Commun. Math. Phys. 92, 309–327 (1984)

Communications in
Mathematical
Physics

The Spectrum of the Transfer Matrix in the C^*-Algebra of the Ising Model at High Temperatures

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Abstract. We investigate the state on the Fermion algebra which gives rise to the thermodynamic limit of the Gibbs ensemble in the two-dimensional Ising model on a half lattice with nearest neighbour interaction. It is shown that the operator P_∞^- in the GNS space, which performs the essential functions of the renormalized transfer matrix, has a quasi-particle structure.

1. Introduction

In lattice models with an interaction potential of finite range, the free energy in a finite volume is determined by the largest eigenvalue of a matrix, known as the transfer matrix. One question which naturally arises is how to normalize the transfer matrix so that it becomes a well-defined operator in the thermodynamic limit. Such a renormalization is easy to make in the domain of Gibbs-state uniqueness (Minlos and Sinai [19]). The limit in this case is a stochastic operator which has a property of asymptotic multiplicativeness which suggests the conjecture that the spectrum of the operator has a quasi-particle structure: there is a grading of the Hilbert space on which the stochastic operator acts into subspaces corresponding to different sets of quasi-particle occupation numbers; these subspaces are invariant under the action of the stochastic operator; on these subspaces the stochastic operator has a simple structure and acts by multiplication. A general analysis of the spectral properties of a stochastic operator arising from a transfer matrix was undertaken by Minlos and Sinai [19] who contructed the single-particle subspace assuming a cluster-property of the transfer-matrix. The first proof of this cluster-property for the two-dimensional Ising model with nearest neighbour interactions was provided by Abdulla-Zade et al. [1]. Malyshev [14, 15] used cluster expansions to make improved estimates of matrix elements and which enabled him to work in arbitrary dimensions, Malyshev and Minlos [17, 18] used these estimates to prove that, for sufficiently small values of β, an operator with the cluster-property has invariant subspaces which are reminiscent of the n-particle subspaces of Fock space; the restriction of the operator to the

n-particle subspace has its spectrum in an interval $[c_1\beta^n, c_2\beta^n]$; these intervals do not overlap.

The analogy of the quasi-particle structure described above to the grading of Fock space suggests that another approach might be used in the case of the two-dimensional Ising model. It is well-known that the Onsager-Kaufmann treatment [20, 7, 8] can be re-formulated in terms of the Fermion algebra (Schultz et al. [22]). In the thermodynamic limit the Gibbs state corresponding to periodic boundary conditions in the finite lattice induces a Fock state ω_β on the CAR algebra $A(l^2(\mathbb{Z}))$ for $0<\beta<\infty$, as was shown by Pirogov [21] and Lewis and Sisson [11, 12]. Because of the translation invariance of this state, all n-point functions are determined by its restriction $\bar{\omega}_\beta$ to the algebra $A(l^2(\mathbb{Z}^+))$ [regarded as a subalgebra of $A(l^2(\mathbb{Z}))$]; the restricted state $\bar{\omega}_\beta$ is a non-Fock quasi-free state. It is primary for $\beta<\beta_c$ and nonprimary for $\beta>\beta_c$ (Lewis and Winnink [13]). The primary decomposition in the $\beta>\beta_c$ regime has been determined and the primary components ω_+ and ω_- identified with the Gibbs states corresponding to $\pm$-boundary conditions (Kuik [9] and Kuik and Winnink [10]). It is conjectured that (at least in the $\beta<\beta_c$ regime) there is a grading of the GNS-space of the state $\bar{\omega}_\beta$ which corresponds to the quasi-particle structure discovered by Minlos and Sinai [19]. In this paper we begin the investigation of this conjecture by investigating the spectrum of the GNS-representation of the renormalized transfer-matrix. In order to do this we develop the theory of Wick-ordering relative to an arbitrary quasi-free state on the CAR algebra, analogous to to the well-known theory for the CCR algebra (see [6, 23] for example). This is described in Sect. 2. In Sect. 3 we give details of the C^*-algebra formulation of the two-dimensional Ising model (following Sisson [24] and Kuik [9]) and define the operator P_∞^- on the GNS-space which performs the essential functions of the renormalized transfer matrix. Our main result is proved in Sect. 4: for $\beta<\beta_c$ the spectrum of the restriction of P_∞^- to F_β^n is contained in the interval $[e^{-2n(K_1^*+K_2)}, e^{-2n(K_1^*-K_2)}]$; thus given $N>0$, there exists a β_N such that for all $\beta<\beta_N$ the spectra of $P_\infty^-|_{F_\beta^n}$, $n=0, 1, \ldots, N$, and $P_\infty^-\Big|\Big(\bigoplus_{n=0}^{N} F_\beta^n\Big)^\perp$ are disjoint. This used the detailed results of Onsager [20] for the two-dimensional Ising model and may be regarded as a sharpening of the results of Malyshev and Minlos [17, 18] for this special case. The results of Sect. 2 on Wick-ordering may be of independent interest.

2. Quasi-Free States on the Clifford Algebra and the Associated Grading

Let H be a real Hilbert space and $s(\cdot,\cdot)$ denoting the real inner product on H. Let $C(H)$ denote the C^*-Clifford algebra [2] generated by self adjoint operators $\{\Gamma(f): f\in H\}$ which satisfy the relations

$$\Gamma(f)\Gamma(g)+\Gamma(g)\Gamma(f)=2s(f,g)1, \qquad f,g\in H.$$

We often identify f with $\Gamma(f)$, and let $C_0(H)$ denote the dense $*$-subalgebra generated by H.

Given a state ω on $C(H)$, there exists an unique covariance operator C_ω on H such that

$$\omega(fg)=s(f,g)+is(C_\omega f,g), \qquad f,g\in H$$

and $\|C_\omega\| \leqq 1$, $C_\omega^* = -C_\omega$. Conversely, given such an operator, one can construct a so-called quasi-free state on $C(H)$, which is completely determined by its two point functions [2]. Here we give an alternative, constructive proof of this, adapted to our need for a grading of the GNS Hilbert space into n-particle spaces, for $n=0,1,2,\ldots,$.

Let A be a skew-adjoint contraction on H, and define a hermitian inner product $\langle\cdot,\cdot\rangle_A$ on H by

$$\langle f,g\rangle_A = s(f,g) + is(Af,g), \qquad f,g\in H.$$

If A is a complex structure, we let $(H^A, \langle\cdot,\cdot\rangle_A)$ denote the complexification of $(H, s(\cdot,\cdot))$ via $(\alpha+i\beta)\phi = \alpha\phi, +\beta A\phi$, $\phi\in H$, $\alpha,\beta\in\mathbb{R}$.

For the skew contraction A, we define a grading $C_0(H) = \sum_{n=0}^{\infty} C_A^{(n)}(H)$ as follows: If $I=\{i_1<\ldots<i_r\}$ is a finite ordered set with cardinality $|I|=r$, we let $\mathscr{D}_I$ denote the set of all subsets of I with the induced ordering. If $J,K\in\mathscr{D}_I$, $J=\{j_1,\ldots,j_s\}$, $K=\{k_1,\ldots,k_l\}$, with $I=I\cup K$, $J\cap K=\emptyset$, let $\varepsilon(J,K)$ denote the signature of the permutation $\begin{pmatrix} & i_1,\ldots,i_r & \\ j_1,\ldots,j_s & & k_1,\ldots,k_l\end{pmatrix}$. If $a_{ij}\in\mathbb{C}$, for $i,j\in I$, with $|I|=2n$ and even, let

$$Pf[a_{ij}] = \sum \varepsilon(J,K) a_{j_1k_1}a_{j_2k_2}\ldots a_{j_nk_n},$$

where the summation is over all disjoint J, K in $\mathscr{D}_I$ with

$$J=\{j_1,\ldots,j_n\}, \quad K=\{k_1,\ldots,k_n\} \quad \text{and} \quad j_m<k_m,\ m=1,\ldots,n.$$

with $Pf[a_{ij}]=1$ if $I=\emptyset$. If $\{f_i : i\in I\}\subseteqq H$, we let $f_I = f_{i_1}\ldots f_{i_r}$, $(r=|I|)$, $f_\phi=1$, and

$$\begin{aligned} \omega_A(f_I) &= 0, && \text{if } |I| \text{ odd},\\ \omega_A(f_I) &= Pf[\langle f_i,f_j\rangle_A : i,j\in I], && \text{if } |I| \text{ even}, \end{aligned}$$

so that $\omega_A(fg) = \langle f,g\rangle_A$. Then define the Wick ordered product by

$$:f_I: = :f_I:_A = \sum (-1)^{|K|/2}\varepsilon(J,K) f_J \omega_A(f_K), \tag{2.1}$$

where the summation is over all disjoint J, K in $\mathscr{D}_I$, with $J\cup K=I$ (cf. [3, 6, 23]). Then define $C_A^{(n)}$ to be the complex subspace of $C_0(H)$ generated by $\{:f_1\ldots f_n:_A : f_i\in H\}$.

