Commun. Math. Phys. 93, 417–436 (1984)

Communications in Mathematical Physics
© Springer-Verlag 1984

# Integrable Euler Equations on SO(4) and their Physical Applications

O. I. Bogoyavlensky

V. A. Steklov Mathematical Institute, The Academy of Sciences of the USSR, SU-117 333 Moscow, USSR

**Abstract.** For the Lie algebra SO(4) (and other six dimensional Lie algebras) we find some Euler's equations which have an additional fourth order integral and are algebraically integrable (in terms of elliptic functions) in a one parameter set of orbits. Integrable Euler's equations having an additional second order integral and generalizing Steklov's case are presented. Equations for rotation of a rigid body having $n$ ellipsoid cavities filled with the ideal incompressible fluid being in a state of homogeneous vortex motion are derived. It is shown that the obtained equations are Euler's equations for the Lie algebra of the group $G_{n+1}=\mathrm{SO}(3)\times \ldots \times \mathrm{SO}(3)$. New physical applications of Euler's equations on SO(4) are discussed.

## 1. Introduction and Summary

We consider two classes of six-dimensional Lie algebras $L$, which are specified by the following commutation relations, that are written down in terms of a basis $X_i$, $Y_k$ $(i,j,k=1,2,3)$, in the first class, $A$,

$$[X_i, X_j]=\varepsilon_{ijk}n_kX_k\,, \qquad [X_i, Y_j]=\varepsilon_{ijk}n_kY_k\,, \qquad [Y_i, Y_j]=\varepsilon_{ijk}n_k\kappa X_k\,, \tag{1.1}$$

and in the second class, $B$

$$[X_i, X_j]=\varepsilon_{ijk}n_kX_k\,, \qquad [X_i, Y_j]=0\,, \qquad [Y_i, Y_j]=\varepsilon_{ijk}m_kY_k\,. \tag{1.2}$$

Here $\varepsilon_{ijk}$ is the totally skew-symmetric tensor, and $n_k$, $m_k$, $\kappa$ are structure constants. The following Lie algebras belong to class $A$: SO(4) $(n_i=1,\ \kappa=1)$, SO(3,1) $(n_1=n_2=1,\ n_3=-1,\ \kappa=-1)$, SO(2,2) $(n_1=n_2=1,\ n_3=-1,\ \kappa=1)$, $E_3$ $(n_i=1,\ \kappa=0)$, $L_3$ $(n_1=n_2=1,\ n_3=-1,\ \kappa=0)$ etc. The Lie algebras $E_3$ and $L_3$ are those corresponding to the groups of motion of the three-dimensional Euclidean and pseudo-Euclidean spaces, respectively. The Lie algebras belonging to class $B$ are $\mathrm{SO}(4)=\mathrm{SO}(3)+\mathrm{SO}(3)$ $(n_i=1,\ m_i=1)$, $\mathrm{SL}(2,R)+\mathrm{SL}(2,R)$ $(n_1=m_1=n_2=m_2=1,\ n_3=m_3=-1)$, $\mathrm{SO}(3)+\mathrm{SL}(2,R)$ $(n_i=1,\ m_1=m_2=1,\ m_3=-1)$ etc.

Let $X_i^*$, $Y_i^*$ be the basis in the co-space $L^*$ to the Lie algebra that is dual to the basis $X_i$, $Y_i$. Vector elements of the space $L^*$ are represented in terms of their components with respect to this basis, $l(t)=\sum M_i(t)X_i^*+K_i(t)Y_i^*)$. Euler's equations are defined in the space $L^*$; in class $A$ they are of the form

$$\begin{gathered}\dot{\mathbf{M}}=\bar{\mathbf{M}}\times\boldsymbol{\omega}+\bar{\mathbf{K}}\times\mathbf{u}, \qquad \dot{\mathbf{K}}=\bar{\mathbf{K}}\times\boldsymbol{\omega}+\kappa\bar{\mathbf{M}}\times\mathbf{u} \\ \omega_i=\partial H/\partial M_i, \qquad u_i=\partial H/\partial K_i, \qquad \bar{M}_i=n_iM_i, \qquad \bar{K}_i=n_iK_i\end{gathered} \tag{1.3}$$

where $\mathbf{X}\times\mathbf{Y}$ stands for the vector product of three-dimensional vectors. For class $B$ Lie algebras Euler's equations are

$$\begin{gathered}\dot{\mathbf{M}}=\bar{\mathbf{M}}\times\mathbf{A}, \qquad \dot{\mathbf{K}}=\bar{\mathbf{K}}\times\mathbf{B}, \\ A_i=\partial H/\partial M_i, \qquad B_i=\partial H/\partial K_i, \qquad \bar{M}_i=n_iM_i, \qquad \bar{K}_i=m_iK_i.\end{gathered} \tag{1.4}$$

Euler's equations (1.3), (1.4) have the following first integrals:

$$\begin{gathered}A: J_1=H, \qquad J_2=\sum_{i=1}^{3}(\kappa n_iM_i^2+n_iK_i^2), \qquad J_3=\sum_{i=1}^{3}n_iM_iK_i, \\ B: J_1=H, \qquad J_2=\sum_{i=1}^{3}n_iM_i^2, \qquad J_3=\sum_{i=1}^{3}m_iK_i^2.\end{gathered} \tag{1.5}$$

Let $G$ be the Lie group associated with the Lie algebra $L$, and $\mathcal{O}$ be the orbits of the co-adjoint representation of the group $G$ in space $L^*$. The orbits $\mathcal{O}$ are determined by the conditions $J_2=c_2$, $J_3=c_3$; they are four-dimensional symplectic manifolds which are invariant with respect to Euler's equations (1.3), (1.4). On the orbits $\mathcal{O}$, the systems of Eqs. (1.3), (1.4) are Hamiltonian systems with a Hamiltonian $H(M_i, K_i)$. In the following we will consider Hamiltonians of the type

$$H=\tfrac{1}{2}\sum_{i=1}^{3}(a_iM_i^2+2c_iM_iK_i+b_iK_i^2+2r_iK_i+2q_iM_i). \tag{1.6}$$

Euler's equations (1.3) for the Lie algebra $E_3$ ($n_i=1$, $\kappa=0$) coincide with the Kirchhoff equations describing the motion of a rigid body in the ideal incompressible fluid; a particular case of these equations is that describing rotations of a massive rigid body with a fixed point [5]. Classical cases of integrability are known for these equations; they were found by Steklov [1], Lyapunov [2], Kowalewski [3], and Chaplygin [4].

According to Liouville, it is sufficient to find an additional first integral $J_4$ to make sure that the Hamiltonian system (1.3) [or the system (1.4)] on the orbits $\mathcal{O}$ is integrable. The method which is used in the present work to construct the first integral $J_4$ is to find conditions for the coefficients present in the Hamiltonian $H$, under which the system of Eq. (1.3) [respectively, (1.4)] implies the pair of equations

$$\dot{z}_+=w_+z_++v_+J_3, \qquad \dot{z}_-=w_-z_-+v_-J_3, \tag{1.7}$$

where the functions $z_+$ and $z_-$ are (complex-valued) polynomials of first or second order, and $w_+=-w_-$. If Eqs. (1.7) hold, the system (1.3) has an additional first integral at the level $J_3=0$ [that is the one-parameter subset of orbits $\mathcal{O}$ ($J_2=c_2$, $J_3=0$)]; the integral is the second- or fourth-order polynomial, $J_4=z_+z_-$. If Eqs. (1.7) hold with $v_+=v_-=0$, the system (1.3) has the first integral $J_4$ in the whole space $L^*$. This construction enables one to obtain all the classical integrable cases mentioned [1–4] for Euler's equations in the Lie algebra $E_3$, as well as their extensions to other Lie algebras, having the commutation relations (1.1) or (1.2).

In Sect. 2 we show that there are two families of Euler's equations (1.3) in the class $A$ Lie algebras [in particular, SO(4)], which are integrable algebraically at the level $J_3=0$. The families are fixed by the conditions

$$\kappa b_1/n_1 = 2a_2/n_2 - a_3/n_3\,, \qquad \kappa b_2/n_2 = 2a_1/n_1 - a_3/n_3\,, \qquad 2b_3/n_3 = b_1/n_1 + b_2/n_2\,, \tag{1.8}$$

$$2\kappa b_i/n_i = a_j/n_j + a_k/n_k\,, \qquad i,j,k=1,2,3\,, \tag{1.9}$$

and $c_i=r_i=q_i=0$. The corresponding additional first integral $J_4$ is a fourth-order polynomial, and at the level $J_3=0$ Euler's equations are integrated explicitly (in Sect. 3) in terms of elliptic functions [for the Lie algebra SO(4)]. Note that, as was shown in [6], the algebraic integrability of Euler's equations in SO(4) (for $c_i=r_i=q_i=0$) is possible in the whole space $L^*$ only under certain special conditions [7]. Meanwhile, new integrable cases, given in Eqs. (1.8), (1.9), do not satisfy the conditions formulated in [7], so they are integrable at the level $J_3=0$ only. The integrable case (1.8) is reduced to the classical system found by Chaplygin [4] as $n_i=1$, $\kappa\to 0$. An extension of the system (1.8) with $r_i\neq 0$ and $a_1=a_2$ is reduced to the case of Kowalewski [3] as $n_i=1$, $\kappa\to 0$. The family of Euler's equations which is specified by Eqs. (1.9) has no classical analogue.

A particular case of Euler's equations for the Lie algebra SO(4) = SO(3) + SO(3) is the system describing rotation of a rigid body with an ellipsoidal cavity filled with an ideal incompressible fluid which is in homogeneous vortex motion [8–10]. These equations were investigated by Steklov [11] as a model for earth rotation; Steklov mentioned integrable cases corresponding to an additional quadratic first integral $J_4$. From the modern point of view, those integrable cases [11] constitute a three-parameter family of integrable Euler's equations for the Lie algebra SO(4) (note that the parameter $M$ in [11, Sects. 3 and 42], is inessential, as it can be excluded by means of a scale transformation applied to parameters $A$, $B$, $C$, $a$, $b$, $c$, which are connected by three equalities in Sect. 42). In Sect. 4 of the present work we present a wider six-parameter family of integrable Euler's equations for SO(4), as well as some of its extensions corresponding to nonzero linear terms in the Hamiltonian $H$. In Sect. 5 new physical applications of the integrable cases for the Lie algebra SO(4) are considered.

