Commun. Math. Phys. 90, 293–315 (1983)

Communications in Mathematical Physics

Surface Models with Nonlocal Potentials: Upper Bounds

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Abstract. The behavior of fluctuations in a class of surface models with exponentially decaying nonlocal potentials in studied. Combining a Mayer expansion with a duality transformation, we demonstrate the equivalence of these models to a class of two dimensional spin systems with nonlocal interactions. The expansions give sufficient control over the potentials to allow the fluctuations to be bounded from above by the means of complex translations in the spin representation of the model.

1. Introduction

In this paper a class of models obtained by introducing nonlocal potentials into the solid-on-solid (SOS) model is studied. We show that for a certain class of potentials there exists a finite positive constant $c(\beta, J)$ such that fluctuations in the interface described by the model may be bounded by

$$\langle (h_0 - h_x)^2 \rangle \leqq c(\beta, J) \ln(1+|x|), \tag{1.1}$$

for all non-zero inverse temperatures β.

The models considered have finite volume partition functions

$$\mathbf{Z}_\Lambda = \sum_{\{h\}} \exp\left(-\beta \sum_{\langle i,j \rangle} |h_i - h_j| + \sum_{X \subset \Lambda} V_X^J(\{h\}|_X) \right). \tag{1.2}$$

Let Λ be a square region in $\mathbb{R}^2$, centered at the origin, of side length $(2m+1)$, $m \in \mathbb{Z}$. The sum over $\{h\}$ runs over all configurations of integer valued fields on $\mathbb{Z}^2$, which obey the boundary conditions $h_i \equiv 0$, for all sites $i \in \mathbb{Z}^2 \cap \Lambda^c$. For technical reasons we require that $|\Lambda| > \Lambda_0$, where $|\Lambda|$ is the number of sites in $\mathbb{Z}^2 \cap \Lambda$, and Λ_0 is some constant defined in the appendix. Throughout this paper, $\langle i, j \rangle$ will denote a pair of nearest neighbor sites in $\mathbb{Z}^2$. Because of our boundary conditions, the sum over nearest neighbor pairs may be thought of as running over all pairs in $\mathbb{Z}^2$, or only over those pairs which intersect Λ. The sum over $X \subset \Lambda$ runs over all connected sets

* The author was an NSF Predoctoral Fellow when this research was begun

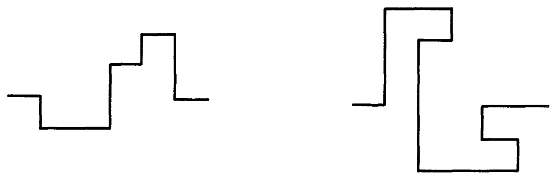

Fig. 1a and b. Sections of interfaces: **a** an interface for the SOS model, **b** because of the overhangs this interface could not appear in the SOS model

of plaquettes in the dual lattice $\mathbb{Z}^{2*}=(\mathbb{Z}+1/2)^2$, which are contained in Λ. The nonlocal potentials $V_X^J(\cdot)$ obey:

(a) Inversion Symmetry. We assume that $V_X^J(\cdot)$ is invariant under the change in the field configuration, $\{h\}\to -\{h\}$.

(b) Translation invariance. We assume that $V_X^J(\cdot)$ depends not on the absolute values of the field $\{h\}|_X$, but only on the differences in $\{h\}$ at adjacent sites in $\mathbb{Z}^2\cap X$. We denote this fact by writing (in a slight abuse of notation)

$$V_X^J(\{h\}|_X)=V_X^J(\{h_i-h_j\}|_X)=V_X^J(\{\partial h\}|_X). \tag{1.3}$$

(c) $\sup_{\{h\}|X} |V_X^J(\{h\}|_X)| \leqq \exp(-J|X|)$.

In (c), $|X|$ = number of plaquettes in X, more generally given a set s, $|s|$ will denote the cardinality of s.

These models arise as follows. Previous studies of surface models, except at very low temperatures, have been limited to either the SOS or discrete Gaussian models [8]. These models attempt to describe the behavior of surfaces at the interface between the two phases of some system. In each case the shape of the interface is specified by associating with each site in $\mathbb{Z}^2$ an integer and assuming that the integer h_i specifies the height of the interface at the point i. These models possess two crucial restrictions. First, they prohibit interactions between lattice sites more than one lattice spacing distant from one another. Second, they don't allow for any overhanging parts of the interface.

The models in (1.2) overcome the first of these restrictions and represent a first step toward removing the second as well.

One more realistic model for interfaces is provided by the three dimensional cubic Ising model. Choose coordinate axes so that the $z=0$ plane lies between the two center planes of spins. Choose boundary conditions so that the boundary spins are forced to be $+1$ if the site under consideration lies above the $z=0$ plane and -1 if it lies below. In this case the interface is the Peierl's contour separating the upper and lower halves of the system. (See [4, 7] for studies of the low temperature behavior of such interfaces.) Such interfaces do have overhangs and one might expect on heuristic grounds that the effect of overhanging configurations would be similar to non-local potentials, just from the observation that the sort of overhangs allowed certainly depends on the shape of the remainder of the interface, because of the requirement that the interface be non-self-intersecting.

The relationship between the non-local potentials and overhanging configurations emerges more clearly in the following example. Consider the class of interfaces, Γ, obtained by "decorating" a SOS interface with overhangs, in such a way that the resulting configuration would be an allowable interface in the Ising model described above. (The SOS interface with no decorations is also an allowed configuration.) Assign to each such interface the Boltzman factor it would have if it had arisen in an anisotropic Ising model with coupling $J=1$ in the x and y directions and J_z in the z direction and with boundary conditions as described above, namely,

$$e^{-\beta\mathscr{H}(\Gamma)}=e^{-\beta\,|\text{number of vertical plaquettes}|}\cdot e^{-\beta J_z\,|\text{number of horizontal plaquettes}|}\,. \tag{1.4}$$

In the partition function for this model we break the sum over all such Γ into two parts, summing first over all overhangs compatible with a given SOS configuration and secondly over all SOS configurations. The sum over overhangs may be rewritten as an exponential using techniques like those of Sect. 3 of the present paper, or [2, 3, 9, 12], and the resulting exponential is interpreted as a sum over non-local potentials. Up to a multiplicative constant, our partition function then has precisely the form of (1.2). One can ask to what extent the conditions (1.3) are verified for the non-local potentials we generate in this procedure. Conditions (1.3) (a) and (1.3) (b) are satisfied. Condition (a) reflects the fact that the number, shape, and size of the overhangs which can be inserted into a given interface are unchanged if one inverts the whole interface. Similarly, condition (b) is just a reflection of the fact that one can translate the whole SOS interface (or some section of it) vertically, without changing the kind of overhangs which one can attach to it. Condition (c) is the one where our estimates break down at present. The assumed exponential decay is valid. It arises from the fact that any overhang must contain at least as many horizontal plaquettes as there are plaquettes in its projection into the $z=0$ plane.

Since $V_X^J(\cdot)$ is associated with sets of plaquettes whose projection into the $z=0$ plane is the region X, and since each horizontal plaquette carries with it a factor of $e^{-\beta J_z}$, we obtain decay of the sort claimed in (2.2) (c), with J proportional to βJ_z. The problem which arises is that this decay is not uniform in $\{h\}$. The reason for this can be seen in Fig. 1. As the vertical "wall" into which we insert the overhang becomes higher and higher, the number of places available to insert the overhang grows, and thus $V_X^J(\cdot)$ should also grow. This is found to be the case. This is the principle restriction which prevents us from applying these results to the anisotropic Ising model.

We note that opposing this growth in the non-local potentials is the fact that the SOS part of the partition function becomes exponentially small as the height of the walls becomes large and it may be possible to use this fact to relax condition (1.3) (c).

Our principle result is

Theorem 1.1 *For every $\beta>0$ there exists a constant $J_0(\beta)>0$ such that for all $J>J_0(\beta)$ there exists a positive constant $k(\beta, J)$ such that for all $x=(n,0)\in\mathbb{Z}^2$, with $0<n<|\Lambda|^{1/8}$ one has*

$$\langle e^{\varepsilon(h_0-h_x)}\rangle_\Lambda \leqq \exp\{\varepsilon^2 k(\beta, J)\ln(1+|x|)\}\,. \tag{1.5}$$

Furthermore, $k(\beta, J)$ may be uniformly bounded by some constant $k(\beta)$ for all $J > J_0(\beta)$.

Note that the bound of (1.2) follows easily from that of (1.5) by subtracting one from both sides of (1.5) and expanding to third order in ε. The term proportional to ε on the left hand side of (1.5) disappears because of the symmetry under $\{h\} \to -\{h\}$. Dividing by ε^2 and taking the limit $\varepsilon \to 0$ yields (1.2). Also, the restriction that the point x lie along the x-axis is not necessary and could be removed at the expense of complicating the proof.

