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Periodic Nonlinear Waves on a Half-Line

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Abstract. Nontrivial solutions of the equation $u_{tt} = u_{xx} - g(u)$ which are 2π-periodic in t and which decay as $x \to \infty$ are shown to exist if $g(a) = 0$ and $g'(0) > 1$. Breather-like solutions, which also decay as $x \to -\infty$, can be interpreted as homoclinic solutions in the x-dynamics; their existence is still in question for general g.

I. Introduction

Let $g: \mathbb{R} \to \mathbb{R}$ be a C^2 function with $g(0) = 0$. We consider the nonlinear wave equation

$$u_{tt} = u_{xx} - g(u) \tag{1}$$

for $x \in [0, \infty)$ and $t \in \mathbb{R}$, where u is real-valued. J.-M. Coron [3] has shown that, if $g'(0) < 1$, then any solution of (1) which is 2π-periodic in t and which decays as $x \to \infty$ in the sense that

$$\int_0^\infty dx \int_0^{2\pi} |u(x,t)| dt < \infty, \tag{2}$$

$$\lim_{x \to \infty} \int_0^{2\pi} (u_t^2(x,t) + u_x^2(x,t)) dt = 0, \tag{3}$$

$$\lim_{x \to \infty} \max_{t \in [0, 2\pi)} |u(x,t)| = 0, \tag{4}$$

must be independent of t.

The purpose of this note is to show that, if $g'(0) > 1$, then there do exist solutions of (1) which are non-constant and 2π-periodic[1] in t and which decay exponentially fast as $x \to \infty$ in the sense that

$$\int_0^{2\pi} (u^2(x,t) + u_t^2(x,t) + u_x^2(x,t)) dt < Ce^{-\lambda x} \text{ for some } \lambda > 0. \tag{5}$$

1 By scaling g, x and t, one can reduce the search for periodic solutions of arbitrary period to the case of period 2π

By the Sobolev inequality, such a u admits a pointwise estimate of the form

$$|u(x,t)| < De^{-\lambda x} \tag{6}$$

as well.

The main idea of our proof is implicit in the second-to-last paragraph of [3], namely to rewrite Eq. (1) as

$$u_{xx} = u_{tt} + g(u) \tag{1'}$$

and to consider (1′) as a dynamical system in the "time" x, while thinking of the "space" variable t as ranging over the circle $\mathbb{R}/2\pi\mathbb{Z}$. (I would like to thank John Rawnsley for suggesting that I take this idea to heart.) We then apply the stable manifold theorem.

If $g(u) = \alpha \sin u$ for $\alpha > 1$, then there is an explicit solution to the sine-Gordon equation (1) which satisfies the specified periodicity and decay conditions, namely the "breather" (see [7]):

$$u_\alpha(x,t) = 4\tan^{-1}\left(\frac{\sqrt{\alpha^2-1}\sin t}{\cosh\sqrt{\alpha^2-1}\xi}\right).$$

This solution has the property of being defined for all $x \in \mathbb{R}$ and decaying as $x \to -\infty$ as well as for $x \to \infty$. It has been remarked [1] that the existence of such periodic solutions might imply that $g(u)$ is a multiple of $\sin u$, while evidence suggesting the contrary has been given in [2] and [4]. At the end of this note, we shall present some thoughts on this question based on the interpretation of breather solutions as homoclinic orbits at 0 for Eq. (1′).

II. Existence of Periodic Waves

We work in the Hilbert space $\mathscr{H}$ of pairs (u,v), where $u \in H^1(\mathbb{R}/2\pi\mathbb{Z})$ and $v \in L_2(\mathbb{R}/2\pi\mathbb{Z})$. Eq. (1′) is equivalent to the system

$$u_x = v, \quad v_x = u_{tt} + g(u). \tag{1''}$$

The vector field determined by (1″) is defined only on a dense subspace of $\mathscr{H}$, but the corresponding local flow is defined on an open subset of $\mathscr{H} \times \mathbb{R}$. In particular, there exists [8] a neighbourhood $U \times I$ of $(0,0,0)$ in $\mathscr{H} \times \mathbb{R}$ and a C^2 family of maps $\phi_x: U \to \mathscr{H}$ such that $\phi_0(u,v) = (u,v)$ and $(u,v) \mapsto \phi_x(u,v)$ is a solution of (1″). The stable manifold of ϕ_x for $x > 0$ in I will provide the decaying solutions we seek.

