both traces of the twist to be on opposite sides of the lattice from $\tau[S_{01}]$, we find

$$G_{L_s/2} \leqq 2\langle (\tfrac{1}{2}\operatorname{tr}\Omega[0])^2(\tfrac{1}{2}\operatorname{tr}\Omega[x])^2 \rangle^{1/2}\langle \tfrac{1}{2}(1 - \tau[S_{01}])\tfrac{1}{2}(1 - \tau[S_{01}']) \rangle^{1/2}$$
$$\leqq 2(\langle \tfrac{1}{4}(1 - \tau[S_{01}]) \rangle_A + \langle \tfrac{1}{4}(\tau[S_{01}']\tau[S_{01}] - \tau[S_{01}']) \rangle_A)^{1/2} .$$

In the second expectation, we may once again flip the signs of the links in the set L_0. This change of variables transforms the second expectation into the first. Therefore, we find

$$G_{L_s/2} \leqq 2\langle \tfrac{1}{2}(1 - \tau[S_{01}]) \rangle^{1/2} = 2\exp(-\tfrac{1}{2}F_A^{\mathrm{elec}}/T) . \qquad (A1.8)$$

Combined with the precious monotonicity of the static quark potential, this result implies that

$$F_A^{q\bar{q}}(|x - x'|)/|x - x'| \geqq F_A^{\mathrm{elec}}/L_s - (2T/L_s)\ln 2 \qquad (A1.9)$$

for $|x - x'| \leqq L_s/2$.

D. 't Hooft Loops and the Magnetic Flux Free Energy

The 't Hooft "loop" $B[S]$ is defined for any set of plaquettes S as [5, 9]

$$B[S] = \langle \tau[S] \rangle_A , \qquad (A1.10)$$

where the operator $\tau[S]$ flips the sign of the coupling on each of the plaquettes in the set S [see Eq. (2.8)]. If S and S' are sets of plaquettes which differ by the coboundary of a set of links, $S - S' = \delta L$, then $B[S] = B[S']$. In the infinite volume

limit, this means that $B[S]$ only depends on the coboundary of S (i.e., the coclosed set of cubes $C = \delta S$). In a non-confining phase the 't Hooft loop is expected to decrease exponentially with the minimal size of the set S, $B[S] \sim \exp - \kappa|S|$, whereas in a confining phase $B[S] \sim \exp - \mu|\delta S|$ [5, 9].

Consider "rectangular" sets of $S_{\vec{j}}$ of the form

$$S_{\vec{j}} \equiv \{p_{01}(\vec{n})|0 \leq n_\mu \leq J_\mu\},$$

where $J_0 = J_1 \equiv 0$ (and $J_\mu \leq L_s$). Applying reflection positivity in essentially the same manner that was described for Wilson loops directly yields the bound

$$B[S_{\vec{0}}] \leq B[S_{\vec{j}}]^{1/|S_{\vec{j}}|}, \tag{A1.11}$$

as well as the stronger statement that $B[S_{\vec{j}}]^{1/|S_{\vec{j}}|}$ is monotonically increasing in each component J_μ (for $2 \leq \mu \leq d$).

Next, note that when $J_\mu = L_s$ for $2 \leq \mu \leq d$, the set $S_{\vec{j}}$ is identical to the coclosed set S_{01} defining the magnetic flux free energy [see Eq. (2.8)], and hence for this set $B[S_{\vec{j}}] = \exp(-F^{\text{mag}}/T)$. Combined with the monotonicity, this means that

$$B[S_{\vec{j}}]^{1/|S_{\vec{j}}|} \leq \exp(-F_A^{\text{mag}}/(TL_s^{d-1})). \tag{A1.12}$$

This shows that non-confining behavior of the magnetic flux free energy $(F_A^{\text{mag}} \sim \kappa L_s^{d-1}, \kappa > 0)$ implies non-confining behavior of 't Hooft loops.

Finally, all of the bounds in this appendix immediately extend from SU(2) to any other gauge group (with a non-trivial center). The only changes involve the use of fourier transforms over the center of the group in the definition of electric and magnetic flux, and the replacement of factors of two by the dimension of the fundamental representation and/or the dimension of the center.

Appendix II. Contour Sums

The following procedure may be used to construct all possible contours (i.e., connected coclosed sets of links intersecting a given path C) of a given size.

1. Order all plaquettes of the lattice in an arbitrary fashion.
2. Pick a link of the path C to be contained in the contour γ.
3. Build the remainder of the contour γ by choosing successive links so as to always remove the lowest ordered plaquette from the coboundary of the previously chosen set of links.