In the limit $J \to \infty$, one recovers the SOS model from our models. Thus Theorem 1.1 guarantees that $\langle (h_0 - h_x)^2 \rangle \leqq k(\beta) \ln(1+|x|)$ for the SOS model. Some easy manipulation of the results of Sects. 2 and 5 show that one can pick $k(\beta) < C_u \beta^{-4}$ (see [16]) for some constant C_u. However, by [8] we know that for β sufficiently small one has $\langle (h_0 - h_x)^2 \rangle \geqq C(\beta) \ln(1+|x|)$ (the results of [16] show that one can take $C(\beta) > C_L |\ln \beta|$ for some $C_L > 0$). Thus one has very precise control of the fluctuations in the "rough" phase of the SOS model, namely the two sided logarithmic bound

$$C(\beta) \ln(1+|x|) \leqq \langle (h_0 - h_x)^2 \rangle \leqq k(\beta) \ln(1+|x|). \tag{1.6}$$

The results presented here also hold if we replace the models (1.2) by perturbations of the discrete Gaussian model, with partition functions

$$\mathbf{Z}_\Lambda = \sum_{\{h\}} \exp\Big(-\beta \sum_{\langle i,j \rangle} (h_j - h_j)^2 + \sum_{X \subset \Lambda} V_J^X(\{h\}|_X) \Big). \tag{1.7}$$

In particular by taking the $J \to \infty$ limit and using the results of [8] one obtains logarithmic growth of fluctuations in the roughened phase of the discrete Gaussian model just as in (1.6).

The chief technical tool used to prove our bounds is a duality transformation which allows us to rewrite these models as two dimensional spin systems. Until now duality transformations have depended on some special feature of the interactions in the system which allowed the sums that arise in the transformation to be explicitly calculated. Using a Mayer expansion we are able to relax this requirement and greatly increase the class of models to which the duality transformation may be applied.

One might expect, in light of the behavior of the SOS model, that the models studied here would undergo a roughening transition at sufficiently high temperatures, resulting in logarithmic lower bounds on the fluctuations. By combining a Mayer expansion with the multipole expansion of Fröhlich and Spencer [8], one can present a formal expansion for these lower bounds. However, *I* do not know how to prove convergence of this expansion, (there is an error in one of the convergence estimates for this expansion presented in [16]), and thus the existence of a roughening transition in these models remains conjectural.

2. The Duality Transformation

Adopting the convention that the bonds (nearest neighbor pairs) $\langle m, k \rangle$ are directed, with the positive direction either up or to the right, and denoting by γ the

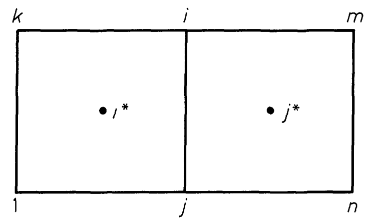

Fig. 2. Representative sites in the original and dual lattice

set of lattice bonds in the path from 0 to x, we may rewrite our unnormalized expectation as:

$$\mathbf{Z}_\Lambda \langle e^{\varepsilon(h_0-h_x)}\rangle = \sum_{\{h\}} \prod_{\langle m,k\rangle\in\gamma} e^{\varepsilon(h_m-h_k)} \prod_{\langle i,j\rangle} e^{-\beta|h_i-h_j|} \exp\left\{\sum_{X\subset\Lambda} V_X^J(\{h_i-h_j\}|_X)\right\}. \quad (2.1)$$

Now exchange the "site" variables $\{h\}$, for "bond" variables $n_{ij}=h_i-h_j$. The variables n_{ij} are not independent. Their sum around any plaquette must be zero. The coboundary operator, ∇, defines a function (on sites of the dual lattice)

$$(\nabla n)(i^*) \equiv \sum_{\langle i,j\rangle} I(i^*,\langle i,j\rangle) n_{ij}, \quad (2.2)$$

where the incidence function, $I(i^*,\langle i,j\rangle)$, is $(+1)$ when the bond $\langle i,j\rangle$ is contained in the boundary of the plaquette associated with the site i^* in the dual lattice and is oriented in the positive direction, (-1) if it is oriented in the negative direction, and 0 if $\langle i,j\rangle$ does not lie in the boundary of i^*. (We choose the counterclockwise orientation to be positive for the boundary of a plaquette. See [1, 11] for a concise explication of these lattice operations.) The constraint on the variables $\{n\}$ now becomes $(\nabla n)(i^*)=0$. Enforcing these constraints with Kronecker δ-functions, (2.1) becomes

$$\begin{aligned}\mathbf{Z}_\Lambda \langle e^{\varepsilon(h_0-h_x)}\rangle_\Lambda &= \sum_{\{n\}} \prod_{\langle k,\ell\rangle\in\gamma} e^{\varepsilon n_{k\ell}} \prod_{\langle i,j\rangle} e^{-\beta|n_{ij}|} \prod_{i^*\in\Lambda^*} (\delta_{0,(\nabla n)(i^*)}) \exp\left\{\sum_{X\subset\Lambda} V_X^J(\{n\}|_X)\right\} \\ &= \sum_{\{n\}} \left[\prod_{i^*\in\Lambda^*} \int_0^{2\pi} \frac{d\theta_{i^*}}{2\pi}\right] \prod_{\langle k,\ell\rangle\in\gamma} e^{\varepsilon n_{k\ell}} \prod_{\langle i,j\rangle} e^{-\beta|n_{ij}|} \prod_{i^*\in\Lambda^*} e^{i\theta_{i^*}(\nabla n)(i^*)} \\ &\cdot \exp\left\{\sum_{X\subset\Lambda} V_X^J(\{n\}|_X)\right\}, \end{aligned} \quad (2.3)$$

where Λ^* is the set of all sites in the dual lattice $(\mathbb{Z}^2)^*=(\mathbb{Z}+1/2)^2$ such that the plaquette (in the original lattice) centered at this site touches Λ. Our boundary conditions then translate into the restriction that n_{ij} vanish for all bonds $\langle i,j\rangle\subset\Lambda^c$. Throughout this section and the next, "starred" quantities (e.g. i^*, $\langle i^*,j^*\rangle$) refer to the dual lattice and unstarred quantities refer to the original lattice.

From Fig. 2 we see that each n_{ij} appears twice in $\prod_{i^*\in\Lambda^*} e^{i\theta_{i^*}(\nabla n)(i^*)}$, once with a positive sign when i^* is the site of the dual lattice immediately above or to the left of $\langle i,j\rangle$, and once with a negative sign when i^* is the site in the dual lattice immediately below or to the right of $\langle i,j\rangle$.

Denoting $\prod_{i^* \in \Lambda^*} \int_0^{2\pi} \frac{d\theta_{i^*}}{2\pi} = \int D\theta$ and defining the function $\delta_{\langle i^*, j^* \rangle, \gamma}$ by

$$\delta_{\langle i^*, j^* \rangle, \gamma} = \begin{cases} 1 & \text{if } \langle i^*, j^* \rangle \text{ intersects some bond in } \gamma \\ 0 & \text{otherwise,} \end{cases} \tag{2.4}$$

we obtain

$$\mathbf{Z}_\Lambda \langle e^{\varepsilon(h_0 - h_x)} \rangle_\Lambda = \int D\theta \sum_{\{n\}} \prod_{\langle i,j \rangle} e^{-\beta |n_{ij}|} e^{i n_{ij}(\theta_{i^*} + i\varepsilon\delta \langle i^*, j^* \rangle, \gamma)} \exp\left\{ \sum_{X \subset \Lambda} V_X^J(\{n\}|_X) \right\}. \tag{2.5}$$

3. The Mayer Expansion

Were it not for the non-local potentials, the sums over $\{n\}$ in (2.5) could be exactly performed, yielding a two dimensional nearest neighbor spin system [8, 17]. We show below that after a Mayer expansion one obtains again a two dimensional spin system, but this time it contains complicated non-local potentials. The form of the Mayer expansion presented here is very similar to that of [13]. Let $\Gamma_1, \ldots, \Gamma_n$ be connected collections of lattice bonds such that for all $i \in \{2, \ldots, n\}$ there exists $j \in \{1, \ldots, i-1\}$ with $\Gamma_i \cap \Gamma_j \neq \emptyset$. Then, define the interpolated potentials

$$V_X^{s_1, \ldots, s_k}(\{n\}|_X; \Gamma_1, \ldots, \Gamma_k) = \begin{cases} s_k V_X^{s_1, \ldots, s_{k-1}}(\{n\}|_X; \Gamma_1, \ldots, \Gamma_{k-1}) & \text{if there exists} \\ \langle i,j \rangle \subset X \quad \text{such that} \quad \langle i,j \rangle \in \Gamma_1 \cup \ldots \cup \Gamma_k & \text{and there exists} \\ \langle i',j' \rangle \subset X \quad \text{such that} \quad \langle i',j' \rangle \notin \Gamma_1 \cup \ldots \cup \Gamma_k & \\ V_X^{s_1, \ldots, s_{k-1}}(\{n\}|_X; \Gamma_1, \ldots, \Gamma_{k-1}) & \text{otherwise.} \end{cases} \tag{3.1}$$

We begin the induction by setting $V_X^{s_1, \ldots, s_{k-1}}(\{n\}|_X; \Gamma_1, \ldots, \Gamma_{k-1}) = V_X(\{n\})|_X)$, on the right hand side of (3.1) in the case when $k=1$. Also, we henceforth suppress the dependence of the potentials on J.