Results of the present work have been announced in our previous publications [12, 13], and were presented to the International Congress of Mathematicians (Warsaw, 1982) [14]. The author learned at the Congress that van Moerbeke had recently found conditions under which some Euler equations for the Lie algebra SO(4) are integrable. However, the corresponding first integrals were not given in van Moerbeke's report [15]. The conditions given in [15] do not hold for the integrable cases which are constructed in the present work.

## 2. Fourth-Order First Integrals $J_4$

Suppose we have Euler's equations (1.3) for the class $A$ Lie algebras, and some coefficients in the Hamiltonian $H$, Eq. (1.6), are zero, namely $c_i=q_i=0$. Let us consider solutions of Eqs. (1.7), for which the functions presented there are of the form

$$\begin{gathered} z_\varepsilon=\alpha_1 M_1^2+2\varepsilon\alpha_3 M_1 M_2+\alpha_2 M_2^2+\beta K_3^2+2\gamma_1 K_1+2\gamma_2 K_2+\sigma\,, \\ w_\varepsilon=2\varepsilon x M_3\,, \qquad v_\varepsilon=2\varepsilon y K_3\,, \end{gathered} \tag{2.1}$$

where $\varepsilon=\pm 1$.

Putting the time derivative of $z_\varepsilon$, as given in Eq. (2.1), into Eq. (1.7) and taking into account the dynamical equations (1.3) we get identities for polynomials of $M_i, K_i$, which are equivalent to the following system of algebraic relations for the coefficients

$$\begin{gathered} \alpha_3(n_3a_1-n_1a_3)=x\alpha_1\,, \qquad \alpha_3(n_2a_3-n_3a_2)=x\alpha_2\,, \\ \alpha_1(n_2a_3-n_3a_2)+\alpha_2(n_3a_1-n_1a_3)=2x\alpha_3\,, \qquad x\beta=-yn_3\,, \\ \alpha_1(n_2b_3-n_3b_2)+\beta(\kappa n_1b_2-n_2a_1)=0\,, \qquad \alpha_3(n_3b_1-n_1b_3)=yn_1\,, \\ \alpha_2(n_3b_1-n_1b_3)+\beta(n_1a_2-\kappa n_2b_1)=0\,, \qquad \alpha_3(n_2b_3-n_3b_2)=yn_2\,, \\ \varepsilon\alpha_3n_3r_1+(n_1\beta\kappa-n_3\alpha_1)r_2=\gamma_2(\kappa n_1b_3-n_3a_1)\,, \\ (n_3\alpha_2-n_2\beta\kappa)r_1-\varepsilon n_3\alpha_3r_2=\gamma_1(n_3a_2-\kappa n_2b_3)\,, \\ \gamma_2(\kappa n_3b_1-n_1a_3)=2x\gamma_1\,, \qquad \gamma_1(n_2a_3-\kappa n_3b_2)=2x\gamma_2\,, \\ \kappa n_3(\gamma_2r_1-\gamma_1r_2)=\varepsilon x\sigma\,. \end{gathered} \tag{2.2}$$

The first three equations in this system are a closed subsystem, and one easily gets from them the coefficients $x$, and $\alpha_1, \alpha_2, \alpha_3$, which are determined up to an inessential common factor. The next five equations are used to find the coefficients $y, \beta$ and three constraints for the coefficients $a_i, b_i$, that are

$$\begin{gathered} \kappa b_1/n_1=2a_2/n_2-a_3/n_3\,, \qquad \kappa b_2/n_2=2a_1/n_1-a_3/n_3\,, \\ 2b_3/n_3=b_1/n_1+b_2/n_2\,. \end{gathered} \tag{2.3}$$

The coefficients $r_i, \gamma_i, \sigma$ are obtained from the remaining five equations in (2.2). Finally, the system (2.2) is reduced to three constraints (2.3) combined with the following relations:

$$\begin{gathered} \alpha_1=n_3a_1-n_1a_3\,, \qquad \alpha_2=n_2a_3-n_3a_2\,, \\ \alpha_3=(\alpha_1\alpha_2)^{1/2}\,, \qquad x=\alpha_3\,, \\ \beta=-n_3^2(b_1/n_1-b_3/n_3)\,, \qquad y=-x\beta/n_3\,, \\ \sigma=\sigma_1+2\varepsilon\sigma_2\,, \\ \sigma_1=-\kappa n_3^2(n_1n_2\alpha_1\alpha_2)^{-1}(n_2^2\alpha_1r_1^2+n_1^2\alpha_2r_2^2)\,, \\ \sigma_2=-\kappa n_3^2r_1r_2x^{-1}\,, \\ \gamma_1=n_3r_1+\varepsilon n_1n_3xr_2(n_2\alpha_1)^{-1}\,, \\ \gamma_2=-n_3r_2-\varepsilon n_2n_3xr_1(n_1\alpha_2)^{-1}\,. \end{gathered} \tag{2.4}$$

The coefficients $\alpha, \beta, x, y$ are defined up to a common factor. Under the conditions (2.3), the system (1.3) has the additional integrals at the level $J_3=0$,

$$\begin{aligned} J_4&=(\alpha_1M_1^2+\alpha_2M_2^2+\beta K_3^2+2n_3(r_1K_1-r_2K_2)+\sigma_1)^2 \\ &\quad-4\alpha_1\alpha_2N^2\,, \\ N&=M_1M_2+(n_1^2n_3\alpha_2r_2K_1-n_2^2n_3\alpha_1r_1K_2)(n_1n_2\alpha_3^2)^{-1}-n_3^2\kappa r_1r_2\alpha_3^{-2}\,, \end{aligned} \tag{2.5}$$

so it is completely integrable in the one-parameter subset of orbits $\mathcal{O}$ $(J_2=c_2, J_3=0)$. In the whole space $L^*(M_i, K_i)$ the function $J_4$ satisfies the equation

$$\dot{J}_4=8\alpha_1\alpha_2 n_3(b_1/n_1-b_3/n_3)NK_3J_3\,. \tag{2.6}$$

If $a_1/n_1=a_2/n_2$, one has $b_i/n_i=b_j/n_j$ because of Eqs. (2.3), so $\dot{J}_4\equiv 0$, and the system (1.3) is integrable for an arbitrary magnitude of the integral $J_3$. If $\kappa=0$, $b_i=0$ we have the classical case of Kowalewski [3], and if $\kappa=0$, $b_i\neq 0$ we are led to Chaplygin's case [4] for the Kirchhoff equations.

If $a_1/n_1\neq a_2/n_2$, the function $J_4$ is an adiabatical invariant for the system (1.3) at $|J_3|\ll 1$. In the obtained family of Euler's equations which are integrable at $J_3=0$, five coefficients $a_1, a_2, a_3, r_1, r_2$ in the Hamiltonian $H$, Eq. (1.6), are arbitrary.

The second family of integrable Euler's equations (1.3) is obtained under the conditions ($\varepsilon=\pm 1$)

$$\begin{gathered} z_\varepsilon=\alpha_1K_1^2+2\varepsilon\alpha_3K_1K_2+\alpha_2K_2^2+\beta K_3^2\,, \qquad w_\varepsilon=\varepsilon xM_3\,, \\ v_\varepsilon=\varepsilon yK_3\,. \end{gathered} \tag{2.7}$$

Putting $\dot{z}_\varepsilon$ into Eq. (1.7), one gets in view of Eqs. (1.3) the following set of algebraic constraints for the coefficients:

$$\begin{gathered} \alpha_3(\kappa n_3b_1-n_1a_3)=x\alpha_1\,, \qquad \alpha_3(n_2a_3-\kappa n_3b_2)=x\alpha_2\,, \\ \alpha_1(n_2a_3-\kappa n_3b_2)+\alpha_2(\kappa n_3b_1-n_1a_3)=2x\alpha_3\,, \qquad x\beta=-yn_3\,, \\ \beta(n_1a_2-\kappa n_2b_1)-\alpha_1(n_3a_2-\kappa n_2b_3)=0\,, \\ \alpha_3(n_3a_1-\kappa n_1b_3)=yn_1\,, \\ \beta(\kappa n_1b_2-n_2a_1)+\alpha_2(n_3a_1-\kappa n_1b_3)=0\,, \\ \alpha_3(\kappa n_2b_1-n_3a_2)=yn_2\,. \end{gathered} \tag{2.8}$$

The system of Eqs. (2.8) is solved in the same manner as that in (2.2); it is reduced to three relations between the coefficients $a_i, b_i$ $(i,j,k=1,2,3)$

$$2\kappa b_i/n_i=a_j/n_j+a_k/n_k\,, \tag{2.9}$$

together with the relations

$$\begin{gathered} \alpha_1=\kappa n_3b_1-n_1a_3\,, \qquad \alpha_2=n_2a_3-\kappa n_3b_2\,, \qquad \alpha_3=(\alpha_1\alpha_2)^{1/2}\,, \\ x=\alpha_3\,, \qquad 2\beta=n_3^2(a_2/n_2-a_1/n_1)\,, \qquad y=-x\beta/n_3\,. \end{gathered} \tag{2.10}$$

If the conditions (2.7) are valid, and $J_3=0$, the system (1.3) has the additional integral

$$J_4=(\alpha_1K_1^2+\alpha_2K_2^2+\beta K_3^2)^2-4\alpha_1\alpha_2K_1^2K_2^2\,, \tag{2.11}$$

so it is completely integrable (at the level $J_3=0$). In the whole space $L^*$ the function $J_4$ satisfies the equation

$$\dot{J}_4=4n_3(a_2/n_2-a_1/n_1)K_1K_2K_3J_3\,;$$

it is an adiabatical invariant for $|J_3|\ll 1$.

Thus we have proved the following

**Theorem 1.** *There exist two families,* (2.3) *and* (2.9), *of Euler's equations* (1.3) *for the class A Lie algebras* ($\kappa\neq 0$), *which are completely integrable, according to*

*Liouville, in the one-parametric subset of orbits* $\mathcal{O}$ ($J_2 = c_2$, $J_3 = 0$), *and have the additional fourth-order integrals, given in Eqs.* (2.5) *and* (2.11), *respectively. For* $a_1/n_1 = a_2/n_2$ *the obtained Euler's equations are completely integrable in the whole space* $L^*$.