Lemma 2.1. *With the above notation:*

$$f_I = \sum_{\substack{J\cup K=I\\ J\cap K=\emptyset}} \varepsilon(J,K) :f_J: \omega_A(f_K), \tag{2.2}$$

$$f:f_I: = :ff_I: + \sum_{s=1}^{r} (-1)^{s+1} :f_{i_1}\ldots\hat{f}_{i_s}\ldots f_{i_r}: \omega_A(ff_{i_s}), \tag{2.3}$$

where $\hat{}$ *over an element means that element is omitted.*

$$:f_{i_1}\ldots f_{i_r}: \textit{ is an anti-symmetric function of } (i_1,\ldots,i_r). \tag{2.4}$$

If B is also a skew contraction then

$$:f_I:_B = \sum_{\substack{J\cup K=I\\ J\cap K=\emptyset}} \varepsilon(J,K):f_J:_A Pf[\langle f_i, f_j\rangle_A - \langle f_i, f_j\rangle_B : i,j\in K]. \tag{2.5}$$

Proof. We first show (2.3). By the definition of Wick ordering we have

$$\begin{aligned}
:ff_I: &= \sum_{\substack{J\cup K=I\\ J\cap K=\emptyset}} (-1)^{|K|/2}\varepsilon(J,K)ff_J\omega_A(f_K)\\
&\quad+\sum_{\substack{J\cup K=I\\ J\cap K=\emptyset}} (-1)^{(|K|+1)/2}(-1)^{|J|}\varepsilon(J,K)f_J\omega_A(f\,f_K)\\
&= f:f_I:\\
&\quad+\sum_{\substack{J\cup K=I\\ J\cap K=\emptyset}} (-1)^{(|K|+1)/2}(-1)^{|J|}\varepsilon(J,K)f_J\omega_A(f\,f_K)
\end{aligned}$$

A Pfaffian expansion of $\omega_A(ff_K)$ now gives the result. Suppose (2.2) holds for $|I|=n$. Then inductively consider

$$\begin{aligned}
ff_I &= \sum_{\substack{J\cup K=I\\ J\cap K=\emptyset}} \varepsilon(J,K)f:f_J:\omega_A(f_K)\\
&= \sum_{\substack{J\cup K=I\\ J\cap K=\emptyset}} \varepsilon(J,K):ff_J:\omega_A(f_K)\\
&\quad+\sum_{t=1}^{s}\sum_{\substack{J\cup K=I\\ J\cap K=\emptyset}} \varepsilon(J,K)(-1)^{t+1}:f_{j_1}\dots\hat{f}_{j_t}\dots f_{j_s}:\omega_A(ff_{j_t})\omega_A(f_K)\\
&\quad[\text{by (2.3), if } J=\{j_1,\dots,j_s\}]\\
&= \sum_{\substack{J\cup K=I\\ J\cap K=\emptyset}} \varepsilon(J,K):ff_J:\omega_A(f_K)\\
&\quad+\sum_{\substack{J_0\cup K_0=I\\ J_0\cap K_0=\emptyset}} \varepsilon(J_0,K_0):f_{J_0}:\omega_A(ff_{K_0})(-1)^{|J_0|}
\end{aligned}$$

again by elementary Pfaffian considerations, which shows that (2.2) holds for $|I|=n+1$.

Assume inductively that $:f_{i_1}\dots f_{i_r}:$ is an anti-symmetric function of $(i_1,\dots,i_r)$ if $r<n$. Then by (2.2), if $I=\{i_1,i_2,\dots,i_n\}$, $I_0=\{i_3,i_4,\dots,i_n\}$, we have

$$\begin{aligned}
f_I &= \sum_{\substack{J\cup K=I\\ J\cap K=\emptyset}} \varepsilon(J,K):f_J:\omega_A(f_K)\\
&= \sum_{\substack{J\cup K=I_0\\ J\cap K=\emptyset}} \varepsilon(J,K)\{:f_{i_1}f_{i_2}f_J:\omega_A(f_K)\\
&\quad+(-1)^{|J|}:f_{i_1}f_J:\omega_A(f_{i_2}f_K)\\
&\quad+(-1)^{(|J|+1)}:f_{i_2}f_J:\omega_A(f_{i_1}f_K)\\
&\quad+:f_J:\omega_A(f_{i_1}f_{i_2}f_K)\}.
\end{aligned}$$

Hence by adding a similar expression for $f_{i_2}f_{i_1}f_{i_3}\dots f_{i_n}$, and using the inductive hypothesis we get:

$$2s(f_{i_1},f_{i_2})f_{I_0} = :f_{i_1}f_{i_2}f_{I_0}: + :f_{i_2}f_{i_1}f_{I_0}: \\ + 2s(f_{i_1},f_{i_2}) \sum_{\substack{J\cup K = I_0 \\ J\cap K=\emptyset}} \varepsilon(J,K) :f_J: \omega_A(f_K).$$

Hence $:f_{i_1}f_{i_2}f_{I_0}: = -:f_{i_2}f_{i_1}f_{I_0}:$, using (2.2) for I_0. In this manner, $:f_{i_1}\dots f_{i_n}:$ is seen to be antisymmetric. Finally (2.5) follows from the definition of $:\ :_B$ and (2.2) for $:\ :_A$, and Pfaffian expansions.

Lemma 2.2. *If* $n\geqq 1$, *then* $((f_i)_{i=1}^n, (g_i)_{i=1}^n)\to\det[\langle f_i, g_j\rangle_A]$ *is positive definite on* $H^n\times H^n$.

Proof. We first show that $(f,g)\to\langle f,g\rangle_A$ is positive definite on $H\times H$. If A is a complex structure, then $\langle\cdot,\cdot\rangle_A$ is the complex inner product on the complexification H^A and is clearly positive definite. In general let $A=U|A|$ be the polar decomposition of A on H. Then on $H_0=\mathrm{Range}(|A|)$, $U^2=-1$, $U^*=-U$, i.e. $U_0=U|_{H_0}$ is a complex structure. Then

$$\langle f,g\rangle_A = s((1-|A|)f,g) + [s(|A|^{1/2}f,|A|^{1/2}g) + is(U|A|^{1/2}f,|A|^{1/2}g)].$$

The first term is a positive definite function of (f,g) because $\|A\|\leqq 1$, and the second is positive definite by considering the complex structure U_0 on $(H_0, s_{|H_0\times H_0})$. It merely remains to show that if $A_{ij}\in M_n(\mathbb{C})$ for $i,j=1,\dots,m$ and $[A_{ij}]$ is positive in $M_m(M_n(\mathbb{C}))$, then $[\det(A_{ij})]$ is positive in $M_m(\mathbb{C})$, (for then consider $(f_r^i)_{r=1}^n\in H^n$, $i=1,\dots,m$ and $A_{ij}=[\langle f_r^i, f_s^j\rangle_A]_{r,s=1}^n$, $i,j=1,\dots,m$). Let $[A_{ij}]=[C_{ij}]^2$, where $[C_{ij}]$ is self adjoint in $M_m(M_n(\mathbb{C}))$. Then

$\det(A_{ij}) = A_{ij}\wedge\dots\wedge A_{ij}$ (n-factors); but

$$\begin{aligned}[A_{ij}\otimes\dots\otimes A_{ij}] &= \sum_{r_1,\dots,r_n=1}^{m} [C_{ir_1}C_{r_1j}\otimes C_{ir_2}C_{r_2j}\otimes\dots\otimes C_{ir_n}C_{r_nj}] \\ &= \sum [(C_{ir_1}\otimes\dots\otimes C_{ir_n})(C_{r_1j}\otimes\dots\otimes C_{r_nj})] \\ &= \sum [(C_{r_1i}\otimes\dots\otimes C_{r_ni})^*(C_{r_1j}\otimes\dots\otimes C_{r_nj})]\geqq 0;\end{aligned}$$

and so by cutting down to $\mathbb{C}^n\wedge\dots\wedge\mathbb{C}^n$:

$$[\det A_{ij}]\geqq 0.$$

Let (C_n, F_A^n) denote the minimal Kolmogorov decomposition [4] of the positive definite kernel $((f_i),(g_i))\to\det[\langle f_i, g_j\rangle_A]$ on $H^n\times H^n$. Then $C_n(f_1,\dots,f_n)$ is an antisymmetric function $(f_1,\dots,f_n)$. Define $F_A=\bigoplus_{n=0}^{\infty} F_A^n$, where F_A^0 is a one-dimensional Hilbert space spanned by a unit vector $\Omega=\Omega_A$. If $f\in H$, then elementary computations with determinants show that

$$\pi_0(f)C_n(f_1,\dots,f_n) = C_{n+1}(f,f_1,\dots,f_n) \\ + \sum_{i=1}^{n} (-1)^{i+1}\langle f,f_i\rangle_A C_{n-1}(f_1,\dots,\hat{f_i},\dots,f_n)$$

