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Reduction Techniques for Infinite-Dimensional Hamiltonian Systems: Some Ideas and Applications*

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Abstract. In the language of tensor analysis on differentiable manifolds, we present a reduction method of integrability structures, and apply it to recover some well-known hierarchies of integrable nonlinear evolution equations.

1. Introduction

In the recent past, starting from the basic works of Lax [1] and of Gardner et al. [2], a lot of remarkable papers have been published which aimed at elucidating the integrability properties of some special classes of infinite-dimensional Hamiltonian systems, expressed by nonlinear evolution equations (NEE's). To describe the properties of such systems, different approaches have been followed: some of them were global, like the inverse scattering method in its various versions [3, 4], which for instance allows one to linearize the associated Cauchy problem and to construct relevant classes of explicit solutions. Some others were local, aiming at achieving an algebraic formulation of the integrability structure of those systems: among them, of prominent importance in our opinion is the one which can be associated with the names of Gel'fand-Dikii et al. [5], where the Hamiltonian structures supported by such systems are obtained from the dual algebra of certain infinite-dimensional Lie algebras, the algebras of pseudo-differential operators of negative degree.

The approach we wish to propose here does not belong to either of the families we have (indeed quite roughly) indicated above. It is in fact of a fairly geometrical nature: it investigates directly the integrability structures defined on some differentiable manifolds, and gives criteria which guarantee the reducibility of such structures on certain regular submanifolds. As special cases, through this approach one is able to recover the integrability structure of the more relevant hierarchies of NEE's studied in the literature. A systematic and exhaustive exposition of this method is given elsewhere [6]; however, it might be worthwhile to emphasize here some of its advantages. First of all, its tensor nature ensures that

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the properties of the integrability structure are preserved by the reduction procedure. Moreover, it shows how a number of (at a first glance) very different hierarchies of NEE's are intimately related to each other, being just different reductions of the same original tensor structures. Finally, it suggests a "canonical" reduction procedure, which actually gives rise to some well known classes of integrable NEE's, and moreover explains the "singular" nature of other integrable systems which, though of course by themselves very interesting, are obtained through a "non-canonical" reduction.

The purpose of this paper is twofold: to illustrate the method and to show, on some familiar examples, how it embodies concrete and effective prescriptions. In this spirit, the paper consists of two main parts: in the first part (Sects. 1–6) there is a general description of the theory, the emphasis being not on the proofs of the theorems (for which the reader is referred to [6]), but on the tools of investigation which they provide; in the second part, we check the effectiveness of the theory, by showing that it explains the integrability structure of the hierarchies of Ablowitz-Kaup-Newell-Segur (AKNS) [7], Heisenberg Spin Chain (HSC) [8], Kaup-Newell (KN) [9], Wadati-Konno-Ichikawa (WKI) [10], and "dulcis in fundo" Korteweg-De Vries (KdV) [11].

2. Poisson-Nijenhuis Manifolds

The main objects dealt with in this paper are the Poisson-Nijenhuis manifolds. Their definition is tersely reviewed in this section, both in global and in local form. Both formulations turn out to be useful, either for theoretical developments or for the applications.

Let M be a differentiable manifold (finite or infinite dimensional) modelled on a Banach space E, and let P and N be two tensor fields on M of type (2,0) and (1,1) respectively. Denoting as usual by $\mathfrak{X}(M)$ the algebra of vector fields on M and by $\mathfrak{X}^*(M)$ the vector space of one-forms on M, we say that the tensor $P: \mathfrak{X}^*(M) \to \mathfrak{X}(M)$ is a *Poisson tensor* on M if it is skewsymmetric, of constant rank and fulfils the condition

$$[P\alpha, P\beta] = P \cdot \{\alpha, \beta\}_p . \tag{2.1}$$

This means that for any pair of one-forms $\alpha, \beta \in \mathfrak{X}^*(M)$, the commutator $[P\alpha, P\beta]$ of the two vector fields $P\alpha, P\beta \in \mathfrak{X}(M)$ is the vector field associated to the "Poisson bracket"[1] defined as:

$$\{\alpha, \beta\}_P := L_{P\alpha}(\beta) - L_{P\beta}(\alpha) + d\langle \alpha, P\beta \rangle , \tag{2.2}$$

where $L_\varphi(\cdot)$ denotes the Lie derivative along the vector field $\varphi \in \mathfrak{X}(M)$.

Analogously, we define N to be a *Nijenhuis tensor* on M if it has constant rank and "zero torsion", i.e., it fulfils the condition:

$$[N\varphi, N\psi] - N[N\varphi, \psi] - N[\varphi, N\psi] + N^2[\varphi, \psi] = 0 \tag{2.3}$$

for any pair of vector fields $\varphi, \psi \in \mathfrak{X}(M)$.

1 We make here a slight abuse of language, as in general the Poisson bracket is defined on closed forms

Finally, we say that the (Poisson) tensor P and the (Nijenhuis) tensor N endow the manifold M with a "*Poisson-Nijenhuis structure*" (in shorthand notation: *PN structure*) if they fulfil the two coupling conditions:[2]

$$NP = \mathrm{PN}^*, \tag{2.4}$$

$$L_{P\alpha}(N)\cdot\varphi - PL_\varphi(N^*\alpha) + PL_{N\varphi}(\alpha) = 0, \tag{2.5}$$

i.e., if the product NP is again a Poisson tensor. In a strict sense, these are all the definitions which are needed further on in the paper. However, when dealing with the construction of a PN manifold, we face two variants of conditions (2.4), (2.5) which naturally lead to introducing two other kinds of manifolds, called $P\Omega$ and PQ manifolds respectively. Their definition relies upon the following remarks. Let Q be a second Poisson tensor on M such that its Schouten bracket [13] with P is identically zero:

$$[P, Q] = 0, \tag{2.6}$$

and assume that Q, as a mapping from $\mathfrak{X}^*(M)$ to $\mathfrak{X}(M)$, is invertible. Then, it can be shown [6] that P and the tensor N

$$N := PQ^{-1} \tag{2.7}$$

endow M with a PN structure. Similarly, let Ω be a presymplectic tensor on M, i.e., a closed skew-symmetric tensor of type (0,2) of constant rank, and assume that the product $\Omega P \Omega$ is again a presymplectic tensor

$$d(\Omega P \Omega) = 0. \tag{2.8}$$

Then, it can be shown that P and the tensor N defined as:

$$N := P\Omega \tag{2.9}$$

endow M with a PN structure.

On the basis of these remarks, we are led to consider manifolds M endowed either with a pair of Poisson tensors P and Q fulfilling condition (2.6) or with a pair of tensors P and Ω fulfilling condition (2.8). Later on these manifolds will be referred to as PQ (or "twofold Hamiltonian" [12]) manifolds, and as $P\Omega$ manifolds respectively.

Let us now turn to the local version of the above definitions. To this end, let us first identify the manifold M with an open set U of a Banach space ("local chart"); the tensor fields Ω, N, P will be then expressed by mappings $\Omega: U \times E \to E^*$, $N: U \times E \to E$, $P: U \times E^* \to E$, linear with respect to the second argument. Denoting again by φ and α arbitrary elements of E and E^* respectively (being now identified with fields and one-forms constant on U), and by u an arbitrary point of U, these mappings can be then written in the form:

$$\alpha = \Omega_u \varphi, \tag{2.10}$$

$$\bar{\varphi} = N_u \varphi, \tag{2.11}$$

$$\varphi = P_u \alpha. \tag{2.12}$$

2 In formula (2.4) by N^* we mean the adjoint of N defined as: $\langle \alpha, N\varphi\rangle = \langle N^*\alpha, \varphi\rangle \forall \alpha \in \mathfrak{X}^*(M)$, $\varphi \in \mathfrak{X}(M)$

Consistently, we shall denote by:

$$\Omega'_u(\varphi;\psi); \qquad N'_u(\varphi;\psi); \qquad P'_u(\alpha;\psi) \tag{2.13}$$

the *Fréchet derivatives* with respect to u (at constant φ and α) of the tensor fields Ω, N, P evaluated at the point u in the ψ direction [14, 15]. In terms of the notations (2.10)–(2.13), the conditions corresponding to the specific properties of the tensor fields Ω, P, N take the form:

$$\langle \Omega'_u(\varphi;\psi),\chi\rangle + \text{cyclic permutation} = 0 \text{ (closure)}, \tag{2.14}$$

$$\langle \alpha, P'_u(\beta; P_u\gamma)\rangle + \text{cyclic permutation} = 0, \tag{2.15}$$

$$N'_u(\varphi; N_u\psi) - N'_u(\psi; N_u\varphi) + N_u(N'_u(\psi;\varphi) - N'_u(\varphi;\psi)) = 0, \tag{2.16}$$

while the coupling conditions which allow one to define the $P\Omega$ and PN manifolds become respectively:

$$\langle P'_u(\Omega_u\varphi;\psi), \Omega_u\chi\rangle + \langle \Omega'_u(\varphi; P_u\Omega_u\psi),\chi\rangle + \text{cyclic perm.} = 0, \tag{2.17}$$

$$\langle \alpha, N_uP_u\beta\rangle + \langle \beta, N_uP_u\alpha\rangle = 0, \tag{2.18}$$

$$\begin{aligned}&\langle \alpha, N'_u(P_u\beta;\varphi) - N'_u(\varphi; P_u\beta)\rangle\\ &\quad + \langle \beta, N'_u(\varphi; P_u\alpha) + N_uP'_u(\alpha;\varphi) - P'_u(\alpha; N_u\varphi)\rangle = 0.\end{aligned} \tag{2.19}$$

As for the property (2.6), since we are not going to use it explicitly in the rest of the paper, we omit here its (rather cumbersome) local version.