For the Lie algebras SO(4) and SO(3, 1), Eqs. (2.3) and (2.9) have an open set of positive solutions $a_i > 0$, $b_i > 0$. The family of the integrable cases, given in Eqs. (2.9), does not admit linear terms in the Hamiltonian $H$, Eq. (1.6), and it is reduced to degenerate equations as $\kappa$ goes to zero. For the family (2.9) the integrability in the whole space $L^*$ for $a_1/n_1 = a_2/n_2$ is evident, while for the family (2.3) and with nonzero $r_i$ it is just a consequence of the existence of the first integral, given in Eq. (2.5), as in the Kowalewski case [3]. Another extension of the Kowalewski case, that was related to the groups of motion of Euclidean spaces, has been proposed by Perelomov [16].

Note that under the conditions $c_i = r_i = q_i = 0$, and

$$\left(\frac{a_1}{n_1} - \kappa\frac{b_3}{n_3}\right)\left(\frac{a_2}{n_2} - \kappa\frac{b_1}{n_1}\right)\left(\frac{a_3}{n_3} - \kappa\frac{b_2}{n_2}\right) = \left(\frac{a_1}{n_1} - \kappa\frac{b_2}{n_2}\right)\left(\frac{a_2}{n_2} - \kappa\frac{b_3}{n_3}\right)\left(\frac{a_3}{n_3} - \kappa\frac{b_1}{n_1}\right). \tag{2.12}$$

Euler's equations (1.3) have the additional first integral

$$J_4 = y_1 K_1^2 + y_2 K_2^2 + y_3 K_3^2, \tag{2.13}$$

where

$$\frac{y_1}{n_1} = \left(\frac{a_2}{n_2} - \kappa\frac{b_1}{n_1}\right)\left(\frac{a_3}{n_3} - \kappa\frac{b_1}{n_1}\right), \quad \frac{y_2}{n_2} = \left(\frac{a_2}{n_2} - \kappa\frac{b_1}{n_1}\right)\left(\frac{a_3}{n_3} - \kappa\frac{b_2}{n_2}\right),$$
$$\frac{y_3}{n_3} = \left(\frac{a_2}{n_2} - \kappa\frac{b_3}{n_3}\right)\left(\frac{a_3}{n_3} - \kappa\frac{b_1}{n_1}\right),$$

so it is completely integrable in Liouville's sense [the existence of the integral (2.13) is verified by means of direct calculation].

For the Lie algebra SO(4) ($n_i = \kappa = 1$) the condition (2.12) leads to the known integrable case found by Manakov [7]. Two families of Euler's equations, specified by the conditions (2.3) and (2.9), do not belong to the five-dimensional set (2.12), their intersections are two-dimensional subsets (axi-symmetric metrics). It is a consequence of the results of [6] therefore that Euler's equations (1.3) under the conditions (2.3) and (2.9) are not algebraically integrable in the whole space $L^*$. It is shown in the next section, however, that at the level $J_3 = 0$ one, nevertheless, has the complete algebraic integrability.

## 3. Explicit Integration of Certain Euler's Equations for the Lie Algebra SO(4)

*I.*

The purpose of this section is to transform Euler's equations (1.3) [under the conditions (2.3) or (2.9)], which are reduced to the invariant two-dimensional tori, fixed by constant values of the first integrals $J_1$, $J_2$, $J_3 = 0$, $J_4$, to appropriate coordinates $s_1, s_2$; the dynamical equations for the latter are integrated explicitly with elliptic functions of the time. The construction of the coordinates $s_1, s_2$ is inspired by the work by Chaplygin [4] dealing with Kirchhoff's equations.

Under the conditions (2.3), the first integral $J_4$, Eq. (2.5), is nonnegative, $J_4 = h^2$, and for $n_i = \kappa = 1$ [the Lie algebra is SO(4)] it has the form

$$J_4 = ((a_3 - a_1)M_1^2 + (a_2 - a_3)M_2^2 + (b_1 - b_3)K_3^2)^2 + 4(a_3 - a_1)(a_3 - a_2)M_1^2 M_2^2 . \tag{3.1}$$

The coordinates $s_1$ and $s_2$ are defined, with $h > 0$, as follows:

$$\begin{gathered} s_1 = (u + h)/v , \qquad s_2 = (u - h)/v , \\ u = (a_3 - a_1)M_1^2 + (a_3 - a_2)M_2^2 , \qquad v = (b_1 - b_3)K_3^2 , \qquad s = a_3 M_3^2 . \end{gathered} \tag{3.2}$$

Hence we get

$$v = 2h/(s_1 - s_2) , \qquad u = (s_1 + s_2)h/(s_1 - s_2) . \tag{3.3}$$

Let us find the coordinates $M_i$, $K_j$ as functions of $s_1, s_2$ and the constants of motion $J_k$. In view of Eq. (3.1) we have

$$u^2 + 2((a_3 - a_1)M_1^2 - (a_3 - a_2)M_2^2)v + v^2 = h^2 .$$

Combining this equality with (3.2), we get

$$2(a_3 - a_1)M_1^2 = (h^2 - (u - v)^2)/2v , \qquad 2(a_3 - a_2)M_2^2 = ((u + v)^2 - h^2)/2v . \tag{3.4}$$

Putting the expressions (3.3) into (3.4) and (3.2), we obtain the desired expressions of the coordinates $M_1, M_2, K_3$ in terms of $s_1, s_2, h$:

$$\begin{gathered} (2(a_3 - a_1))^{1/2} M_1 = \left( \frac{h(1 - s_2)(s_1 - 1)}{s_1 - s_2} \right)^{1/2} , \\ (2(a_3 - a_2))^{1/2} M_2 = \left( \frac{h(s_1 + 1)(s_2 + 1)}{s_1 - s_2} \right)^{1/2} , \\ (b_1 - b_2)^{1/2} K_3 = -2(h/(s_1 - s_2))^{1/2} . \end{gathered} \tag{3.5}$$

Next we shall find expressions for other coordinates $K_1, K_2, M_3$ in terms of $s_1, s_2$ and the integrals $J_i$. Putting the expressions (3.2) for $M_3, K_3$, as functions of $v$ and $s$, and the expressions (3.4) into the first integrals $J_1, J_2$, Eqs. (1.5), we get

$$\begin{aligned} (b_1 - b_2)K_1^2 &= -u - v + J_1 - b_2 J_2 + \frac{(a_1 - a_2)((u + v)^2 - h^2)}{2v(a_3 - a_2)} - \frac{2(a_3 - a_1)}{a_3} s , \\ (b_1 - b_2)K_2^2 &= u - v - J_1 + b_1 J_2 + \frac{(a_1 - a_2)(h^2 - (u - v)^2)}{2v(a_3 - a_1)} + \frac{2(a_3 - a_2)}{a_3} . \end{aligned} \tag{3.6}$$

Thus one has just to find an expression for $s$ as a function of $s_1, s_2, J_k$. Note that as $J_3 = 0$, we have in view of Eq. (1.5),

$$M_3^4 K_3^4 - 2(M_1^2 K_1^2 + M_2^2 K_2^2) M_3^2 K_3^2 + (M_1^2 K_1^2 - M_2^2 K_2^2)^2 = 0 . \tag{3.7}$$

Putting the above expressions into Eq. (3.7), one gets the following equation for the variable $s$,

$$4(hva_3^{-1})^2 s^2 - 2va_3^{-1} Ps + Q^2/4 = 0 , \tag{3.8}$$

where

$$\begin{gathered}\gamma P=(h^2-u^2-v^2)(X_1-X_2-h^2(a_1-a_2))+2uv(X_1+X_2-h^2(2a_3-a_1-a_2)),\\ \gamma Q=(h^2-u^2-v^2)(Y_1+Y_2)+2uv(Y_1-Y_2),\\ X_1=(a_3-a_2)(u+v)(J_1-b_2J_2),\qquad X_2=(a_3-a_1)(u-v)(J_1-b_1J_2),\\ Y_1=(a_3-a_2)(-u-v+J_1-b_2J_2),\qquad Y_2=(a_3-a_1)(u-v-J_1+b_1J_2).\end{gathered} \tag{3.9}$$

Because of Eq. (3.8) we have

$$4h^2vs/a_3=P\pm(P^2-h^2Q^2)^{1/2}. \tag{3.10}$$

Hence we obtain ($s/a_3=M_3^2$)

$$2h(2v)^{1/2}M_3=(P+hQ)^{1/2}\pm(P-hQ)^{1/2}. \tag{3.11}$$

Using Eqs. (3.9) we find

$$\begin{gathered}\gamma(P-hQ)=((h-u)^2-v^2)(\alpha_1v-\beta_1(u+h)),\\ \gamma(P+hQ)=((h+u)^2-v^2)(\alpha_2v-\beta_2(u-h)),\end{gathered} \tag{3.12}$$

where

$$\begin{gathered}\alpha_1=m_1(J_1+h)-m_2J_2,\qquad \alpha_2=m_1(J_1-h)-m_2J_2,\\ \beta_1=(a_1-a_2)(J_1+h)-m_3J_2,\qquad \beta_2=(a_1-a_2)(J_1-h)-m_3J_2,\\ m_1=2a_3-a_1-a_2,\qquad m_2=(a_3-a_2)b_2+(a_3-a_1)b_1,\\ m_3=(a_3-a_2)b_2-(a_3-a_1)b_1.\end{gathered} \tag{3.13}$$

Putting the expressions given in (3.12) into Eq. (3.11), and using relations (3.2), we get

$$M_3=[((s_1^2-1)(\alpha_2-\beta_2s_2))^{1/2}+((s_2^2-1)(\alpha_1-\beta_1s_1))^{1/2}](s_1-s_2)^{-1}(2\gamma)^{-1/2}. \tag{3.14}$$

Hence we get an expression for $s$.