If the final set of links is a valid topologically trivial contour, then in order for $\gamma[C]$ to equal 1, γ must enclose one of the endpoints of the path C. The minimal size of a contour which intersects a link of C at a distance k from the endpoint it encloses is $2(d-1)k + 2$. Therefore at step 2, no more than $2(|\gamma|-2)/2(d-1)$ choices can lead to a valid contour. If the final set is topologically non-trivial contour, then the number of valid choices at step 2 equals the length of the path C. The minimal size of a topologically non-trivial contour is $(L_s)^{d-1}$, and the maximal length path we ever consider is $L_s/2$. Therefore, provided $L_s \geq 4$ (which we will henceforth assume), $|C| \leq (L_s^{d-1} - 2)/(d-1)$ so that the number of valid choices at step 2 is always bounded by $(|\gamma|-2)/(d-1)$.

Each succeeding link of the contour is determined by selecting one of the links in the boundary of a particular plaquette (the lowest ordered plaquette in the coboundary of the partially chosen contour). Since at least one of the links in the boundary of the plaquette has already been selected, one has at most 3 choices for each successive link. Furthermore, in order for the resulting set of links to be a coclosed contour, the final $d-1$ links must be chosen in a unique way. Therefore, the total number of choices satisfies

$$N(|\gamma|) \leqq \frac{(|\gamma|-2)}{(d-1)} 3^{|\gamma|-d}. \qquad (A2.1)$$

Appendix III. Transfer Matrix Bounds

Propagation in euclidean "time" is described by the transfer matrix T whose matrix elements are given by

$$T^L(\{U'[l]\}, \{U[l]\}; \beta_t, \beta_s) \equiv \int \prod_{t=1}^{L-1} \prod_{l \in \Lambda_s} dU_t[l]$$

$$\cdot \prod_{t=1}^{L} \exp\left\{ \beta_t \sum_{l \in \Lambda_s} \mathrm{tr}(U_t[l] U_{t-1}^{\dagger}[l]-1) + \beta_s \sum_{p \in \Lambda_s} \mathrm{tr}(U_t[\partial p]-1) \right\}. \qquad (A3.1)$$

Here the link variables at the initial and final times are fixed, $U_0[l] \equiv U[l]$, $U_L[l] \equiv U'[l]$. For later convenience we assume that L is an integral power of 2, $L \equiv 2^\alpha$.

A. Upper Bound

To derive an upper bound, we may simply set β_s to zero. Then, in the absence of spacelike couplings, one can exactly integrate over every other spacelike link variable (i.e., $U_t[l]$ for t odd). Thus,

$$T^L(\beta_t, 0) = \prod_{l \in \Lambda_s} \left\{ \int \prod_{t=1}^{L-1} dU_t[l] \prod_{t=1}^{L} \exp \beta_t \, \mathrm{tr}(U_t[l] U_{t-1}^{\dagger}[l]-1) \right\}$$

$$= \prod_{l \in \Lambda_s} \left\{ \int \prod_{\substack{t=2 \\ t\,\mathrm{even}}}^{L-1} dU_t[l] \prod_{\substack{t=2 \\ t\,\mathrm{even}}}^{L} I(U_t[l], U_{t-2}[l]; \beta_t) \right\},$$

where

$$I(U', U; \beta) \equiv \int dU'' \exp \beta \, \mathrm{tr}(U''(U+U')^{\dagger}-2) = \exp[-4\beta + f(2\beta\|U+U'\|)].$$

Here, the function $f(z)$ is defined as

$$f(z) \equiv \ln\left[\int_0^\pi \frac{d\theta}{\pi/2} \sin^2\theta \exp[z\cos\theta] \right] = \ln(2I_1(z)/z) \qquad (A3.2)$$

and $\|U+U'\| \equiv (\frac{1}{2}\mathrm{tr}(U+U')(U+U')^{\dagger})^{1/2}$. From this integral representation, one may immediately see that $f(z)$ is convex ($f''(z) \geqq 0$) and that $f(0)=0$. Therefore $f(z(1-s)) \leqq (1-s)f(z)$ for $0 \leqq s \leqq 1$.

Since $\|U+U'\| = \sqrt{\mathrm{tr}(1+U'U^{\dagger})} \leqq 2(1-\frac{1}{8}\mathrm{tr}(1-U'U^{\dagger}))$, we find that

$$I(U', U; \beta) \leqq \exp[f(4\beta)-4\beta+\tfrac{1}{8}f(4\beta)\,\mathrm{tr}(U'U^{\dagger}-1)].$$

This result has the same exponential dependence on $\mathrm{tr}(U'U^{\dagger}-1)$ as does the original nearest neighbor interaction. Consequently, we find that the length L transfer matrix may be bounded by a length $L/2$ transfer matrix,

$$T^{2^{\alpha}}(\beta_t, \beta_s) \leqq T^{2^{\alpha-1}}(\beta_>^{(1)}, 0) \exp\left[2^{\alpha-1} \sum_{l \in \Lambda_s} (8\beta_>^{(1)} - 4\beta_t)\right],$$

where the "renormalized" coupling $\beta_>^{(1)} \equiv \frac{1}{8} f(4\beta_t)$. Iterating this result yields our desired upper bound,

$$T^{2^{\alpha}}(\beta_t, \beta_s) \leqq T(\beta_>^{(\alpha)}, 0) \exp(|\Lambda_s| F_>^{(\alpha)}) = \exp \sum_{l \in \Lambda_s} \beta_>^{(\alpha)} \mathrm{tr}(U'[l]U[l]^{\dagger} - 1) \exp(|\Lambda_s| F_>^{(\alpha)}),$$