Next, let us consider a change of local chart on M, that is a local diffeomorphism $f: U \to \bar{U}$ between open sets U and $\bar{U}$ of E. According to the elementary transformation laws for vectors φ and for one-forms α, given by:

$$\bar{\varphi} = f'_u \cdot \varphi; \qquad \alpha = f'^*_u \cdot \bar{\alpha} \tag{2.20}$$

it is readily seen that the representatives Ω_u, N_u, P_u of the tensor fields Ω, N, P on M obey the transformation laws:

$$\bar{\Omega}_u = (f'^*_u)^{-1} \cdot \Omega_u \cdot (f'_u)^{-1}, \tag{2.21}$$

$$\bar{N}_{\bar{u}} \cdot f'_u = f'_u \cdot N_u, \tag{2.22}$$

$$\bar{P}_{\bar{u}} = f'_u \cdot P_u \cdot f'^*_u, \tag{2.23}$$

where f'^*_u is the dual of the Fréchet derivative f'_u, defined as:

$$\langle \alpha, f'_u \cdot \varphi\rangle = \langle f'^*_u \cdot \alpha, \varphi\rangle. \tag{2.24}$$

Taking into account the above local definitions and properties, one can thus define a *local* PN *manifold* as an open set U of a Banach space E, endowed with two mappings $N: E \to E$, $P: E^* \to E$, which fulfil the "closure conditions" (2.15), (2.16), the "coupling conditions" (2.18) and (2.19), and which, under local diffeomorphisms, obey the transformation laws (2.22) and (2.23) respectively. Obviously, one can also define a *local $P\Omega$ manifold* in an analogous way.

3. The Basic PN Manifold

Throughout this paper, a single example of manifold M endowed with different PN structures will be considered, in order to recover different classes of integrable NEE's. This manifold is modelled on a Fréchet space rather than a Banach space, in contrast with the standard assumptions of the theory. A priori, this would require some care in dealing with it, since, as is well known, not all the results for Banach manifolds hold for Fréchet manifolds. In our case, however, a general discussion of the difficulties arising from this extension can be avoided, as the simple nature of our manifold will allow us to verify directly, on the example, the validity of the construction described on a theoretical ground for Banach manifolds.

Let then $\mathbb{F}$ be the Fréchet space $\mathbb{F}:=C^{\infty}(\mathbb{R}, \mathrm{gl}^*(2,\mathbb{C}))$. As the manifold M, we consider the affine hyperplane of $\mathbb{F}$ formed by the matrix-valued C^{∞} functions $u:\mathbb{R}\to \mathrm{gl}^*(2,\mathbb{C})$, obeying given asymptotic conditions.

In other words, the points of our manifolds will be, in the applications, 2×2 matrices whose entries are scalar C^{∞} functions defined on the whole real axis $\mathbb{R}$ and obeying preassigned asymptotic conditions. (We notice parenthetically that, with no essential extra-troubles, one could even consider the case of $n\times n$ matrices whose entries are themselves matrices: this would be the so-called "non-abelian" case [6, 16]).

The vectors φ and the covectors α will be then C^{∞} matrix-valued functions $\varphi:\mathbb{R}\to \mathrm{gl}^*(2,\mathbb{C})$ and $\alpha:\mathbb{R}\to \mathrm{gl}(2,\mathbb{C})$ which fulfil the asymptotic conditions:

$$\lim_{|x|\to\infty} \varphi(x)=0\,, \qquad \lim_{|x|\to\infty} \alpha(x)=0\,,$$

the value of the covector α on the vector φ being given by

$$\langle \alpha, \varphi\rangle = \mathrm{Tr}\int_{-\infty}^{+\infty} dx\alpha(x)\varphi(x)\,. \tag{3.2}$$

On this manifold we shall consider the following four *Poisson tensors*:

$$P_1\alpha=\alpha_x\,, \tag{3.3}$$

$$P_2\alpha=[u,\alpha]\,, \tag{3.4}$$

$$P_3\alpha=\alpha_x+[u,\alpha]\,, \tag{3.5}$$

$$P_4\alpha=[a,\alpha]\,, \tag{3.6}$$

where, in formula (3.6), a is an arbitrary x-independent matrix.

Considered in pairs, (P_1, P_2) and (P_3, P_4) define on M two different PQ structures which according to the literature will be denoted as *chiral structure* and AKNS *structure*. Moreover, since P_1 and P_3 are invertible, to both such structures is naturally associated a Nijenhuis tensor (mostly denoted as "*recursion operator*" [17] in the literature), given by:

$$N_1\varphi:=[u, \int_{-\infty}^{x} \varphi dx]\,, \quad \text{(chiral)}\,, \tag{3.7}$$

$$N_2\varphi:=P_4\cdot P_3^{-1}\varphi\,, \quad \text{(AKNS)}\,. \tag{3.8}$$

Although different, the above two structures are actually closely related (such a relation being usually called "gauge-equivalence"). One can indeed show that all the Poisson structures (3.3)–(3.6) can be derived, through a systematic procedure, by a single "group-theoretical" tensor structure; here, however, we are neither going to discuss the group theoretical origin of the tensor fields so far introduced, referring to [6] for such derivation, nor to show explicitly that they fulfill all the required "closure" and "coupling" conditions. As an example, we will just show that N_1 (3.7) is indeed a Nijenhuis tensor, mainly to elucidate how the local conditions (2.14)–(2.19) can be handled in a concrete case. To this aim, we notice first that:

$$N'_{1u}(\varphi;\psi)=[\psi, \int_{-\infty}^{x} \varphi dx], \tag{3.9}$$

whence it follows:

$$\begin{aligned} &N'_{1u}(\varphi;N_{1u}\psi)-N'_{1u}(\psi;N_{1u}\varphi)\\ &=\left[\left[u, \int_{-\infty}^{x} \psi\right], \int_{-\infty}^{x} \varphi\right]-\left[\left[u, \int_{-\infty}^{x} \varphi\right], \int_{-\infty}^{x} \psi\right]=\left[u,\left[\int_{-\infty}^{x} \psi, \int_{-\infty}^{x} \varphi\right]\right]. \end{aligned} \tag{3.10}$$

On the other hand, we have:

$$\begin{aligned} &N_{1u}(N'_{1u}(\varphi;\psi)-N'_{1u}(\psi;\varphi))\\ &=\left[u, \int_{-\infty}^{x}\left[\psi, \int_{-\infty}^{x} \varphi dx'\right]dx\right]-\left[u, \int_{-\infty}^{x}\left[\varphi, \int_{-\infty}^{x'} \psi dx'\right]dx\right]\\ &=\left[u, \int_{-\infty}^{x} \psi dx\cdot \int_{-\infty}^{x} \varphi dx-\int_{-\infty}^{x}\left(\int_{-\infty}^{x'} \psi dx'\right)\varphi dx-\int_{-\infty}^{x} \varphi dx\cdot \int_{-\infty}^{x} \psi dx\right.\\ &\quad\left.+\int_{-\infty}^{x} \varphi\left(\int_{-\infty}^{x'} \psi dx'\right)dx\right]-\left[u, \int_{-\infty}^{x}\left[\varphi, \int_{-\infty}^{x'} \psi dx'\right]dx\right]\\ &=\left[u,\left[\int_{-\infty}^{x} \psi dx, \int_{-\infty}^{x} \varphi dx\right]\right]. \end{aligned}$$

(Thus condition (2.16) is fulfilled.) (3.11)

The validity of the remaining relations can be directly checked by the reader through a straightforward, although tedious, calculation.

Our purpose will be now to show that a systematic investigation of the previous PN structures will naturally lead to deriving some of the main classes of nonlinear evolution equations (in one space dimension) solvable by the Inverse Spectral Transform. To achieve this goal, we still need an essential tool, namely the "*theory of reduction*", which will be the subject of the following section.

4. The Theory of Restriction

There are several reduction techniques for a given $P\Omega$ or PN structure [6]; here we shall confine ourselves to illustrate the simplest among them, relying on the so-called "restriction method". The problem amounts to determine the regular submanifolds $S\subset M$, which inherit from M a $P\Omega$ or PN structure in the same way as any surface embedded in a Euclidean space inherits a Riemannian structure. However, in contrast with the Euclidean case, it is easy to show, by concrete examples, that not any submanifold $S \subset M$ can inherit the structures of M: to guarantee such "inheritage" one has to impose simple compatibility conditions between S and the structures of M.

To express them, let us first introduce the following notations:

$\mathfrak{X}(S, M)$: the algebra of the vector fields defined on S, taking values in TM,

$\mathfrak{X}^*(S, M)$: the vector space of the one-forms defined on S and taking values in T^*M,

$\mathfrak{X}(S)$: the algebra of the vector fields tangent to S,

$\mathfrak{X}(S)^0$: the "annihilator" of $\mathfrak{X}(S)$, i.e., the set of one-forms $\in \mathfrak{X}^*(S, M)$ which vanish on $\mathfrak{X}(S)$,

$\mathfrak{X}_P^*(S)$: the subspace of one-forms belonging to $\mathfrak{X}^*(S, M)$ mapped by P into vector fields tangent to S.

Moreover, let us introduce a *parametrization* of S, consisting of a new manifold M' and of a differentiable injection $f: M' \to M$, whose image is exactly S:

$$f(M') = S, \tag{4.1}$$

and whose differential $df(m')$: $T_m M' \to T_m S$ is everywhere injective (technically speaking, f is an "immersion" of M' in M [18]).

We will denote by:

$$df: \varphi' \in \mathfrak{X}(M') \mapsto \varphi \in \mathfrak{X}(S, M) \qquad \varphi(f(m')) = df(m') \cdot \varphi'(m'), \tag{4.2}$$

$$\delta f: \alpha \in \mathfrak{X}^*(S, M) \mapsto \alpha' \in \mathfrak{X}^*(M') \qquad \alpha'(m') = \delta f(m') \cdot \alpha(f(m')), \tag{4.3}$$

the linear mappings induced by f between the vector fields and (respectively) the one-forms defined on M' and those defined on S. By definition, df is an injective mapping whose image is $\mathfrak{X}(S)$, while δf is a surjective mapping whose kernel is $\mathfrak{X}(S)^0$.

We are now able to state the following *theorem*, which provides *sufficient conditions* in order that a given $P\Omega$ structure can be restricted on S.