Substituting in Eqs. (3.6) the notations of (3.13) for $\alpha_i, \beta_i$, we can find

$$\begin{aligned}(b_1-b_2)K_1^2&=\frac{(\beta_1s_1-\alpha_1)(s_2+1)-(\beta_2s_2-\alpha_2)(s_1+1)}{2(a_3-a_2)(s_1-s_2)}-\frac{2(a_3-a_1)}{a_3}s,\\ (b_1-b_2)K_2^2&=\frac{(\beta_1s_1-\alpha_1)(1-s_2)+(\beta_2s_2-\alpha_2)(s_1-1)}{2(a_3-a_1)(s_1-s_2)}+\frac{2(a_3-a_2)}{a_3}s.\end{aligned} \tag{3.15}$$

Next we put into Eqs. (3.15) the quantity $s/a_3=M_3^2$, as given in Eq. (3.14), and obtain

$$\begin{aligned}(b_1-b_2)^{1/2}K_1&=(2(a_3-a_2))^{-1/2}(s_1-s_2)^{-1}[((1+s_1)(1-s_2)\\ &\quad(\alpha_2-\beta_2s_2))^{1/2}-((1-s_1)(1+s_2)(\alpha_1-\beta_1s_1))^{1/2}],\\ (b_1-b_2)^{1/2}K_2&=(2(a_3-a_1))^{-1/2}(s_1-s_2)^{-1}[((1+s_1)(s_2-1)\\ &\quad(\alpha_1-\beta_1s_1))^{1/2}+((s_1-1)(1+s_2)(\alpha_2-\beta_2s_2))^{1/2}].\end{aligned} \tag{3.16}$$

Thus Eqs. (3.5), (3.14), and (3.16) provide with expressions for all the coordinates $M_i, K_j$ in terms of the new coordinates $s_1, s_2$ and the constants of motion

$J_1, J_2, J_4 = h^2$ (at the level $J_3 = 0$). Note that the formulae admit a simultaneous sign inversion for any coordinate pair $M_i, K_i$.

Next we transform Euler's equations (1.3) to equations for the coordinates $s_1, s_2$. Let us find the time derivatives of the expressions for $s_1, s_2$ in (3.2), applying the equations of motion (1.3). The result is

$$\begin{aligned} \dot{s}_1/2 &= (a_3 - a_2)(s_1 + 1)M_1K_2/K_3 + (a_3 - a_2)(1 - s_1)M_2K_1/K_3, \\ \dot{s}_2/2 &= (a_3 - a_1)(s_2 + 1)M_1K_2/K_3 + (a_3 - a_2)(1 - s_2)M_2K_1/K_3. \end{aligned} \tag{3.17}$$

Substituting in these equations the obtained expressions for the coordinates $M_i, K_j$ Eqs. (3.5), (3.16), we obtain a system of equations, which is closed,

$$\dot{s}_1 = -((1 - s_1^2)(\alpha_1 - \beta_1 s_1))^{1/2}, \qquad \dot{s}_2 = -((1 - s_2^2)(\alpha_2 - \beta_2 s_2))^{1/2}. \tag{3.18}$$

Because of equations (3.18), $s_i(t)$ are elliptic functions of the time $t$. Substituting $s_i(t)$ in Eqs. (3.5), (3.14), (3.16) we obtain an explicit representation for the dynamics determined by Euler's equations (1.3) in terms of the elliptic functions. The degenerate case $J_4 = 0$ is integrated with elementary functions.

*II.*

Our next purpose is to perform the integration of Euler's equations (1.3) under the conditions (2.9), $b_i = \frac{1}{2}(a_j + a_k)$. The integral $J_4$ is

$$J_4 = h^2 = ((a_3 - a_2)K_1^2 + (a_1 - a_3)K_2^2 + (a_1 - a_2)K_3^2)^2 + 4(a_3 - a_1)(a_3 - a_2)K_1^2K_2^2. \tag{3.19}$$

Let us introduce new coordinates $s_1, s_2$,

$$\begin{aligned} s_1 &= (u + h)/v, \qquad s_2 = (u - h)/v, \\ u &= (a_3 - a_2)K_1^2 + (a_3 - a_1)K_2^2, \qquad v = (a_1 - a_2)K_3^2, \end{aligned} \tag{3.20}$$

As in the first subsection, we shall find expressions for the coordinates $M_i, K_j$ in terms of the new coordinates $s_1, s_2$ and the constants $J_k$. Using Eq. (3.19) we get

$$2(a_3 - a_2)K_1^2 = (h^2 - (u - v)^2)/2v, \qquad 2(a_3 - a_1)K_2^2 = ((u + v)^2 - h^2)/2v. \tag{3.21}$$

Substituting here the expressions for $u, v$ as functions of $s_1, s_2$, Eqs. (3.3), we have

$$\begin{aligned} (2(a_3 - a_2))^{1/2}K_1 &= (h(1 - s_2)(s_1 - 1)(s_1 - s_2)^{-1})^{1/2}, \\ (2(a_3 - a_1))^{1/2}K_2 &= (h(s_1 + 1)(s_2 + 1)(s_1 - s_2)^{-1})^{1/2}, \\ (a_1 - a_2)^{1/2}K_3 &= -(2h(s_1 - s_2)^{-1})^{1/2}. \end{aligned} \tag{3.22}$$

Substituting (3.20), (3.21), we get from Eqs. (1.5), with the notation $s = a_3M_3^2$,

$$\begin{aligned} (a_1 - a_2)M_1^2 &= J_1 - a_2J_2 - \frac{u + v}{2} - \frac{(a_1 - a_2)((u + v)^2 - h^2)}{4v(a_3 - a_1)} - \frac{a_3 - a_2}{a_3}s, \\ (a_1 - a_2)M_2^2 &= -J_1 + a_1J_2 + \frac{u - v}{2} - \frac{(a_1 - a_2)(h^2 - (u - v)^2)}{4v(a_3 - a_2)} + \frac{a_3 - a_1}{a_3}s. \end{aligned} \tag{3.23}$$

Putting the resulting expressions for the coordinates $M_i, K_j$ into Eq. (3.7), ($J_3 = 0$), we obtain an equation determining $s$,

$$4(hva_3^{-1})^2s^2 - 2va_3^{-1}P_1s + Q_1^2/4 = 0, \tag{3.24}$$

where

$$\begin{aligned}
\gamma_1 P_1 &= (h^2-u^2-v^2)(x_1+x_2+h^2(a_1-a_2)/2) \\
&\quad + 2uv(x_1-x_2-h^2(2a_3-a_1-a_2)/2), \\
\gamma_1 Q_1 &= (h^2-u^2-v^2)(y_1+y_2)+2uv(y_1-y_2), \\
\gamma_1 &= (a_3-a_1)(a_3-a_2), \\
x_1 &= (a_3-a_1)(u+v)(J_1-a_2J_2), \quad x_2=(a_3-a_2)(u-v)(a_1J_2-J_1), \\
y_1 &= (a_3-a_1)(J_1-a_2J_2-(u+v)/2), \quad y_2=(a_3-a_2)(a_1J_2-J_1+(u-v)/2).
\end{aligned} \tag{3.25}$$

From Eq. (3.24) it follows

$$4vh^2 s/a_3 = P_1 \pm (P_1^2-h^2Q_1^2)^{1/2},$$

and, as $M_3^2=s/a_3$, we have

$$2(2v)^{1/2}hM_3=(P_1+hQ_1)^{1/2}+(P_1-hQ_1)^{1/2}. \tag{3.26}$$

Equations (3.25) lead to

$$\begin{aligned}
P_1-hQ_1 &= ((u-h)^2-v^2)(v\alpha_1^0-(u+h)\beta_1^0)\gamma_1^{-1}, \\
P_1+hQ_1 &= ((u+h)^2-v^2)(v\alpha_2^0-(u-h)\beta_2^0)\gamma_1^{-1},
\end{aligned} \tag{3.27}$$

where

$$\begin{aligned}
\alpha_1^0 &= (J_1+h/2)n_1+J_2n_2, \quad \alpha_2^0=(J_1-h/2)n_1+J_2n_2, \\
\beta_1^0 &= (a_1-a_2)(J_2a_3-J_1-h/2), \quad \beta_2^0=(a_1-a_2)(J_2a_3-J_1+h/2), \\
n_1 &= 2a_3-a_1-a_2, \quad n_2=2a_1a_2-a_3(a_1+a_2).
\end{aligned} \tag{3.28}$$

Using Eqs. (3.3), the expression in (3.26) is transformed to

$$M_3=[((s_1^2-1)(\alpha_2^0-\beta_2^0 s_2))^{1/2}+((s_2^2-1)(\alpha_1^0-\beta_1^0 s_1))^{1/2}](s_1-s_2)^{-1}(2\gamma_1)^{-1/2}. \tag{3.29}$$

Putting the corresponding expression for $s=a_3M_3^2$ into Eq. (3.23), we have

$$\begin{aligned}
(a_1-a_2)^{1/2}M_1 &= (2(a_3-a_1))^{-1/2}(s_1-s_2)^{-1}[((1+s_1)(1-s_2)(\alpha_2^0-\beta_2^0 s_2))^{1/2} \\
&\quad -((1-s_1)(1+s_2)(\alpha_1^2-\beta_1^0 s_1))^{1/2}], \\
(a_1-a_2)^{1/2}M_2 &= (2(a_3-a_2))^{-1/2}(s_1-s_2)^{-1}[((1+s_1)(s_2-1) \\
&\quad (\alpha_1^0-\beta_1^0 s_1))-((s_1-1)(s_2+1)(\alpha_2^0-\beta_2^0 s_2))^{1/2}].
\end{aligned} \tag{3.30}$$

Equations (3.22), (3.29), and (3.30) present expressions for all the original coordinates $M_i, K_j$ in terms of the new coordinates $s_1, s_2$ and the constants of motion $J_k$; as in the preceding case, for any pair $M_i, K_i$ the sign can be inversed.

Calculating the time derivatives of the coordinates $s_1, s_2$, as given in Eqs. (3.20), by means of Euler's equations (1.3), we have

$$\begin{aligned}
\dot{s}_1 &= (a_3-a_2)(1-s_1)M_1K_2/K_3+(a_3-a_2)(s_1+1)M_2K_1/\dot{K}_3, \\
\dot{s}_2 &= (a_3-a_2)(1-s_2)M_1K_2/K_3+(a_3-a_2)(s_2+1)M_2K_1/K_3.
\end{aligned}$$

The system of dynamical equations is obtained when we put here the expressions for $M_i$, $K_j$ as functions of $s_1, s_2$,

$$\dot{s}_1 = -((1-s_1^2)(\alpha_1^0 - \beta_1^0 s_1)/2)^{1/2}, \qquad \dot{s}_2 = -((1-s_2^2)(\alpha_2^0 - \beta_2^0 s_2)/2)^{1/2}. \quad (3.31)$$

Thus, under conditions (2.9) Euler's equations are also integrated explicitly in terms of elliptic functions.