We must analyze the linearization $T_0\phi_x$ of ϕ_x at the equilibrium point $(0,0)$. The maps $T_0\phi_x$, are determined by solving the linearized equations

$$u_x = v, \quad v_x = u_{tt} + g'(0)u. \tag{$1''_L$}$$

Let L be the linear operator defind by $L(u,v) = (v, u_{tt} + g'(0)u)$. Then E is decomposed into L-invariant subspaces E_k for $k = 0,1,2,\ldots$, where E_0 is two-dimensional and spanned by $(1,0)$ and $(0,1)$, while E_k for $k > 0$ is four-dimensional and spanned by $(\sin kt, 0)$, $(0, \sin kt)$, $(\cos kt, 0)$, and $(0, \cos kt)$. The eigenvalues $\omega_k^\pm$ of L on E_k are the solutions of the dispersion relation $\omega^2 = -k^2 + g'(0)$. Each E_k is also invariant under $T_0\phi_x$, with eigenvalues $e^{\pm\omega k}$. It follows that $T_0\phi_x$ is elliptic on the infinite-dimensional space $E^c = \bigoplus_{k \geqq g'(0)} E_k$ and hyperbolic on the finite-dimensional space $E^h = \bigoplus_{k<g'(0)} E_k$; moreover, E^h splits into expanding and contracting subspaces E^u and E^s of equal dimension.

By the stable manifold theorem (see [5] for a version which applies in the present context), there is a piece of submanifold $\Sigma^s \subset \mathscr{H}$ tangent to E^s at 0 such that $\phi_x(\Sigma^s) \subset \Sigma^s$ for $x>0$ sufficiently small, and hence for all $x>0$. Since the differential at 0 of $\phi_x|\Sigma^s$ has norm <1, for (u,v) sufficiently close to 0 in Σ^s we have the inequality $\|\phi_x(u,v)\| \leqq e^{-k}\|(u,v)\|$ for some $k>0$, where the norm is that in $\mathscr{H}$. In particular, if $u(x,t)$ is the first component of $\phi_x(u,v)$, then u satisfies Eq. (1) and inequalities (5) and (6).

If $g'(0) \leqq 1$, then E^s and hence Σ^s consists of at most the functions constant in t (and not even these, if $g'(0) \leqq 0$). This is essentially the case considered by Coron [3]. On the other hand, if $g'(0)>1$, then E^s has dimension at least 3, and so there exist solutions decaying in x which are not constant in t. In fact, there are solutions asymptotic to $e^{-\mu x}\sin t$ as $x \to 0$, similar to the sine-Gordon breathers.

III. Are There Solutions Decaying as $x \to \pm\infty$?

The argument in Sect. II may be applied just as well to Eq. (1) on the half line $-\infty < x \leqq 0$, yielding solutions 2π-periodic in t which decay as $x \to -\infty$. As stated in the introduction, it is interesting to know whether there are periodic solutions defined for all $x \in \mathbb{R}$ and decaying as $x \to \pm\infty$. In terms of the dynamical system (1″), there is a stable manifold Σ^s and an unstable manifold Σ^u through $(0,0)$, and the question is how they intersect.

For simplicity, assume that $1 < g'(0) < 4$, so that Σ^s and Σ^u are three-dimensional. (If $g'(0)$ is larger, Σ^s and Σ^u have higher dimension, but the general picture should be the same.) It seems to me highly unlikely that these manifolds should intersect in the infinite-dimensional space $\mathscr{H}$, except possibly along the one-dimensional manifold corresponding to the functions independent of t. (This one-dimensional intersection may be considered, in a sense, "forced" by the symmetry of the equation under translations in t, for which the functions independent of t are the fixed point manifold. It occurs, for example, in the case $g(u) = \alpha(u-u^3)$.)

The intersection of Σ^s and Σ^u corresponding to the sine-Gordon breathers may be attributed to the complete integrability of that equation. (For the sine-Gordon equation, the solutions independent of t are not homoclinic at $(0,0)$ but are rather heteroclinic, connecting $(0,0)$ to the equilibria at $(\pm 2\pi, 0)$.) It seems likely that the intersection will disappear along with integrability for all but very special perturbations of $g(u)$ from $\alpha \sin u$. To check whether this is actually the case, a "Melnikov" integration with respect to x (see [6]) along breather solutions of the sine-Gordon equation may be instructive. I hope to carry this out in the near future.

Meanwhile, one is left with the question of interpreting the numerical and asymptotic results in [2] and [4]. For the asymptotic results, the simplest explanation may be that the series obtained do not converge. The numerically observed solutions, on the other hand, may not be truly periodic in t but only approximately so, so that they would represent long-lived rather than permanent bound states.

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