Theorem 4.1 (Restriction Theorem for $P\Omega$ Manifolds). *Let M be a $P\Omega$ manifold and S a regular submanifold of M, parametrized by $(M', f: M' \to M)$. If the following conditions:*

$$\mathfrak{X}_P^*(S) + \mathfrak{X}(S)^0 = \mathfrak{X}^*(S, M), \tag{4.4}$$

$$\Omega(\mathfrak{X}(S)) \subset \mathfrak{X}_P^*(S), \tag{4.5}$$

are fulfilled on S, then S inherits from M a restricted $P\Omega$ structure, which in the given parametrization is defined on M' by the tensors:

$$P' := df^{-1} \cdot P \cdot \delta f|_{\mathfrak{X}_P(S)}^{-1}, \tag{4.6}$$

$$\Omega' := \delta f \cdot \Omega \cdot df. \tag{4.7}$$

Without proving this theorem, we make a few comments in order to explain the meaning of the conditions (4.4)–(4.5) and the use of the formulas (4.6), (4.7). The first condition ensures that the restriction of δf on $\mathfrak{X}_P^*(S)$ is again a surjective mapping, hence possessing a right-inverse, $\delta f|_{\mathfrak{X}_P^*(S)}^{-1}: \mathfrak{X}^*(M') \to \mathfrak{X}_P^*(S, M)$, such that:

$$\delta f|_{\mathfrak{X}_P^*(S)} \cdot \delta f|_{\mathfrak{X}_P^*(S)}^{-1} = \mathrm{id}|_{M'} . \tag{4.8}$$

Concretely, such a right-inverse can be constructed in the following way: first of all, one determines the subspace $\mathfrak{X}_P^*(S)$ of the "constrained one-forms", by solving the equation:

$$P\mathfrak{X}_P^*(S) = \mathfrak{X}(S) . \tag{4.9}$$

Then, one considers the equation:

$$\delta f \cdot \alpha = \alpha', \qquad \alpha' \in \mathfrak{X}^*(M') , \tag{4.10}$$

α' being an arbitrary one-form $\in \mathfrak{X}^*(M')$, and constructs its general solution in the standard way:

$$\alpha = \alpha_0 + \mathfrak{X}(S)^0 , \tag{4.11}$$

where by α_0 we mean a particular solution of (4.10) while $\mathfrak{X}(S)^0$ is of course, by definition, the general solution of the associated homogeneous equation: such a general solution can be written explicitly, in parametric form, by taking advantage of the Lagrange-multipliers technique. Finally, one determines the Lagrange multipliers (or possibly just some of them) by requiring that the solution (4.11) fulfils the constraint conditions (4.9). Furthermore, condition (4.4) also entails that $\mathrm{Ker}\, \delta f|_{\mathfrak{X}_P^*(S)} \subseteqq \mathrm{Ker}\, P$, so that the product $P \cdot \delta f|_{\mathfrak{X}_P^*(S)}^{-1}$ does not depend on the choice of the right-inverse. Thus, the definition (4.6) is well posed, and in [6] one has shown explicitly that P is again a Poisson tensor. So the first condition ensures that P is reducible on S. As for the second condition, we notice first that, as is well known [19], formula (4.7) defines a pre-symplectic tensor on M': thus, the role of condition (4.5) is just to guarantee that P' and Ω' define again a $P\Omega$ structure on M'. According to (2.8), this amounts to show that the product $\Omega' P' \Omega'$ is again a presymplectic tensor. To this aim, let us notice that, due to (4.6), (4.7), we have

$$\Omega' P' \Omega' = \delta f \cdot (\Omega P) \cdot (\delta f|_{\mathfrak{X}_P^*(S)}^{-1} \cdot \delta f) \cdot (\Omega df) , \tag{4.12}$$

and that (4.5) entails:

$$(\delta f|_{\mathfrak{X}_P^*(S)}^{-1} \cdot \delta f) \cdot (\Omega df) = \Omega df . \tag{4.13}$$

Therefore, the closure property required on $\Omega' P' \Omega'$ follows from the analogous property of $\Omega P \Omega$.

However, we have to point out that condition (4.5) is only a *sufficient condition*, since what is really required is that $\Omega P(\delta f|^{-1} \delta f)\Omega$ be a presymplectic tensor, not necessarily equal to $\Omega P \Omega$ on $\mathfrak{X}(S)$. So, it might happen that the reduced tensors Ω' and P' define again a $P\Omega$ structure on M' although condition (4.5) is not fulfilled: we shall come back to this point when discussing the applications (Sect. 10).

An analogous restriction theorem holds for PN manifolds: we give here just the statement of this theorem, referring to [6] for the proof.

Theorem 4.2 (Restriction Theorem for PN manifolds). *Let M be a* PN *manifold and S a regular submanifold of M, parametrized by $(M', f: M' \to M)$. If condition* (4.4) *is* *fulfilled on S and moreover it holds that*

$$N(\mathfrak{X}(S)) \subset \mathfrak{X}(S), \tag{4.14}$$

then S inherits from M a reduced PN *structure, defined, in the given parametrization, by the tensors:*

$$P' := df^{-1} \cdot P \cdot \delta f|^{-1}_{\mathfrak{X}^*_P(S)}, \tag{4.15}$$

$$N' := df^{-1} \cdot N \cdot df. \tag{4.16}$$

To make the above theorems effective, one has to assign some criteria in order to select regular submanifolds S which fulfil the required conditions. To go further on this subject, one has to look somewhat deeper at the geometrical features of PN manifolds (we shall just consider this case, since, as already remarked, any $P\Omega$ manifold is a PN manifold as well). This will be done in the next section.

5. Some Elements of the Geometry of PN Manifolds

Let M be a PN manifold. The first essential element of the geometry of such manifolds is the integrable distribution (in the Frobenius sense [19]) defined as:

$$\mathscr{D}_m = P_m(T^*_m M), \quad m \in M. \tag{5.1}$$

$\mathscr{D}_m$ is called the characteristic distribution of the Poisson tensor P, while its integral manifolds are called the characteristic leaves of P.

Let then S be any one of such leaves, and let us restrict N to S (i.e., let us consider N as acting just on the vectors φ tangent to S). Due to the coupling condition (2.4), we have:

$$N(\mathfrak{X}(S)) \subset \mathfrak{X}(S), \tag{5.2}$$

which implies that the leaf S is invariant with respect to N. We can thus iterate the action of N on $\mathfrak{X}(S)$, hence defining the two sequences of distributions given, for each point $m \in S$, by the subspaces:

$$\operatorname{Im} N^K_m := \{\psi(m) \in T_m S : \psi(m) = N^K_m \varphi(m), \text{ for some } \varphi(m) \in T_m S\}, \tag{5.3}$$

$$\operatorname{Ker} N^K_m := \{\chi(m) \in T_m S : N^K_m \chi(m) = 0\}. \tag{5.4}$$

The first sequence fulfils the obvious inclusion relations:

$$\operatorname{Im} N \supset \operatorname{Im} N^2 \supset \ldots \supset \operatorname{Im} N^K \supset \ldots, \tag{5.5}$$

while for the second one we have the reversed relations:

$$\operatorname{Ker} N \subset \operatorname{Ker} N^2 \subset \ldots \subset \operatorname{Ker} N^K \subset \ldots. \tag{5.6}$$

We shall *assume* that, $\forall m \in S$, there exist a finite index $r(m) \equiv \operatorname{ind}(N)(m)$, called the (Riesz) index of the tensor N at the point m, such that for $k = r(m)$, both sequences (5.5) and (5.6) become stationary [20]:

$$\operatorname{Im} N^r_m = \operatorname{Im} N^{r+1}_m; \quad \operatorname{Ker} N^r_m = \operatorname{Ker} N^{r+1}_m. \tag{5.7}$$

We shall also assume $r(m)$ to be constant (i.e.: m-independent) on the leaf S.

Under the above assumptions one can show [6] that all the previous distribution (5.3) and (5.4) are integrable, and that the characteristic distributions of N, namely the distributions $\mathrm{Im}\,N_m^r$ and $\mathrm{Ker}\,N_m^r$, intersect transversally on S, fulfilling the relation ("Splitting Theorem", [20]):

$$T_m M = \mathrm{Im}\,N_m^r \oplus \mathrm{Ker}\,N_m^r\,. \tag{5.8}$$

Let then S' be any integral submanifold of the distribution $\mathrm{Im}\,N_m^r$ (referred to, henceforth, as a characteristic leaf of N on S). A simple argument shows that both S and S' fulfil the conditions of the restriction theorem (4.2), and thus support a reduced PN structure. Indeed the very definition of S and the skew-symmetry of P entail:

$$\mathfrak{X}_P^*(S) = \mathfrak{X}^*(S, M) \Leftrightarrow S\text{: char. leaf of } P\,, \tag{5.9}$$

$$\mathfrak{X}(S)^0 = \mathrm{Ker}\,P\,. \tag{5.10}$$

Thus, condition (4.4) is (by (5.9)) trivially fulfilled together with (4.14). Hence it follows that S supports a reduced PN structure. As for S', it suffices to remark that:

(i) The reduced Poisson tensor P' on S is *kernel-free* (this property being a consequence of the reduction law (4.4) and of (5.10), which implies $\mathrm{Ker}\,P = \mathrm{Ker}\,\delta f$);

(ii) P' and N' fulfil the coupling condition (2.4) on S;

(iii) The Splitting Theorem (5.8) entails:

$$\mathfrak{X}(S, M) = \mathrm{Im}\,N^r \oplus \mathrm{Ker}\,N^r\,. \tag{5.11}$$

Indeed, (i) and (ii) imply:

$$\mathfrak{X}(S') = \mathrm{Im}\,N'^r = P'(\mathrm{Im}(N'^*)^r)\,, \tag{5.12}$$

whence:

$$\mathfrak{X}_P^*(S') = \mathrm{Im}(N'^*)^r\,. \tag{5.13}$$

Thus, the reduction condition (4.4) is an immediate consequence of (iii) once it is noticed that

$$\mathfrak{X}(S')^0 = \mathrm{Ker}(N'^*)^r\,. \tag{5.14}$$

It is moreover readily seen that also the reduced tensors on S', say P'' and N'', are both kernel-free (this being again a consequence of the Splitting Theorem), so that the whole procedure amounts to deducing a reduced, kernel-free PN structure from the original one (under the only condition of a *finite* Riesz index for N). To derive this reduced structure it is sufficient to perform two subsequent restrictions: the first one will be performed on a characteristic leaf of P, the second one on a characteristic leaf of N (lying on the previous leaf). Of course, the procedure may well end just at the first step: indeed, if $\mathrm{ind}(N') = 0$ there is no further restriction to be performed.