(A3.3)

where

$$\beta_>^{(k)} \equiv \frac{1}{8} f(4\beta_>^{(k-1)})$$

(A3.4)

with $\beta_>^{(0)} \equiv \beta_t$, and

$$F_>^{(\alpha)} = \sum_{k=0}^{\alpha-1} 2^{\alpha-k} \frac{d}{2} (f(4\beta_>^{(k)}) - 4\beta_>^{(k)}).$$

(A3.5)

As $T \to \infty$,

$$\beta_>^{(\alpha)} \sim \frac{1}{16}(8\beta_>^{(\alpha-1)} - 3\ln 4\beta_>^{(\alpha-1)} - \ln \pi/2) \sim 2^{-\alpha}\beta_t + O(\ln \beta_t),$$

and $F_>^{(\alpha)} \sim -\dfrac{d}{2}(2^{\alpha}-1)\ln 4\pi\beta_t^3 - \dfrac{\alpha d}{2}\ln 8$.

B. Lower Bound

To prove a similar lower bound, we begin by inserting a set of projection operators which will force spacelike links at neighboring times (i.e., $U_t[l]$ and $U_{t+1}[l]$) to be close together. Let

$$P_{\varepsilon}(U) \equiv \begin{cases} 1 & \text{if } \frac{1}{2}\mathrm{tr}\, U \geqq \cos\varepsilon \\ 0 & \text{otherwise}, \end{cases}$$

where the constant ε will be chosen later. Note that the condition $P_{\varepsilon}(U) \neq 0$ is equivalent to the requirement that the rotation angle, or geodesic distance between U and the identity be no greater than ε. Now $T^L(\beta_t, \beta_s) \geqq T_{\varepsilon}^L(\beta_t, \beta_s)$, where

$$T_{\varepsilon}^L(\beta_t, \beta_s) \equiv \int \prod_{\substack{t=1}}^{L-1} \prod_{l \in \Lambda_s} dU_t[l] \prod_{t=1}^{L} \left\{ \prod_{l \in \Lambda_s} P_{\varepsilon}(U_t[l]U_{t-1}^{\dagger}[l]) \right.$$
$$\left. \cdot \exp\left[\beta_t \sum_{l \in \Lambda_s} \mathrm{tr}(U_t[l]U_{t-1}^{\dagger}[l] - 1) + \beta_s \sum_{p \in \Lambda_s} \mathrm{tr}(U_t[\partial p] - 1)\right] \right\}.$$

The condition $P_{\varepsilon}(U'U^{\dagger}) \neq 0$ implies that

$$\|U - U'\| = \sqrt{\mathrm{tr}(1 - U'U^{\dagger})} \leqq 2\sin(\varepsilon/2) \leqq \varepsilon.$$

Furthermore, if $\|U_t[l] - U_{t+1}[l]\| \leqq \varepsilon$ for all $l \in \Lambda_s$, then $\|U_t[\partial p] - U_{t+1}[\partial p]\| \leqq 4\varepsilon$. Therefore

$$T_{\varepsilon}^L(\beta_t, \beta_s) \geqq \int \prod_{\substack{t=2 \\ t \text{ even}}}^{L-1} \prod_{l \in \Lambda_s} dU_t[l] \exp 2\beta_s \sum_{\substack{t=2 \\ t \text{ even}}}^{L} \sum_{p \in \Lambda_s} (\mathrm{tr}(U_t[\partial p]) - 1) - 4\varepsilon)$$
$$\cdot \prod_{\substack{t=2 \\ t \text{ even}}}^{L} \prod_{l \in \Lambda_s} J_{\varepsilon}(U_{t+2}[l], U_t[l]; \beta_t),$$

where

$$J_{\varepsilon}(U', U; \beta) \equiv \int dU'' P_{\varepsilon}(U''U^{\dagger}) P_{\varepsilon}(U''U'^{\dagger}) \exp \beta\, \mathrm{tr}(U''(U + U')^{\dagger} - 2).$$