defines a bounded operator $\pi_0(f)$ on F_A. It is easy to check that $\pi_0(f)$ is self-adjoint, and $\pi_0(f)\pi_0(g)+\pi_0(g)\pi_0(f)=2s(f,g)$, $f,g\in H$. Hence there exists an unique representation $\pi=\pi_A$ of $C(H)$ on F_A such that $\pi(\Gamma(f))=\pi_0(f)$. Moreover

$$\pi(:f_1,\dots,f_n:)\Omega=C_n(f_1,\dots,f_n), \qquad f_i\in H. \tag{2.6}$$

Assume, inductively, that this is so for $n-1$. Then

$$\pi(:f_1\dots f_n:)\Omega=\pi(f_1)\pi(:f_2\dots f_n:)\Omega-\sum_{i=2}^{n}(-1)^i\langle f_1,f_i\rangle\pi(:f_2\dots\hat{f}_i\dots f_n:)\Omega$$

by (2.3)

$$=\pi(f_1)C_{n-1}(f_2,\dots,f_n)\Omega-\sum_{i=2}^{n}(-1)^i\langle f_1,f_i\rangle_A C_{n-2}(f_2,\dots,\hat{f}_i,\dots,f_n)$$

$$=C_n(f_1,\dots,f_n) \text{ by definition of } \pi(f_1).$$

Thus (π_A,F_A,Ω_A) is a cyclic representation of the Clifford algebra $C(H)$. Define a state ω_A on $C(H)$ by $\omega_A(x)=\langle\pi_A(x)\Omega_A,\Omega_A\rangle$, for $x\in C(H)$. Claim that

$$\begin{aligned}\omega_A(f_1f_2\dots f_n)&=0, & n\text{ odd},\\ &=Pf[\langle f_i,f_j\rangle_A], & n\text{ even},\end{aligned} \tag{2.7}$$

$$\omega_A(:f_m\dots f_1::g_1\dots g_n:)=\det[\langle f_i,g_j\rangle_A]\delta_{nm}. \tag{2.8}$$

(2.7) follows from (2.2), and (2.8) is a consequence of (2.6), and $:f_m\dots f_1:^*=:f_1\dots f_m:$.

We summarise this by

Proposition 2.3. *If A is a skew contraction on H, there exists an unique state ω_A on $C(H)$ such that*

$$\omega_A(f_1\dots f_n)=Pf[\langle f_i,f_j\rangle_A] \quad \text{if } n \text{ is even},$$

$$\omega_A(f_1\dots f_n)=0 \quad \text{if } n \text{ is odd},$$

$$\omega_A(:f_m,\dots,f_1::g_1\dots g_n:)=\det[\langle f_i,f_j\rangle_A]\delta_{nm}.$$

There is a grading $F_A=\bigoplus_{n=0}^{\infty}F_A^n$ of the GNS *Hilbert space of ω_A such that the* GNS *vector Ω_0 spans F_A^0, and if π_A is the* GNS *representation then $(f_1,\dots,f_n)\to\pi_A(:f_1\dots f_n:)\Omega_A$ is the minimal Kolmogorov decomposition of the positive definite kernel $((f_i),(g_i))\to[\det\langle f_i,g_i\rangle_A]$.*

Remark 2.4. Note that the theory of quasi-free completely positive maps developed in [3, 5] can be transformed into the real setting, e.g. if T is a contraction between real Hilbert spaces H and K intertwining with skew contractions A and B, then there exists an unique unital completely positive map $C_A(T):C(H)\to C(K)$ such that

$$C_A(T)(:f_1\dots f_n:_A)=:(Tf_1)\dots(Tf_n):_B, \qquad f_i\in H.$$

Moreover there exists an unique contraction $F_{A,B}(T)=\bigoplus_{n=0}^{\infty} F^n_{A,B}(T)$ from F_A into F_B, where $F^n_{A,B}(T):F^n_A\to F^n_B$ is given by

$$F^n_{A,B}(T)\pi_A(:f_1\ldots f_n:_A)\Omega_A=\pi_B(:(Tf_1)\ldots,(Tf_n):_B)\Omega_B\,,\qquad f_i\in H\,.$$

Remark 2.5. If A is a complex structure on H, let $a_A(f)=\frac{1}{2}[\Gamma(f)+i\Gamma(Af)]$, $a^*_A(f)=a_A(f)^*$, $f\in H$, denote the associated annihilation and creation operators. Then

$$\pi_A(:f_1\ldots f_n:)\Omega_A=\pi_A(a^*_A(f_1)\ldots a^*_A(f_n))\Omega_A\,,$$

so that F^n_A is the usual n-particle space. Moreover if T is a contraction commuting with A, then $F^n_A(T)=F^n_{A,A}(T)$ is the usual n-particle operator, and $F_A(T)=F_{A,A}(T)$ the usual second quantization.

3. The C^*-Algebra of the Ising Model

In order to establish our notation, we summarise here the C^*-formulation of the two dimensional Ising model with periodic boundary conditions. Full details may be found in [24, 11–13, 9, 10].

The two dimensional classical Ising model with nearest neighbour interactions can be reduced to a non-commutative one-dimensional system by means of the transfer matrix method. For a finite lattice

$$\Lambda=\Lambda_{LN}=\{(i,j)\in\mathbb{Z}^2:1\leqq i\leqq L,\ -N\leqq j\leqq N\}\,,$$

$P(\Lambda)$ denotes the space $\{-1,+1\}^\Lambda$ of all configurations and the algebra of observables is $C(P(\Lambda))$, the space of all complex valued functions on $P(\Lambda)$. We will always impose periodic boundary conditions on our nearest neighbour Hamiltonians. The transfer matrix method takes us from observables in the commutative $C(P(\Lambda))$ and Gibbs states $\langle\cdot\rangle_{LN}$ on $C(P(\Lambda))$ to observables and certain states associated with a (non-commutative) Paulion algebra $\mathcal{A}_L$ of $2^L\times 2^L$ complex matrices, or equivalently, a Clifford algebra $C(H_L)$ on a L-dimensional complex Hilbert space H_L. Thus if f is a local observable in $C(P(\Lambda_{LN_0}))$, say, there exists an element a_f in $C(H_L)$ and a state ϱ_{LN} on $C(H_L)$ such that $\langle f\rangle_{LN}=\varrho_{LN}(a_f)$ for all $N>N_0$. In fact, [identifying $C(H_L)$ with $M_{2^L}(\mathbb{C})$], ϱ_{LN} is given by an operator $(V_L)^{2N+1}$:

$$\varrho_{LN}=\mathrm{tr}(\cdot V_L^{2N+1})/\mathrm{tr}(V_L^{2N+1})\,.$$

This reduction leads us to study the states ϱ_{LN} on $C(H_L)$, and the thermodynamic limit ϱ on $C(H)$, if $H=\lim_L H_L$. The transfer matrix is the (normalised) limit of V_L, as $L\to\infty$. Our aim is to show the existence of this normalised limit in a suitable C^*-setting, and obtain some information on its spectrum for high temperatures. We now describe this set up in a little more detail.

First, in order to describe the Clifford algebra setting, let J be a fixed complex structure on a real infinite dimensional Hilbert space H, with inner product $s(\cdot,\cdot)$, Let $\{e_n:n=1,2,\ldots\}$ be a complete orthonormal basis for $(H^J,\langle\cdot,\cdot\rangle_J)$ so that $\{e_n, Je_n:n=1,\ldots\}$ is a complete orthonormal basis for (H,s), and let E be the closed subspace of (H,s) spanned by $\{e_n : n=1,2,\ldots\}$. Then $H=E\oplus JE$, and Λ the conjugation determined by J defined by $\Lambda\phi=\phi$, $\Lambda J\phi=-J\phi$, $\phi\in E$, satisfies $\Lambda^2=1$, $\Lambda J=-J\Lambda$ and $\tilde{P}=(1+\Lambda)/2$, $\tilde{Q}=(1-\Lambda)/2$ are the orthogonal projections on E, JE, respectively.

Let $H_L\subset H$ be the subspace spanned by $\{e_n, Je_n : n=1,2,\ldots,L\}$, and $s_L(\cdot,\cdot)$ (respectively, J_L, Λ_L, etc.) denote the restriction of $s(\cdot,\cdot)$ (respectively, J, Λ, etc.) to H_L.

The transformation of the classical theory to the Clifford algebras is done via Pauli algebras. Let $\mathscr{A}_L$ be the Paulion algebra generated by $\{\sigma_j^\alpha : j=1,\ldots,L, \alpha=x,y,z\}$ which obey mixed commutation relations $[\sigma_j^\alpha, \sigma_k^{\alpha'}]_-=0$, $j\neq k$, $\sigma_j^x\sigma_j^y=i\sigma_j^z$ and cyc., $(\sigma_j^\alpha)^2=1$. Let $\mathscr{H}$ be a two-dimensional Hilbert space with orthonormal basis $e(+)=\begin{pmatrix}1\\0\end{pmatrix}$, $e(-)=\begin{pmatrix}0\\1\end{pmatrix}$, and $\mathscr{H}_L=\bigotimes_1^L\mathscr{H}$. Let π_L be the representation of $\mathscr{A}_L$ as bounded operators on $\mathscr{H}_L$ by $\pi_L(\sigma_i^\alpha)=1\otimes\ldots\otimes\underset{i^{\text{th}}\text{ position}}{\sigma^\alpha}\otimes\ldots\otimes 1$, $\alpha=x,y,z$ where $\sigma^x=\begin{pmatrix}0&1\\1&0\end{pmatrix}$, $\sigma^y=\begin{pmatrix}0&-i\\i&0\end{pmatrix}$.