The method of reduction outlined above is in some sense canonical, but it is not the only possible one at all. One can easily check, through concrete examples, that there are regular submanifolds S which, though not being characteristic manifolds of P and N, are nevertheless endowed with a PN structure inherited from the ambient space M. This fact is intimately related to the existence of a further *reduction technique ("reduction by projection")* which exploits the existence of the second class of characteristic manifolds of N, namely the integral manifolds of the distribution $\mathrm{Ker}\,N^r$. A description of this technique is outside the scope of the present paper. Referring again for details to [6], we shall give here a *modified version* of it, which does not explicitly require the use of the projection formalism. The key idea of this version is very simple and amounts to show that, under suitable assumptions on S, it is possible to modify the given PN structure so to obtain a novel structure which has S as a characteristic manifold and thus is reducible on S. The main drawback of this technique is that it does not suggest any criterion to select a priori the submanifolds S satisfying the required assumptions. However, once such a submanifold has been somehow determined, the technique we are going to describe turns out to be quite effective and fast in leading to the final result. For the sake of simplicity, it is convenient to consider just the Nijenhuis tensor (omitting the study of the Poisson tensor). This will be done in the next section.

6. A Reduction Method for Nijenhuis Manifolds

Let M be a Nijenhuis manifold and S a regular submanifold of M. Let us assume that:

(i) N has a finite Riesz index r, which is constant on S, so that:

$$T_mM = \operatorname{Im} N_m^r \oplus \operatorname{Ker} N_m^r, \qquad \forall m \in S. \tag{6.1}$$

(ii) S is *transversal* to the distribution $\operatorname{Ker} N^r$, so that:

$$T_mM = T_mS \oplus \operatorname{Ker} N_m^r, \qquad \forall m \in S. \tag{6.2}$$

Under the above assumptions, S inherits from M a Nijenhuis structure which can be constructed in the following way. First of all, one considers the canonical projection

$$\pi(m): T_mM \to \operatorname{Im} N_m^r, \qquad m \in S \tag{6.3}$$

associated to the decomposition (6.1), and notices that, due to (6.2), the restriction $\pi_s(m)$ of $\pi(m)$ on T_mS is a bijection. It is then possible to consider its inverse $\pi_s(m)^{-1}$: $\operatorname{Im} N_m^r \to T_mS$ and to construct the mapping $\bar{N}_m: T_mM \to T_mS$, $m \in S$, defined as:

$$\bar{N}_m := \pi_s(m)^{-1} \cdot N_m \cdot \pi(m). \tag{6.4}$$

One can then show that the mapping (6.4) defines a new Nijenhuis tensor on $\mathfrak{X}(S, M)$ for which S is a characteristic manifold. To construct the reduced Nijenhuis structure we have just to introduce an arbitrary parametrization $(M', f: M' \to M)$ of S and to follow the procedure given in Theorem (4.2). The reduced structure will be thus given by the formula:

$$N' = df^{-1} \cdot \bar{N} \cdot df. \tag{6.5}$$

The proofs of the assertions made in this section are given in [6]: here we will just show the effectiveness of this reduction technique by applying it to some special cases, discussed in Sects. 10 and 11.

7. The Integrability Structure of the AKNS Hierarchy

Let M be the affine hyperplane of $\mathbb{F} = C^\infty(\mathbb{R}, \mathrm{gl}^*(2, \mathbb{C}))$ defined in Sect. 3, endowed with the PN structure induced by the Poisson tensors:

$$P\alpha := [a, \alpha], \tag{7.1}$$

$$Q\alpha := \alpha_x + [u, \alpha], \tag{7.2}$$

where a is a x-independent matrix, arbitrary in principle, which however throughout this section will be identified with the Pauli matrix σ_3:

$$a=\sigma_3=\operatorname{diag}(1,-1). \tag{7.3}$$

As explained in Sect. 2, the pair (P, Q) induces on M a *canonical* Nijenhuis tensor given by:

$$N:=PQ^{-1}. \tag{7.4}$$

The purpose of the present section is the reduction of the above PN structure first on a characteristic leaf of P and then on a characteristic leaf of N, according to the "standard scheme" given in Sect. 5: through this twofold reduction we shall obtain in a simple way the "integrability structure" of the AKNS hierarchy.

Let us first notice that the characteristic distribution of P is given by:

$$\operatorname{Im} P=\{\varphi\in\mathfrak{X}(M):\varphi_D=0\}, \tag{7.5}$$

when the subscript D stands for the diagonal part of a matrix. Its integral manifolds are then the affine hyperplanes given by:

$$u_D=C, \tag{7.6}$$

when C is any constant (matrix), to be determined from the initial conditions. Let us make the simplest choice, namely $C=0$, so that:

$$S=\{u:u_D=0\}. \tag{7.7}$$

To determine the reduced PN structure on S we have just to choose a suitable parametrization of S, applying then formulas (4.15) and (4.16). The most natural choice amounts to taking as a parameter space M' the affine hyperplane of off-diagonal matrices (matrices with zero entries on the main diagonal), and as parametrization $f:M'\to M$ the canonical immersion:

$$f:u'\to u=u'. \tag{7.8}$$

Since the vector fields $\varphi'\in\mathfrak{X}(M')$ and the one-forms $\alpha'\in\mathfrak{X}^*(M')$ are simply given by off-diagonal matrices, we readily get:

$$df:\varphi'\to\varphi=\varphi', \tag{7.9}$$

$$\delta f:\alpha\to\alpha'=\alpha_{\mathrm{OD}}, \tag{7.10}$$

where the subscript OD denotes the off-diagonal part of a matrix.

Formula (7.10) follows from:

$$\langle\delta f\cdot\alpha,\varphi'\rangle=\langle\alpha, df\varphi'\rangle=\langle\alpha_D,\varphi'\rangle+\langle\alpha_{\mathrm{OD}},\varphi'\rangle=\langle\alpha_{\mathrm{OD}},\varphi'\rangle. \tag{7.11}$$

The simplest choice for the right-inverse of δf is thus:

$$\delta f^{-1}:\alpha'\to\alpha=\alpha'. \tag{7.12}$$

Applying then formulas (4.15) and (4.16) we get:

$$P'\alpha'=df^{-1}\cdot P\cdot\delta f^{-1}\cdot\alpha'=[\sigma_3,\alpha'], \tag{7.13}$$

$$\bar{\varphi}'=N'\varphi'=df^{-1}\cdot PQ^{-1}\cdot df\cdot\varphi'. \tag{7.14}$$

Equation (7.14) can be obviously replaced by the equivalent system:

$$\begin{aligned} Q\alpha = df\cdot\varphi' &\Leftrightarrow \alpha_x + [u,\alpha] = \varphi', \\ P\alpha = df\cdot\bar{\varphi}' &\Leftrightarrow [\sigma_3,\alpha] = \bar{\varphi}', \end{aligned} \tag{7.15}$$

which yields

$$(\alpha_{Dx} + [u', \alpha_{\mathrm{OD}}]) + (\alpha_{\mathrm{OD},x} + [u', \alpha_D]) = \varphi', \tag{7.16}$$

$$[\sigma_3, \alpha_{\mathrm{OD}}] = \bar{\varphi}'. \tag{7.17}$$

Equation (7.16) in turn, splits into:

$$\alpha_D = -\int_{-\infty}^{x} [u', \alpha_{\mathrm{OD}}]dx, \tag{7.18a}$$

$$\varphi' = \alpha_{\mathrm{OD},x} - \left[u', \int_{-\infty}^{x} [u', \alpha_{\mathrm{OD}}]dx\right], \tag{7.18b}$$

so that the reduced Nijenhuis tensor N' is obtained by getting rid of α_{OD} from (7.17) and (7.18b). Turning to the usual notation by components:

$$u' = \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}; \quad \varphi' = \begin{pmatrix} 0 & \varphi_1 \\ \varphi_2 & 0 \end{pmatrix}; \quad \alpha' = \begin{pmatrix} 0 & \alpha_2 \\ \alpha_1 & 0 \end{pmatrix}, \tag{7.19}$$

we get:

$$\varphi_1 = \frac{1}{2}\bar{\varphi}_{1x} - q\int_{-\infty}^{x} (q\bar{\varphi}_2 + r\bar{\varphi}_1)dx, \tag{7.20}$$

$$\varphi_2 = -\frac{1}{2}\bar{\varphi}_{2x} + r\int_{-\infty}^{x} (q\bar{\varphi}_2 + r\bar{\varphi}_1)dx, \tag{7.21}$$

which is the well known "recursion operator" for the AKNS hierarchy. Such operator is invertible, hence it has Riesz index $r=0$. Thus the reduction procedure ends at the first step, as pointed out in Sect. 5.

As a final remark, we just notice that the reduction procedure given in this section can be easily generalized to $n\times n$ matrices, replacing σ_3 by a diagonal matrix with distinct entries, and even to the so-called non-abelian case.

8. The Recursion Operator for the Heisenberg Chain

Let us consider again the manifold M of the previous example. Now, we shall think of it as endowed with the "chiral" PN structure, defined by the tensors:

$$P\alpha = \alpha_x, \tag{8.1}$$

$$N\varphi = \left[u, \int_{-\infty}^{x} \varphi dx\right]. \tag{8.2}$$

In this case the only characteristic leaf of P is given by the whole hyperplane M (since P is invertible); thus the only non-trivial reduction can be performed on a characteristic leaf of N. We notice that

$$\mathrm{Im}\,N = \{\varphi : \mathrm{Tr}\,u^k\varphi = 0, k=0,1\}. \tag{8.3}$$

The integral leaves are then given by:

$$S_C=\{u:\mathrm{Tr}\,u^k=C_k, k=1,2\}. \tag{8.4}$$

We make the special choice:

$$S=\{u:\mathrm{Tr}\,u=0, \mathrm{Tr}\,u^2=1\}, \tag{8.5}$$

corresponding to the integral leaf passing through the Pauli matrix σ_3.