## 4. Second-Order First Integrals $J_4$

In the present section we will investigate Euler's equations (1.3), (1.4) for general-type Hamiltonians (1.6). We derive conditions for the coefficients $a_i$, $b_i$, $c_i$, $r_i$, $q_i$, under which Euler's equations are equivalent to a pair of linear equations [cf. Eqs. (1.7)],

$$\dot{z}_+ = w_+ z_+, \qquad \dot{z}_- = w_- z_-, \quad (4.1)$$

where

$$z_\varepsilon = \alpha_1 M_1 + \varepsilon\alpha_2 M_2 + \beta_1 K_1 + \varepsilon\beta_2 K_2 + n_3\gamma_1 + \varepsilon n_3 \gamma_2,$$
$$w_\varepsilon = \varepsilon x M_3 + \varepsilon y K_3, \qquad \varepsilon = \pm 1.$$

In this case the second-order polynomial $J_4 = z_+ z_-$ is an integral of Eqs. (1.3), or (1.4). There is a special case of Eqs. (4.1), where $z = z_+$ is a complex-valued function, $w_+$ is pure imaginary; then $z_- = \bar{z}_+$, $w_- = \bar{w}_+$, and the integral is $J_4 = z\bar{z}$.

*I.*

For the class $A$ Lie algebras, (1.1), Eqs. (4.1) result in a set of algebraic relations for the coefficients, which are obtained when the derivatives $\dot{z}_\varepsilon$ are calculated by means of the dynamical equations (1.3), as $M_i$, $K_i$ are arbitrary. So we have

$$\begin{aligned}
&\alpha_1(n_2a_3 - n_3a_2) + \kappa\beta_1(n_2c_3 - n_3c_2) = x\alpha_2, \\
&\alpha_1(n_2b_3 - n_3b_2) + \beta_1(n_2c_3 - n_3c_2) = y\beta_2, \\
&\alpha_1(n_2c_3 - n_3c_2) + \beta_1(\kappa n_2b_3 - n_3a_2) = y\alpha_2, \\
&\alpha_1(n_2c_3 - n_3c_2) + \beta_1(-\kappa n_3b_2 + n_2a_3) = x\beta_2, \\
&\alpha_2(n_3a_1 - n_1a_3) + \kappa\beta_2(n_3c_1 - n_1c_3) = x\alpha_1, \\
&\alpha_2(n_3b_1 - n_1b_3) + \beta_2(n_3c_1 - n_1c_3) = y\beta_1, \\
&\alpha_2(n_3c_1 - n_1c_3) + \beta_2(-\kappa n_1b_3 + n_3a_1) = y\alpha_1, \\
&\alpha_2(n_3c_1 - n_1c_3) + \beta_2(\kappa n_3b_1 - n_1a_3) = x\beta_1, \\
&\alpha_1q_2 + \kappa\beta_1r_2 = -x\gamma_2, \qquad \alpha_1r_2 + \beta_1q_2 = -y\gamma_2, \\
&\alpha_2q_1 + \kappa\beta_2r_1 = x\gamma_1, \qquad \alpha_2r_1 + \beta_2q_1 = y\gamma_1, \qquad r_3 = q_3 = 0.
\end{aligned} \quad (4.2)$$

The system of Eqs. (4.2) is solved explicitly with respect to the coefficients $a_i$, $b_i$, $c_i$, $r_i$, $q_i$. The first four pairs of equations in (4.2) (these eight equations are a closed subsystem) are used to exclude the variables of the type $n_ic_j - n_jc_i$, and we get four closed equations for three quantities $\sigma_1, \sigma_2, \sigma_3$, where $\sigma_i = a_i - \kappa b_i$. Solving the latter equations, we get a single relation for the coefficients $\alpha_i$, $\beta_i$,

$$\alpha_1\beta_1n_2 + \alpha_2\beta_2n_1 = 0, \quad (4.3)$$

and the expressions for $\sigma_i$ in terms of $\alpha_j$, $\beta_k$, $x$, $y$,

$$
\begin{gathered}
n_3(a_1-\kappa b_1)=(xC+yB)(2\alpha_2\beta_2)^{-1}, \qquad n_3(a_2-\kappa b_2)=(xC-yB)(2\alpha_1\beta_1)^{-1}, \\
n_1(a_3-\kappa b_3)=(-xA+yD)(2\alpha_2\beta_2)^{-1}, \\
A=\alpha_1\beta_2+\alpha_2\beta_1, \qquad B=\alpha_1\alpha_2-\kappa\beta_1\beta_2, \\
C=\alpha_1\beta_2-\alpha_2\beta_1, \qquad D=\alpha_1\alpha_2+\kappa\beta_1\beta_2.
\end{gathered} \tag{4.4}
$$

We consider next two pairs of equations in (4.2): the first and the third, the fifth and the seventh. Putting there $\kappa b_i=a_i-\sigma_i$ and using (4.4), one easily gets the following expressions

$$
\begin{aligned}
&n_2a_3-n_3a_2=(x\alpha_1\alpha_2-\kappa\beta_1(xA+yB)/2\alpha_1)(\alpha_1^2-\kappa\beta_1^2)^{-1}, \\
&n_3a_1-n_1a_3=(x\alpha_1\alpha_2-\kappa\beta_2(xA+yB)/2\alpha_2)(\alpha_2^2-\kappa\beta_2^2)^{-1}, \\
&n_2c_3-n_3c_2=(xC+yB)(\alpha_1^2-\kappa\beta_1^2)^{-1}/2, \\
&n_3c_1-n_1c_3=(-xC+yB)(\alpha_2^2-\kappa\beta_2^2)^{-1}/2.
\end{aligned} \tag{4.5}
$$

The remaining two pairs of the equations are used to express $r_i, q_i$ via the parameters $\gamma_i$, $\alpha_j$, $\beta_k$, $x$, $y$,

$$
\begin{gathered}
r_1=\gamma_1\frac{\alpha_2y-\beta_2x}{\alpha_2^2-\kappa\beta_2^2}, \qquad r_2=-\gamma_2\frac{\alpha_1y-\beta_1x}{\alpha_1^2-\kappa\beta_1^2}, \\
q_1=\gamma_1\frac{\alpha_2x-\kappa\beta_2y}{\alpha_2^2-\kappa\beta_2^2}, \qquad q_2=-\gamma_2\frac{\alpha_1x-\kappa\beta_1y}{\alpha_1^2-\kappa\beta_1^2}, \qquad r_3=q_3=0.
\end{gathered} \tag{4.6}
$$

Equations (4.4) and (4.5) are invariant under the transformation $\alpha_i\to C\alpha_i$, $\beta_i\to C\beta_i$, so their right-hand sides depend on four free parameters, under the constraint (4.3). For $\kappa\neq 0$ the coefficients $a_i, b_i, c_i$ are determined by Eqs. (4.4) and (4.5) up to two-parameter transformations,

$$
a_i\to a_i+n_i\kappa T_1, \qquad b_i\to b_i+n_iT_1, \qquad c_i\to c_i+n_iT_2,
$$

where $T_1$ and $T_2$ are the parameters. So 9 coefficients $a_i, b_i, c_i$, constrained by the relations (4.3)–(4.5) depend on 6 free parameters. The parameters $\alpha_1, \beta_1, \gamma_1$ acquire arbitrary real values, the parameters $\alpha_2$, $\beta_2$, $\gamma_2$, $x$, $y$ may be either all real, or all imaginary, in both cases the expressions in (4.4)–(4.6) are real; except for this restriction the parameters $\gamma_1, \gamma_2$ are arbitrary.

A consequence of the above consideration is

**Theorem 2.** *Euler's equations* (1.3) *(for the class A Lie algebras at $\kappa\neq 0$), with the coefficients $a_i$, $b_i$, $c_i$, $r_i$, $q_i$ expressed by means of Eqs.* (4.4)–(4.6) *via the parameters $\alpha_i$, $\beta_i$, $\gamma_i$, $x$, $y$, which satisfy constraint* (4.3), *have an additional first integral,*

$$
J_4=z_+z_-=(\alpha_1M_1+\beta_1K_1+n_3\gamma_1)^2-(\alpha_2M_2+\beta_2K_2+n_3\gamma_2)^2, \tag{4.7}
$$

*and are therefore completely integrable, in Liouville's sense, on the orbits $\mathcal{O}$ $(J_2=c_2, J_3=c_3)$.*

If $\kappa=0$, it follows from Eqs. (3.2) that the coefficients $a_i, b_i, c_i$ must satisfy the following relations $(i,j,k=1,2,3)$

$$
\begin{gathered}
n_j(n_ia_i^{-1}-n_ja_j^{-1})(n_jc_i-n_ic_j)^{-1}=n_k(n_ia_i^{-1}-n_ka_k^{-1})(n_kc_i-n_ic_k)^{-1}, \\
n_in_j^2n_k^2b_i-n_i^3(n_jc_k-n_kc_j)^2a_i^{-1}=n_i^2n_jn_k^2b_j-n_j^3(n_ic_k-n_kc_i)^2a_j^{-1}.
\end{gathered} \tag{4.8}
$$

The coefficients $\alpha_i$, $\beta_i$, $x$, $y$ are, correspondingly,

$$\begin{gathered}\alpha_1 = C(n_3a_1 - n_1a_3)^{1/2}, \quad \alpha_2 = C(n_2a_3 - n_3a_2)^{1/2}, \quad x = \alpha_1\alpha_2C^{-2}, \\ y = (n_1a_2 + n_2a_1)(n_3c_1 - n_1c_3)^{1/2}(n_2c_3 - n_3c_2)^{1/2}(n_1n_2a_1a_2)^{-1/2}, \\ \beta_1 = -y\alpha_2n_1n_3^{-1}(n_1a_2 + n_2a_1)^{-1}, \quad \beta_2 = -\beta_1\alpha_1n_2\alpha_2^{-1}n_1^{-1}.\end{gathered} \tag{4.9}$$

If the relations (4.8), (4.9), and (4.6) are valid, Eqs. (1.3) have the additional integral (4.7) and are therefore completely integrable. For $n_i = 1$ the constraints (4.8) determine the cases of integrability which were found by Steklov [1] and Lyapunov [2] for the Kirchhoff equations.