The Jordan-Wigner transformation is a $*$-isomorphism $\eta_L:\mathscr{A}_L\to C(H_L)$ and is defined by

$$\eta(\sigma_1^z)=\Gamma(e_1),$$

$$\eta(\sigma_1^y)=-\Gamma(Je_1),$$

$$\eta(\sigma_k^z)=\prod_{n=1}^{k-1}[-i\Gamma(e_n)\Gamma(Je_n)]\Gamma(e_k),\qquad k>1,$$

$$\eta(\sigma_k^y)=-\prod_{n=1}^{k-1}[-i\Gamma(e_n)\Gamma(Je_n)]\Gamma(Je_k),\qquad k>1.$$

For each finite subset $\Theta\subset\mathbb{Z}^2$, let $U(\Theta)$ denote the C^*-algebra generated by $\{\sigma_\theta^\alpha:\theta\in\Theta, \alpha=x,y,z\}$ which obey $[\sigma_\theta^\alpha, \sigma_\phi^{\alpha'}]_-=0$, $\theta\neq\phi$, $\sigma_\theta^x\sigma_\theta^y=i\sigma_\theta^z$ and cyc., $(\sigma_\theta^\alpha)^2=1$. Thus if $\Theta_j=\{(i,j):1\leqq i\leqq L\}$, $U(\Theta_j)\simeq\mathscr{A}_L$ for each j.

Taking $\Theta=\Lambda=\Lambda_{LN}$, the finite lattice described previously, the classical algebra $C(P(\Lambda))$ is isomorphic to the C^*-algebra generated by the third component Pauli matrices $\{\sigma_\theta^z:\theta\in\Lambda\}\subset U(\Lambda)$. Moreover, imposing nearest neighbour interactions, with periodic boundary conditions, the Hamiltonian of the finite system is the observable

$$H_{LN}=-\sum_{i=1}^{L}\sum_{j=-N}^{N}[J_2\sigma_{(i,j)}^z\sigma_{(i+1,j)}^z+J_1\sigma_{(i,j)}^z\sigma_{(i,j+1)}^z]$$

[where with abuse of notation, $(\sigma_{(L+1,j)}^z, \sigma_{(i,N+1)}^z)$ are identified with $(\sigma_{(1,j)}^z, \sigma_{(i,-N)}^z)$]. Here J_1, J_2 are constants greater than zero.

Now any configuration $X=\{x_{ij}\}$, can be broken up as

$$X=\begin{pmatrix}y_L^N(X)\\ \vdots\\ y_L^{-N}(X)\end{pmatrix}$$

if $y_L^i(x)=\{x_{1,i},\dots,x_{L,i}\}\in\{-1,+1\}^L$, $-N\leqq i\leqq N$. We then have a decomposition

$$H(X)=\sum_{j=-N}^{N} S(y_L^j)+\sum_{j=-N}^{N} I(y_L^{j+1},y_L^j)$$

in terms of the internal energies of the rows and the interaction energies between neighbouring rows if

$$S(y_L^j)=-\sum_{i=1}^{L} J_2 x_{ij}x_{i+1,j},$$

$$I(y_L^m,y_L^n)=-\sum_{i=1}^{L} J_1 x_{im}x_{in},$$

identifying $x_{L+1,j}$ with $x_{1,j}$ and y_L^{N+1} with y_L^{-N} as usual.

The expectation value of any observable f is given by the Gibbs formula

$$\langle f\rangle_{LN}^P=Z_{LN}^{-1}\sum_{X\in P(\Lambda)}\{f(X)\exp[-\beta H_{LN}(X)]\},$$

where the partition function

$$Z_{LN}=\sum_{X\in P(\Lambda)}\exp[-\beta H_{LN}(X)],$$

and $\beta\geqq 0$ is the inverse temperature.

We now express this using the transfer matrix formalism. First, the partition function or free energy is given by

$$\begin{aligned}Z&=\sum_{X\in P(\Lambda)}\exp[-\beta H_{LN}(X)]\\&=\sum T_L(y_L^{-N},y_L^{-N+1})T_L(y_L^{-N+1},y_L^{-N+2})\dots T_L(y_L^{N-1},y_L^N)T_L(y_L^N,y_L^{-N})\\&=\operatorname{tr} T_L^{2N+1}\end{aligned}$$

if T, the transfer matrix is defined as the array

$$T(y,y')=\exp-\beta\{\tfrac{1}{2}[S(y)+S(y')]+I(y,y')\},$$

which is a $2^L\times 2^L$ matrix, if $y,y'\in\{-1,+1\}^L$. Then T_L defines an element V_L in the Paulion algebra $\mathscr{A}_L$ by

$$\left\langle \pi(V_L)\bigotimes_{i=1}^{L} e(\alpha_i),\bigotimes_{j=1}^{L} e(\alpha_j')\right\rangle_L=T_L(y_L^m,y_L^n),$$

where

$$\begin{aligned}\alpha_i&=\pm\quad\text{if}\quad x_{i,m}=\pm1\qquad y_L^m=\{x_{1,m},\dots,x_{L,m}\}\\ \alpha_j'&=\pm\quad\text{if}\quad x_{j,n}=\pm1\qquad y_L^n=\{x_{1,n},\dots,x_{L,n}\}.\end{aligned}$$

Then $Z=\operatorname{tr}_{\mathscr{H}_L}\pi_L(V_L^{2N+1})$.

Similarly $\sum f(X)\exp[-\beta H(X)]$ can be computed for a local observable as follows. It will be enough to consider $f=\prod_{m=-N_0}^{N_0} f_m\in C(P(\Lambda_{LN_0}))$, where each f_m is a function of the m^{th} row alone. Thus using the canonical basis

$$\left\{\bigotimes_{i=1}^{L} e(\alpha_i) : \alpha_i \in \{\pm\}, i=1,\ldots,L\right\}$$

for $\mathscr{H}_L$, each f_m determines a multiplication operator on $\mathscr{H}_L$, and hence an element $\hat{f}_m$ in the Pauli algebra $\mathscr{A}_L$. Then for $N > N_0$:

$$\begin{aligned}
&\sum_{X \in P(\Lambda_{LN}))} f(X)\exp[-\beta H(X)] \\
&= \sum T_L(y_L^{-N}, y_L^{-N+1}) \ldots T_L(y_L^{-N_0+1}, y_L^{-N_0}) f_{-N_0}(y_L^{-N_0}) \\
&\quad \cdot T_L(y_L^{-N_0}, y_L^{-N_0+1}) f_{-N_0+1}(y_L^{-N_0+1}) \ldots T_L(y_L^{N_0-1}, y_L^{N_0}) f_{N_0}(y_L^{N_0}) \\
&\quad \cdot T_L(y_L^{N_0}, y_L^{N_0+1}) \ldots T_L(y_L^{N-1}, y_L^{N}) T_L(y_L^{N}, y_L^{-N}) \\
&= \operatorname{tr} \mathscr{H}_L[\pi_L(V_L^{N-N_0} \hat{f}_{-N_0} V_L \hat{f}_{-N_0+1} \ldots \hat{f}_{N_0} V_L^{N-N_0+1}] \\
&= \operatorname{tr} \mathscr{H}_L \pi_L(V_L^{2N+1} a_f),
\end{aligned}$$

if $a_f = V_L^{-N_0} \hat{f}_{-N_0} V_L \ldots \hat{f}_{N_0} V_L^{-N_0} \in \mathscr{A}_L$.