Defining an arbitrary order for the bonds which intersect Λ, and choosing $\Gamma_1 = \{\langle i_1, j_1 \rangle\}$, where $\langle i_1, j_1 \rangle$ is the first bond with respect to this order, the fundamental theorem of calculus yields

$$\begin{aligned} \exp\left\{ \sum_{X \subset \Lambda} V_X(\{n\}|_X) \right\} &= \exp\left\{ \sum_{\substack{X \subset \Lambda: \\ \langle i_1, j_1 \rangle \subset X}} V_X(\{n\}|_X) \right\} + \int_0^1 ds_1 \left[\sum_{X_1 \subset \Lambda} \chi_{X_1}(\emptyset; \Gamma_1; \emptyset) V_{X_1}(\{n\}|_{X_1}) \right] \\ &\quad \cdot \exp\left\{ \sum_{X \subset \Lambda} V_X^{s_1}(\{n\}|_X; \{\langle i_1, j_1 \rangle\}) \right\}, \end{aligned} \tag{3.2}$$

where

$$\chi_X(\Gamma_1, \ldots, \Gamma_{m-1}; \Gamma_m; \Gamma_{m+1}, \ldots, \Gamma_k) = \begin{cases} 1 & \text{if there exists } \langle i,j \rangle \subset X \text{ such that } \langle i,j \rangle \subset \Gamma_m, \\ & \langle i',j' \rangle \subset X \text{ such that } \langle i',j' \rangle \notin \Gamma_1 \cup \ldots \cup \Gamma_k \textit{ and } \text{no} \\ & \langle i'',j'' \rangle \subset X \text{ such that } \langle i'',j'' \rangle \in \Gamma_1 \cup \ldots \cup \Gamma_{m-1}, \\ 0 & \text{otherwise.} \end{cases} \tag{3.3}$$

To proceed further we use.

Lemma 3.1. *For any $k \geqq 1$*

$$\frac{d}{ds_k}\Big[\sum_X V_X^{s_1,\dots,s_k}(\{n\}|_X;\Gamma_1,\dots,\Gamma_k)\Big]$$
$$= \sum_{\substack{X:\,\langle i,j\rangle\subset X \text{ and } \langle i',j'\rangle\subset X\\ \text{with } \langle i,j\rangle\notin\Gamma_1\cup\dots\cup\Gamma_k\\ \text{and } \langle i',j'\rangle\in\Gamma_1\cup\dots\cup\Gamma_k}} V_X^{s_1,\dots,s_{k-1}}(\{n\}|_X;\Gamma_1,\dots,\Gamma_{k-1})$$
$$= \sum_X \sum_{j=1}^{k} (s_j,\dots,s_{k-1})\chi_X(\Gamma_1,\dots,\Gamma_{j-1};\Gamma_j;\Gamma_{j+1},\dots,\Gamma_k)V_X(\{n\}|_X). \tag{3.4}$$

We have adopted the convention that

$$\chi_X(\Gamma_1,\dots,\Gamma_{k-1};\Gamma_k;\Gamma_{k+1},\dots,\Gamma_k)=\chi_X(\Gamma_1,\dots,\Gamma_{k-1};\Gamma_k;\emptyset).$$

Proof. The first equality follows immediately from (3.1). The second equality is also obvious in the case $k=1$, from (3.3). Assume that it holds for $k<m$. In the case $k=m$, we rewrite the right hand side of the first equality in (3.4), using (3.3), as

$$\sum_{\substack{X:\,\langle i,j\rangle\subset X \text{ and } \langle i',j'\rangle\subset X\\ \text{with } \langle i',j'\rangle\notin\Gamma_1\cup\dots\cup\Gamma_m\\ \text{and } \langle i,j\rangle\in\Gamma_1\cup\dots\cup\Gamma_{m-1}}} V_X^{s_1,\dots,s_{m-1}}(\{n\}|_X;\Gamma_1,\dots,\Gamma_{m-1})$$
$$+\sum_X \chi_X(\Gamma_1,\dots,\Gamma_{m-1};\Gamma_m;\emptyset)V_X^{s_1,\dots,s_{m-1}}(\{n\}|_X;\Gamma_1,\dots,\Gamma_{m-1})$$
$$=s_{m-1}\sum_X\sum_{j=1}^{m-1}(s_j,\dots,s_{m-2})\chi_X(\Gamma_1,\dots,\Gamma_{j-1};\Gamma_j;\Gamma_{j+1},\dots,\Gamma_m)V_X(\{n\}|_X)$$
$$+\sum_X \chi_X(\Gamma_1,\dots,\Gamma_{m-1};\Gamma_m;\emptyset)V_X^{s_1,\dots,s_{m-1}}(\{n\}|_X;\Gamma_1,\dots,\Gamma_{m-1}), \tag{3.5}$$

where the first term on the right hand side of (3.5) results from applying (3.1) and then the induction hypothesis, coupled with the observation that there must always be a bond in X exterior to $\Gamma_1\cup\dots\cup\Gamma_m$. The second term is handled by the observation that if

$$\Gamma(X)\cap\{\Gamma_1\cup\dots\cup\Gamma_{m-1}\}=\emptyset, \quad \text{then} \quad V_X^{s_1,\dots,s_{m-1}}(\{n\}|_X;\Gamma_1,\dots,\Gamma_{m-1})=V_X(\{n\}|_X),$$

where $\Gamma(X)=\{$set of lattice bonds contained in $X\}$. (This follows from the definition of the interpolated potentials.) With this comment the second term in (3.5) becomes $\sum_X \chi_X(\Gamma_1,\dots,\Gamma_{m-1};\Gamma_m;\emptyset)V_X(\{n\}|_X)$, which when combined with the first term yields (3.4) and completes the proof.

We now complete the expansion. The second term in (3.2) is reexpanded by introducing a decoupling parameter s_2, and choosing $\Gamma_2=\Gamma(X_1)$. This yields

$$\int_0^1 ds_1 \sum_{X_1\subset\Lambda}\chi_{X_1}(\emptyset;\Gamma_1;\emptyset)V_{X_1}(\{n\}|_X)\exp\Big\{\sum_{X\subset\Lambda}V_X^{s_1}(\{n\}|_X:\{\langle i_1,j_1\rangle\})\Big\}$$
$$=\int_0^1 ds_1 \sum_{X_1\subset\Lambda}\chi_{X_1}(\emptyset;\Gamma_1;\emptyset)V_{X_1}(\{n\}|_{X_1})\Big[\exp\Big\{\sum_{X\subset\Lambda}V_X^{s_1s_2=0}(\{n\}|_X;\{\langle i_1,j_1\rangle\})\Big\}$$

$$+\int_0^1 ds_2 \frac{d}{ds_2} \exp\left\{\sum_{X\subset\Lambda} V_X^{s_1s_2}(\{n\}|_X;\{\langle i_1,j_1\rangle\})\right\}\Bigg]$$

$$=\int_0^1 ds_1 \sum_{X_1\subset\Lambda} \chi_{X_1}(\emptyset;\Gamma_1;\emptyset)V_{X_1}(\{n\}|_{X_1})$$

$$\cdot\exp\left\{\sum_{\substack{X\subset\Lambda\\ \Gamma(X)\cap(\Gamma_2\cup\Gamma_1)=\emptyset}} V_X(\{n\}|_X)+\sum_{X\subset X_1} V_X^{s_1}(\{n\}|_X;\Gamma_1)\right\}$$

$$+\int_0^1 ds_1\int_0^1 ds_2 \sum_{X_1\subset\Lambda}\sum_{X_2\subset\Lambda}\sum_{j=1}^{2}(s_j,\dots,s_1)\chi_{X_2}(\Gamma_1,\dots,\Gamma_{j-1};\Gamma_j;\Gamma_{j+1},\dots,\Gamma_2)$$

$$\chi_{X_1}(\emptyset;\Gamma_1;\phi)\cdot V_{X_1}(\{n\}|_{X_1})V_{X_2}(\{n\}|_{X_2})\exp\left\{\sum_{X\subset\Lambda} V_X^{s_1s_2}(\{n\}|_X;\Gamma_1,\Gamma_2)\right\}, \tag{3.6}$$

where in the second step we have used the facts that $V_X^{s_1s_2=0}(\{n\}|_X;\Gamma_1,\Gamma_2)=0$ for any X with $\Gamma(X)\cap\Gamma(X_1)=\emptyset$, *and* $\Gamma(X)\cap(\Gamma_\Lambda\backslash\Gamma(X))=\emptyset$ ($\Gamma_\Lambda=\{$set of bonds intersecting $\Lambda\}$), $V_X^{s_1}(\{n\}|_X;\Gamma_1)=V_X(\{n\}|_X)$ for $\Gamma(X)\cap\Gamma_1=\emptyset$, and $V_X^{s_1s_2=0}(\{n\}|_X;\Gamma_1,\Gamma_2)=V^{s_1}(\{n\}|_X;\Gamma_1)$ if $X\subset X_1$, all of which follow from the definition of the interpolated potentials. The general induction step sets $\Gamma_k=\Gamma(X_{k-1})$ and introduces the interpolated potentials $V_X^{s_1,\dots,s_k}(\{n\}|_X;\Gamma_1,\dots,\Gamma_k)$. Continuing the decoupling process until Γ_Λ is exhausted one obtains

Lemma 3.2.

$$\exp\left[\sum_{X\subset\Lambda} V_X(\{n\}|_X)\right]=\sum_{k=1}^{\infty}\sum_{X_1}\dots\sum_{X_{k-1}}\sum_{\eta}\int_0^1 ds_1,\dots,ds_{k-1}$$

$$\cdot\prod_{\ell=2}^{k}[s_{\eta(\ell)},\dots,s_{\ell-2}\chi_{X_{\ell-1}}(\Gamma_1,\dots,\Gamma_{\eta(\ell)-1};\Gamma_{\eta(\ell)};\Gamma_{\eta(\ell)+1},\dots,\Gamma_{\ell-1})V_{X_{\ell-1}}(\{n\}|_{X_{\ell-1}})]$$

$$\cdot\exp\left\{\sum_{X\subset X_1\cup\dots\cup X_{k-1}} V_X^{s_1,\dots,s_{k-1}}(\{n\}|_X;\Gamma_1\cup\dots\cup\Gamma_{k-1})+\sum_{\substack{X\subset\Lambda\\ \Gamma(X)\cap(\Gamma_1\cup\dots\cup\Gamma_k)=\emptyset}} V_X(\{n\}|_X)\right\} \tag{3.7}$$

Here the sum over η runs over all tree functions (see [5]), i.e. functions on the integers mapping $\{1,\dots,k\}$ into itself satisfying $\eta(j)<j$, and is determined by which term in the sum over j we pick in (3.4). We adopt the convention that empty products are set equal to one and empty sums are set equal to zero. (For a more detailed exposition of the general induction step see [13].) Note that there are no convergence problems with (3.7) since on a finite lattice only a finite number of the terms are non-zero.