When u belongs to S, we have:

$$\mathrm{Ker}\,N=\{\chi:\chi=\lambda_x I+(\mu u)_x; \lambda,\mu\in C^\infty(\mathbb{R},\mathbb{C})\}. \tag{8.6}$$

Thus $\mathrm{Im}\,N$ and $\mathrm{Ker}\,N$ are transversal on S: hence S is a characteristic leaf of N and $\mathrm{ind}(N)|_S=1$.

To get the reduction of the starting PN structure on S, the most natural choice of the parameter space M' is the manifold S^2 given by:

$$S^2:=\{\mathbf{q}\in\mathbb{R}^3:\mathbf{q}\cdot\mathbf{q}=1\}. \tag{8.7}$$

Identifying $T_q(S^2)$ and $T_q^*(S^2)$ with the space of vectors which are tangent to S^2 at the point $\mathbf{q}$, namely setting:

$$T_q(S^2):=\{\boldsymbol{\psi}\in\mathbb{R}^3:\boldsymbol{\psi}\cdot\mathbf{q}=0\}, \tag{8.8a}$$

$$T_q^*(S^2):=\{\boldsymbol{\beta}\in\mathbb{R}^3:\boldsymbol{\beta}\cdot\mathbf{q}=0\}, \tag{8.8b}$$

the parametrization $(S^2, f:S^2\to M)$ of the submanifold S is then given by:

$$f:\mathbf{q}\to u=\mathbf{q}\cdot\hat{\sigma}, \tag{8.9}$$

where $\hat{\sigma}=(\sigma_1,\sigma_2,\sigma_3)$, σ_i being the Pauli matrices, and the mappings $df:\mathfrak{X}(S^2)\to\mathfrak{X}(S,M)$, $\delta f:\mathfrak{X}^*(S,M)\to\mathfrak{X}^*(S^2)$ take the explicit form:

$$df:\boldsymbol{\psi}\to\varphi=\boldsymbol{\psi}\cdot\hat{\sigma}, \tag{8.10}$$

$$\delta f:\alpha\to\boldsymbol{\beta}=2\boldsymbol{\alpha}-2(\boldsymbol{\alpha}\cdot\mathbf{q})\mathbf{q}, \tag{8.11}$$

where

$$\boldsymbol{\alpha}=\frac{1}{2}\sum_{j=1}^{3}\mathrm{Tr}(\alpha\sigma_j)\mathbf{e}^{(j)}. \tag{8.12}$$

Formula (8.11) follows from the very definition of δf, which entails:

$$\begin{aligned}\langle\delta f\alpha,\boldsymbol{\psi}\rangle&=\langle\alpha, df\boldsymbol{\psi}\rangle\\&=\mathrm{Tr}\int_{-\infty}^{+\infty}\sum_j\alpha(\boldsymbol{\psi}\cdot\mathbf{e}^{(j)})\sigma_j dx\\&=\int_{-\infty}^{+\infty}\boldsymbol{\psi}\cdot\sum_j\mathrm{Tr}(\alpha\sigma_j)\mathbf{e}^{(j)}dx=\int_{-\infty}^{+\infty}2\boldsymbol{\psi}\cdot\boldsymbol{\alpha}dx.\end{aligned} \tag{8.13}$$

The constraint (8.8a) on $\boldsymbol{\psi}$ implies:

$$\boldsymbol{\beta}=\delta f\cdot\alpha=2\boldsymbol{\alpha}+\lambda\mathbf{q}, \tag{8.14}$$

and thus the constraint (8.8b) on $\boldsymbol{\beta}$ yields:

$$\lambda=-2\boldsymbol{\alpha}\cdot\mathbf{q}. \tag{8.15}$$

Let us first consider the reduction on S of the Nijenhuis tensor N given by (8.2). The reduced tensor $N': \mathfrak{X}(S^2) \to \mathfrak{X}(S^2)$ is defined as:

$$N' := df^{-1} \cdot N \cdot df\,, \tag{8.16}$$

which yields:

$$\begin{aligned}\boldsymbol{\psi}' &= N'\boldsymbol{\psi} = df^{-1} \cdot N \cdot (\boldsymbol{\psi} \cdot \hat{\sigma}) \\ &= df^{-1}\left[u, \int_{-\infty}^{x} \boldsymbol{\psi} \cdot \hat{\sigma} dx\right] = df^{-1}\left(2i\mathbf{q} \wedge \int_{-\infty}^{x} \boldsymbol{\psi} dx\right) \cdot \hat{\sigma} \\ &= 2i\mathbf{q} \wedge \int_{-\infty}^{x} \boldsymbol{\psi} dx\,, \end{aligned} \tag{8.17}$$

where we have used the well-known relation:

$$\begin{aligned}[u, \boldsymbol{\varphi} \cdot \hat{\sigma}] &= [\mathbf{q} \cdot \hat{\sigma}, \boldsymbol{\varphi} \cdot \hat{\sigma}] = \sum_{i,j=1}^{3} q_i \varphi_j [\sigma_i, \sigma_j] \\ &= 2i \sum_{ij} \varepsilon_{ijk} q_i \varphi_j \sigma_k = 2i\mathbf{q} \wedge \boldsymbol{\varphi} \cdot \hat{\sigma}\,. \end{aligned} \tag{8.18}$$

To invert N', getting in this way the recursion operator generating the Heisenberg chain hierarchy, we notice that, from (8.17), it follows:

$$\frac{1}{2i}(\boldsymbol{\psi}' \wedge \mathbf{q})_x = \boldsymbol{\psi} - \left[\mathbf{q}\left(\mathbf{q} \cdot \int_{-\infty}^{x} \boldsymbol{\psi} dx\right)\right]_x\,. \tag{8.19}$$

Thanks to the constraint (8.8a), Eq. (8.19) can be explicitly (and uniquely) solved with respect to $\boldsymbol{\psi}$, yielding:

$$\boldsymbol{\psi} = -\frac{1}{2i}\left[\boldsymbol{\psi}' \wedge \mathbf{q} - \mathbf{q}\left(\int_{-\infty}^{x} (\boldsymbol{\psi}' \wedge \mathbf{q})_x \cdot \mathbf{q} dx\right)\right]_x\,. \tag{8.20}$$

Let us now turn to the reduction of the Poisson tensor P (8.1). To this aim, we construct the right-inverse of the adjoint mapping δf, which reads:

$$\delta f^{-1}: \boldsymbol{\beta} \to 2\alpha = \lambda I + \mu \mathbf{q} \cdot \hat{\sigma} + \boldsymbol{\beta} \cdot \hat{\sigma}\,, \tag{8.21}$$

where λ and μ are "Lagrange multipliers" to be determined by imposing the constraints induced by the choice of the leaf (8.5). So, by requiring:

$$\operatorname{Tr} \alpha_x = \operatorname{Tr}(\alpha u)_x = 0\,, \tag{8.22}$$

we get:

$$\lambda = 0\,, \qquad \mu = -\int_{-\infty}^{x} (\boldsymbol{\beta}_x \cdot \mathbf{q}) dx\,, \tag{8.23}$$

and thus:

$$\delta f|^{-1}_{\mathfrak{X}^*_P(S)}: \boldsymbol{\beta} \to 2\alpha = \boldsymbol{\beta} \cdot \hat{\sigma} - \left(\int_{-\infty}^{x} (\boldsymbol{\beta}_x \cdot \mathbf{q}) dx\right) \mathbf{q} \cdot \hat{\sigma}\,. \tag{8.24}$$

The corresponding reduced Poisson tensor

$$P': \mathfrak{X}^*(S^2) \to \mathfrak{X}(S^2) = df^{-1} \cdot P \cdot \delta f|^{-1}_{\mathfrak{X}^*_P(S)} \tag{8.25}$$

can be then evaluated explicitly, yielding:

$$\begin{aligned}\boldsymbol{\psi}=P'\boldsymbol{\beta}&=\frac{1}{2}\operatorname{Tr}(\varphi\sigma_j)\mathbf{e}^{(j)}=\frac{1}{2}\operatorname{Tr}(\alpha_x\sigma_j)\mathbf{e}^{(j)}\\&=\frac{1}{4}\operatorname{Tr}\left\{\left[\boldsymbol{\beta}\cdot\hat{\sigma}-\left(\int_{-\infty}^{x}\boldsymbol{\beta}_x\cdot\mathbf{q}dx\right)\mathbf{q}\cdot\hat{\sigma}\right]_x\sigma_j\right\}\mathbf{e}^{(j)}\\&=\frac{1}{2}\left[\boldsymbol{\beta}-\left(\int_{-\infty}^{x}\boldsymbol{\beta}_x\cdot\mathbf{q}dx\right)\mathbf{q}\right]_x.\end{aligned}\tag{8.26}$$

9. Integrability Structure for KN Hierarchy

The manifold M considered in the previous examples will be now thought of as endowed with the $P\Omega$ structure, defined by the Poisson tensor P:

$$P\alpha=\alpha_x+[u,\alpha],\tag{9.1}$$

and by the symplectic tensor Ω:

$$\Omega\varphi=\int_{-\infty}^{x}\varphi dx,\tag{9.2}$$

and thus by the Nijenhuis tensor $N=P\Omega$:

$$N\varphi=\varphi+\left[u,\int_{-\infty}^{x}\varphi dx\right].\tag{9.3}$$

We shall show in this section that the above $P\Omega$ (or equivalently PN) structure can be reduced on the submanifold

$$S:=\{u:u_D=\sigma_3\},\tag{9.4}$$

although S *is not a characteristic manifold* of P and that the resulting reduced structure is the integrability structure for KN hierarchy.