Define states ϱ_{LN} on $\mathscr{A}_L$ by

$$\varrho_{LN}(a) = \operatorname{tr} \mathscr{H}_L[\pi_L(a)(V_L)^{2N+1}] / \operatorname{tr} \mathscr{H}_L \pi_L(V_L)^{2N+1}.$$

By linearity if f is a local observable, in $C(P(\Lambda_{LN_0}))$ say, then there exists $a_f \in \mathscr{A}_L$ such that

$$\langle f \rangle_{LN} = \varrho_{LN}(a_f) \quad \text{for all large enough } N.$$

Now

$$V_L = [2\sinh(2K_1)]^{L/2} (V_{2,L})^{1/2} V_{1,L} (V_{2,L})^{1/2},$$

where

$$V_{1,L} = \exp\left(K_1^* \sum_{i=1}^{L} \sigma_i^x\right),$$

$$V_{2,L} = \exp\left(K_2 \sum_{i=1}^{L} \sigma_i^z \sigma_{i+1}^z\right), \qquad \sigma_{L+1}^z = \sigma_1^z,$$

and

$$e^{-2K_1} = \tanh K_1^* \qquad K_i = \beta J_i. \tag{3.1}$$

Let $U_L = \prod_{k=1}^{L} [-i\Gamma(e_k)\Gamma(J_L e_k)] \in C(H_L)$, which is a self adjoint unitary such that $U_L \Gamma(\phi) = -\Gamma(\phi) U_L$, $\phi \in H_L$, with spectral projections $\bar{P}_L = (1+U_L)/2$, $\bar{Q}_L = (1-U_L)/2$. Define operators $W_L^{\pm}$ on H_L by

$$\begin{gathered}
W_L^{\pm} e_j = e_{j+1}, \quad W_L^{\pm} J_L e_j = J_L e_{j+1}, \quad 1 \leqq j \leqq L-1, \\
W_L^{\pm} e_L = \pm e_1, \quad W_L^{\pm} J_L e_L = \pm J_L e_1.
\end{gathered} \tag{3.2}$$

Define

$$\eta(V_{2,L}^{\pm}) = \exp\left\{-iK_2 \sum_{k=1}^{L} \Gamma(J_L e_k) \Gamma(W_L^{\pm} e_k)\right\}. \tag{3.3}$$

Then $\eta(V_L)=(2\sinh 2K_1)^{L/2}[\eta(V_L^-)\bar{P}_L+\eta(V_L^+)\bar{Q}_L]$, where

$$V_L^{\pm}=(V_{2,L}^{\pm})^{1/2}V_{1,L}(V_{2,L}^{\pm})^{1/2}. \tag{3.4}$$

Define operators $\gamma_L^{\pm}$, $\delta_L^{*\pm}$, $A_L^{\pm}$, $\theta_L^{\pm}$, $S_L^{\pm}$ on H_L by

$$\cosh\gamma_L^{\pm}=\cosh 2K_1^*\cosh 2K_2 1-\sinh 2K_1^*\sinh 2K_2(W_L^{\pm}+(W_L^{\pm})^{-1})/2, \tag{3.5}$$

$$\sinh\gamma_L^{\pm}\cos\delta_L^{*\pm}=\cosh 2K_1^*\sinh 2K_2 1-\sinh 2K_1^*\cosh 2K_2(W_L^{\pm}+(W_L^{\pm})^{-1})/2, \tag{3.6}$$

$$\sinh\gamma_L^{\pm}\sin\delta_L^{*\pm}=\sinh 2K_1^*[(W_L^{\pm}-(W^{\pm})^{-1})/2](-J_L), \tag{3.7}$$

$$\begin{aligned}A_L^{\pm}&=-J_L\exp[J_L\Lambda_L\delta_L^{*\pm}][(W_L^{\pm})^{-1}\tilde{P}_L+W_L^{\pm}\tilde{Q}_L]\\&=J_L\exp[2J_L\Lambda_L\theta_L^{\pm}]=S_L^{\pm}J_L(S_L^{\pm})^{-1},\end{aligned} \tag{3.8}$$

$$S_L^{\pm}=\exp[-J_L\Lambda_L\theta_L^{\pm}]. \tag{3.9}$$

Then

$$\eta(V_L^{\pm})\Gamma(x)\eta(V_L^{\pm})^{-1}=\Gamma(\cosh\gamma_L^{\pm}x)+i\Gamma(\sinh\gamma_L^{\pm}A_L^{\pm}x);\qquad x\in H_L. \tag{3.10}$$

On the complexification $H_L^{J_L}$, the spectra of $W_L^{\pm}$ are:

$$\sigma(W_L^+)=\{\exp(i\omega_{k,L}^+)\in\mathbb{C}:\omega_{k,L}^+=2k\pi/L, k=1,\dots,L\},$$
$$\sigma(W_L^-)=\{\exp i\omega_{k,L}^-\in\mathbb{C}:\omega_{k,L}^-=(2k+1)\pi/L, k=1,\dots,L\},$$

and

$$W_L^{\pm}g_{k,L}^{\pm}=e^{i\omega_{k,L}^{\pm}}g_{k,L}^{\pm}, \tag{3.11}$$

if

$$g_{k,L}^{\pm}=L^{-1/2}\sum_{n=1}^{L}e^{-J_L\omega_{k,L}^{\pm}n}e_n, \tag{3.12}$$

so that $\{g_{k,L}^{\pm}, J_Lg_{k,L}^{\pm}\}_{k=1}^{L}$ are orthonormal bases for H_L.

If we let $a_{J_L}^*(\cdot)$ denote the creation operators of the complex structure J_L, as in Remark 2.5, and $\Omega_L=\bigotimes_{k=1}^{L}e$, where $e=[e(+)+e(-)]/\sqrt{2}$, then $\pi\eta^{-1}a_{J_L}(f)\Omega_L=0$, $f\in H_L$, and so $(\pi\eta^{-1},\mathscr{H}_L,\Omega_L)$ can be identified with the GNS decomposition of ω_{J_L}. Moreover

$$\pi\eta^{-1}(\bar{P}_L)\Omega_L=\Omega_L,\qquad \pi\eta^{-1}(\bar{Q}_L)\Omega_L=0. \tag{3.13}$$

The Bogoliubov automorphisms $\alpha(S_L^{\pm}):a_{J_L}(f)\to a_{A_L^{\pm}}(S_L^{\pm}f)$ are implemented by

$$S_L^{\pm}=\exp\left\{i\sum_{0\leqq\omega_k^{\pm}\leqq\pi}\theta(\omega_{k,L}^{\pm})[a_{J_L}^*(g_{k,L}^{\pm})a_{J_L}^*(\Lambda_Lg_{k,L}^{\pm})-a_{J_L}(g_{k,L}^{\pm})a_{J_L}(\Lambda_Lg_{k,L}^{\pm})]\right\},$$

where

$$\cosh\gamma(\omega)=2\cosh 2K_1^*\cosh 2K_2-\sinh 2K_1^*\sinh 2K_2\cos\omega, \tag{3.14}$$

$$\sinh\gamma(\omega)\cos\delta^*(\omega)=\cosh 2K_1^*\sinh 2K_2-\sinh 2K_1^*\cosh 2K_2\cos\omega, \tag{3.15}$$

$$\sinh\gamma(\omega)\sin\delta^*(\omega)=\sinh 2K_1^*\sin\omega, \tag{3.16}$$

$$2\theta(\omega)=\delta^*(\omega)+\omega-\pi. \tag{3.17}$$

For $\beta<\beta_c$ (i.e. $K_2<K_1^*$), the principal eigenvalue of $\pi_L(V_L)$ is asymptotically non-degenerate and its eigenvector is $\psi_L^-=\pi_L\eta^{-1}(S_L^-)\Omega_L$. Thus

$$\begin{aligned}\lim_{N\to\infty}\varrho_{LN}(\cdot)&=\langle\pi_L\eta^{-1}(\cdot)\psi_L^-,\psi_L^-\rangle\\&=\omega_{J_L}\circ\alpha S_L^-\circ\eta^{-1}\\&=\omega_{A_L^-}\circ\eta^{-1},\quad \text{by (3.7), [12, Theorem 1].}\end{aligned}$$

Then for a local observable f,

$$\langle f\rangle_\infty=\lim_{L\to\infty}\lim_{N\to\infty}\langle f\rangle_{LN}=\lim_{L\to\infty}\omega_{A_L^-}(\eta^{-1}(a_f)).$$

The weak $\lim\limits_{L\to\infty}A_L^\pm$ exists and can be described as follows. Let L_2 be the real Hilbert space of complex-valued square integrable functions on $[0,2\pi]$ with inner product $\check{s}(f,g)=\mathrm{re}\frac{1}{2\pi}\int_0^{2\pi}f\bar{g}$, and complexification $(if)(x)=if(x)$. Then L_2^i is the complexification with inner product $\langle\cdot,\cdot\rangle$ say. If $\chi_n(p)=e^{inp}$, $p\in[0,2\pi]$, then $\{\chi_n:n\in\mathbb{Z}\}$ (respectively $\{\chi_n,\ i\chi_n:n\in\mathbb{Z}\}$) is a complete orthonormal basis for L_2^i (respectively L_2). Define $L_{2,+}^i$ (respectively $L_{2,+}$) to be the closed linear span of $\{\chi_n:n=1,2,\ldots\}$ (respectively $\{\chi_n,\ i\chi_n:n=1,2,\ldots\}$) in L_2^i (respectively L_2). Then $F(e_n)=\chi_n$ defines a unitary operator F of (H,S) onto $(L_{2,+},\check{s})$ and (H^J,s) onto $(L_{2,+}^i,\langle\cdot,\cdot\rangle)$. If A is a bounded linear operator on H or H^J, let $\check{A}=FAF^{-1}$.