Now, given any two group elements, $U, U' \in \mathrm{SU}(2)$, let $\bar{U} = (U + U')/\|U + U'\|$ and note that if

$$\tfrac{1}{2}\mathrm{tr}(\bar{U}U^\dagger) \geq \cos(\lambda\varepsilon/2) \quad \text{and} \quad \tfrac{1}{2}\mathrm{tr}(U''\bar{U}^\dagger) \geq \cos(1-\lambda/2)\varepsilon$$

for some $\lambda \in [0, 2]$, then the triangle inequality applied to the geodesic distances between U, $\bar{U}$, and U'' implies that $\tfrac{1}{2}\mathrm{tr}(U''U^\dagger) \geq \cos\varepsilon$. In other words, the domain where $P_\varepsilon(U''U^\dagger)P_\varepsilon(U''U'^\dagger) \neq 0$ contains the region where $P_{\lambda\varepsilon}(U'U^\dagger)$ and $P_{(1-\lambda/2)\varepsilon}(U''\bar{U}^\dagger)$ are simultaneously non-zero. Therefore,

$$J_\varepsilon(U', U; \beta) \geq P_{\lambda\varepsilon}(U'U^\dagger)\exp[-4\beta + g_\varepsilon(2\beta\|U + U'\|)],$$

where the function $g_\varepsilon(z)$ is defined as

$$g_\varepsilon(z) = \ln\left[\int_0^{(1-\lambda/2)\varepsilon} \frac{d\theta}{\pi/2}\sin^2\theta\exp(z\cos\theta)\right]. \tag{A3.6}$$

Since $g_\varepsilon(z)$ is a convex function,

$$g_\varepsilon(2\beta\|U + U'\|) \geq g_\varepsilon(4\beta) + 2\beta g'_\varepsilon(4\beta)(\|U + U'\| - 2).$$

Furthermore, $P_{\lambda\varepsilon}(U'U^\dagger) \neq 0$ implies that

$$\|U + U'\| = \sqrt{\mathrm{tr}(1 + U'U^\dagger)} \geq 2 - (2\cos(\lambda\varepsilon/4))^{-2}\mathrm{tr}(1 - U'U^\dagger).$$

Combining these bounds and noting that $g'_\varepsilon(z) \leq 1$, we find that

$$J_\varepsilon(U', U; \beta) \geq P_{\lambda\varepsilon}(U'U^\dagger)\exp[g_\varepsilon(4\beta) - 4\beta + \beta(2\cos^2\lambda\varepsilon/4)^{-1}\mathrm{tr}(U'U^\dagger - 1)].$$

Once again, this result has the same form as the original nearest neighbor interaction. Consequently, we have the lower bound

$$T_\varepsilon^{2^\alpha}(\beta_t, \beta_s) \geq T_{\lambda\varepsilon}^{2^{\alpha-1}}(\beta_<^{(1)}, 2\beta_s) \cdot \exp\left[2^{\alpha-1}\left(\sum_{l \in \Lambda_s}(g_\varepsilon(4\beta_t) - 4\beta_t) - \sum_{p \in \Lambda_s} 8\varepsilon\beta_s\right)\right],$$

where $\beta_<^{(1)} \equiv (2\cos^2\lambda\varepsilon/4)^{-1}\beta_t$. Iterating this result yields

$$T^{2^\alpha}(\beta_t, \beta_s) \geq T_{\lambda^\alpha\varepsilon}(\beta_<^{(\alpha)}, 2^\alpha\beta_s)\exp(|\Lambda_s|F_<^{(\alpha)})$$

$$= \prod_{l \in \Lambda_s} P_{\lambda^\alpha\varepsilon}(U'[l]U[l]^\dagger)\exp\sum_{l \in \Lambda_s}\beta_<^{(\alpha)}\mathrm{tr}(U'[l]U[l]^\dagger - 1)$$

$$\cdot \exp\sum_{p \in \Lambda_s} 2^\alpha\beta_s\mathrm{tr}(U'[\partial p] - 1)\exp(|\Lambda_s|F_<^{(\alpha)}), \tag{A3.7}$$

where

$$\beta_<^{(k)} \equiv (2\cos^2(\lambda^k\varepsilon/4))^{-1}\beta_<^{(k-1)} \tag{A3.8}$$

with $\beta_<^{(0)} \equiv \beta_t$, and

$$F_<^{(\alpha)} \equiv \sum_{k=0}^{\alpha-1}\left[2^{\alpha-k}\frac{d}{2}(g_{\lambda^k\varepsilon}(4\beta_<^{(k)}) - 4\beta_<^{(k)}) - 2^\alpha d(d-1)2\beta_s\lambda^k\varepsilon\right]. \tag{A3.9}$$

If we choose $\lambda^\alpha\varepsilon \sim \xi/\sqrt{\beta_t/L_t}$ for fixed ξ as $T \to \infty$, and $1 < \lambda < \sqrt{2}$, then for high temperature,

$$\beta_<^{(k)} \sim 2^{-k}\beta_t = 2^{\alpha-k}(\beta_t/L_t)$$

and

$$F^{(\alpha)}_< - F^{(\alpha)}_> \sim -\sum_{j=1}^{\alpha} \frac{d}{2} 2^j R(\xi(2-\lambda)(\sqrt{2}/\lambda)^j) - 2d(d-1)\xi \frac{(\beta_s L_t)}{\sqrt{\beta_t/L_t}} \frac{(1-\lambda^{-\alpha})}{(\lambda-1)},$$