*II.*

For the class $B$ Lie algebras, Eqs. (1.2), the Eqs. (4.1) are reduced to the following set of algebraic relations for the coefficients, because of Euler's equations (1.4),

$$\begin{aligned} &\alpha_2(n_3a_1 - n_1a_3) = x\alpha_1, && \alpha_1(n_2a_3 - n_3a_2) = x\alpha_2, \\ &\beta_2(m_3b_1 - m_1b_3) = y\beta_1, && \beta_1(m_2b_3 - m_3b_2) = y\beta_2, \\ &\alpha_2n_3c_1 - \beta_2m_1c_3 = x\beta_1, && -\alpha_2n_1c_3 + \beta_2m_3c_1 = y\alpha_1, \\ &-\alpha_1n_3c_2 + \beta_1m_2c_3 = x\beta_2, && \alpha_1n_2c_3 - \beta_1m_3c_2 = y\alpha_2. \end{aligned} \tag{4.10}$$

Besides, they lead to explicit expressions for the coefficients $r_i$, $q_i$,

$$\begin{gathered} r_1 = yn_3\gamma_1(m_3\beta_2)^{-1}, \quad r_2 = -yn_3\gamma_2(m_3\beta_1)^{-1}, \\ q_1 = x\gamma_1/\alpha_2, \quad q_2 = -x\gamma_2/\alpha_1, \quad r_3 = q_3 = 0. \end{gathered} \tag{4.11}$$

The coefficients $x$, $y$ and $\alpha_i$, $\tau\beta_i$ (with an indefinite factor $\tau$) are easily obtained from the first two pairs of relations in (4.10). Then one gets $c_i$ and the factor $\tau$ from the third and the fourth pair of the equations in (4.10). Finally, we have three relations for the coefficients $a_i$, $b_i$, $c_i$,

$$\begin{gathered} c_i = (n_i\varphi_i - m_i\varphi_i^{-1})q\varphi_1\varphi_2\varphi_3, \\ \varphi_i = ((n_ja_k - n_ka_j)/(m_jb_k - m_kb_j))^{1/2}, \\ q = (m_ib_j - m_jb_i)(n_jm_i\varphi_j^2 - n_im_j\varphi_i^2)^{-1} \end{gathered} \tag{4.12}$$

(it is easily verified that $q$ is independent of the subscripts $i, j, i \neq j$). Other coefficients, up to a common factor at $\alpha_i$, $\beta_i$, are as follows:

$$\begin{gathered} \alpha_1 = (n_3a_1 - n_1a_3)^{1/2}, \quad \alpha_2 = (n_2a_3 - n_3a_2)^{1/2}, \quad x = \alpha_1\alpha_2, \\ \beta_1 = n_3m_3^{-1}\varphi_3(m_3b_1 - m_1b_3)^{1/2}, \quad \beta_2 = n_3m_3^{-1}\varphi_3(m_2b_3 - m_3b_2)^{1/2}, \\ y = (m_3b_1 - m_1b_3)^{1/2}(m_2b_3 - m_3b_2)^{1/2}. \end{gathered} \tag{4.13}$$

To conclude, if $a_i$, $b_i$, $c_i$ satisfy the relations (4.12), and $r_i$, $q_i$ are given by Eqs. (4.11), Eqs. (4.1) have the solutions $z_\varepsilon$, $w_\varepsilon$, which are determined by Eqs. (4.13).

The above discussion is summed up in the following statement:

**Theorem 3.** *Euler's equations* (1.4) *for the class B Lie algebras, if relations* (4.12), (4.11) *hold, have an additional first integral* $J_4 = z_+z_-$, *as given in Eq.* (4.7), *and they are completely integrable in Liouville's sense, on the orbits* $\mathcal{O}$.

*III.*

The integrable case (4.12) for the Lie algebra SO(4) ($n_i = m_i = 1$), containing six arbitrary parameters $a_i, b_i$, is a generalization of the three-parameter family of integrable cases obtained by Steklov [11, Sect. 42]. Formulae which were presented in [11] are too complicated, and that integrable case has not been investigated at all. We shall describe some properties of a more general case, specified by Eqs. (4.12), for $r_i = q_i = 0$.

As the coefficients $c_i$ must be real and satisfy Eqs. (4.12), for $a_i > a_j > a_k$ one has either $b_i > b_j > b_k$, or $b_i < b_j < b_k$. Suppose $a_1 > a_2 > a_3$, then $\alpha_1$ and $\beta_1$ are real because of (4.13), while $\alpha_2, \beta_2, x, y$ are imaginary. In this case the integral $J_4$ is

$$J_4 = \left((a_1 - a_3)^{1/2} M_1 + \left(\frac{(a_1 - a_2)(b_1 - b_3)}{b_1 - b_2}\right)^{1/2} K_1\right)^2 + \left((a_2 - a_3)^{1/2} M_2 + \left(\frac{(a_1 - a_2)(b_2 - b_3)}{b_1 - b_2}\right)^{1/2} K_2\right)^2 . \tag{4.14}$$

The conditions (4.12) (for $n_i = m_i = 1$) are not changed if the order of the axes is rearranged. Therefore, for given values of the $a_i, b_i, c_i$, Eq. (4.1) has two other pairs of solutions, corresponding to the substitutions $2 \to 3$, $3 \to 2$ and $1 \to 3$, $3 \to 1$. Respectively, one has two other integrals of Euler's equations (1.4),

$$\begin{aligned} J_5 &= \left((a_1 - a_2)^{1/2} M_1 + \left(\frac{(a_1 - a_3)(b_1 - b_2)}{b_1 - b_3}\right)^{1/2} K_1\right)^2 - \left((a_2 - a_3)^{1/2} M_3 + \left(\frac{(a_1 - a_3)(b_2 - b_3)}{b_1 - b_3}\right)^{1/2} K_3\right)^2 , \\ J_6 &= \left((a_1 - a_2)^{1/2} M_2 + \left(\frac{(a_2 - a_3)(b_1 - b_2)}{b_2 - b_3}\right)^{1/2} K_2\right)^2 + \left((a_1 - a_3)^{1/2} M_3 + \left(\frac{(a_2 - a_3)(b_1 - b_3)}{b_2 - b_3}\right)^{1/2} K_3\right)^2 . \end{aligned} \tag{4.15}$$

The integrals $J_5$ and $J_6$ are linear combinations of the integrals $J_1, J_2, J_3, J_4$; if the constraints (4.12) hold, the following identities are valid

$$\begin{gathered} 2J_1 - (a_2 - c_2 \varphi_1 \varphi_3^{-1}) J_2 - (b_2 - c_2 \varphi_3 \varphi_1^{-1}) J_3 - p_1 (c_2 \varphi_3^{-1} J_4 - c_3 \varphi_2^{-1} J_5) = 0 , \\ 2J_1 - (a_1 - c_1 \varphi_2 \varphi_3^{-1}) J_2 - (b_1 - c_1 \varphi_3 \varphi_2^{-1}) J_3 \\ - p_2 (c_1 \varphi_3^{-1} J_4 + c_3 \varphi_1^{-1} J_6) = 0 , \\ p_1 = ((a_2 - a_3)(b_2 - b_3))^{-1/2} , \qquad p_2 = ((a_1 - a_3)(b_1 - b_3))^{-1/2} . \end{gathered} \tag{4.16}$$

In the case of the Lie algebra SO(4) the orbits $\mathcal{O}$ ($J_2 = c_2, J_3 = c_3$) are compact manifolds $\mathcal{M}^4 = S^2 \times S^2$. The sets of minima of the integrals $J_4, J_6$, and saddle points of the integral $J_5$ are specified by two conditions, and they are three two-dimensional tori $\mathbf{T}_i^2$ in the manifold $\mathcal{M}^4$ ($i = 1, 2, 3$). Intersections of these tori with the equipotential surface of the Hamiltonian, $J_1 = \text{const}$, are closed trajectories for

Eqs. (1.4), which can be integrated explicitly, in terms of elliptic functions of the time parameter. In the three-dimensional manifold of the common level of the integrals $J_1, J_2, J_3$, the manifold $\mathscr{M}^3$, these trajectories are, in view of identities (4.16), the sets of minima, maxima, and saddle points of the function $J_4$.

In the limiting case where $b_1 = b_2 = b_3 = b$, there are integrable cases with two constraints,

$$c_1c_2c_3(c_3^{-2} - c_1^{-2}) = a_1 - a_3, \qquad c_1c_2c_3(c_2^{-2} - c_3^{-2}) = a_3 - a_2, \tag{4.17}$$

and so containing five free parameters. The coefficients $\alpha_i$, $x$ are given in Eqs. (4.13), $\beta_1 = c_3\alpha_1/c_2$, $\beta_2 = c_3\alpha_2/c_1$, $y=0$. As in the preceding case, Euler's equations (1.4) with the relations (4.17) have the additional integral,

$$J_4 = (a_1 - a_3)(M_1 + c_3K_1/c_2)^2 + (a_2 - a_3)(M_2 + c_3K_2/c_1)^2, \tag{4.18}$$

so they are completely integrable on the orbits $\mathcal{O}$.

## 5. On Physical Applications of Euler's Equations for the Lie Algebra SO(4)

*I.*

Let us consider Euler's equations (1.4) for the Lie algebra SO(4) = SO(3) + SO(3) $(n_i = m_i = 1)$ with the general quadratic Hamiltonian,

$$2H = \sum_{i,j=1}^{3} (a_{ij}M_iM_j + 2c_{ij}M_iK_j + b_{ij}K_iK_j). \tag{5.1}$$

The Hamiltonian depends on 21 parameters $a_{ij}$, $b_{ij}$, $c_{ij}$.

A special case of Euler's equations (1.4) are the classical equations describing motion of a rigid body with an ellipsoid cavity filled with the ideal incompressible fluid which is in homogeneous vortex motion [8–10]. In the most general case, these equations contain 12 parameters which determine components of the tensor of inertia of the rigid body $I_{ik}$ and the position of the ellipsoid cavity, $D_{ik}$. In particular, in the case of diagonal matrices $a_{ij}, b_{ij}, c_{ij}$, which constitute a nine-dimensional space $V^9$, the Hamiltonians describing the dynamics of the object in view constitute a six-dimensional submanifold $V^6$ in $V^9$. The new integrable cases presented in Sect. 2 [Eqs. (2.3) and (2.9)] depend on three arbitrary parameters, and their intersection with the manifold $V^6$ is trivial, so these cases has nothing to do with the physical problem mentioned above. The integrable cases specified in Eqs. (4.12) depend on six arbitrary parameters, so their intersection with the submanifold $V^6$ has no more than three dimensions. This is just the three-dimensional family of integrable cases which was indicated by Steklov [11].