If $\phi\in L_{\mathbb{C}}^\infty[0,2\pi]$, let $M(\phi)$ denote the corresponding multiplication operator on L_2 (or L_2^i). If E denotes the orthogonal projection of L_2 on $L_{2,+}$ (or L_2^i on $L_{2,+}^i$), and $\phi\in L_{\mathbb{C}}^\infty[0,2\pi]$ let $T_\phi=T(\phi)$ denote the Toeplitz operator which is the restriction of $EM(\phi)$ to $L_{2,+}$ (or $L_{2,+}^i$, respectively). Let $t(p)=\exp(2i\theta(p))$, $p\in[0,2\pi]$. Then $A=wk\lim A_L^\pm$, where

$$\check{A}=\check{J}T_{t^{-1}}\check{\check{p}}+\check{J}T_t\check{\check{Q}}.\tag{3.18}$$

The phase transition manifests itself by a jump in the mod-2 index of A [24, 12, 13, 9]. For $\beta<\beta_c$ (i.e. $K_2<K_1^*$), index $A=0$ and ω_A is primary, and for $\beta>\beta_c$ (i.e. $K_2>K_1^*$), index $A=1$ and ω_A is non-primary.

4. The Spectrum of the Transfer Matrix in the Thermodynamic Limit at High Temperature

Let $C_{00}(H)$ denote the *-sub-algebra of $C(H)$ generated by $\bigcup\limits_L H_L$, so that $C_{00}(H)=\bigcup\limits_L C(H_L)$. Suppressing the representation of $C(H_L)$ on $\mathscr{H}_L$, we can write

$$\omega_{A_L^-}=\langle(\cdot)\Omega_L,\Omega_L\rangle,$$

and similarly we let $C(H)$ act on F_A, the GNS Hilbert space of ω_A, and write

$$\omega_A=\langle(\cdot)\Omega,\Omega\rangle,\quad\text{where}\quad\Omega=\Omega_A.$$

Proposition 4.1. *There exists self adjoint contractions P_∞, P_∞^- on F_A such that*

$$\lim_{L\to\infty}\left\langle\frac{\eta(V_L)}{\lambda_L}x\Omega_L, y\Omega_L\right\rangle=\langle P_\infty x\Omega, y\Omega\rangle, \tag{4.1}$$

$$\lim_{L\to\infty}\left\langle\frac{\eta(V_L^-)}{\lambda_L}x\Omega_L, y\Omega_L\right\rangle=\langle P_\infty^- x\Omega, y\Omega\rangle \tag{4.2}$$

for all $x, y\in C_{00}(H)$, and where λ_L denotes the maximum eigenvalue of V_L.

Proof. Ω_L is the eigenvector of $\eta(V_L)$ with eigenvalue λ_L (Sect. 3) so that

$$\langle\eta(V_L)x\Omega_L, y\Omega_L\rangle/\lambda_L=\langle\eta(V_L)x\eta(V_L)^{-1}\Omega_L, y\Omega_L\rangle.$$

We claim that $\lim\limits_{L\to\infty}\langle\eta(V_L)x\eta(V_L^{-1})\Omega_L, y\Omega_L\rangle$ exists for all x, y in $C_{00}(H)$. Now

$$\eta(V_L)=(2\sinh 2K_1)^{L/2}[\eta(V_L^+)\bar{Q}_L+\eta(V_L^-)\bar{P}_L] \tag{3.4}$$

and $\bar{P}_L\Gamma(\phi)=\Gamma(\phi)\bar{Q}_L$ for all ϕ in H_L.

Let $x=\Gamma(\phi_1)\dots\Gamma(\phi_m)$, $y=\Gamma(\psi_n)\dots\Gamma(\psi_1)$, where ϕ_i, $\psi_j\in H_{L_0}$, and $L_0<\infty$. Then

$$\begin{aligned}
&\langle\eta(V_L)x\eta(V_L^{-1})\Omega_L, y\Omega_L\rangle\\
&=\langle(\eta(V_L^+)\bar{Q}_L+\eta(V_L^-)\bar{P}_L)x\eta(V_L^-)^{-1}\Omega_L, y\Omega_L\rangle \quad \text{by (3.13)}\\
&=\begin{cases}\langle\eta(V_L^-)\bar{P}_Lx\eta(V_L^-)^{-1}\Omega_L, y\Omega_L\rangle & \text{if } m \text{ even}\\ \langle\eta(V_L^+)\bar{Q}_Lx\eta(V_L^-)^{-1}\Omega_L, y\Omega_L\rangle & \text{if } m \text{ odd}\end{cases}\\
&=\begin{cases}\langle\eta(V_L^-)x(V_L^-)^{-1}\Omega_L, y\Omega_L\rangle & \text{if } m \text{ and } n \text{ even}\\ \langle\eta(V_L^+)x(V_L^-)^{-1}\Omega_L, y\Omega_L\rangle & \text{if } m \text{ and } n \text{ odd}\\ 0 & \text{otherwise}.\end{cases}
\end{aligned}$$

Case (i). m and n even.

Then

$$\begin{aligned}
&\langle V_L^-\phi_1\dots\phi_n(V_L^-)^{-1}\Omega_L, \psi_s\dots\psi_1\Omega_L\rangle\\
&=\left\langle\prod_{j=1}^{m}[\cosh\gamma_L^-\phi_j+iA_L^-\sinh\gamma_L^-\phi_j]\Omega_L, \psi_n\dots\psi_1\Omega_L\right\rangle
\end{aligned}$$

by (3.10). Expanding this as a Pfaffian (2.7), one has a finite sum of products where each factor is one of the following three kinds:

$$\omega_{A_L^-}(\psi_j\psi_k), \text{ which converges to } \omega_A(\psi_j\psi_k) \text{ as } L\to\infty, \tag{4.2a}$$

$$\omega_{A_L^-}(\psi_j\cosh\gamma_L^-\phi_k)=s(\psi_j, \cosh\gamma_L^-\phi_k)+is(A_L^-\psi_j, \cosh\gamma_L^-\phi_k). \tag{4.2b}$$

Proceeding as in [9], take $\psi_j=e_r$, $\phi_k=e_s$, where $e_r=\frac{1}{L^{1/2}}\sum\limits_{l=1}^{L}e^{J_L\omega_{\bar{l},L}r}g_{l,L}^-$, and using $A_L^-=J_L\{\cos 2\theta_L^-+J_L\Lambda_L\sin 2\theta_L^-\}$, we have:

$$s(A_L^- e_r, \cosh\gamma_L^- e_s)$$
$$= \frac{1}{L}\sum_{l,t} s((J_L\cos 2\theta_L^- - \sin 2\theta_L^-)e^{J\omega_{l,L}^- r} g_{l,L}^-, \cosh(\gamma_L^-)e^{J\omega_{t,L}^- s} g_{t,L}^-)$$
$$= \frac{1}{L}\sum_{l,t} s(\cosh(\gamma_L^-)[J_L\cos 2\theta_L^- - \sin 2\theta_L^-]e^{J(\omega_{l,L}^- r - \omega_{t,L}^- s)} g_{1,L}^-, g_{t,L}^-)$$
$$= \frac{1}{L}\sum_{l,t} s(\cosh\gamma_L^-[J_L\cos 2\theta_L^- - \sin 2\theta_L^-][\cos(\omega_{l,L}^- r - \omega_{t,L}^- s)$$
$$- J\sin(\omega_{l,L}^- r - \omega_{t,L}^- s)] g_{1,L}^-, g_{l,L}^-)$$
$$= -\frac{1}{L}\sum_{l,t} \cosh\gamma(\omega_{l,L}^-)[\sin 2\theta(\omega_{l,L}^-)\cos(\omega_{l,L}^- r - \omega_{t,L}^- s)$$
$$- \cos 2\theta(\omega_{l,L}^-)\sin(\omega_{l,L}^- r - \omega_{t,L}^- s)]\delta_{l,t}$$
$$= -\frac{1}{L}\sum \cosh\gamma(\omega_{l,L}^-)[\sin(2\theta(\omega_{l,L}^-) + \omega_{l,L}^-(r-s)]$$
$$\to \frac{-1}{2\pi}\int_0^{2\pi} \cosh\gamma(\omega)\sin[2\theta(\omega) + \omega(r-s)]d\omega,$$

a Riemann integral as $L\to\infty$.

In this way one sees as in [9] for the computation of wk limit A_L^- that

$$s(A_L^-\phi, \cosh\gamma_L^-\psi) \to s(B\phi, \psi) \text{ as } L\to\infty, \quad \text{for} \quad \phi, \psi \in H_{L_0},$$

where $\check{B} = \check{J}T(\cosh(\gamma)t^{-1})\check{\check{P}} + \check{J}T(\cosh(\gamma)t)\check{\check{Q}}$. Similarly $s(\psi, \cosh\gamma_L^-\phi) \to s(C\psi, \phi)$, where $\check{C} = T(\cosh\gamma)$

$$\omega_{A_L^-}(\psi_j A_L^- \sinh\gamma_L^-\phi_k). \tag{4.2c}$$

This is similar to the previous case.

$$[\omega_{A_L^-}(\Gamma(\cosh\gamma_L^-\phi_j) i\Gamma(A_L^- \sinh\gamma_L^-\phi_k)) + \omega_{A_L^-}(i\Gamma(A_L^- \sinh\gamma_L^-\phi_j)\Gamma(\cosh\gamma_L^-\phi_k))] = 0, \tag{4.2d}$$

$$[\omega_{A_L^-}\Gamma(\cosh\gamma_L^-\phi_j)\Gamma(\cosh\gamma_L^-\phi_k) + \omega_{A_L^-}(i\Gamma(A_L^- \sinh\gamma_L^-\phi_j) i\Gamma(A_L^- \sinh\gamma_L^-\phi_k))]$$
$$= \omega_{A_L^-}(\phi_j\phi_k), \tag{4.2e}$$

and so is the same as case (4.2a).