(A3.10)

where

$$R(z) \equiv -\ln\left[1 - \sqrt{2/\pi} \int_z^\infty d\theta\, \theta^2 \exp(-\theta^2/2)\right].$$

(A3.11)

Given the definitions (2.2) of β_t and β_s, this result shows that $F^{(\alpha)}_< - F^{(\alpha)}_>$ is bounded as the temperature $T \to \infty$ with the bare coupling g^2 and L_t fixed. Regretably, $F^{(\alpha)}_< - F^{(\alpha)}_>$ does not remain bounded as $L_t \to \infty$, and consequently our bounds on confinement criteria do not have uniform $L_t \to \infty$ limits.[12]

Appendix IV. Partition Function Bounds

A. XY Model Upper Bound

To find a useful upper bound on the xy model partition function,

$$Z_L^{xy}(\beta) \equiv \int \prod_{\underset{\sim}{n}} \frac{d\theta(n)}{2\pi} \exp\sum_{\underset{\sim}{n}} \sum_{\mu=1}^{d} \beta\left(\cos(\theta(\underset{\sim}{n}) - \theta(\underset{\sim}{n}+\hat{\mu})) - 1\right),$$

we will use a simplified version of the Migdal-Kadanoff recursion relations [26]. Let

$$\Delta_v = \sum_{\substack{\underset{\sim}{n} \\ n_v \text{ even}}} \sum_{\mu \neq v} [\cos(\theta(\underset{\sim}{n}) - \theta(\underset{\sim}{n}+\hat{\mu})) - \cos(\theta(\underset{\sim}{n}+\hat{v}) - \theta(\underset{\sim}{n}+\hat{\mu}+\hat{v}))],$$

and define $Z[\tau] \equiv Z^{xy}\langle \exp\tau\Delta_1\rangle_{xy}$. Note that

$$\frac{\partial}{\partial\tau} Z[\tau]|_{\tau=0} = 0$$

and

$$\frac{\partial^2}{\partial\tau^2} Z[\tau] = Z^{xy}\langle \Delta_1^2 \exp\tau\Delta_1\rangle_{xy} \geq 0.$$

Therefore $Z^{xy} = Z[0] \leq Z[\beta]$. Now, in $Z[\beta]$ the effect of the operator Δ_1 is to either double or cancel the coupling on links in such a way that half of the variables become coupled to only two nearest neighbors. These variables may then be integrated out as follows.

$$Z[\beta] = \int \prod_{\substack{\underset{\sim}{n} \\ n_1 \text{ even}}} \frac{d\theta(n)}{2\pi} \exp\sum_{\underset{\sim}{n}} \sum_{\mu \neq 1} 2\beta(\cos(\theta(\underset{\sim}{n}) - \theta(\underset{\sim}{n}+\hat{\mu})) - 1) \prod_{\substack{\underset{\sim}{n} \\ n_1 \text{ even}}} K(\theta(\underset{\sim}{n}), \theta(\underset{\sim}{n}+2e_1)),$$

where

$$K(\theta, \theta') \equiv \int \frac{d\theta''}{2\pi} \exp\beta(\cos(\theta''-\theta) + \cos(\theta''-\theta') - 2)$$

$$= \exp[-2\beta + h(\beta|e^{i\theta} + e^{i\theta'}|)],$$

and

$$h(z) \equiv \ln\left[\int_0^{2\pi} \frac{d\theta}{2\pi} e^{z\cos\theta}\right] = \ln I_0(z).$$

(A4.1)

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12 We thank the referee for drawing attention to this point

Since $h(z)$ is convex, $h(0)=0$, and $|e^{i\theta}+e^{i\theta'}|\leq 2-(1-\cos(\theta-\theta'))/2$, we find

$$K(\theta,\theta')\leq\exp[h(2\beta)-2\beta+\tfrac{1}{4}h(2\beta)(\cos(\theta-\theta')-1)],$$

so that

$$Z^{xy}\leq\int\sum_{\substack{n\\ n_1\text{ even}}}\frac{d\theta(n)}{2\pi}\exp\sum_{\substack{n\\ n_1\text{ even}}}\left[(h(2\beta)-2\beta)+\frac{1}{4}h(2\beta)(\cos(\theta(\underline{n})-\theta(\underline{n}+2e_1))-1)\right.$$

$$\left.+\sum_{\mu\neq 1}2\beta(\cos(\theta(\underline{n})-(\underline{n}+\hat{\mu}))-1)\right].$$

Repeating this procedure for the other $(d-1)$ directions (i.e., inserting $e^{2\beta\Delta_2}$ and integrating out variables on sites with n_2 odd, etc.) yields

$$Z^{xy}\leq\int\prod_{\substack{n\\ \text{even}}}\frac{d\theta(n)}{2\pi}\exp\sum_{\substack{n\\ \text{even}}}\sum_{v}2^{d-v}$$

$$\cdot\left[h(2^v\beta)-2^v\beta+\frac{1}{4}h(2^v\beta)(\cos(\theta(\underline{n})-\theta(\underline{n}+2v))-1)\right].$$