In order that P be reducible on S, it is sufficient that it fulfils the condition (4.4) of Theorem (4.1), namely:

$$\mathfrak{X}_P^*(S)+\mathfrak{X}(S)^0=\mathfrak{X}^*(S,M).\tag{9.5}$$

Since this condition entails

$$\mathfrak{X}(S)^0\cap\mathfrak{X}_P^*(S)\subset\operatorname{Ker}P,\tag{9.6}$$

and since, in the case under scrutiny, P is kernel-free, we have to show that

$$\mathfrak{X}^*(S,M)=\mathfrak{X}(S)^0\oplus\mathfrak{X}_P^*(S),\tag{9.7}$$

i.e., that any $\alpha\in\mathfrak{X}^*(S,M)$ can be uniquely written in the form:

$$\alpha=\beta+\gamma,\quad \beta\in\mathfrak{X}(S)^0,\quad \gamma\in\mathfrak{X}_P^*(S).\tag{9.8}$$

But the validity of (9.8) follows immediately from the structure of $\mathfrak{X}(S)^0$ and $\mathfrak{X}_P^*(S)$. Indeed, we have:

$$\mathfrak{X}(S)^0:=\{\beta:\beta_{\mathrm{OD}}=0\},\tag{9.9}$$

$$\mathfrak{X}_P^*(S):=\{\gamma:\gamma_{D,x}+[u_{\mathrm{OD}},\gamma_{\mathrm{OD}}]=0\}.\tag{9.10}$$

So, we can uniquely write $\alpha=\beta+\gamma$, with:

$$\begin{aligned}\beta&=\alpha_D+\int_{-\infty}^{x}[u_{\mathrm{OD}},\alpha_{\mathrm{OD}}]dx;\\ \gamma&=\alpha_{\mathrm{OD}}-\int_{-\infty}^{x}[u_{\mathrm{OD}},\alpha_{\mathrm{OD}}]dx.\end{aligned}\tag{9.11}$$

To get the explicit reduction of the $P\Omega$ structure (9.1), (9.2) we take advantage of the parametrization already introduced in Sect. 7, which yields:

$$u=u'+\sigma_3;\qquad \varphi=\varphi';\qquad \alpha'=\alpha_{\mathrm{OD}},\tag{9.12}$$

where u', α', φ' are 2×2 off-diagonal matrices. The restricted right-inverse of the mapping δf is then given by:

$$\delta f|^{-1}_{\mathfrak{X}^*_P(S)}:\alpha'\to\gamma=\alpha'-\int_{-\infty}^{x}[u',\alpha']dx.\tag{9.13}$$

Thus, the reduced tensor $P':\mathfrak{X}^*(M')\to\mathfrak{X}(M')$ reads:

$$\begin{aligned}\varphi'=P'\alpha':\varphi'&=\mathrm{d}\,\mathrm{f}^{-1}\cdot P\cdot\delta\,\mathrm{f}\,|^{-1}_{\mathfrak{X}^*_P(S)}\cdot\alpha'\\ &=df^{-1}\cdot P\left(\alpha'-\int_{-\infty}^{x}[u',\alpha']dx\right)\\ &=df^{-1}\cdot\left(\alpha'_x-\left[u',\int_{-\infty}^{x}[u',\alpha']dx\right]+[\sigma_3,\alpha']\right)\\ &=\alpha'_x-\left[u',\int_{-\infty}^{x}[u',\alpha']dx\right]+[\sigma_3,\alpha'].\end{aligned}\tag{9.14}$$

On the other hand, the symplectic tensor Ω is obviously reducible on S', its reduction being simply given by:

$$\Omega'\varphi'=\delta f\cdot\Omega\cdot df\cdot\varphi'=\delta f\cdot\int_{-\infty}^{x}\varphi'dx=\int_{-\infty}^{x}\varphi'dx.\tag{9.15}$$

Hence, the tensor $N'=P'\Omega'$ is given by:

$$\bar{\varphi}'=N'\varphi':\bar{\varphi}'=\varphi'-\left[u',\int_{-\infty}^{x}\left[u',\int_{-\infty}^{x}\varphi'dx\right]dx\right]+\left[\sigma_3,\int_{-\infty}^{x}\varphi'dx\right].\tag{9.16}$$

However, in this case, the condition (4.5) of Theorem (4.1) is not fulfilled, as we have:

$$\Omega(\mathfrak{X}(S))\cap\mathfrak{X}^*_P(S)=\emptyset.\tag{9.17}$$

Thus, we cannot be sure, a priori, that the tensor N' (9.16) is a Nijenhuis tensor. On the other hand, as already remarked in Sect. 4, condition (4.5) is only a sufficient condition: therefore, it is again possible that the tensor $\Omega'P'\Omega'=\Omega'N'$ be a presymplectic tensor (hence ensuring N' to be Nijenhuis), but one has to check it directly. Since $\Omega'P'\Omega'$ is clearly skew-symmetric, we have just to check the closure condition $d(\Omega'P'\Omega')=0$. For this purpose, it is convenient to write:

$$\Omega'P'\Omega'=\Omega_1+\Omega_2,\tag{9.18}$$

where

$$\Omega_1\varphi' := \int_{-\infty}^{x} \varphi' dx + \int_{-\infty}^{x} \left[\sigma_3, \int_{-\infty}^{x} \varphi' dx\right] dx, \tag{9.19}$$

$$\Omega_2\varphi' := -\int_{-\infty}^{x} N^2\varphi' dx, \tag{9.20}$$

N being the Nijenhuis tensor:

$$N\varphi := \left[u, \int_{-\infty}^{x} \varphi dx\right] \tag{9.21}$$

associated to the integrability structure of the Heisenberg chain hierarchy.

The tensor Ω_1, being (skew-symmetric and) constant on S, is obviously presymplectic. As for Ω_2, it is presymplectic because, as already discussed in Sect. 8, the Nijenhuis tensor of the Heisenberg chain is "well-coupled" with the symplectic operator $\int_{-\infty}^{x} dx$. Hence, the tensor N' (9.16) is again a Nijenhuis tensor, and, together with P', defines a PN structure on the submanifold S. Finally, we notice that $N' = \mathbb{1} + \bar{N}$, which implies that $\bar{N}$ is a Nijenhuis tensor too, obviously well-coupled with Ω'. The integrability structure of KN hierarchy is just given by $\bar{N}^{-1}$ and $\Omega'^{-1} = \partial_x$. To get their explicit expression, we use the notation by components (7.19). We can thus write:

$$\begin{aligned} \bar{\varphi}' = \bar{N}\varphi' : \bar{\varphi}_1 &= 2\int_{-\infty}^{x} \varphi_1 dx + 2q\int_{-\infty}^{x}\left(q\int_{-\infty}^{x}\varphi_2 dx - r\int_{-\infty}^{x}\varphi_1 dx\right)dx, \\ \bar{\varphi}_2 &= -2\int_{-\infty}^{x} \varphi_2 dx - 2r\int_{-\infty}^{x}\left(q\int_{-\infty}^{x}\varphi_2 dx - r\int_{-\infty}^{x}\varphi_1 dx\right)dx. \end{aligned} \tag{9.23}$$

The tensor $\bar{N}$ can be easily inverted, yielding:

$$\begin{aligned} \varphi' = \bar{N}^{-1}\bar{\varphi}' : 2\varphi_1 &= \left(\bar{\varphi}_1 + q\int_{-\infty}^{x}(q\bar{\varphi}_2 + r\bar{\varphi}_1)dx\right)_x \\ 2\varphi_2 &= \left(-\bar{\varphi}_2 + r\int_{-\infty}^{x}(q\bar{\varphi}_2 + r\bar{\varphi}_1)dx\right)_x . \end{aligned} \tag{9.24}$$

The tensor Ω'^{-1} is clearly given by:

$$\varphi' = \Omega'^{-1}\alpha' : \varphi_1 = \alpha_{2x}, \varphi_2 = \alpha_{1x} . \tag{9.25}$$

The first equation in the KN hierarchy is obtained by setting in (9.24) $\bar{\varphi}_1 = q_x$, $\bar{\varphi}_2 = r_x$.

Remark. The natural question now arises whether, analogously to the AKNS structure, even the KN structure could be generalized to $n \times n$ matrices, by replacing again σ_3 by a diagonal matrix with distinct entries. It turns out that for the KN structure this generalization does not take place, because, although the Poisson tensor P is always reducible, the tensor $\Omega'P'\Omega'$ is no longer presymplectic. Indeed in this case:

$$\Omega'P'\Omega' = \Omega_1 + \Omega_2 + \Omega_3 , \tag{9.27}$$

where the extra-term Ω_3

$$\Omega_3\varphi := -\int_{-\infty}^{x}\left[\sigma_D, \int_{-\infty}^{x}\left[u, \int_{-\infty}^{x}\varphi dx\right]dx\right]dx \tag{9.28}$$

is not presymplectic. Only for 2×2 matrices Ω_3 vanishes identically, since, in this case

$$[\sigma_D, [u_{\mathrm{OD}}, \varphi_{\mathrm{OD}}]] = 0 . \tag{9.29}$$

Hence, as noticed in the Introduction, through this "non-canonical" reduction procedure we have selected a somehow "singular" case.

10. The Recursion Operator for WKI Hierarchy

As an example of the reduction method for Nijenhuis manifolds described in Sect. 6, we will get here the recursion operator associated to the so-called WKI hierarchy.

