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Recurrence of Random Walks in the Ising Spins

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Abstract. Consider the 1/2-Ising model in Z^2. Let σ_j be the spin at the site $(j,0)\in Z^2$ $(j=0, \pm 1, \pm 2, \ldots)$. Let $\{X_n\}_{n=0}^{+\infty}$ be a random walk with the random transition probabilities such that

$$P(X_{n+1}=j\pm 1|X_n=j)=p_j^{\pm}\equiv 1/2\pm\nu(\sigma_j-\mu)/2.$$

We show a case where $E[p_j^+]\gtreqless E[p_j^-]$, but $\lim_{n\to\infty} X_n=-\infty$ a.s. or X_n is recurrent a.s.

Let $\{\sigma_j\}_{j=-\infty}^{+\infty}$ be an ergodic random sequence of ± 1 spins with the mean $E[\sigma_j]=m$. Considering $-\sigma_j$ if $m<0$, we may assume $0\leqq m<1$. Let $\{X_n\}_{n=0}^{+\infty}$ be a random walk with random transition probabilities such that

$$P(X_{n+1}=j+1|X_n=j)=p_j^+\equiv 1/2+\nu(\sigma_j-\mu)/2,$$
$$P(X_{n+1}=j-1|X_n=j)=p_j^-\equiv 1/2-\nu(\sigma_j-\mu)/2,$$

where ν and μ are constants with

$$|\nu|(1+|\mu|)<1.$$

We are interested in the recurrence of the random walk $\{X_n\}_{n=0}^{+\infty}$. Since the recurrence is trivial if $\nu=0$, let us assume $\nu\neq 0$. We apply Chung's results, which are summarized in the following

Lemma 1 (Sect. 12, Part I in [1]). *Let $\{X_n\}_{n=0}^{+\infty}$ be a random walk with non-random positive transition probabilities $p_j^{\pm}$ $(p_j^+ + p_j^- = 1)$ which depend on j, i.e.,*

$$P(X_{n+1}=j\pm 1|X_n=j)=p_j^{\pm}.$$

i) *If* $\sum_{r=1}^{+\infty} p_1^- p_2^- \ldots p_r^-/(p_1^+ p_2^+ \ldots p_r^+) = \sum_{r=-\infty}^{0} p_r^+ p_{r+1}^+ \ldots p_0^+/(p_r^- p_{r+1}^- \ldots p_0^-) = +\infty$, *then $\{X_n\}_{n=0}^{+\infty}$ is recurrent a.s.*

ii) *If* $\sum_{r=1}^{+\infty} p_1^- p_2^- \dots p_r^- / (p_1^+ p_2^+ \dots p_r^+) < +\infty$ *and*
$\sum_{r=-\infty}^{0} p_r^+ p_{r+1}^+ \dots p_0^+ / (p_r^- p_{r+1}^- \dots p_0^-) = +\infty$, *then* $\lim_{n\to+\infty} X_n = +\infty$ *a.s.*

iii) *If* $\sum_{r=1}^{+\infty} p_1^- p_2^- \dots p_r^- / (p_1^+ p_2^+ \dots p_r^+) = +\infty$ *and*
$\sum_{r=-\infty}^{0} p_r^+ p_{r+1}^+ \dots p_0^+ / (p_r^- p_{r+1}^- \dots p_0^-) < +\infty$, *then* $\lim_{n\to+\infty} X_n = -\infty$ *a.s.*

In a case when transition probabilities are random as in ours, Lemma 1 shows that the condition

$$E[\log(p_j^-/p_j^+)] = 0 \tag{1}$$

is critical [5, 6]. If $p_j^{\pm} = 1/2 \pm \nu(\sigma_j - \mu)/2$, it is easy to see that

$$\begin{aligned} p_j^-/p_j^+ &= A_\nu(\mu)^{1/2} B_\nu(\mu)^{-\sigma_j/2} \\ &= \exp[-\{\sigma_j - \log A_\nu(\mu)/\log B_\nu(\mu)\} \log B_\nu(\mu)/2], \end{aligned}$$

where

$$\begin{aligned} A_\nu(\mu) &= \{(1+\nu\mu)^2 - \nu^2\}/\{(1-\nu\mu)^2 - \nu^2\}, \\ B_\nu(\mu) &= \{(1+\nu)^2 - \nu^2\mu^2\}/\{1-\nu)^2 - \nu^2\mu^2\}. \end{aligned}$$

Concerning condition (1), we have

Lemma 2. *The equation for* μ

$$A_\nu(\mu) = B_\nu(\mu)^m, \tag{2}$$

which is equivalent to (1), *has a unique solution* $\mu = \mu_\nu(m)$ *in an interval* $(-|\nu|^{-1}+1, |\nu|^{-1}-1)$.