*II.*

In order to find physical applications for a wider class of Euler's systems for the Lie algebra SO(4) we will consider the dynamics of a rigid body with $n$ ellipsoid cavities filled with the ideal incompressible fluid. We choose the coordinate frame fixed to the rigid body, $S$, and put the origin 0 to the center of mass. The cavities are enumerated by an index $\alpha = 1, \ldots, n$; let $r^1_\alpha, r^2_\alpha, r^3_\alpha$ be coordinates of the cavity centers, and $D_\alpha$ be symmetrical operators transforming the unit sphere into the ellipsoid $\alpha$-cavity. The eigenvalues $d_{\alpha 1}, d_{\alpha 2}, d_{\alpha 3}$ of the operator $D_\alpha$ are the semi-axes of the $\alpha$-cavity, and $d_\alpha = \det D_\alpha = d_{\alpha 1} d_{\alpha 2} d_{\alpha 3}$. Every cavity is filled completely with the ideal incompressible fluid with a constant density $\varrho_\alpha$ and the total mass $m_\alpha = 4\pi \varrho_\alpha d_\alpha / 3$.

The motion of the rigid body with the fixed center of mass is determined by an orthogonal matrix $Q(t)$. Motions of the fluid in every cavity satisfies the equations of hydrodynamics,

$$\varrho d\mathbf{v}/dt = -\operatorname{grad} p, \qquad \operatorname{div} \mathbf{v} = 0, \tag{5.2}$$

where $\varrho$ is the density of the fluid, $\mathbf{v}$ is its velocity, and $p$ is the pressure. In the following we assume that the motion of the fluid in every cavity is a motion with homogeneous deformation (see in [17]); in other words, it is given by a transformation from the Lagrange coordinates $a^k$ [which are within the unit sphere, $(a^1)^2 + (a^2)^2 + (a^3)^2 \leqq 1$] to the Euler coordinates $x^i$,

$$x^i = \sum_{k=1}^{3} (F^i_{\alpha k}(t) a^k + Q^i_k r^k_\alpha), \qquad F_\alpha = Q D_\alpha Q_\alpha, \tag{5.3}$$

where $Q_\alpha(t)$ is an orthogonal matrix determining the rotation of the fluid in the $\alpha^{\text{th}}$ cavity with respect to the rigid body.

The equations of motion for the rigid body with $n$ cavities filled with the ideal incompressible fluid are the hydrodynamical equations (5.2) for every cavity and the conservation law for the total angular momentum. Let us introduce the notation

$$\dot{Q} = Q A_0, \qquad \dot{Q}_\alpha = -B_{0\alpha} Q_\alpha, \tag{5.4}$$

and use the isomorphism of vectors with the components $v^i$ in $R^3$ and skew-symmetrical $3 \times 3$ matrices with the components $V_{jk}$,

$$v^i \to V_{jk} = -\sum_{i=1}^{3} v^i \varepsilon_{ijk}. \tag{5.5}$$

At this isomorphism the vector product of two vectors, $\mathbf{x} \times \mathbf{y}$, is mapped to the commutator of the corresponding matrices, $[XY] = XY - YX$. Let the skew-symmetrical matrices $A_0$ and $B_{0\alpha}$ be isomorphic to vectors with components $A^i$, $B^i_\alpha$, $i = 1, 2, 3$.

The angular momentum of the fluid in the $\alpha^{\text{th}}$ cavity (with respect to the center of mass 0) is given by

$$\begin{gathered} M^i_\alpha = \varrho_\alpha \int (\mathbf{x} \times \mathbf{v})^i dx^1 dx^2 dx^3 = \sum_{j,k=1}^{3} \left(-\tfrac{1}{2} \varepsilon_{ijk} M_{\alpha jk} + Q^i_j I_{\alpha jk} A^k\right), \\ M_\alpha = \mu_\alpha^{-1} (\dot{F}_\alpha F^t_\alpha - F_\alpha \dot{F}^t_\alpha) = \mu_\alpha^{-1} Q (D^2_\alpha A_0 + A_0 D^2_\alpha - 2 D_\alpha B_{0\alpha} D_\alpha) Q^t, \\ I_{\alpha jk} = m_\alpha \left( \delta_{jk} \sum_{\ell=1}^{3} (r^\ell_\alpha)^2 - r^j_\alpha r^k_\alpha \right), \qquad \mu_\alpha^{-1} = m_\alpha / 5 \end{gathered} \tag{5.6}$$

(the integral is over the cavity volume). Replacing the matrix notations by their vector counterparts, we get from Eqs. (5.6) that the total angular momentum vector of the system in the reference frame $S$ is

$$\begin{aligned} &\mathbf{M}=I\cdot\mathbf{A}+\sum_{\alpha=1}^{n}\mu_\alpha^{-1}(C_\alpha\mathbf{A}-2d_\alpha D_\alpha^{-1}\mathbf{B}_\alpha),\\ &I_{ik}=I_{0ik}+I_{1ik}+\ldots+I_{nik},\qquad C_\alpha=\mathrm{Tr}(D_\alpha^2)E-D_\alpha^2, \end{aligned}\tag{5.7}$$

where $I_{0ik}$ is the tensor of inertia of the rigid body in the reference frame $S$.

The conservation law for the total angular momentum in the reference frame $S$ is written down as

$$\dot{\mathbf{M}}=\mathbf{M}\times\mathbf{A}.\tag{5.8}$$

For the fluid motions with homogeneous deformations, Eqs. (5.3), the continuity equation $\mathrm{div}\,\mathbf{v}=0$, Eq. (5.2), is satisfied identically; the dynamical equation $d\mathbf{v}/dt=-\mathrm{grad}\,p$ is equivalent to the Helmholtz law (the vortex is frozen, $\boldsymbol{\omega}=\mathrm{curl}\,\mathbf{v}$, so $d\boldsymbol{\omega}/dt=0$ along the fluid trajectories). The vortex vector in the $\alpha^{\text{th}}$ cavity, $\boldsymbol{\omega}_\alpha$, is mapped by the isomorphism (5.5) to the matrix

$$K_{0\alpha}=\dot{F}_\alpha^tF_\alpha-F_\alpha^t\dot{F}_\alpha=Q_\alpha^tK_{1\alpha}Q_\alpha,\qquad K_{1\alpha}=D_\alpha^2B_{0\alpha}+B_{0\alpha}D_\alpha^2-2D_\alpha A_0D_\alpha.\tag{5.9}$$

The Helmholtz law ($\dot{K}_{0\alpha}=0$) is written as

$$\dot{K}_{1\alpha}=[K_{1\alpha},B_{0\alpha}].\tag{5.10}$$

Hence using the isomorphism (5.5), $K_{1\alpha}\to\mathbf{K}_\alpha$ and multiplying by $\mu_\alpha^{-1}$, we get

$$\dot{\mathbf{K}}_\alpha=\mathbf{K}_\alpha\times\mathbf{B}_\alpha,\qquad \mathbf{K}_\alpha=\mu_\alpha^{-1}(C_\alpha B_\alpha-2d_\alpha D_\alpha^{-1}\mathbf{A}).\tag{5.11}$$

Equations (5.8), (5.11) describe completely the dynamics of the rigid body with $n$ ellipsoid cavities filled with the ideal incompressible fluid being in a state of homogeneous vortex motion. After the Legendre transformation $\mathbf{A},\mathbf{B}_\alpha\to\mathbf{M},\mathbf{K}_\alpha$ (which is evidently symmetrical) these equations are reduced to the system

$$\begin{aligned} &\dot{\mathbf{M}}=\mathbf{M}\times\mathbf{A},\qquad \dot{\mathbf{K}}_\alpha=\mathbf{K}_\alpha\times\mathbf{B}_\alpha,\\ &A_i=\partial H/\partial M_i,\qquad B_{\alpha i}=\partial H/\partial K_{\alpha i}, \end{aligned}\tag{5.12}$$

where the Hamiltonian $H$ is the total kinetic energy of the rotation of the rigid body and the fluid in every cavity. The Hamiltonian is

$$\begin{aligned} &2H=(\mathbf{M},\mathbf{A})+\sum_{\alpha=1}^{n}(\mathbf{K}_\alpha,\mathbf{B}_\alpha)=(G^{-1}\mathbf{N},\mathbf{N})+\sum_{\alpha=1}^{n}\mu_\alpha(\mathbf{K}_\alpha,C_\alpha^{-1}\mathbf{K}_\alpha),\\ &\mathbf{N}=\mathbf{M}+\sum_{\alpha=1}^{n}2d_\alpha C_\alpha^{-1}D_\alpha^{-1}\mathbf{K}_\alpha,\\ &G=I+\sum_{\alpha=1}^{n}\mu_\alpha^{-1}C_\alpha^{-1}(C_\alpha^2-4d_\alpha^2D_\alpha^{-2}). \end{aligned}\tag{5.13}$$

Equations (5.12) are determined in the space $R^{3n+3}$; they are a special case of Euler's equations in the space $L_{n+1}^*$ which is conjugate to the Lie algebra of the group $G_{n+1}=\mathrm{SO}(3)\times\ldots\times\mathrm{SO}(3)$ ($n+1$ factors). The Hamiltonian $H$ contains $6n+6$ independent parameters which are components of the symmetrical matrices $I,D_1,\ldots,D_n$ (note that the parameters $\mu_\alpha$, or the densities $\varrho_\alpha$, are inessential, since they are excluded by means of the substitution $\bar{D}_\alpha = \mu_\alpha^{-1/2} D_\alpha$). Scalar products combining all the pairs $\mathbf{K}_\alpha, \mathbf{K}_\beta$ and $\mathbf{K}_\alpha, \mathbf{M}$ are present in the quadratic form $H$ in Eq. (5.13); that is to say, the form $H$ is essentially non-diagonal in the considered natural basis.

Equations (5.12) have $n+1$ geometrical integrals, besides the Hamiltonian $H$; they are

$$J_0 = (\mathbf{M}, \mathbf{M}), \qquad J_\alpha = (\mathbf{K}_\alpha, \mathbf{K}_\alpha), \qquad \alpha = 1, \ldots, n, \tag{5.14}$$

where $J_0$ is the integral corresponding to the total angular momentum squared. The level surfaces for the integrals given in (5.14) are orbits $\mathcal{O}$ of the co-adjoint representation of the Lie group $G_{n+1}$ in the space $L_{n+1}^*$. The manifolds $\mathcal{O} = S^2 \times \ldots \times S^2$ ($n+1$ factors) have the standard symplectic structure with respect to which Eqs. (5.12) are Hamiltonian equations with the Hamiltonian function $H$, Eq. (5.13).