Let M be the manifold considered in the previous sections, endowed with the Nijenhuis tensor:

$$N\varphi := \left[u, \int_{-\infty}^{x}\varphi dx\right]. \tag{10.1}$$

Let S be the submanifold:

$$S := \{u : u_D = \sigma_3\} . \tag{10.2}$$

We shall show that N is reducible on S since $\operatorname{ind}(N)|_S = 1$, and:

$$\mathfrak{X}(S, M) = \mathfrak{X}(S) \oplus \operatorname{Ker} N . \tag{10.3}$$

In terms of the usual notation by components:

$$u \in S : u = \begin{pmatrix} 1 & q \\ r & -1 \end{pmatrix}, \tag{10.4}$$

$$\varphi \in \mathfrak{X}(S, M) : \varphi = \begin{pmatrix} \varphi_1 & \varphi_2 \\ \varphi_3 & \varphi_4 \end{pmatrix}, \tag{10.5}$$

$$\alpha \in \mathfrak{X}^*(S, M) : \alpha = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix}, \tag{10.6}$$

the distributions $\operatorname{Im} N$ and $\operatorname{Ker} N$, at any point u of S, are given by

$$\operatorname{Im} N := \{\psi : \psi_1 + \psi_4 = 0; 2\psi_1 + q\psi_3 + r\psi_2 = 0\} , \tag{10.7}$$

$$\operatorname{Ker} N := \{\chi : \chi_1 = (\lambda + \mu)_x; \chi_2 = (\mu q)_x; \chi_3 = (\mu r)_x; \chi_4 = (\lambda - \mu)_x\} . \tag{10.8}$$

One can then easily see that the two distributions $\operatorname{Im} N$ and $\operatorname{Ker} N$ are transversal, as $\chi \in \operatorname{Ker} N \cap \operatorname{Im} N$ implies:

$$\chi_1 + \chi_4 = 0 \Rightarrow \lambda_x = 0 , \tag{10.9}$$

$$2\chi_1 + q\chi_3 + r\chi_2 = 0 \Rightarrow \mu = 0 . \tag{10.10}$$

Moreover, since we have:

$$\mathfrak{X}(S) := \{\varphi : \varphi_1 = \varphi_4 = 0\}, \tag{10.11}$$

it follows from (10.8) and (10.11) that $\mathfrak{X}(S)$ and $\operatorname{Ker} N$ are also transversal.

The reducibility conditions for N given in Sect. 6 are thus all satisfied.

To perform this reduction explicitly, one has first to determine the projection $\pi : \mathfrak{X}(S, M) \to \operatorname{Im} N$ and its restriction $\pi_s : \mathfrak{X}(S) \to \operatorname{Im} N$. Since $\operatorname{ind}(N) = 1$, any $\varphi \in \mathfrak{X}(S, M)$ can be decomposed uniquely as:

$$\varphi = \psi + \chi ; \qquad \psi \in \operatorname{Im} N , \qquad \chi \in \operatorname{Ker} N . \tag{10.12}$$

Taking into account (10.7) and (10.8), we get:

$$\begin{aligned} \psi_1 + \lambda_x + \mu_x &= \varphi_1 , \\ -\psi_1 + \lambda_x - \mu_x &= \varphi_4 , \\ \psi_2 + (\mu q)_x &= \varphi_2 , \\ \psi_3 + (\mu r)_x &= \varphi_3 , \\ q\psi_3 + r\psi_2 + 2\psi_1 &= 0 . \end{aligned} \tag{10.13}$$

From the first two equalities in (10.13) it follows:

$$2\lambda_x = \varphi_1 + \varphi_4 ; \qquad 2\psi_1 = \varphi_1 - \varphi_4 - 2\mu_x , \tag{10.14}$$

which, inserted in the last 3 equalities, yield:

$$\mu = \frac{1}{2} a \int_{-\infty}^{x} a(\varphi_1 - \varphi_4 + q\varphi_3 + r\varphi_2) dx ; \qquad a = (1 + qr)^{-1/2} . \tag{10.15}$$

Thus, the projection π is given by

$$\begin{aligned} & 2\psi_1 = \varphi_1 - \varphi_4 - \left(a \int_{-\infty}^{x} a(\varphi_1 - \varphi_4 + q\varphi_3 + r\varphi_2) dx \right)_x , \\ \pi : \varphi \to \psi : \ & 2\psi_2 = 2\varphi_2 - \left(aq \int_{-\infty}^{x} a(\varphi_1 - \varphi_4 + q\varphi_3 + r\varphi_2) dx \right)_x , \\ & 2\psi_3 = 2\varphi_3 - \left(ar \int_{-\infty}^{x} a(\varphi_1 - \varphi_4 + q\varphi_3 + r\varphi_2) dx \right)_x , \\ & 2\psi_4 = -\varphi_1 + \varphi_4 + \left(a \int_{-\infty}^{x} a(\varphi_1 - \varphi_4 + q\varphi_3 + r\varphi_2) dx \right)_x . \end{aligned} \tag{10.16}$$

Its restriction π_s [which one obtains by setting $\varphi_1 = \varphi_4 = 0$ into (10.16)] reads:

$$\begin{aligned} 2\psi_1 &= -\left(a \int_{-\infty}^{x} a(q\varphi_3 + r\varphi_2) dx \right)_x , \\ 2\psi_2 &= 2\varphi_2 - \left(aq \int_{-\infty}^{x} a(q\varphi_3 + r\varphi_2) dx \right)_x , \\ 2\psi_3 &= 2\varphi_3 - \left(ar \int_{-\infty}^{x} a(q\varphi_3 + r\varphi_2) dx \right)_x , \\ 2\psi_4 &= \left(a \int_{-\infty}^{x} a(q\varphi_3 + r\varphi_2) dx \right)_x . \end{aligned} \tag{10.17}$$

From (10.17) it follows that the inverse $\pi_s^{-1}: \mathrm{Im}N \to \mathfrak{X}(S)$ is given by:

$$\pi_S^{-1}: \psi \to \varphi_S: \quad \begin{aligned} 2\varphi_2 &= 2\psi_2 + \left(q \int_{-\infty}^{x} (q\psi_3 + r\psi_2)dx\right)_x, \\ 2\varphi_3 &= 2\psi_3 + \left(r \int_{-\infty}^{x} (q\psi_3 + r\psi_2)dx\right)_x. \end{aligned} \tag{10.18}$$

The Nijenhuis tensor N', reduced on the submanifold S, is then given by:

$$\bar{\varphi}_S = N'\varphi_S : \bar{\varphi}_S = \pi_S^{-1} N \pi_S \varphi_S = \pi_S^{-1}\left[u, \int_{-\infty}^{x} \pi_S \varphi_S dx\right]. \tag{10.19}$$

The explicit expression of N' is very involved: much simpler is the explicit expression of its inverse, which will be easily recognized to be the recursion operator associated to the WKI hierarchy.

We have:

$$\varphi_S = N'^{-1}\bar{\varphi}_S : \varphi_S = \pi_S^{-1} N_I^{-1} \pi_S \bar{\varphi}_S, \tag{10.20}$$

where N_I^{-1} is the inverse of the restriction of N on $\mathrm{Im}N$, which uniquely exists since $\mathrm{ind}(N) = 1$.

To evaluate N_I^{-1}, we notice that:

$$\bar{\psi} = N_I \psi : \quad \begin{aligned} \bar{\psi}_2 &= 2\int_{-\infty}^{x} \psi_2 + q \int_{-\infty}^{x} (q\psi_3 + r\psi_2)dx \\ \bar{\psi}_3 &= -2\int_{-\infty}^{x} \psi_3 - r \int_{-\infty}^{x} (q\psi_3 + r\psi_2)dx, \end{aligned} \tag{10.21}$$

whence it follows:

$$\begin{aligned} r\bar{\psi}_{2x} - q\bar{\psi}_{3x} &= 2(1+qr)(r\psi_2 + q\psi_3) \\ &\quad + (1+qr)_x \int_{-\infty}^{x} (r\psi_2 + q\psi_3)dx, \end{aligned} \tag{10.22}$$

and thus:

$$r\psi_2 + q\psi_3 = \left(\frac{a}{2} \int_{-\infty}^{x} a(r\bar{\psi}_{2x} - q\bar{\psi}_{3x})dx\right)_x. \tag{10.23}$$

From (10.21) and (10.23) we derive the explicit expression of N_I^{-1}, which reads:

$$\psi = N_I^{-1}\bar{\psi} : \quad \begin{aligned} 2\psi_1 &= \bar{\psi}_{2x} - \left(\frac{a}{2} q \int_{-\infty}^{x} a(r\bar{\psi}_{2x} - q\bar{\psi}_{3x})dx\right)_x \\ 2\psi_2 &= -\bar{\psi}_{3x} - \left(\frac{a}{2} r \int_{-\infty}^{x} a(r\bar{\psi}_{2x} - q\bar{\psi}_{3x})dx\right)_x, \end{aligned} \tag{10.24}$$

Finally, the reduced inverse N'^{-1} is obtained by noticing that:

$$\begin{aligned} 2\varphi_2 &\overset{(10.18)}{=} 2\psi_2 + \left(q \int_{-\infty}^{x} (q\psi_3 + r\psi_2)dx\right)_x \\ &\overset{(10.23),(10.24)}{=} \bar{\psi}_{2x} \\ &\overset{(10.17)}{=} \bar{\varphi}_{2x} - \frac{1}{2}\left(aq \int_{-\infty}^{x} a(q\bar{\varphi}_3 + r\bar{\varphi}_2)dx\right)_{xx}, \end{aligned} \tag{10.25}$$

and, analogously:

$$\begin{aligned} 2\varphi_3 &= 2\psi_3 + \left(r \int_{-\infty}^{x} (q\psi_3 + r\psi_2)dx \right)_x \\ &= -\tilde{\psi}_{3x} \qquad\qquad (10.26) \\ &= -\bar{\varphi}_{3x} + \frac{1}{2}\left(ar \int_{-\infty}^{x} a(q\bar{\varphi}_3 + r\bar{\varphi}_2)dx \right)_{xx} . \end{aligned}$$

The operator given by (10.25), (10.26) is the recursion operator for the WKI hierarchy.