Definition. A cluster Y consists of

(a) An integer $k\geqq 1$.

(b) A collection of connected subsets of Λ, $\{X_1,\dots,X_{k-1}\}$.

(c) A collection of sets of lattice bonds

$$\{\{\langle i_1,j_1\rangle\},\Gamma(X_1),\dots,\Gamma(X_{k-1})\}\equiv\{\Gamma_1,\Gamma_2,\dots,\Gamma_k\},\quad\text{with}\quad\langle i_1,j_1\rangle$$
$$\text{the minimal bond in }\bigcup_1^k\Gamma_j\equiv\tilde\Gamma(Y). \tag{3.8}$$

(d) A tree function η on k vertices.
(e) An interpolation parameter s_m for $m=1,\ldots,k-1$.

Definition. Given a cluster Y we define the cluster function;

$$Q(Y,\theta,\varepsilon)=\sum_{\{n\}|_{\Gamma(Y)}}\prod_{\substack{\langle i,j\rangle:\\ \langle i,j\rangle\in\Gamma(Y)}} e^{-\beta|n_{ij}|}e^{in_{ij}(\theta_{i*}-\theta_{j*}+\varepsilon\delta_{\langle i*,j*\rangle,\gamma})}$$
$$\cdot\prod_{\ell=2}^{k}[s_{\eta(\ell)},\ldots,s_{\ell-2}\varkappa_{X_{\ell-1}}(\Gamma_1,\ldots,\Gamma_{\eta(\ell)-1};\Gamma_{\eta(\ell)};\Gamma_{\eta(\ell)+1},\ldots,\Gamma_{\ell-1})V_{X_{\ell-1}}(\{n\}|_{X_{\ell-1}})]$$
$$\cdot\exp\Bigg[\sum_{X\subset X_1\cup\ldots\cup X_{k-1}} V_X^{s_1,\ldots,s_{k-1}}(\{n\}|_X;\Gamma_1\cup\ldots\cup\Gamma_{k-1})\Bigg]. \tag{3.9}$$

With these two definitions we may write

$$\sum_{\{n\}}\Bigg[\prod_{\langle i,j\rangle} \mathrm{e}^{-\beta|n_{ij}|}\mathrm{e}^{in_{ij}(\theta_{i*}-\theta_{j*}+i\varepsilon\delta_{\langle i*,j*\rangle,\gamma})}\exp\Bigg\{\sum_{X\subset\Lambda}V_X(\{n\}|_X)\Bigg\}\Bigg]$$
$$=\sum_{\substack{Y:\Gamma_1=\{\langle i_1,j_1\rangle\}\\ \Gamma(Y)\subset\Gamma_\Lambda}} Q(Y,\theta,\varepsilon)\Bigg[\sum_{\{n\}|_{\Gamma_\Lambda\setminus\tilde\Gamma(Y)}}\prod_{\substack{\langle i,j\rangle:\\ \langle i,j\rangle\in\Gamma_\Lambda\setminus\Gamma(Y)}} e^{in_{ij}(\theta_{i*}-\theta_{j*}+i\varepsilon\delta_{\langle i*,j*\rangle,\gamma})}$$
$$\cdot\exp\sum_{\substack{X\subseteq\Lambda\\ \Gamma(X)\cap\tilde\Gamma(Y)=\emptyset}} V_X(\{n\}|_X)\Bigg], \tag{3.10}$$

where we have compressed our notation so that

$$\sum_{\substack{Y:\Gamma_1=\{\langle i_1,j_1\rangle\}\\ \tilde\Gamma(Y)\subset\Gamma_\Lambda}} = \sum_{k=1}^{\infty}\sum_{X_1}\cdots\sum_{X_{k-1}}\sum_{\eta}\int_0^1 ds_1,\ldots,ds_{k-1}\,.$$

Repeat the expansion process, this time applying it to the bracketed quantity on the right hand side of (3.10). We choose $\langle i_1',j_1'\rangle$ so that Γ_1 is the smallest bond (with respect to the previously defined ordering) in $\Gamma_\Lambda\setminus\Gamma(Y)$. The clusters generated in this expansion must satisfy a compatibility condition with respect to those generated in the first expansion. Specifically, we require that $\tilde\Gamma(Y_1)\cap\tilde\Gamma(Y_2)=\emptyset$. Continuing this process until Γ_Λ is exhausted we arrive finally at the expression,

Lemma 3.3

$$\sum_{\{n\}}\Bigg(\prod_{\langle i,j\rangle} e^{-\beta|n_{ij}|}e^{in_{ij}(\theta_{i*}-\theta_{j*}+i\varepsilon\delta_{\langle i*,j*\rangle,\gamma})}e^{\sum_{X\subseteq\Lambda}V_X(\{n\}|_X)}\Bigg)$$
$$=\sum_{\substack{\{Y_1,\ldots,Y_k\}:\\ \tilde\Gamma(Y_m)\cap\tilde\Gamma(Y_j)=\emptyset;\,j\neq m\\ \Gamma_1(Y_1)<\ldots<\Gamma_1(Y_k)\\ \bigcup_1^k\Gamma(Y_m)=\Gamma_\Lambda}}\prod_{m=1}^{k} Q(Y_m,\theta,\varepsilon). \tag{3.11}$$

By $\Gamma_1(Y_j)$ we mean Γ_1 for the cluster Y_j. Now remove the various restrictions on the allowed sets of clusters. First remove the requirement that $\bigcup_1^k\tilde\Gamma(Y_m)=\Gamma_\Lambda$. Define

$$I_\beta(\theta)=\sum_{n=-\infty}^{\infty} e^{-\beta|n|}e^{in\theta}=\frac{1-e^{-2\beta}}{1+e^{-2\beta}-2e^{-\beta}\cos\theta}. \tag{3.12}$$

Then for ε sufficiently small we have

$$|I_\beta(\theta_{i^*}-\theta_{j^*}+i\varepsilon\delta_{\langle i^*,j^*\rangle,\gamma})|\geqq\lambda(\beta,\varepsilon)>0.$$

Also, note that for any cluster Y, with $k=1$, $\tilde{\Gamma}(Y)=\{\langle i,j\rangle\}$ for some $\langle i,j\rangle$, and $Q(Y,\theta,\varepsilon)=I_\beta(\theta_{i^*}-\theta_{j^*}+i\varepsilon\delta_{\langle i^*,j^*\rangle,\gamma})$. (Recall that $\langle i^*,j^*\rangle$ is just the bond in the dual lattice which bisects $\langle i,j\rangle$.) Defining

$$\hat{Q}(Y,\theta,\varepsilon)=\Big[\prod_{\langle i,j\rangle\in\tilde{\Gamma}(Y)} I_\beta(\theta_{i^*}-\theta_{j^*}+i\varepsilon\delta_{\langle i^*,j^*\rangle\gamma})\Big]^{-1}Q(Y,\theta,\varepsilon), \tag{3.13}$$

we may rewrite the right hand side of (3.11) as

$$\sum_{\substack{\{Y_1,\dots,Y_k\}:\\ \tilde{\Gamma}(Y_m)\cap\tilde{\Gamma}(Y_j)=\emptyset;\,m\neq j\\ \Gamma_1(Y_1)<\dots<\Gamma_1(Y_k)\\ \bigcup\Gamma(Y_m)=\Gamma_\Lambda}}' \Big[\prod_{\langle i,j\rangle\in\Gamma_\Lambda} I_\beta(\theta_{i^*}-\theta_{j^*}+i\varepsilon\delta_{\langle i^*,j^*\rangle,\gamma})\Big]\prod_1^k\hat{Q}(Y_m,\theta,\varepsilon)\Big], \tag{3.14}$$

where $\sum'$ indicates that the sum runs only over clusters with $k>1$. Expression (3.14) is equivalent to (3.11) since we may regard any bond in $\Gamma_\Lambda\Big\backslash\Big\{\bigcup_1^k\tilde{\Gamma}(Y_m)\Big\}$ in (3.14) as belonging to some cluster with $k=1$, for which $\hat{Q}(Y,\theta,\varepsilon)=1$.