For this $\mu_\nu(m)$, *it holds that*
$\mu_\nu(m) = \mu_{-\nu}(m)$,
$\mu_\nu(0) = 0$,
$0 < \mu_\nu(m) < m$, *if* $m > 0$,
$\mu_\nu(m)$ *is strictly monotone increasing in m.*

We say that the sequence $\{\sigma_j\}_{j=-\infty}^{+\infty}$ *generates weakly recurrent partial sums*, if almost surely

$$\underline{\lim}_{n\to+\infty} \sum_{j=1}^{n} (\sigma_j - m), \quad \underline{\lim}_{n\to-\infty} \sum_{j=n}^{0} (\sigma_j - m) \lneqq \infty,$$
$$\overline{\lim}_{n\to+\infty} \sum_{j=1}^{n} (\sigma_j - m), \quad \overline{\lim}_{n\to-\infty} \sum_{j=n}^{0} (\sigma_j - m) \gneqq -\infty.$$

Our aim is to prove the following

Theorem. *Assume that* $\{\sigma_j\}_{j=-\infty}^{+\infty}$ *generates weakly recurrent partial sums.*

i) *If* $1 - |\nu|^{-1} < \mu < \mu_\nu(m)$, *then* $\lim_{n\to+\infty} X_n = (\operatorname{sgn}\nu)\infty$ *a.s.*

ii) *If* $\mu = \mu_\nu(m)$, *then* X_n *is recurrent a.s.*

iii) *If* $\mu_\nu(m) < \mu < |\nu|^{-1} - 1$, *then* $\lim_{n\to+\infty} X_n = -(\operatorname{sgn}\nu)\infty$ *a.s.*

Remark. Assume $m>0$, $\nu>0$. Then, $\mu_\nu(m)\leqq\mu<m$ implies $E[p_j^+]>E[p_j^-]$, i.e., the probability p_j^+ that X_n steps to the right is greater in the mean than the probability p_j^- to the left. But, our Theorem says that in this case X_n is recurrent or $\lim_{n\to+\infty} X_n=-\infty$ according as $\mu=\mu_\nu(m)$ or $\mu_\nu(m)<\mu<m$.

Let σ_j be the spin at $(j,0)\in Z^d$ in the ferromagnetic Ising model in Z^2 with the nearest neighbour interactions. Let the probability measure P be the limiting Gibbs distribution with the $+$ boundary conditions. Then, all the assumptions on $\{\sigma_j\}_{j=-\infty}^{+\infty}$ in our Theorem are satisfied by this $\{\sigma_j\}_{j=-\infty}^{+\infty}$, i.e., we have

Proposition. *The sequence of the Ising spins $\{\sigma_j\}_{j=-\infty}^{+\infty}$ stated above generates weakly recurrent partial sums.*

Let us prove our results. At first we carry out

Proof of Lemma 2. In case $m=0$, $\mu=0$ is the unique solution of (2). Since $A_{-\nu}(\mu)=A_\nu(\mu)^{-1}$, and $B_{-\nu}(\mu)=B_\nu(\mu)^{-1}$, we may assume $\nu>0$ and $m>0$.

Put

$$\begin{aligned}F_\nu(\mu)&=\{(1-\nu)^2-\nu^2\mu^2\}^m/\{(1-\nu\mu)^2-\nu^2\}\\&\quad-\{(1+\nu)^2-\nu^2\mu^2\}^m/\{(1+\nu\mu)^2-\nu^2\}\\&=(1-\nu+\nu\mu)^m/\{(1-\nu-\nu\mu)^{1-m}(1-\nu\mu+\nu)\}\\&\quad-\{(1+\nu)^2-\nu^2\mu^2\}^m/\{(1+\nu\mu)^2-\nu^2\}\,.\end{aligned}$$

Equation (2) is equivalent to $F_\nu(\mu)=0$. It is easy to see that $F_\nu(\mu)$ is monotone increasing in $\mu\in[0,\nu^{-1}-1)$ and that

$$F_\nu(0)<0\,,\qquad F(\nu^{-1}-1-0)=+\infty\,.$$

Therefore, (2) has a unique solution in $(0,\nu^{-1}-1)$. Since $A_\nu(\mu)<1$ and $B_\nu(\mu)>1$ for $\mu<0$, (2) has no negative solution.