11. A Further Example of Reduction of a Nijenhuis Structure: The KdV Hierarchy

As a final application of the reduction method, we will show in this section how the well-known recursion operator for the Korteweg-De Vries (KdV) hierarchy can be obtained by reducing, according to the technique displayed in Sect. 6, the Nijenhuis tensor:

$$N := PQ^{-1}, \qquad (11.1)$$

where P and Q are the Poisson tensors defined as:

$$Q\alpha := \alpha_x + [u, \alpha], \qquad (11.2)$$

$$P\alpha := [a, \alpha]. \qquad (11.3)$$

In formulas (11.2), u is again a point in the manifold M considered in the previous sections and a is the constant matrix $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. The submanifold $S \subset M$ on which we are going to reduce N is the manifold of 2×2 traceless Frobenius matrices, namely:

$$S := \left\{ u_F : \begin{pmatrix} 0 & u \\ 1 & 0 \end{pmatrix} \right\}, \qquad (11.4)$$

so that

$$\mathfrak{X}(S) = \left\{ \varphi_F : \begin{pmatrix} 0 & \varphi \\ 0 & 0 \end{pmatrix} \right\}. \qquad (11.5)$$

In contrast with the case treated in Sect. 10, we have now $\operatorname{ind}(N) = 2$ for any u_F.
In fact

$$\operatorname{Im} N = \operatorname{Im} P = \{\tilde{\psi} : \tilde{\psi}_1 + \tilde{\psi}_4 = 0, \tilde{\psi}_3 = 0\}, \qquad (11.6)$$

$$\operatorname{Ker} N = Q \operatorname{Ker} P = \{\tilde{\chi} : \tilde{\chi}_3 = 0, \tilde{\chi}_{4x} - \tilde{\chi}_{1x} = 2\tilde{\chi}_2\}, \qquad (11.7)$$

so that $\operatorname{Im} N \cap \operatorname{Ker} N \neq \emptyset$.
On the other hand, taking into account that:

$$\operatorname{Ker} N^2 = \{\chi : \chi = Q\alpha, P\alpha \in \operatorname{Ker} N\}, \qquad (11.8)$$

$$\operatorname{Im} N^2 = \{\psi : \psi = P\alpha, Q\alpha \in \operatorname{Im} N\}, \qquad (11.9)$$

one gets:

$$\operatorname{Ker} N^2 = \left\{ \chi : 4\chi_2 + 2\chi_{1x} - 2\chi_{4x} - \chi_{3xx} = 0 \right\}, \qquad (11.10)$$

$$\operatorname{Im} N^2 = \{\psi : \psi_1 + \psi_4 = 0, \psi_3 = 0, \psi_2 = \psi_{1x}\}, \qquad (11.11)$$

which easily yield $\operatorname{Im} N^2 \oplus \operatorname{Ker} N^2 = \mathfrak{X}(S, M)$. Thus, denoting by φ any arbitrary element of $\mathfrak{X}(S, M)$, the decomposition $\varphi = \psi + \chi$ implies:

$$\begin{aligned}
&\varphi_1 = \psi_1 + \chi_1 , \\
&\varphi_2 = \psi_{1x} + \tfrac{1}{4}\chi_{3xx} + \tfrac{1}{2}(\chi_4 - \chi_1)_x , \\
&\varphi_3 = \chi_3 , \\
&\varphi_4 = -\psi_1 + \chi_4 .
\end{aligned} \tag{11.12}$$

From (11.12) it follows that:

$$2\psi_{1x} = -\tfrac{1}{4}\varphi_{3xx} + \varphi_2 - \tfrac{1}{2}(\varphi_4 - \varphi_1)_x , \tag{11.13}$$

the projection $\pi : \mathfrak{X}(S, M) \to \operatorname{Im} N^2$ is then given by:

$$\begin{aligned}
\pi : \varphi \to \psi : \quad &\psi_1 = -\frac{1}{8}\varphi_{3x} + \frac{1}{2}\int_{-\infty}^{x} \varphi_2 dx - \frac{1}{4}(\varphi_4 - \varphi_1) \\
&\psi_2 = -\frac{1}{8}\varphi_{3xx} + \frac{1}{2}\varphi_2 - \frac{1}{4}(\varphi_4 - \varphi_1)_x , \\
&\psi_3 = 0 , \\
&\psi_4 = -\psi_1 = \frac{1}{8}\varphi_{3x} - \frac{1}{2}\int_{-\infty}^{x} \varphi_2 dx + \frac{1}{4}(\varphi_4 - \varphi_1) ,
\end{aligned} \tag{11.14}$$

the restriction $\pi_s : \mathfrak{X}(S) \to \operatorname{Im} N^2$ reads:

$$\pi_S : \varphi_F \to \psi : \psi = \frac{1}{2}\begin{pmatrix} \int_{-\infty}^{x} \varphi dx & \varphi \\ 0 & -\int_{-\infty}^{x} \varphi dx \end{pmatrix} , \tag{11.15}$$

so that $\pi_s^{-1} : \operatorname{Im} N^2 \to \mathfrak{X}(S)$ is given by:

$$\pi_S^{-1} : \psi \to \varphi_F : \varphi_F = \begin{pmatrix} 0 & 2\psi_{1x} \\ 0 & 0 \end{pmatrix} . \tag{11.16}$$

To conclude, the Nijenhuis tensor N', reduced on the submanifold of 2×2 Frobenius matrices (11.4) and defined as:

$$\bar{\varphi}_F = N'\varphi_F : \bar{\varphi}_F = \pi_S^{-1} P Q^{-1} \pi_S \varphi_F \tag{11.17}$$

is obtained by eliminating α from the equations:

$$\bar{\varphi}_F = \pi_S^{-1} P\alpha , \tag{11.18a}$$

$$\pi_S \varphi_F = Q\alpha . \tag{11.18b}$$

From (11.18a) we get:

$$\bar{\varphi} = 2(\alpha_4 - \alpha_1) , \tag{11.19a}$$

Scheme

Hierarchy of NEE's	Integrability structure	Reduction technique	Reduction submanifold	"Recursion operator" of the hierarchy
AKNS	$P\alpha=[a,\alpha]$ $Q\alpha=\alpha_x+[u,\alpha]$ $N\varphi=P\cdot Q^{-1}\varphi$	Geometric (Sect. 5)	$S:=\{u:u_D=0\}$	$\varphi_1=\frac{1}{2}\bar{\varphi}_{1x}-q\int_{-\infty}^{x}(q\bar{\varphi}_2+r\bar{\varphi}_1)dx$ $\varphi_2=-\frac{1}{2}\bar{\varphi}_{2x}+r\int_{-\infty}^{x}(q\bar{\varphi}_2+r\bar{\varphi}_1)dx$ Eqs. (7.20–21)
HSC	$P\alpha=\alpha_x$ $N\varphi=\left[u,\int_{-\infty}^{x}\varphi dx\right]$	Geometric (Sect. 5)	$S^2:=\{\mathbf{q}\in\mathbb{R}^3:\lvert\mathbf{q}\rvert=1\}$	$\boldsymbol{\psi}=-\frac{1}{2i}\left[\boldsymbol{\psi}'\wedge\mathbf{q}-\mathbf{q}\left(\int_{-\infty}^{x}(\boldsymbol{\psi}'\wedge\mathbf{q})_x\cdot\mathbf{q}\,dx\right)\right]_x$ Eq. (8.20)
KN	$P\alpha=\alpha_x+[u,\alpha]$ $\Omega\varphi=\int_{-\infty}^{x}\varphi dx$ $N\varphi=\varphi+\left[u,\int_{-\infty}^{x}\varphi dx\right]$	Geometric (Sect. 5)	$S:=\{u:u_D=\sigma_3\}$	$\varphi_1=\frac{1}{2}\left[\bar{\varphi}_1+q\int_{-\infty}^{x}(r\bar{\varphi}_1+q\bar{\varphi}_2)dx\right]_x$ $\varphi_2=-\frac{1}{2}\left[\bar{\varphi}_2-r\int_{-\infty}^{x}(r\bar{\varphi}_1+q\bar{\varphi}_2)dx\right]_x$ Eq. (9.23)
WKI	$N\varphi=\left[u,\int_{-\infty}^{x}\varphi dx\right]$	Algebraic (Sect. 6)	$S:=\{u:u_D=\sigma_3\}$	$\varphi_2=\frac{1}{2}\bar{\varphi}_{2x}-\frac{1}{4}\left[aq\int_{-\infty}^{x}a(q\bar{\varphi}_3+r\bar{\varphi}_2)dx\right]_{xx}$ $\varphi_3=-\frac{1}{2}\bar{\varphi}_{3x}+\frac{1}{4}\left[ar\int_{-\infty}^{x}a(q\bar{\varphi}_3+r\bar{\varphi}_2)dx\right]_{xx}$ (Eqs. (10.25–26))
KdV	$P\alpha=[a,\alpha]$ $Q\alpha=\alpha_x+[u,\alpha]$ $N\varphi=P\cdot Q^{-1}\varphi$	Algebraic (Sect. 6)	$S:=\left\{u_F:\begin{pmatrix}0&u\\1&0\end{pmatrix}\right\}$	$\varphi=-\frac{1}{4}\bar{\varphi}_{xx}+u\bar{\varphi}+\frac{1}{2}u_x\int_{-\infty}^{x}\bar{\varphi}dx$ Eq. (11.21)

while (11.18b) yields:

$$\int_{-\infty}^{x} \varphi dx = 2[\alpha_{1x} + u\alpha_3 - \alpha_2],$$

$$\varphi = 2[\alpha_{2x} + u(\alpha_4 - \alpha_1)] \tag{11.19b}$$

$$0 = \alpha_{3x} + (\alpha_1 - \alpha_4) - \int_{-\infty}^{x} \varphi dx = 2[\alpha_{4x} + \alpha_2 - u\alpha_3].$$

From Eqs. (11.19b) it follows:

$$\varphi = -\frac{1}{2}(\alpha_4 - \alpha_1)_{xx} + 2u(\alpha_4 - \alpha_1) + u_x \int_{-\infty}^{x} (\alpha_4 - \alpha_1) dx, \tag{11.20}$$

and thus, using (11.19a):

$$\varphi = N'^{-1}\bar{\varphi} = -\frac{1}{4}\bar{\varphi}_{xx} + u\bar{\varphi} + \frac{1}{2}u_x \int_{-\infty}^{x} \bar{\varphi} dx, \tag{11.21}$$

which is the well-known recursion operator of the KdV hierarchy.

For a more general discussion, explaining in particular the origin of the submanifold S of traceless Frobenius matrices, the reader is referred to [6].

In the previous scheme, the examples considered in Sects. 7–11 are briefly summarized. The reduction technique described in Sect. 5 and based on the restriction Theorem 4.2 is called "geometric", while the reduction for Nijenhuis manifolds described in Sect. 6 is denoted as "algebraic". In the last column ("recursion operators") we give the forms of the reduced Nijenhuis tensors as they usually appear in the literature.

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