One eliminates the requirement that $\Gamma_1(Y_1)<\dots<\Gamma_1(Y_k)$ by summing over all ordered sets of clusters $(Y_1,\dots,Y_k)$ and dividing by $1/k!$ to cancel the overcounting. Finally, remove the restriction that $\tilde{\Gamma}(Y_m)\cap\tilde{\Gamma}(Y_j)=\emptyset$, by introducing functions $U(Y_i,Y_j)$ with

$$U(Y_i,Y_j)=\begin{cases}1 & \text{if } \Gamma(Y_i)\cap\Gamma(Y_j)\neq 0\\ 0 & \text{otherwise}.\end{cases} \tag{3.15}$$

With these modifications (3.14) becomes,

$$\prod_{\langle i,j\rangle} I_\beta(\theta_{i^*}-\theta_{j^*}+i\varepsilon\delta_{\langle i^*,j^*\rangle,\gamma})\sum_{m=0}^\infty\frac{1}{m!}\sum_{(Y_1,\dots,Y_m)}'\prod_1^m\hat{Q}(Y_m,\theta,\varepsilon)\prod_{1\leqq i<j\leqq m}U(Y_i,Y_j). \tag{3.16}$$

Standard manipulations (see [2, 3, 9, 12]) combined with convergence estimates presented in Sect. 5 yield:

Lemma 3.4. *There exists $\varepsilon_0(\beta)$ and $J_0(\beta)>0$ such that for all $0<\varepsilon<\varepsilon_0(\beta)$ and all $J>J_0(\beta)$ one has*

$$\begin{aligned}\mathbf{Z}_\Lambda\langle e^{\varepsilon(h_0-h_x)}\rangle_\Lambda=&\int D\theta\Big(\prod_{\langle i^*,j^*\rangle} I_\beta(\theta_{i^*}-\theta_{j^*}+i\varepsilon\delta_{\langle i^*,j^*\rangle,\gamma})\Big)\\ &\cdot\exp\Big[\sum_{m=1}^\infty\frac{1}{m!}\sum_{(Y_1,\dots,Y_m)}'\Big(\sum_{g\in G_c(Y_1,\dots,Y_m)}\prod_{\ell\in g}A(\ell)\Big)\prod_{j=1}^m Q(Y_j,\theta,\varepsilon)\Big].\end{aligned} \tag{3.17}$$

Here, $G_c(Y_1,\dots,Y_m)$ is the set of connected graphs on $(Y_1,\dots,Y_m)$, $A(\ell)=A(Y_i,Y_j)$ $=U(Y_i,Y_j)-1$, where ℓ is the leg in the graph g with endpoints at Y_i and Y_j. Note also that one obtains a representation for the partition function in terms of the variables, $\{\theta\}$, if one sets $\varepsilon=0$, everywhere on the right hand side of (3.17).

4. Complex Translations and an Upper Bound

The analyticity and periodicity of the integrand on the right hand side of (3.17) permit us to make the change of variables (see [14, 15]),

$$\theta_{i*} \to \theta_{i*} + i a_{i*}. \tag{4.1}$$

Let γ_L^* be the set of sites in the dual lattice closest to and above γ. Let $\chi_L(\cdot)$ be the characteristic function of γ_L^*. We pick

$$a_{i*} = \varepsilon \sum_{j*} C_D(i^*, j^*)(\partial_2 \chi)(j^*). \tag{4.2}$$

Here, ∂_2 is the y-component of the lattice gradient, and $C_D(i^*, j^*)$ is the covariance of the negative of the two dimensional lattice laplacian with Dirichlet boundary conditions imposed outside of Λ^*. Under the change of variables (4.1)

$$\begin{aligned}
\mathbf{Z}_\Lambda \langle e^{\varepsilon(h_0 - h_x)} \rangle_\Lambda &= \int D\theta \Big(\prod_{\langle i^*, j^* \rangle} I_\beta(\theta_{i*} - \theta_{j*} + i a_{i*} - i a_{j*} + i\varepsilon \delta_{\langle i^*, j^* \rangle, \gamma}) \Big) \\
&\quad \cdot \exp \left[\sum_{m=1}^{\infty} \frac{1}{m!} \sum'_{(Y_1, \ldots, Y_m)} \Big(\sum_{g \in G_c(Y_1, \ldots, Y_m)} \prod_{\ell \in g} A(\ell) \Big) \prod_{j=1}^{m} \hat{Q}(Y_j, \theta + ia, \varepsilon) \right] \\
&\leqq \sup_{\{\theta\}} \left\{ \left| \frac{\prod_{\langle i^*, j^* \rangle} I_\beta(\theta_{i*} - \theta_{j*} + i a_{i*} - i a_{j*} + i\varepsilon \delta_{\langle i^*, j^* \rangle, \gamma})}{\prod_{\langle i^*, j^* \rangle} I_\beta(\theta_{i*} - \theta_{j*})} \right| \right. \\
&\quad \left. \cdot \left| \exp \left[\sum_{m=1}^{\infty} \frac{1}{m!} \sum'_{(Y_1, \ldots, Y_m)} \Big(\sum_{g \in G_c} \prod_{\ell} A(\ell) \Big) \left\{ \prod_{j=1}^{m} \hat{Q}(Y_j, \theta + ia, \varepsilon) - \hat{Q}(Y_j, \theta, \varepsilon = 0) \right\} \right] \right| \right\} \\
&\quad \cdot \int D\theta \Big[\prod_{\langle i^*, j^* \rangle} I_\beta(\theta_{i*} - \theta_{j*}) \Big] \exp \left[\sum_{m=1}^{\infty} \frac{1}{m!} \sum'_{(Y_1, \ldots, Y_m)} \Big(\sum_{g \in G_c} \prod_{\ell} A(\ell) \Big) \prod_{j=1}^{m} \hat{Q}(Y_j, \theta, \varepsilon = 0) \right].
\end{aligned} \tag{4.3}$$

This inequality results from multiplying and dividing the integrand on the right hand side of the first equality by the untranslated I_β and $\hat{Q}$ functions, and then extracting a supremum of the quantity in curly brackets. This last step uses implicitly the fact that $\hat{Q}(Y, \theta, \varepsilon = 0)$ is real to show that the integrand in the last inequality is positive. The reality of $\hat{Q}(Y, \theta, \varepsilon = 0)$ follows by noting that the $I_\beta(\cdot)$ functions are all real, and $Q(Y, \theta, \varepsilon = 0) = Q^*(Y, \theta, \varepsilon = 0)$ just by changing variables from $\{n\} \to -\{n\}$ in the Definition (3.9).

The integral on the right hand side of (4.3) is the partition function which allows us to bound $\langle e^{\varepsilon(h_0 - h_x)} \rangle$ by the quantity in curly brackets. To estimate that quantity we need the following two lemmas, the first of which is proved in the appendix, and the second of which we prove in the next section.

Lemma 4.1. *There exists a constant c, such that for any nearest neighbor pair* $\langle i^*, j^* \rangle$, $|a_{i*} - a_{j*}| < c \cdot \varepsilon$. *Furthermore, there exists a constant* c' *such that*

$$\Big| \sum_{\langle i^*, j^* \rangle} (a_{i*} - a_{j*} + i\varepsilon \delta_{\langle i^*, j^* \rangle, \gamma})^2 \Big| \leqq c' \varepsilon^2 \ln(1 + |x|). \tag{4.4}$$

Lemma 4.2. For every $\beta>0$, there exists $J_0(\beta)>0$ and $\varepsilon_0(\beta)>0$ such that for $J>J_0(\beta)$ and $0<\varepsilon<\varepsilon_0(\beta)$, there exists a constant $C''(\beta, J)$ such that

$$\sup_{\{\theta\}}\left|\exp \sum_{m=1}^{\infty} \frac{1}{m!} \sum_{(Y_1,\dots,Y_m)}' \left(\sum_{g\in G_c} \prod_{\ell\in g} A(\ell)\right)\left[\prod_{j=1}^{m} \hat{Q}(Y_j, \theta+ia, \varepsilon) - \prod_{j=1}^{m} \hat{Q}(Y_j, \theta, \varepsilon=0)\right]\right|$$
$$\leqq \exp\left[C''(\beta, J) \sum_{\langle i^*, j^*\rangle} (a_{i^*}-a_{j^*}+\varepsilon\delta_{\langle i^*, j^*\rangle, \gamma})^2\right]. \tag{4.5}$$

Furthermore, $C''(\beta, J)$ can be made arbitrarily small by choosing J sufficiently large, and is uniformly bounded for $J>J_0(\beta)$.

Note that Lemma 4.1, the explicit form of $I_\beta(\theta)$ and a simple application of Taylor's theorem imply the existence of a constant $C'''(\beta)>0$ such that

$$\sup_{\{\theta\}}\left|\frac{I_\beta(\theta_{i^*}-\theta_{j^*}+ia_{i^*}+i\varepsilon\delta_{\langle i^*, j^*\rangle, \gamma})}{I_\beta(\theta_{i^*}-\theta_{j^*})}\right| \leqq e^{c'''(\beta)(a_{i^*}-a_{j^*}+i\varepsilon\delta_{\langle i^*, j^*\rangle, \gamma})^2}, \tag{4.6}$$

for ε sufficiently small. Combining this with (4.5) and (4.3) we obtain the bound

$$\langle e^{\varepsilon(h_0-h_x)}\rangle_\Lambda \leqq \exp\{(c'''(\beta)+c''(\beta, J)) \sum_{\langle i^*, j^*\rangle} (a_{i^*}-a_{j^*}+\varepsilon\delta_{\langle i^*, j^*\rangle, \gamma})^2\}$$
$$\leqq \exp\{k(\beta, J)\varepsilon^2 \ln(1+|x|)\}, \tag{4.7}$$

where in the last inequality we have used Lemma 4.1, and defined $k(\beta, J) = c'[c''(\beta, J)+c'''(\beta)]$. This completes the proof of Theorem 1.1.