From $A_{-\nu}(\mu)=A_\nu(\mu)^{-1}$ and $B_{-\nu}(\mu)=B_\nu(\mu)^{-1}$, it follows that $\mu_\nu(m)=\mu_{-\nu}(m)$. Differentiating $\log A_\nu(\mu_\nu(m))-m\log B_\nu(\mu_\nu(m))\equiv0$ in m, we have

$$4\nu\{1-\nu^2(\mu^2+2m\mu+1)\}\mu'/[\{(1+\nu\mu)^2-\nu^2\}\{(1-\nu\mu)^2-\nu^2\}]=\log B_\nu(\mu)\,.$$

Since $\nu(1+|\mu|)<1$ and $\nu\log B_\nu(\mu)>0$, we have $\dfrac{d\mu_\nu(m)}{dm}>0$, i.e., $\mu_\nu(m)$ is strictly monotone increasing in m.

Let us prove $\mu_\nu(m)<m$ for $0<m<1$. If $m\geqq\nu^{-1}-1$, then $\mu_\nu(m)<\nu^{-1}-1\leqq m$. Assume $m<\nu^{-1}-1$, i.e., $\nu<(m+1)^{-1}$. Let us introduce a function

$$G(\nu)=\log\{A_\nu(m)B_\nu(m)^{-m}\}\quad\text{for}\quad 0<\nu<(m+1)^{-1}\,.$$

We have

$$\frac{dG(\nu)}{d\nu}=8m(1-m^2)\nu^2/[\{(1+\nu m)^2-\nu^2\}\{(1-\nu m)^2-\nu^2\}]>0\,,$$

$$(0<\nu<(m+1)^{-1})\,.$$

On the other hand, $G(0)=0$, hence $G(\nu)>0$, i.e., $A_\nu(m)>B_\nu(m)^m$. Therefore, $F_\nu(m)>0$. Since $F_\nu(\mu)$ is monotone increasing, we have $\mu_\nu(m)<m$.

Proof of Theorem. Assume $\nu > 0$.

1) Let $\mu \neq \mu_\nu(m)$. We have

$$p_1^- p_2^- \cdots p_r^- / (p_1^+ p_2^+ \cdots p_r^+) \\ = \exp[-\{\log B_\nu(\mu)/2\} \sum_{j=1}^{r} \{\sigma_j - \log A_\nu(\mu)/\log B_\nu(\mu)\}].$$

Since $\sum_{j=1}^{r} \{\sigma_j - \log A_\nu(\mu)/\log B_\nu(\mu)\} \sim r\{m - \log A_\nu(\mu)/\log B_\nu(\mu)\}$ as $r \to +\infty$ by the point-wise ergodic theorem, we have

$$\sum_{r=1}^{\infty} p_1^- p_2^- \cdots p_r^- / (p_1^+ p_2^+ \cdots p_r^+) \begin{cases} < +\infty, & \text{if } m - \log A_\nu(\mu)/\log B_\nu(\mu) > 0, \\ = +\infty, & \text{if } m - \log A_\nu(\mu)/\log B_\nu(\mu) < 0. \end{cases}$$

Our results in case $\mu \neq \mu_\nu(m)$ follow from Lemma 1.

2) Let $\mu = \mu_\nu(m)$. We have

$$p_1^- p_2^- \cdots p_r^- / (p_1^+ p_2^+ \cdots p_r^+) = \exp\left[-\{\log B_\nu(\mu)/2\} \sum_{j=1}^{r} (\sigma_j - m) \right].$$

Since $\{\sigma_j\}$ generates weakly recurrent partial sums, the sequence $\sum_{j=1}^{r} (\sigma_j - m)$ hits a bounded set infinitely often as $r \to +\infty$. Therefore

$$\sum_{r=1}^{+\infty} p_1^- p_2^- \cdots p_r^- / (p_1^+ p_2^+ \cdots p_r^+) = +\infty \quad \text{a.s.}$$

Our results also follow from Lemma 1.

Let us proceed to the

Proof of Proposition. Let β and h be the reciprocal temperature and the external field, respectively.

1) Case $\beta \leqq \beta_c$ and $h = 0$. In this case, $m = 0$. Suppose

$$p\left(\varliminf_{n \to +\infty} \sum_{j=1}^{n} \sigma_j = +\infty \right) > 0.$$

Since $\left\{ \varliminf_{n \to +\infty} \sum_{j=1}^{n} \sigma_j = +\infty \right\}$ is a tail event, $P\left(\varliminf_{n \to +\infty} \sum_{j=1}^{r} \sigma_j = +\infty \right) = 1$ [4]. It is well known that the Gibbs measure P is invariant under the transformation $\sigma_x \to -\sigma_x$ $(x \in Z^2)$. Therefore, $P\left(\varlimsup_{n \to +\infty} \sum_{j=1}^{n} \sigma_j = -\infty \right) = 1$, which is a contradiction. Hence, $\varliminf_{n \to +\infty} \sum_{j=1}^{n} \sigma_j < +\infty$ a.s.