*III.*

Euler's equations for the Lie algebra $\mathrm{SO}(4) = \mathrm{SO}(3) \times \mathrm{SO}(3)$ are derived from (5.12) in two particular cases. The first one is the classical case; it is $n = 1$ and there are 12 free parameters which are the components $I_{ik}, D_{ik}$. The second case corresponds to $n = 2$ and $J_0 = 0$ (i.e. $\mathbf{M} = 0$); here the equations given in (5.12) are reduced to

$$\dot{\mathbf{K}}_1 = \mathbf{K}_1 \times \mathbf{B}_1, \qquad \dot{\mathbf{K}}_2 = \mathbf{K}_2 \times \mathbf{B}_2. \tag{5.15}$$

These equations [the corresponding Hamiltonian is that of (5.13) for $M = 0$] contain 18 free parameters which are components of $I_{ik}$, $D_{1ik}$, and $D_{2ik}$. Clearly, they are Euler's equations for SO(4). In the case where the matrices $I, D_1, D_2$ are diagonal, we have a nine-dimensional region of homogeneous Hamiltonians of the type (1.6), with $r_i = q_i = 0$.

Let us study the possibility of applying the above physical interpretation to the integrable cases specified in Eqs. (2.3) and (2.9). For $\kappa = 1$ the class $A$ Lie algebras are transformed to the class $B$ Lie algebras with $n_i = m_i$ by means of the substitution $\bar{X}_i = \frac{1}{2}(X_i + Y_i)$, $\bar{Y}_i = \frac{1}{2}(X_i - Y_i)$. Thus we get the decomposition of the Lie algebra $\mathrm{SO}(4) = \mathrm{SO}(3) + \mathrm{SO}(3)$ ($n_i = 1$, $\kappa = 1$), and Euler's equations (1.3) are transformed to Eqs. (1.4), namely, the dynamical equations of (5.15), where $K_{1i} = M_i + K_i$, $K_{2i} = M_i - K_i$. This transformation applied to the Hamiltonian of Eq. (1.6), for $c_i = r_i = q_i = 0$, that has been considered in Sect. 2, leads to the Hamiltonian

$$2H = \sum_{i=1}^{3} (\alpha_i (K_{1i}^2 + K_{2i}^2) + 2\beta_i K_{1i} K_{2i}), \qquad 4\alpha_i = a_i + b_i, \qquad 4\beta_i = a_i - b_i. \tag{5.16}$$

In terms of the new coordinates, the integrability cases (2.3) and (2.9) are given by the conditions

$$\beta_1 = -2\alpha_1 + \alpha_2 + \alpha_3, \qquad \beta_2 = \alpha_1 - 2\alpha_2 + \alpha_3, \qquad \beta_3 = -\alpha_1 - \alpha_2 + 2\alpha_3, \tag{5.17}$$

and

$$\beta_i = 3\alpha_i - \alpha_1 - \alpha_2 - \alpha_3, \tag{5.18}$$

respectively. The corresponding Euler's equations (5.15) are integrable at the level of the first integrals $(\mathbf{K}_1, \mathbf{K}_1) = (\mathbf{K}_2, \mathbf{K}_2)$ [i.e. $J_3 = (\mathbf{M}, \mathbf{K}) = 0$].

Let us consider a rigid body with two ellipsoid cavities, the symmetry axes of which are parallel to the principal axes of the tensor of inertia $I_{ik}$; in other words, three matrices $I, D_1, D_2$ are diagonal simultaneously in the reference frame $S$ which is related to the axes of the tensor $I_{ik}$. Suppose the fluid densities $\varrho_1, \varrho_2$ and the semi-axes of the cavities satisfy the similitude conditions,

$$d_{1i}/d_{2i}=(\varrho_2/\varrho_1)^{1/5}. \tag{5.19}$$

Then we have $\mu_1^{-1/2}d_{1i}=\mu_2^{-1/2}d_{2i}=d_i$. The corresponding Hamiltonian $H$ of the form (5.13) is reduced to that of (5.16), where

$$\begin{gathered}\beta_i=4d_j^2d_k^2(d_j^2+d_k^2)^{-2}(I_i+2(d_j^2-d_k^2)^2(d_j^2+d_k^2)^{-1})^{-1}>0,\\ \alpha_i=\beta_i+(d_j^2+d_k^2)^{-1}, \qquad i,j,k=1,2,3.\end{gathered} \tag{5.20}$$

The integrable case Eq. (2.9), or equivalently, Eq. (5.18), does not satisfy the physical conditions (5.20), as it follows from (5.18) that $\beta_1+\beta_2+\beta_3=0$, while $\beta_i>0$ in (5.20).

After the substitution (5.20), the conditions for the existence of the integrable case (2.3) [Eq. (5.17)] acquire the following form

$$\begin{aligned}
&d_1^4+d_2^4=2d_3^4,\\
&R_2I_2=I_1(d_1^2+d_3^2)^{-1}(16d_1^2d_3^2(d_2^2+d_3^2)+6(d_1^2-d_3^2)^2(d_2^2-d_1^2))\\
&\quad+(d_2^2-d_1^2)\left(12\frac{(d_1^2-d_3^2)^2(d_2^2-d_3^2)^2}{(d_1^2+d_3^2)(d_2^2+d_3^2)}\right.\\
&\quad\left.+32\frac{d_3^2(2d_1^2d_2^2-d_3^2(d_1^2+d_2^2))}{d_1^2+d_2^2}\right),\\
&R_3I_3=I_1(d_1^2+d_2^2)^{-1}(8d_1^2d_2^2(d_2^2+d_3^2)+6(d_1^2-d_2^2)^2(d_3^2-d_1^2))\\
&\quad+\frac{d_2^2-d_1^2}{d_2^2+d_3^2}\left(12\frac{(d_2^2-d_3^2)^2(d_2^2-d_1^2)(d_3^2-d_1^2)}{d_1^2+d_2^2}\right.\\
&\quad\left.+8d_2^2(d_1^2d_2^2+d_3^2(3d_1^2-4d_2^2))\right),\\
&R_j=16\frac{d_2^2d_3^2}{d_2^2+d_3^2}(d_1^2+d_2^2+d_3^2-d_j^2)+3I_1(d_1^2-d_j^2)\\
&\quad+6(d_2^2-d_3^2)^2\frac{d_1^2-d_j^2}{d_2^2+d_3^2}, \qquad j=2,3.
\end{aligned} \tag{5.21}$$

Conditions (5.21) determine a relation between the parameters $d_1, d_2, d_3$ and give expressions for the components $I_2, I_3$ in terms of $I_1, d_1, d_2, d_3$. For $d_1\approx d_2$, Eq. (5.21) leads to $d_1\approx d_2\approx d_3$, $I_1\approx I_2\approx 2I_3$, so the necessary physical condition $I_i+I_j>I_k$ is fulfilled.

Thus the integrable case (2.3) [or (5.17)] describes the rotation of a rigid body with two ellipsoid cavities filled with the ideal incompressible fluid under the conditions (5.19)–(5.21) at the level of the first integrals $(\mathbf{M},\mathbf{M})=0$, $(\mathbf{K}_1,\mathbf{K}_1)=(\mathbf{K}_2,\mathbf{K}_2)$. As it was shown in Sect. 3, in this case Euler's equations (1.3), (5.15) are integrated explicitly in terms of elliptic functions of the time variable.

## References


1. Steklov, V.A.: On motion of rigid body in a fluid (in Russian). Kharkov: Darre 1893
2. Lyapunov, M.A.: A new case of integrability of differential equations describing rigid body in a fluid (in Russian). Trans. Kharkov Math. Soc. **4**, 10–14 (1893)
3. Kowalewski, S.V.: Sur la probleme de la rotation d'un corps solide autour d'un point fixé. Acta Math. **12**, 177–232 (1889)
4. Chaplygin, S.A.: A new particular solution for the problem of motion of a rigid body in a fluid (in Russian). Proc. Depart. Physical Sciences, Imper. Soc. Naturalists **11**, 56–62 (1902)
5. Novikov, S.P.: Variational methods and periodical solutions of Kirchhoff-type equations. II (in Russian). Funkt. Anal. Pril. **15**, 37–53 (1981)
6. Adler, M., van Moerbeke, P.: Kowalewski's asymptotic method, Kac-Moody Lie algebras and regularization. Commun. Math. Phys. **83**, 83–106 (1982)
7. Manakov, S.V.: A note on integration of Euler's equations for dynamics of $n$-dimensional rigid body. Funkt. Anal. Pril. **10**, 93–94 (1976)
8. Greenhill, A.G.: On the general motion of a liquid ellipsoid. Proc. Cambridge Philos. Soc. **4**, No. 4 (1880)
9. Zhukovsky, N.E.: On the motion of a rigid body with cavities filled with a homogeneous droplet liquid (in Russian). J. Russ. Phys. Chem. Soc., Part Phys. **17**, 81–113 (1885)
10. Poincaré, H.: Sur la precession des corps deformables. Bull. Astronom. **27**, 21–34 (1910)
11. Steklov, V.A.: Sur la mouvement d'un corps solide ayant une cavité de forme ellipsoidale remplie par un liquide incompressible et sur les variations des latitudes. Ann. Fac. Sci. Toulouse, Ser. 3, **1**, 145–256 (1909)
12. Bogoyavlensky, O.I.: Integrable Euler's equations on six-dimensional Lie algebras. Dokl. Akad. Nauk USSR **268**, 11–15 (1983)
13. Bogoyavlensky, O.I.: Fourth-order integrals for Euler's equations on six-dimensional Lie algebras. Dokl. Akad. Nauk USSR **273**, 15–19 (1983)
14. Bogoyavlensky, O.I.: Model of pulsar rotation and Euler's equations on Lie algebras. International Congress of Mathematicians 1982, Warsaw. Short Communications (Abstracts) **11**, 27 (1983)
15. van Moerbeke, P.: Algebraic complete integrability of Hamiltonian systems and Kac-Moody Lie algebras. Invited lecture at the International Congress of Mathematicians 1982, Warsaw. (Preprint 1983)
16. Perelomov, A.M.: Lax representation for the system of S. Kovalevskaya type. Commun. Math. Phys. **81**, 239–243 (1981)
17. Bogoyavlensky, O.I.: Methods of qualitative theory of dynamical systems in astrophysics and gas dynamics (in Russian). Moscow: Nauka 1980 [Engl. transl.: Berlin, Heidelberg, New York, Tokyo: Springer 1984 (to be published)]


Communicated by Ya. G. Sinai

Received November 20, 1983