5. Convergence Estimates

In this chapter we prove Lemma 4.2 which yields convergence of the expansion for the upper bound. As a first step, consider differences of products of cluster functions.

Lemma 5.1. *There exists $\varepsilon_0(\beta)>0$ such that for all $0<\varepsilon<\varepsilon_0(\beta)$*

$$\left[\prod_{j=1}^{m} \hat{Q}(Y_j, \theta+ia, \varepsilon) - \prod_{j=1}^{m} \hat{Q}(Y_j, \theta, \varepsilon=0)\right] = iQ_1(Y_1, \dots, Y_m, \theta, a, \varepsilon) + Q_2(Y_1, \dots, Y_m, \theta, a, \varepsilon), \tag{5.1}$$

where $Q_1(Y_1, \dots, Y_m, \theta, a, \varepsilon)$ is a real function for all values of its arguments. There exists $J_0(\beta)>0$ such that for all $J>J_0(\beta)$,

$$|Q_2(Y_1, \dots, Y_m, \theta, a, \varepsilon)| \leqq \sup_{\langle i,j\rangle \in \bigcup_1^m \Gamma(Y_k)} (a_{i^*}-a_{j^*}+\varepsilon\delta_{\langle i^*, j^*\rangle, \gamma})^2 \prod_{k=1}^{m} S(Y_k), \tag{5.2}$$

with

$$S(Y) = \left(\prod_{j=2}^{k} [s_{\eta(j)}, \dots, s_{j-2}]\chi_{X_{j-2}}(\Gamma_1, \dots, \Gamma_{\eta(j)-1}; \Gamma_{\eta(j)}; \Gamma_{\eta(j)+1}, \dots, \Gamma_{j-1})\right) e^{-(3/4)J|X_{j-1}|}, \tag{5.3}$$

where we again recall that $\langle i^*, j^*\rangle$ is the bond dual to $\langle i, j\rangle$.

Proof. For clusters with $k=1$, the result is immediate. Otherwise, define

$$G_\lambda(\theta, a, \varepsilon) = \prod_{j=1}^{m} \hat{Q}(Y_j, \theta + i\lambda a, \lambda\varepsilon) - \prod_{j=1}^{m} \hat{Q}(Y_j, \theta, \varepsilon = 0).$$

By Taylor's theorem

$$G_{\lambda=1}(\theta, a, \varepsilon) = G_{\lambda=0}(\theta, a, \varepsilon) + \left.\frac{dG_\lambda}{d\lambda}\right|_{\lambda=0} + \frac{\lambda_0^2}{2}\left.\frac{d^2 G_\lambda}{d\lambda^2}\right|_{\lambda=\lambda_0} \tag{5.4}$$

for some $\lambda_0 \in [0,1]$.

Define $Q_1(Y_1, \ldots, Y_m, \theta, a, \varepsilon) = (-i)\left.\frac{dG_\lambda}{d\lambda}\right|_{\lambda=0}$. We prove that $(-i)\left.\frac{dQ}{d\lambda}\right|_{\lambda=0}$ is real. Since we have already shown that $Q(Y_j, \theta, 0)$ is real, that suffices to prove that $(-i)\left.\frac{dG_\lambda}{d\lambda}\right|_{\lambda=0}$ and hence $Q_1(Y_1, \ldots, Y_m, \theta, a, \varepsilon)$ is real.

$$\begin{aligned}
&\frac{d\hat{Q}}{d\lambda}(Y, \theta + i\lambda a, \lambda\varepsilon)|_{\lambda=0} = \Bigg[\prod_{\substack{\langle i,j\rangle \\ \langle i,j\rangle \in \tilde{\Gamma}(Y)}} I_\beta(\theta_{i^*} - \theta_{j^*})\Bigg]^{-1} \\
&\cdot\left[\{Q(Y, \theta, 0) \times \sum_{\substack{\langle i,j\rangle \\ \langle i,j\rangle \in \tilde{\Gamma}(Y)}} \frac{\frac{dI_\beta}{d\lambda}(\theta_{i^*} - \theta_{j^*} + i\lambda a_{i^*} - i\lambda a_{j^*} + i\lambda\varepsilon\delta_{\langle i^*, j^*\rangle, \gamma})|_{\lambda=0}}{I_\beta(\theta_{i^*} - \theta_{j^*})}\right] \\
&+\Bigg\{\sum_{\{n\}|_{\Gamma(Y)}}\Bigg[\sum_{\langle i,j\rangle \in \tilde{\Gamma}(Y)} e^{-\beta|n_{ij}|}(-n_{ij})(a_{i^*} - a_{j^*} + \varepsilon\delta_{\langle i^*, j^*\rangle, \gamma})e^{in_{ij}(\theta_{i^*} - \theta_{j^*})}\Bigg] \\
&\cdot\Bigg[\prod_{\substack{\langle i',j'\rangle \in \tilde{\Gamma}(Y) \\ \langle i',j'\rangle \neq \langle i,j\rangle}} e^{-\beta|n_{i'j'}|} e^{in_{i'j'}(\theta_{i'^*} - \theta_{j'^*})}\Bigg] \\
&\cdot\prod_{\ell=2}^{k} s_{\eta(\ell)}, \ldots, s_{\ell-2}\chi_{X_{\ell-1}}(\Gamma_1, \ldots, \Gamma_{\eta(\ell)-1}; \Gamma_{\eta(\ell)}; \Gamma_{\eta(\ell)+1}, \ldots, \Gamma_{\ell-1})V_{X_{\ell-1}}(\{n\}|_{X_{\ell-1}}) \\
&\cdot\exp\Bigg[\sum_{X \subset X_1 \cup \ldots \cup X_{k-1}} V_X^{s_1, \ldots, s_{k-1}}(\{n\}|_X; \Gamma_1, \ldots, \Gamma_{k-1})\Bigg]\cdot\Bigg\}
\end{aligned} \tag{5.5}$$

The first term is pure imaginary since $Q(Y_j, \theta, \varepsilon=0)$ and $I_\beta(\theta_{i^*} - \theta_{j^*})$ are real, while $\frac{dI_\beta}{d\lambda}(\theta_{i^*} - \theta_{j^*} + i\lambda a_{i^*} - i\lambda a_{j^*} + i\lambda\delta_{\langle i^*, j^*\rangle, \gamma})|_{\lambda=0}$ is pure imaginary, by explicit computation. In the second term we may ignore the factors of $I_\beta(\theta_{i^*} - \theta_{j^*})$ since they are real. Changing variables in the sum over $\{n\}|_{\tilde{\Gamma}(Y)}$ to $\{m\} = \{-n\}$ and using the invariance of $e^{-\beta|n|}$ and $V_X(\{n\}|_X)$ under this change of variables, one finds that this term is pure imaginary too. This proves that $\frac{d\hat{Q}}{d\lambda}(Y, \theta + i\lambda a, \lambda\varepsilon)|_{\lambda=0}$ is pure imaginary. Thus $Q_1(Y_1, \ldots, Y_m, \theta, a, \varepsilon)$ is a real function as claimed in the first half of Lemma 5.1.

Now define

$$Q_2(Y_1, \ldots, Y_m, \theta, a, \varepsilon) \equiv \frac{\lambda_0^2}{2} \left. \frac{d^2 G_\lambda}{d\lambda^2} \right|_{\lambda=\lambda_0} . \tag{5.6}$$

We first prove some bounds on $Q(Y_\ell, \theta + i\lambda_0 a, \lambda_0 \varepsilon)$ and its derivatives.

$$|V_X^{s_1, \ldots, s_{k-1}}(\{n\}|_X ; \Gamma_1, \ldots, \Gamma_{k-1})| \leqq e^{-J|X|} . \tag{5.7}$$

This is true by assumption for $k=0$, and since the interpolation procedure either leaves $V_X^{s_1, \ldots, s_k}(\cdot\,;\cdot)$ unchanged, or multiplies it by some constant of magnitude less than one, it holds for all larger k as well. By the standard Peierl's argument, there exists $J_0(\beta) > 0$ such that for all $J > J_0(\beta)$ one has

$$\left| \sum_{X : x_0 \in X} V_X^{s_1, \ldots, s_{k-1}}(\{n\}|_X ; \Gamma_1, \ldots, \Gamma_{k-1}) \right| \leqq \sum_{|X|=3}^{\infty} e^{-J|X|} \cdot c^{|X|} \leqq e^{-(3/4)J} . \tag{5.8}$$

Thus,

$$\begin{aligned} &\left| \sum_{X \subset X_1 \cup \ldots \cup X_{k-1}} V_X^{s_1, \ldots, s_{k-1}}(\{n\}|_X ; \Gamma_1, \ldots, \Gamma_{k-1}) \right| \\ &\leqq \sum_{x \in X_1 \cup \ldots \cup X_{k-1}} \left| \sum_{X : x \in X} V_X^{s_1, \ldots, s_{k-1}}(\{n\}|_X ; \Gamma_1, \ldots, \Gamma_{k-1}) \right| \\ &\leqq e^{-(3/4)J} \left| \bigcup_{m=1}^{k-1} X_m \right| . \end{aligned} \tag{5.9}$$