The anisotropy in the renormalized couplings may be removed by noting that $2^{-v}h(2^v\beta)\geq h(2\beta)/2$. Thus, we find

$$Z^{xy}_L(\beta)\leq Z^{xy}_{L/2}(2^{d-3}h(2\beta))\exp\left[L^d\sum_v(2^{-v}h(2^v\beta)-\beta)\right].$$

If $L=2^\alpha$, then iterating this bound yields

$$Z^{xy}_L(\beta)\leq(z_{xy}(\beta))^{L^d},\tag{A4.2}$$

where

$$z_{xy}(\beta)\equiv\exp\sum_{k=0}^{\alpha-1}\sum_{v=1}^{d}2^{-kd}(2^{-v}h(2^v\beta^{(k)})-\beta^{(k)}),\tag{A4.3}$$

and

$$\beta^{(k)}\equiv 2^{d-3}h(2\beta^{(k-1)}).\tag{A4.4}$$

The convexity of $h(z)$ implies that

$$\beta^{(k)}\leq 2^{(d-2)k}\beta,$$

and

$$\beta^{(k)}\geq 2^{(d-2)k}\beta+\sum_{j=1}^{k}2^{d-3}(h(2^{(d-2)j}2\beta)-2^{(d-2)j}2\beta).$$

Using this, one may show that

$$\ln z_{xy}=\left[\sum_{k=0}^{\alpha-1}\sum_{v=1}^{d}-\tfrac{1}{2}2^{-kd-v}\ln(2^{v+(d-2)k}2\pi\beta)\right]+O(\ln\beta/\beta)$$

$$=\tfrac{1}{2}\{(1-2^{-\alpha d})[-\ln 8\pi\beta+(2^d-1)^{-1}\ln 4]$$

$$+\alpha(d-2)2^{-\alpha d}\ln 2+O(\ln\beta/\beta)\},$$

uniformly in $L=2^\alpha$.

B. Pure Gauge Lower Bound

To find a lower bound on the pure gauge theory partition function,

$$Z_\Lambda(\beta_t,\beta_s)=\int\prod_{l\in\Lambda}dU[l]\exp\left(\beta_s\sum_{p_s\in\Lambda}\text{tr}(U[\partial p]-1)+\beta_t\sum_{p_t\in\Lambda}\text{tr}(U[\partial p]-1)\right),$$

we first write Z in terms of the transfer matrix,

$$Z_\Lambda(\beta_t, \beta_s) = \int \prod_{l \in \Lambda_s} dU[l] \prod_{x \in \Lambda_s} d\Omega[x] T^{L_t}(\{U^\Omega[l]\}, \{U[l]\}; \beta_t, \beta_s),$$

and then use the lower bound on the transfer matrix derived in Appendix III. This yields

$$Z_\Lambda \geq \int \prod_{x \in \Lambda_s} d\Omega[x] \prod_{l \in \Lambda_s} dU[l] \prod_{l = \langle yy' \rangle \in \Lambda_s} P_{\lambda^\alpha \varepsilon}(\Omega[y]U[l]\Omega[y']^\dagger U[l]^\dagger)$$

$$\cdot \exp\left(\sum_{p \in \Lambda_s} L_t \beta_s \, \text{tr}(U[\partial p] - 1) + \sum_{l \in \Lambda_s} \beta_<^{(\alpha)} \, \text{tr}(\Omega[y]U[l]\Omega[y']^\dagger U[l]^\dagger - 1) \right)$$

$$\cdot \exp|\Lambda_s| F_<^{(\alpha)},$$

where $\alpha \equiv \ln_2 L_t$, λ and ε are arbitrary constants (which will be chosen presently), and the numbers $\beta_<^{(\alpha)}$ and $F_<^{(\alpha)}$ (depending on β_t, β_s, L_t, λ, and ε) are defined in Eqs. (A3.8) and (A3.9).

We now parameterize the twist and the remaining spacelike link variables as follows,

$$\Omega[x] \equiv V[x] e^{i\tau_3 \omega[x]} V[x]^\dagger,$$

$$U[l] \equiv V[y](\cos\theta[l] e^{i\tau_3 \phi[l]} + i\tau_1 \sin\theta[l] e^{i\tau_3 \chi[l]}) V[y']^\dagger$$

(where $l = \langle yy' \rangle$). $\phi[l]$ and $\chi[l]$ range from 0 to 2π, $\omega[x]$ lies in $[0, \pi]$, and $\theta[l]$ is in $[0, \pi/2]$. Note that

$$d\Omega[x] = dV[x]d\omega[x]\sin^2\omega[x]/(\pi/2),$$

and $$dU[l] = d\theta[l]d\phi[l]d\chi[l]\sin 2\theta[l]/(2\pi)^2,$$

$$\text{tr}(\Omega[y]U[l]\Omega[y']^\dagger U[l]^\dagger) = 2(\cos\omega[y]\cos\omega[y'] + \cos 2\theta[l]\sin\omega[y]\sin\omega[y']).$$