2) Case $\beta > \beta_c$ or $h \neq 0$. Since the correlations decay exponentially in this case, condition (3) in the following Lemma 3 holds for $\{\sigma_n\}$ in place of $\{\xi_n\}$ ([2]). Therefore, our result in Proposition is a corollary to

Lemma 3. *Let $\{\xi_n\}_{n=-\infty}^{+\infty}$ be a stationary sequence of bounded random variables with $E[\xi_n]=0$. For $n<m$, let $\mathscr{B}_n^m$ be the σ-algebra generated by $\{\xi_j; n\leqq j\leqq m\}$. Put*

$$\alpha(n)=\sup\{|P(A\cap B)-P(A)P(B)|; A\in\mathscr{B}_{-\infty}^0, B\in\mathscr{B}_n^{+\infty}\}.$$

If

$$\sum_{n=1}^{+\infty}\alpha(n)<+\infty, \tag{3}$$

then $\overline{\lim_{n\to+\infty}}\sum_{j=1}^{n}\xi_j=+\infty$ *and* $\underline{\lim_{n\to+\infty}}\sum_{j=1}^{n}\xi_j=-\infty$ a.s.

Proof. The central limit theorem holds for this $\{\xi_n\}$ [3], i.e.,

$$\lim_{n\to+\infty}P\left(\sum_{j=1}^{n}\xi_j/(\sigma\sqrt{n})>z\right)=1/\sqrt{2\pi}\int_z^{+\infty}e^{-x^2/2}dx,$$

where $\sigma^2=E[\xi_0^2]+2\sum_{j=1}^{+\infty}E[\xi_0\xi_j]<+\infty$. Putting $z=1/\sigma$, we can find $N_0\geqq 1$ such that for any $n\geqq N_0$

$$P\left(\sum_{j=1}^{n}\xi_j>\sqrt{n}\right)\geqq\frac{1}{2}\frac{1}{\sqrt{2\pi}}\int_{1/\sigma}^{+\infty}e^{-x^2/2}dx\equiv\delta.$$

Let $c\geqq 1$ be a constant such that $|\xi_n|\leqq c$. Put $n_1=N_0$. A sequence $\{n_k, m_k\}_{k=1}^{+\infty}$ is defined recursively in the following way;

$$\begin{cases} m_k=n_k+k, \\ n_{k+1}=m_k+(cm_k+k)^2. \end{cases}$$

Remark that $\sqrt{n_{k+1}-m_k}=cm_k+k\geqq N_0$. Put

$$E_k=\left\{\sum_{j=m_k+1}^{n_{k+1}}\xi_j>\sqrt{n_{k+1}-m_k}\right\}.$$

We have

$$\begin{aligned}P\left(\bigcap_{k=K}^{+\infty}E_k^c\right)&=P\left(E_K^c\cap\bigcap_{k=K+1}^{+\infty}E_k^c\right)\\&\leqq P(E_K^c)P\left(\sum_{k=K+1}^{+\infty}E_k^c\right)+\alpha(K+1)\\&\leqq(1-\delta)P\left(\bigcap_{k=K+1}^{+\infty}E_k^c\right)+\alpha(K+1).\end{aligned}$$

Letting $K\to+\infty$, we have $\lim_{K\to+\infty}P\left(\bigcap_{k=K}^{+\infty}E_k^c\right)\leqq(1-\delta)\lim_{K\to+\infty}P\left(\bigcap_{k=K}^{+\infty}E_k^c\right)$, hence

$$P\left(\bigcup_{K=1}^{+\infty}\bigcap_{k=K}^{+\infty}E_k^c\right)=\lim_{K\to+\infty}P\left(\bigcap_{k=K}^{+\infty}E_k^c\right)=0.$$

Therefore, $P\left(\bigcap_{K=1}^{+\infty}\bigcup_{k=K}^{+\infty}E_k\right)=1$, i.e., infinitely many E_k's occur a.s. If E_k occurs, then

$$\sum_{j=1}^{n_{k+1}}\xi_j=\sum_{j=1}^{m_k}\xi_j+\sum_{j=m_k+1}^{n_{k+1}}\xi_j\geqq -cm_k+\sqrt{n_{k+1}-m_k}=k.$$

Thus, $\overline{\lim_{n\to+\infty}}\sum_{j=1}^{n}\xi_j=+\infty$ *a.s.*

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