By Lemma 4.1, $|a_{i^*} - a_{j^*}| \leqq c \cdot \varepsilon$. Pick $\varepsilon_0(\beta)$ so that $\varepsilon(c+1) \leqq (1/2)\beta$ for all $0 < \varepsilon < \varepsilon_0(\beta)$. Then for $\lambda_0 \in [0, 1]$,

$$\left| \sum_n e^{-\beta|n|} e^{in\theta} e^{-\lambda_0 n(a_{i^*} - a_{j^*} + \varepsilon\delta_{\langle i^*, j^* \rangle, \gamma})} \right| \leqq 2 \sum_{n=0}^{\infty} e^{(-1/2)\beta n} = k_1(\beta) . \tag{5.10}$$

Combining (5.7), (5.9), (5.10), and the definition of $Q(Y, \theta, \varepsilon)$ yields

$$\begin{aligned} &|Q(Y, \theta + i\lambda_0 \varepsilon, \lambda_0 \varepsilon)| \\ &\leqq \left(\prod_{\ell=2}^{k} s_{\eta(\ell)}, \ldots, s_{\ell-2} \chi_{X_{\ell-1}}(\Gamma_1, \ldots, \Gamma_{\eta(\ell)-1} ; \Gamma_{\eta(\ell)} ; \Gamma_{\eta(\ell)+1}, \ldots, \Gamma_{\ell-1}) e^{-J|X_{\ell-1}|} \right) \\ &\quad \cdot \exp\left(e^{-(3/4)J} \left| \bigcup_{m=1}^{k-1} X_m \right| \right) \times k_1(\beta)^{|\tilde{\Gamma}(Y)|} . \end{aligned} \tag{5.11}$$

Now bound derivatives of $Q(Y, \theta + i\lambda a, \lambda\varepsilon)$ with respect to λ. The derivative produces $|\tilde{\Gamma}(Y)|$ terms [see the second term in curly brackets in (5.5)], each of which may be controlled with the bounds we used for $Q(Y, \theta + i\lambda a, \varepsilon)$, supplemented by

$$\left| \sum_n e^{-\beta|n|} (n) e^{in\theta} e^{-n\lambda_0(a_{i^*} - a_{j^*} + \varepsilon\delta_{\langle i^*, j^* \rangle, \gamma})} \right| \leqq k_2(\beta) . \tag{5.12}$$

This yields

$$\left|\frac{dQ}{d\lambda}(Y,\theta+i\lambda a,\lambda\varepsilon)\right| \leqq \left(\sup_{\langle i,j\rangle\in\Gamma(Y)}|a_{i^*}-a_{j^*}+\varepsilon\delta_{\langle i^*,j^*\rangle,\gamma}|\right)$$
$$\cdot|\tilde{\Gamma}(Y)|\cdot k_2(\beta)\cdot k_1(\beta)^{|\tilde{\Gamma}(Y)|}\exp\left(e^{-(3/4)J}\left|\bigcup_1^{k-1}X_m\right|\right)$$
$$\cdot\left[\prod_{\ell=2}^{k} s_{\eta(\ell)},\dots,s_{\ell-2}\chi_{X_{\ell-1}}(\Gamma_1,\dots,\Gamma_{\eta(\ell)-1};\Gamma_{\eta(\ell)};\Gamma_{\eta(\ell)+1},\dots,\Gamma_{\ell-1})e^{-(3/4)J|X_{\ell-1}|}\right]. \quad (5.13)$$

Second derivatives are bounded in analogous fashion, using in addition

$$\left|\sum_n e^{-\beta|n|}(n^2)e^{in\theta}e^{-n\lambda_0(a_{i^*}-a_{j^*}+i\varepsilon\delta_{\langle i^*,j^*\rangle,\gamma})}\right| \leqq k_3(\beta), \quad (5.14)$$

and results in

$$\left|\frac{d^2Q}{d\lambda}(Y,\theta+i\lambda a,\lambda\varepsilon)|_{\lambda=\lambda_0}\right| \leqq \left(\sup_{\langle i,j\rangle\in\tilde{\Gamma}(Y)}|a_{i^*}-a_{j^*}+\varepsilon\delta_{\langle i^*,j^*\rangle,\gamma}|\right)^2$$
$$\cdot\{|\tilde{\Gamma}(Y)|k_3(\beta)k_1(\beta)+|\tilde{\Gamma}(Y)|^2k_2(\beta)^2\}k_1(\beta)^{|\tilde{\Gamma}(Y)|-2}$$
$$\cdot\exp\left\{e^{-3/4J}\left|\bigcup_1^m X_m\right|\right\}\times\left[\prod_{\ell=2}^{k} s_{\eta(\ell)},\dots,s_{\ell-2}\right.$$
$$\left.\cdot\chi_{X_{\ell-1}}(\Gamma_1,\dots,\Gamma_{\eta(\ell)-1};\Gamma_{\eta(\ell)};\Gamma_{\eta(\ell)+1},\dots,\Gamma_{\ell-1})e^{-(3/4)J|X_{\ell-1}|}\right]. \quad (5.15)$$

In this formula, the factor of $|\tilde{\Gamma}(Y)|$ comes from estimating the number of terms in which both derivatives act on a single exponential, and the factor $|\tilde{\Gamma}(Y)|^2$ comes from estimating the number of terms in which the derivatives act on different exponential factors in $Q(Y,\theta+i\lambda a,\lambda\varepsilon)$.

Now consider derivatives of $\hat{Q}(Y,\theta+i\lambda a,\lambda\varepsilon)$. Combining our bounds on derivatives of $Q(Y,\theta+i\lambda a,\lambda\varepsilon)$ with the bounds

$$[I_\beta(\theta_{i^*}-\theta_{j^*}+i\lambda_0(a_{i^*}-a_{j^*}+\varepsilon\delta_{\langle i^*,j^*\rangle,\gamma}))]^{-1}\leqq k_4(\beta),$$
$$\left|\frac{dI_\beta}{d\lambda}(\theta_{i^*}-\theta_{j^*}+i\lambda(a_{i^*}-a_{j^*}+\varepsilon\delta_{\langle i^*,j^*\rangle,\gamma}))\right|_{\lambda=\lambda_0} \leqq k_2(\beta)|a_{i^*}-a_{j^*}+\varepsilon\delta_{\langle i^*,j^*\rangle,\gamma}|, \quad (5.16)$$

and

$$\left|\frac{d^2I_\beta}{d\lambda}(\theta_{i^*}-\theta_{j^*}+i\lambda(a_{i^*}-a_{j^*}+\varepsilon\delta_{\langle i^*,j^*\rangle,\gamma}))\right|_{\lambda=\lambda_0} \leqq k_3(\beta)|a_{i^*}-a_{j^*}+\varepsilon\delta_{\langle i^*,j^*\rangle,\gamma}|^2,$$

which are easily obtained from the definition of $I_\beta(\cdot)$, we find

$$\left|\frac{d\hat{Q}}{d\lambda}(Y,\theta+i\lambda a,\lambda\varepsilon)|_{\lambda=\lambda_0}\right| \leqq \left\{\left|\prod_{\langle i,j\rangle\in\tilde{\Gamma}(Y)}I_\beta(\dots)\right|\right\}^{-1}\left\{\left|\frac{dQ}{d\lambda}(Y,\theta+i\lambda a,\lambda\varepsilon)|_{\lambda=\lambda_0}\right|\right\}$$
$$+|Q(Y,\theta+i\lambda_0 a,\lambda_0\varepsilon)|\left\{\left|\prod_{\langle i,j\rangle\in\tilde{\Gamma}(Y)}I_\beta(\dots)\right|\right\}^{-1}\left\{\left|\sum_{\langle i,j\rangle\in\tilde{\Gamma}(Y)}\frac{dI_\beta}{d\lambda}(\dots)[I_\beta(\dots)]^{-1}\right|\right\}$$

$$\leqq \sup_{\langle i,j\rangle\in\tilde{\Gamma}(Y)} |a_{i*}-a_{j*}+\varepsilon\delta_{\langle i*,j*\rangle,\gamma}|$$

$$\cdot\left[\prod_{\ell=2}^{k}(s_{\eta(\ell)},\ldots,s_{\ell-2})\chi_{X_{J}-2}(\Gamma_1,\ldots,\Gamma_{\eta(\ell)-1};\Gamma_{\eta(\ell)};\Gamma_{\eta(\ell)+1},\ldots,\Gamma_{\ell-1})e^{-J|X_{\ell-1}|}\right]$$

$$\cdot\left\{\exp\left(e^{-(3/4)J}\left|\bigcup_1^{k-1}X_m\right|\right)k_4(\beta)^{|\tilde{\Gamma}(Y)|}[|\tilde{\Gamma}(Y)|k_2(\beta)k_1(\beta)^{|\tilde{\Gamma}(Y)|}+|\tilde{\Gamma}(Y)|k_1(\beta)^{|\tilde{\Gamma}(Y)|+1}]\right\}, \tag{5.17}$$

where we have omitted the arguments of the $I_\beta(\cdot)$ functions to save space. Similarly one has

$$\left|\frac{d^2Q}{d\lambda^2}(Y,\theta+i\lambda a,\lambda\varepsilon)|_{\lambda=\lambda_0}\right| \leqq \sup_{\langle i,j\rangle\in\tilde{\Gamma}(Y)}(a_{i*}-a_{j*}+\varepsilon\delta_{\langle i*,j*\rangle,\gamma})^2$$

$$\left[\prod_{\ell=2}^{k}(s_{\eta(\ell)},\ldots,s_{\ell-2})\chi_{X_{\ell-1}}(\Gamma_1,\ldots,\Gamma_{\eta(\ell)-1};\Gamma_{\eta(\ell)};\Gamma_{\eta(\ell)+1},\ldots,\Gamma_{\ell-1})e^{-J|X_{\ell-1}|}\right]$$

$$\cdot\left\{\exp\left(e^{-(3/4)J}\left|\bigcup_1^{k-1}X_m\right|\right)\cdot k_4(\beta)^{|\tilde{\Gamma}(Y)|}\times[|\tilde{\Gamma}(Y)|k_3(\beta)k_1(\beta)^{|\tilde{\Gamma}(Y)|-1}\right.$$

$$+|\tilde{\Gamma}(Y)|^2k_2^2(\beta)k_1(\beta)+|\tilde{\Gamma}(Y)|^2k_2(\beta)k_1(\beta)^{|\tilde{\Gamma}(Y)|+1}k_4(\beta)$$

$$\left.+k_1(\beta)^{|\tilde{\Gamma}(Y)|}|\tilde{\Gamma}(Y)|(k_2(\beta)k_4(\beta)+k_1^2(\beta)k_4^2(\beta))+k_1(\beta)^{|\tilde{\Gamma}(Y)|}|\tilde{\Gamma}(Y)|^2k_4^2(\beta)k_1^2(\beta)]\right\}. \tag{5.18}$$