Hence case (i) is established.

Case (ii) m and n odd.

We compute

$$\langle V_L^+ x(V_L^-)^{-1}\Omega_L, y\Omega_L\rangle = \langle V_L^+(V_L^-)^{-1}[V_L^- x(V_L^-)^{-1}]\Omega_L, y\Omega_L\rangle,$$

where

$$V_L^+(V_L^-)^{-1} = (V_{2,L}^+)^{1/2} V_{1,L}(V_{2,L}^+)^{1/2}(V_{2,L}^-)^{-1/2} V_{1,L}^{-1}(V_{2,L}^-)^{-1/2}.$$

Now

$$\eta(V_{2,L}^\pm) = \prod_{k=1}^{L} \exp - iK_2[\Gamma(J_L e_k)\Gamma(W_L^\pm e_k)],$$

$$\eta[(V_{2,L}^+)^{1/2}(V_{2,L}^-)^{-1/2}] = \exp - iK_2[\Gamma(J_L e_L)\Gamma(e_1)],$$

and $\eta(V_{1,L}) = \prod_{k=1}^{L} \exp - iK_1^*[\Gamma(e_k)\Gamma(J_L e_k)]$. Now if ϕ, ψ are orthogonal unit vectors, $\alpha \in \mathbb{C}$, then $\mathrm{Ad}(\exp\alpha\Gamma(\phi)\Gamma(\psi))\Gamma(f) = \Gamma(g)$, if

$$g = f + \sin 2\alpha[s(\psi, f)\phi - s(\phi, f)\psi] - (1 - \cos 2\alpha)[s(\psi, f)\psi + s(\phi, f)\phi]. \tag{4.3}$$

Hence

$$\mathrm{Ad}[\exp - iK_1^*\Gamma(e_1)\Gamma(J_L e_1)](\Gamma(e_1)) = \cosh(2K_1^*)\Gamma(e_1) + i\sinh(2K_1^*)\Gamma(J_L e_1),$$

and

$$\mathrm{Ad}[\exp - iK_1^*\Gamma(e_L)\Gamma(J_L e_L)](\Gamma(J_L e_L)) = \cosh 2K_1^*\Gamma(J_L e_L) - i\sinh 2K_1^*\Gamma(e_L).$$

Thus

$$\begin{aligned}&\eta(V_{1,L}(V_{2,L}^+)^{1/2}(V_{2,L}^-)^{-1/2}V_{1,L}^{-1})\\ &\quad = \exp - iK_2\{[\Gamma(\cosh 2K_1^* J_L e_L - i\sinh 2K_1^* e_L)][\Gamma(\cosh 2K_1^* e_1\\ &\quad + i\sinh 2K_1^* J_L e_1)]\}.\end{aligned}$$

Similarly,

$$\begin{aligned}&\mathrm{Ad}\left[\exp\left(\frac{-iK_2}{2}\Gamma(J_L e_1)\Gamma(e_2)\right)\exp\left(\frac{-iK_2}{2}\Gamma(J_L e_{L-1})\Gamma(e_L)\right)\right]\\ &\{\eta[V_{1,L}(V_{2,L}^+)^{1/2}(V_{2L}^-)^{-1/2}V_{1,L}^{-1}]\} = \exp - iK_2\Gamma(f_L)\Gamma(\theta_1) \quad \text{for} \quad L > 2,\end{aligned}$$

if

$$\begin{aligned}f_L &= \cosh 2K_1^* J_L e_L - i\sinh 2K_1^*(\cosh K_2 e_L - i\sinh K_2 J_L e_{L-1}),\\ \theta_1 &= \cosh 2K_1^* e_1 + i\sinh 2K_1^*(\cosh K_2 J_L e_1 + i\sinh K_2 e_2).\end{aligned}$$

Hence

$$\eta(V_L^+(V_L^-)^{-1}) = \exp - \frac{iK_2}{2}\Gamma(J_L e_L)\Gamma(W_L^+ e_L),$$

$$\begin{aligned}&\mathrm{Ad}\left\{\prod_{k=1}^{L-1}\left(\exp - \frac{iK_2}{2}\Gamma(J_L e_k)\Gamma(W_L^+ e_k)\right)\right\}\\ &\quad\cdot[\eta(V_{1,L}(V_{2,L}^+)^{1/2}(V_{2,L}^-)^{-1/2}V_{1,L}^{-1})]\exp + \frac{iK_2}{2}\Gamma(J_L e_L)\Gamma(W_L^- e_L)\\ &= \exp\left[-i\frac{K_2}{2}\Gamma(J_L e_L)\Gamma(e_1)\right]\exp[-iK_2\Gamma(f_L)\Gamma(\theta_1)]\\ &\quad\cdot\exp\left[\frac{-iK_2}{2}\Gamma(J_L e_L)\Gamma(e_1)\right].\end{aligned}$$

Now $\|f_L\|^2 = \|\theta_1\|^2 = a^2$ say, which is independent of L, and if f, g are orthogonal unit vectors in H, then $\exp\alpha\Gamma(f)\Gamma(g) = \cos\alpha + \sin\alpha\Gamma(f)\Gamma(g)$. Thus

$$\begin{aligned}\eta(V_L^+(V_L^-)^{-1}) &= (\cosh(K_2/2 - i\sinh(K_2/2)\Gamma(J_L e_L)\Gamma(e_1)\\ &\quad\cdot(\cosh(K_2 a^2) + \sinh(K_2 a^2)a^{-2}\Gamma(f_L)\Gamma(\theta_1))\\ &\quad\cdot(\cosh(K_2/2) - i\sinh(K_2/2)\Gamma(J_L e_L)\Gamma(e_1))\\ &= [\cosh K_2 - i\sinh K_2\Gamma(J_L e_L)\Gamma(e_1)]\\ &\quad\cdot[\cosh(K_2 a^2) - i\sinh(K_2 a^2)a^{-2}\Gamma(g_L)\Gamma(\alpha_1)],\end{aligned} \tag{4.4}$$

where

$$\begin{aligned} g_L &= \cosh 2K_1^*(\cosh K_2 J_L e_L - i\sinh K_2 e_1) \\ &\quad - i\sinh 2K_1^*(\cosh K_2 e_L - i\sinh K_2 J_L e_{L-1}), \\ \alpha_1 &= \cosh 2K_1^*(\cosh K_2 e_1 + i\sinh K_2 J_L e_L) \\ &\quad + i\sinh 2K_1^*(\cosh K_2 J_L e_1 + i\sinh K_2 e_2). \end{aligned}$$

Hence

$$\begin{aligned} \langle V_L^+ x (V_L^-)^{-1}\Omega_L, y\Omega_L\rangle &= \langle (\cosh K_2 - i\sinh K_2 \Gamma(J_L e_L)\Gamma(e_1))(\cosh(K_2 a^2) \\ &\quad - i\sinh(K_2 a^2) a^{-2}\Gamma(g_L)\Gamma(\alpha_1)], \\ &\quad \cdot \prod [\Gamma(\cosh\gamma_L^- \phi_j) + i\Gamma(A_L^- \sinh\gamma_L^- \phi_j)\Omega_L, \prod \Gamma(\psi_k)\Omega_L\rangle. \end{aligned}$$

Using the Pfaffian expansion, we see that we must consider the limits in the previous expressions (4.2a)–(4.2e), where ϕ and or ψ are replaced by one of $e_L, J_L e_L, J_L e_{L-1}$: e.g.

$$\begin{aligned} s(A_L^- e_r, \cosh\gamma_L^- e_L) &= \frac{1}{L}\sum_l \cosh\gamma(\omega_{l,L}^-)\,[\sin(2\theta(\omega_{l,L}^-) - \omega_{l,L}^-(r-L)] \\ &= \frac{-1}{L}\sum_l \cosh\gamma(\omega_{l,L}^-)\,[\sin(2\theta(\omega_{l,L}^-) - \omega_{l,L}^- r] \quad \text{using (3.11)} \\ &\to -\frac{1}{2\pi}\int_0^{2\pi} \cosh\gamma(\omega)\sin[2\theta(\omega) - \omega r]d\omega. \end{aligned}$$

The details are left to the reader.