We now restrict the integration region to the domain

$$\theta[l] \leq \eta, \qquad \omega[x] \leq \lambda^\alpha \varepsilon,$$

with $0 < \eta < \pi/6$, and $\lambda^\alpha \varepsilon < \pi/2$. Therefore $\|U[l] - e^{i\tau_3 \phi[l]}\| = 2\sin(\theta[l]/2) \leq \eta$, and

$$\tfrac{1}{2}\text{tr}(\Omega[y]U[l]\Omega[y']^\dagger U[l]^\dagger) \leq \cos(\lambda^\alpha \varepsilon) \leq 1 - \tfrac{1}{2}(\lambda^\alpha \varepsilon)^2.$$

Consequently,

$$Z_\Lambda \geq \left[\int_0^{\lambda^\alpha \varepsilon} \frac{d\omega}{\pi/2} \sin^2\omega \right]^{|\Lambda_s|} \left[\int_0^\eta d\theta \sin 2\theta \right]^{d|\Lambda_s|}$$

$$\cdot \exp|\Lambda_s^{(\alpha)}| \left[F_<^{(\alpha)} - d\beta_<^{(\alpha)}(\lambda^\alpha \varepsilon)^2 - 4d(d-1)L_t \beta_s \eta \right]$$

$$\geq z(\beta_t, \beta_s; \xi, \lambda, \eta)^{|\Lambda_s|}, \tag{A4.5}$$

where

$$\xi \equiv \lambda^\alpha \varepsilon \sqrt{\beta_t/L_t}, \tag{A4.6}$$

and

$$z(\beta_t, \beta_s; \xi, \lambda, \eta) \equiv \frac{2}{3\pi}(\sin(\xi\sqrt{L_t/\beta_t}))^3(\sin\eta)^{2d}$$

$$\cdot \exp\left[-d\xi^2(\beta_<^{(\alpha)}L_t/\beta_t) - 4d(d-1)L_t \beta_s \eta + F_<^{(\alpha)} \right]. \tag{A4.7}$$

Finally, using (A3.10) note that

$$z(\beta_s, \beta_t; \xi, \lambda, \eta)e^{-F_>^{(\alpha)}} \sim cT^{-3/2}$$

as $T \rightarrow \infty$ for some positive constant c depending on g^2, L_t, ξ, λ, and η. (In principle, ξ, λ, and η could be chosen to optimize this bound.)

Appendix V. One Dimensional Fermions

We wish to compute the one dimensional fermion integral,

$$d(\Omega) = \int \prod_{t=1}^{L_t} d\psi_t d\bar\psi_t \exp\tfrac{1}{2} \sum_{t=1}^{L_t} (\bar\psi_{t-1}\gamma_0\Omega^{1/L_t}\psi_t - \bar\psi_t\gamma_0\Omega^{-1/L_t}\psi_{t-1}),$$

where $\psi_0 \equiv -\psi_{L_t}$ and $\bar\psi_0 \equiv -\bar\psi_{L_t}$ (due to the antiperiodic boundary conditions for fermions). If $\Omega = u e^{i\omega\tau_3} u^\dagger$, then by a change of variables $\psi_t \rightarrow u\psi_t$, and a fourier transform we obtain,

$$d(\Omega) = \int \prod_{n=1}^{L_t} d\psi_n d\bar\psi_n \exp \sum_{n=1}^{L_t} \bar\psi_n (i\gamma_0 \sin(v_n + \omega\tau_3/L_t))\psi_n$$

$$= \prod_{n=1}^{L_t} \det(i\gamma_0 \sin(v_n + \omega\tau_3/L_t)),$$

where $v_n \equiv (2n-1)\pi/L_t$. Since (for L even)

$$\prod_{k=0}^{L-1} \sin(x + 2k\pi/L) = -(-)^{L/2} 2^{-L}(e^{ixL} + e^{-ixL} - 2),$$

we find

$$d(\Omega) = \det(\gamma_0 2^{-L_t}|1 + e^{i\omega\tau_3}|^2) = 2^{-vL_t}(\mathrm{tr}(1+\Omega))^v. \tag{A5.1}$$

References

1. Tomboulis, E., Yaffe, L.: Nonconfinement at high temperatures. Phys. Rev. D **29**, 780 (1984); Chiral symmetry restoration at finite temperature. Phys. Rev. Lett. **52**, 2115 (1984)
2. Glimm, J., Jaffe, A., Spencer, T.: Phase transitions for ϕ_2^4 quantum fields. Commun. Math. Phys. **45**, 203 (1975)
   Fröhlich, J., Lieb, E.: Phase transitions in anisotropic lattice spin systems. Commun. Math. Phys. **60**, 233 (1978)
3. See, for example, Simon, B.: The $P(\phi)_2$ Euclidean (quantum) field theory. Princeton, NJ: Princeton University Press 1974
4. See, for example, Aizenman, M.: Geometric analysis of ϕ^4 field and Ising models. Commun. Math. Phys. **86**, 1–48 (1982); or
   Brydges, D., Fröhlich, J., Spencer, T.: The random walk representation of classical spin systems and correlation inequalities. Commun. Math. Phys. **83**, 123–150 (1982)
5. See, for example, Ukawa, A., Windey, P., Guth, A.: Dual variables for lattice gauge theories and the phase structure of $Z(N)$ systems. Phys. Rev. D **21**, 1013–1036 (1980)
6. Osterwalder, K., Seiler, E.: Gauge field theories on a lattice. Ann. Phys. (New York) **110**, 440 (1978)
7. Brydges, D., Fröhlich, J., Seiler, E.: On the construction of quantized gauge fields. I. General results. Ann. Phys. (New York) **121**, 227–284 (1979)

8. Glimm, J., Jaffe, A.: Quantum physics. A functional integral point of view. Berlin, Heidelberg, New York: Springer 1981
9. Mack, G., Petkova, V.: Comparison of lattice gauge theories with gauge groups Z_2 and SU(2). Ann. Phys. (New York) **123**, 442 (1979)
   Yaffe, L.: Confinement in SU(N) lattice gauge theories. Phys. Rev. D **21**, 1574 (1980)
10. Borgs, C., Seiler, E.: Lattice Yang-Mills theory of non-zero temperature and the confinement problem. Commun. Math. Phys. **91**, 329–380 (1983)
11. Wilson, K.: Confinement of quarks. Phys. Rev. D **10**, 2445–2459 (1974)
12. Fröhlich, J., Israel, R., Lieb, E., Simon, B.: Phase transitions and reflection positivity. I. General theory and long range lattice models. Commun. Math. Phys. **62**, 1–34 (1978)
    Fröhlich, J., Israel, R., Lieb, E., Simon, B.: Phase transitions and reflection positivity. II. Lattice systems with short-range and Coulomb interactions. J. Stat. Phys. **22**, 297–347 (1980)
13. Polyakov, A.: Thermal properties of gauge fields and quark liberation. Phys. Lett. **72B**, 477–480 (1978)
    McLerran, L., Svetitsky, B.: Quark liberation at high temperature: Monte Carlo study of SU(2) gauge theory. Phys. Rev. D **24**, 450–460 (1981)
14. Weiss, N.: Effective potential for the order parameter of gauge theories at finite temperature. Phys. Rev. D **24**, 475–480 (1981)
15. See, for example, Fröhlich, J.: Phase transitions, goldstone bosons, and topological superselection rules. Acta. Phys. Austriaca [Suppl. **XV**] [Schladming 1976] 133–269 (1976); or
    Dyson, F., Lieb, E., Simon, B.: Phase transitions in quantum spin systems with isotropic and nonisotropic interactions. J. Stat. Phys. **18**, 335–383 (1978)
16. 't Hooft, G.: A property of electric and magnetic flux in non-abelian gauge theories. Nucl. Phys. B **153**, 141–160 (1979)
17. See for example, Fontaine, J., Gruber, C.: Surface tension and phase transition for lattice systems. Commun. Math. Phys. **70**, 243–269 (1979)
18. Yaffe, L.: See [9]
19. Susskind, L.: Lattice fermions. Phys. Rev. D **16**, 3031–3939 (1977)
20. Sharatchandra, H., Thun, H., Weisz, P.: Susskind fermions on a Euclidean lattice. Nucl. Phys. B **192**, 205–236 (1981)
21. Berezin, F.: The method of second quantization. New York: Academic 1966
22. Glimm, J., Jaffe, A.: Constructive quantum field theory, pp. 199–242. Velo, G., Wightman, A. (eds.). Berlin, Heidelberg, New York: Springer 1973
23. See, for example, Gross, D.: Methods in field theory, pp. 141–250. Balian, R., Zinn-Justin, J. (eds.). Amsterdam, New York: North-Holland 1976
24. Seiler, E.: Upper bound on the color-confining potential. Phys. Rev. D **18**, 482–483 (1978)
25. 't Hooft, G.: On the phase transition towards permanent quark confinement. Nucl. Phys. B **138**, 1–25 (1978)
26. Migdal, A.: Zh. Eksp. Teor. Fiz. **69**, 810; **69**, 1457 (1975) [Recursion equations in gauge field theories. Sov. Phys. JETP **42**, 413–418 (1976); Phase transitions in gauge and spin-lattice systems. Sov. Phys. JETP **42**, 743–746 (1976)]
    Kadanoff, L.: Notes on Migdal's recursion formulas. Ann. Phys. (New York) **100**, 359–394 (1976)