We have thus established that $\lim\limits_{L\to\infty}\left\langle \frac{\eta(V_L)}{\lambda_L} x\Omega_L, y\Omega_L\right\rangle$ exists for all $x, y \in C_{00}(H)$. But $\|V_L\| \leqq \lambda_L$, hence

$$\begin{aligned} \left|\lim_{L\to\infty}\left\langle \frac{\eta(V_L)}{\lambda_L} x\Omega_L, y\Omega_L\right\rangle\right| &\leqq \lim_{L\to\infty} \|x\Omega_L\|\,\|y\Omega_L\| \\ &= \lim_{L\to\infty} \omega_{A_L^-}(x^*x)^{1/2}\omega_{A_L^-}(y^*y)^{1/2} \\ &= \omega_A(x^*x)^{1/2}\omega_A(y^*y)^{1/2} \quad \text{as} \quad A = wk\text{-}\lim A_L^- \\ &= \|x\Omega\|\,\|y\Omega\|. \end{aligned}$$

Since Ω is cyclic $C_{00}(H)$, it follows from the Riesz representation theorem that there exists a self adjoint contraction P_∞ on F_A such that (4.1) holds. The remainder is now clear.

With the grading of Sect. 2 we can now show:

Theorem 4.2.

$$P_\infty^- F_A^n \subseteqq F_A^n \quad \textit{for all} \quad n \geqq 1, \tag{4.5}$$

$$P_\infty F_A^n \subseteqq F_A^n \quad \textit{for} \quad n \ \textit{even}, \tag{4.6}$$

and

$$P_\infty F_A^n \subseteqq F_A^{n-4} \oplus F_A^{n-2} \oplus F_A^n \oplus F_A^{n+2} \oplus F_A^{n+4}, \quad \text{for} \quad n \text{ odd}, \tag{4.7}$$

with $F_A^n = 0$ *if* $n < 0$.

Proof. Now

$$\eta(V_L^-) a^*_{A_{\bar{L}}}(f) \eta(V_L^-)^{-1} = a^*_{A_{\bar{L}}}(e^{-\gamma_{\bar{L}}} f) \quad [\text{by (3.10)}]. \tag{4.8}$$

Let $\phi_1, \ldots, \phi_m, \psi_1, \ldots, \psi_n \in H_{L_0}$, $L_0 < \infty$.
Then

$$\begin{aligned}
&\langle P_\infty^- : \phi_1 \ldots \phi_m :_A \Omega, :\psi_n \ldots \psi_1 :_A \Omega \rangle \\
&= \lim_{L \to \infty} \langle \eta(V_L^-) : \phi_1 \ldots \phi_m :_A \Omega_L, :\psi_n \ldots \psi_1 :_A \Omega_L \rangle / \lambda_L \\
&= \lim_{L \to \infty} \sum \varepsilon(J, K) \varepsilon(J', K') \omega_{A_{\bar{L}}}(\psi(J') \eta(V_L^-) \phi(J)) \omega_A(\phi(K)) \omega_A(\psi(K')) / \lambda_L \\
&= \lim_{L \to \infty} \sum \varepsilon(J, K) \varepsilon(J', K') \omega_{A_{\bar{L}}}(\psi(J') \eta(V_L^-) \phi(J)) \omega_{A_{\bar{L}}}(\phi(K)) \omega_{A_{\bar{L}}}(\psi(K')) / \lambda_L \\
&= \lim_{L \to \infty} \sum \langle \eta(V_L^-) : \phi_1 \ldots \phi_m :_{A_{\bar{L}}} \Omega_L, :\psi_n \ldots \psi_1 :_{A_{\bar{L}}} \Omega_L \rangle / \lambda_L \\
&= \lim_{L \to \infty} \langle \eta(V_L^-) a^*_{A_{\bar{L}}}(\phi_1) \ldots a^*_{A_{\bar{L}}}(\phi_m) \Omega_L, a^*_{A_{\bar{L}}}(\psi_n) \ldots a^*_{A_{\bar{L}}}(\psi_1) \Omega_L \rangle / \lambda_L \quad \text{by Remark 2.5} \\
&= \lim_{L \to \infty} \langle a^*_{A_{\bar{L}}}(e^{-\gamma_{\bar{L}}} \phi_1) \ldots a^*_{A_{\bar{L}}}(e^{-\gamma_{\bar{L}}} \phi_m) \Omega_L, a^*_{A_{\bar{L}}}(\psi_n) \ldots a^*_{A_{\bar{L}}}(\psi_1) \Omega_L \rangle \quad \text{by (4.8)} \\
&= 0 \quad \text{if} \quad m \neq n.
\end{aligned}$$

Thus $P_\infty^- F_A^n \subseteqq F_A^n$. Then by similarly considering

$$\lim_{L \to \infty} \langle \eta(V_L^+ (V_L^-)^{-1}) \eta(V_L^-) : \phi_1 \ldots \phi_m :_{A_{\bar{L}}} \Omega_L, :\psi_n \ldots \psi_1 :_{A_{\bar{L}}} \Omega_L \rangle / \lambda_L$$

and using (4.4) and (2.3), one gets (4.7). The theorem then follows.
We now concentrate on P_∞^-, noting that $P_\infty^-|_{F_A^n} = P_\infty|_{F_A^n}$ if n is even.

Theorem 4.3. *For* $\beta < \beta_c$,

$$\sigma(P_\infty^-|_{F_A^n}) \subseteqq [\exp - 2n(K_1^* + K_2), \exp - 2n(K_1^* - K_2)].$$

Then given $N > 0$, *there exists* β_N *such that for all* $\beta < \beta_N$, $\sigma(P_\infty^-|_{F_A^n})$, $n = 0, \ldots, N$, *and* $\sigma\left(P_\infty^- \Big| \left(\bigotimes_{n=0}^{N} F_A^n\right)^\perp\right)$ *are disjoint.*

Proof. From (3.5) we have on $H_L^{J_L}$:

$$\cosh 2K_1^* \cosh 2K_2 - \sinh 2K_1^* \sinh 2K_2 \leqq \cosh(\gamma_L^-) \leqq \cosh 2K_1^* \cosh 2K_2 + \sinh 2K_1^* \sinh 2K_2,$$

i.e.

$$\cosh 2(K_1^* - K_2) \leqq \cosh \gamma_L^- \leqq \cosh 2(K_1^* + K_2).$$

Hence for $\beta < \beta_c$, $2(K_1^* - K_2) \leqq \gamma_L^- \leqq 2(K_1^* + K_2)$ on $H_L^{J_L}$. $S_L^- : (H^{J_L}, \langle \cdot, \cdot \rangle_{J_L}) \to (H^{A_{\bar{L}}}, \langle \cdot, \cdot \rangle_{A_{\bar{L}}})$ is isometric and commutes with γ_L^-, hence

$$2(K_1^* - K_2) \leqq \gamma_L^- \leqq 2(K_1^* + K_2) \quad \text{on} \quad H_L^{A_{\bar{L}}}.$$

Thus

$$e^{-2n(K_1^*+K_2)} \leqq F^n_{A_{\bar{L}}}(e^{-\gamma_{\bar{L}}}) \leqq e^{-2n(K_1^*+K_2)}.$$

Let $x = \sum_f \lambda_f : f :_A$, be a finite linear combination of Wick ordered products where $\lambda_f \in \mathbb{C}$, $f = f_1 \dots f_n$ and $f_i \in H_{L_0}$, $L_0 < \infty$. Let $x_L = \sum \lambda_f : f :_{A_{\bar{L}}}$.

Then

$$\|x\Omega\| = \lim_{L \to \infty} \|x_L \Omega_L\|.$$

From the proof of Theorem 4.2:

$$\langle P^-_\infty x\Omega, x\Omega \rangle = \lim_{L \to \infty} \langle F^n_{A_{\bar{L}}}(e^{-\gamma_{\bar{L}}}) x_L \Omega_L, x_L \Omega_L \rangle.$$

Hence $\exp[-2n(K_1^* + K_2)] \leqq P^-_\infty|_{F^n_A} \leqq \exp[-2n(K_1^* - K_2)]$.

For $\sigma(P^-_\infty|_{F^n_A})$ to be disjoint from $\sigma(P^-_\infty|_{F^{n+1}})$ it is sufficient that $(2n+1)K_2 < K_1^*$, i.e. $\beta \ll \beta_c$. The theorem follows.

Remark 4.4.

$$K_1^* = \tanh^{-1}(e^{-2K_1}) = \tfrac{1}{2}\log\left(\frac{1+e^{-2K_1}}{1-e^{-2K_1}}\right),$$

so that

$$e^{-2(K_1^* \pm K_2)} = \left(\frac{1-e^{-2K_1}}{1+e^{-2K_1}}\right) e^{\pm 2K_2} = O(\beta) \quad \text{as} \quad \beta \to 0.$$

Thus Theorem 4.3 could be regarded as a strengthening of [14–18] where spectra in disjoint intervals of the type $[c_1\beta^n, c_2\beta^n]$ were obtained.

Acknowledgement. The first named author would like to thank the Dublin Institute for Advanced Studies for their warm hospitality and support during visits when this work was carried out.

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Communicated by H. Araki

Received September 20, 1982; in revised form May 16, 1983