The first two terms in square brackets come from estimating the case when both derivatives act on the function $Q(Y,\theta+i\lambda a,\lambda\varepsilon)$, the third term from the case where one derivative acts on Q and one on $I_\beta(\cdot)$, and the last two come from the case where both derivatives act on the product of $I_\beta(\cdot)$ functions.

The important point is that there exist constants $K_1(\beta)$ and $K_2(\beta)$ such that the bracketed qunatities in (5.17) and (5.18) may be bounded from above by $\exp(K_1(\beta)+e^{-(3/4)J})\left|\bigcup_1^{k-1}X_m\right|$ and $\exp\left\{(K_2(\beta)+e^{-(3/4)J})\left|\bigcup_1^{k-1}X_m\right|\right\}$, respectively. $\left(\text{Recall that for } k\geqq 1,\ |\tilde{\Gamma}(Y)|\leqq\left|\bigcup_1^{k-1}X_m\right|.\right)$ Using (5.11) and (5.16) we also have

$$|\hat{Q}(Y,\theta+i\lambda_0a,\lambda_0\varepsilon)|\leqq\exp\left\{(K_3(\beta)+e^{-(3/4)J})\left|\bigcup_1^{k-1}X_m\right|\right\}$$

$$\cdot\prod_{\ell=2}^{k}[s_{\eta(\ell)},\ldots,s_{\ell-2}]\chi_{X_{\ell-2}}(\Gamma_1,\ldots,\Gamma_{\eta(\ell)-1};\Gamma_{\eta(\ell)};\Gamma_{\eta(\ell)+1},\ldots,\Gamma_{\ell-1})e^{-J|X_{\ell-1}|}. \tag{5.19}$$

Pick $J_0(\beta)>8\cdot\max\{K_1(\beta)+e^{-(3/4)J},\ K_2(\beta)+e^{-(3/4)J},\ K_3(\beta)+e^{-(3/4)J}\}$, then

$$\left|\frac{\lambda_0^2}{2}\frac{d^2G_\lambda}{d\lambda^3}\bigg|_{\lambda=\lambda_0}\right|\leqq\sup_{\substack{\langle i*,j*\rangle\in\Gamma(Y_\ell)\\ \ell=1,\ldots,m}}(a_{i*}-a_{j*}+\varepsilon\delta_{\langle i*,j*\rangle,\gamma})^2(m+2m^2)$$

$$\cdot\prod_{\ell=1}^{m}\left(\prod_{j=2}^{k_\ell}[s_{\eta(j)},\ldots,s_{j-2}]\chi_{X_{j-1}}(Y_\ell)(\Gamma_1(Y_\ell),\ldots,\Gamma_{\eta(j)-1}(Y_\ell);\right.$$

$$\left.\Gamma_{\eta(j)}(Y_\ell);\Gamma_{\eta(j)+1}(Y_\ell),\ldots,\Gamma_{j-1}(Y_\ell)\right)e^{-(7/8)J|X_{j-1}(Y_\ell)|}. \tag{5.20}$$

By k_ℓ we mean the integer k associated with the cluster Y_ℓ. Similarly $\Gamma_j(Y_\ell)$ and $X_j(Y_\ell)$ are the sets Γ_j and X_j associated with Y_ℓ. Since [possibly by enlarging $J_0(\beta)$] one has

$$(m+2m^2)\prod_{\ell=1}^{m}\left(\prod_{j=1}^{k_\ell} e^{-(1/8)J|X_{j-1}(Y_\ell)|}\right) \leqq 1, \tag{5.21}$$

for all $J>J_0(\beta)$, Eq. (5.20) leads immediately to (5.2).

We now estimate sums of the new cluster functions $S(Y)$, defined in (5.3). The principle result is

Lemma 5.2. *For every $\beta>0$ there exists $J_0(\beta)>0$ such that for all $J>J_0(\beta)$,*

$$\sum_{\substack{(Y_1,\ldots,Y_m):\\ \langle i,j\rangle\in\bigcup_1^m\tilde{\Gamma}(Y_k)}}' \left|\sum_{g\in G_c}\left[\prod_{\ell\in g}A(\ell)\right]\prod_{k=1}^{m}S(Y_k)\right| \leqq m!\,e^{-(1/8)Jm}. \tag{5.22}$$

Recall that $\sum'$ just means that we sum only over clusters with $k\geqq 2$.

This lemma implies Lemma 4.2. By Lemma 5.1

$$\begin{aligned}
&\left|\exp\left\{\sum_{m=1}^{\infty}\frac{1}{m!}\sum_{(Y_1,\ldots,Y_m)}'\left(\sum_{g\in G_c}\left[\prod_{\ell}A(\ell)\right]\right)\left(\prod_{j=1}^{m}\hat{Q}(Y_j,\theta+ia,\varepsilon)-\prod_{j=1}^{m}\hat{Q}(Y,\theta,0)\right)\right\}\right|\\
&=\left|\exp\sum_{m=1}^{\infty}\frac{1}{m!}\sum_{(Y_1,\ldots,Y_m)}'\left(\sum_{g\in G_c}\left[\prod_{\ell}A(\ell)\right]\right)Q_2(Y_1,\ldots,Y_m,\theta,a,\varepsilon)\right|\\
&\leqq\exp\left\{\sum_{\langle i^*,j^*\rangle}\left(\sum_{m=1}^{\infty}\frac{1}{m!}\sum_{\substack{(Y_1,\ldots,Y_m):\\ \langle i,j\rangle\in\bigcup_1^m\Gamma(Y_k)}}'\left|\sum_{g\in G_c}\left[\prod_{\ell}A(\ell)\right]\prod_{k=1}^{m}S(Y_k)\right|\right)\right.\\
&\qquad\left.\cdot(a_{i^*}-a_{j^*}+\varepsilon\delta_{\langle i^*,j^*\rangle,\gamma})^2\right\}.
\end{aligned}\tag{5.23}$$

In the last inequality we have applied (5.3) and insured that the $\sup(a_{i^*}-a_{j^*}+\varepsilon\delta_{\langle i^*,j^*\rangle,\gamma})^2$ is attained by summing over all $\langle i^*,j^*\rangle$ with $\langle i,j\rangle$ in $\bigcup_1^m\tilde{\Gamma}(Y_k)$. Lemma 5.2 bounds this expression by

$$\exp\left(\sum_{m=1}^{\infty}e^{-(1/8)Jm}\right)\left(\sum_{\langle i^*,j^*\rangle}(a_{i^*}-a_{j^*}+\varepsilon\delta_{\langle i^*,j^*\rangle,\gamma})^2\right). \tag{5.24}$$

This is the bound of Lemma 4.2 if we make the identification

$$C''(\beta,J)=\sum_{m=1}^{\infty}e^{-(1/8)Jm}.$$

Note that Lemma 5.2 combined with (5.19) also insures that the sums over $(Y_1,\ldots,Y_m)$ in (3.17) are absolutely convergent and hence that the manipulations leading to the exponentiated form of the expansion were justified.

Lemma 5.3. *There exists $J_0(\beta)>0$ such that for $J>J_0(\beta)$*

$$\sum'_{\substack{(Z_1,\ldots,Z_m)\\ \tilde{\Gamma}(Z_s)\cap\tilde{\Gamma}(Y_1)\neq\emptyset\\ s=1,\ldots,m\\ \sum_s|\tilde{\Gamma}(Z_s)|=M}} \prod_{s=1}^{m} S(Z_s)\leqq|\tilde{\Gamma}(Y_1)|^m\cdot e^{-(3/16)JM}, \tag{5.25}$$

for any cluster Y_1.

Proof. We first show that for J sufficiently large

$$\sum'_{\substack{Y:\langle i,j\rangle\in\tilde{\Gamma}(Y)\\ |\tilde{\Gamma}(Y)|=N}} S(Y)\leqq e^{-(1/4)JN}. \tag{5.26}$$

Assume that the cluster Y has $|\Gamma_2|=n_2,\ldots,|\Gamma_k|=n_k$. (One has $|\Gamma_1|=1$ always.) There are at most 2^N ways of partitioning the bonds in $\tilde{\Gamma}(Y)$ among the Γ_k's. Expression (5.26) is then bounded by

$$2^N\cdot\sum_{k=2}^{\infty}(k)e^{-(5/16)JN}\sup_{\{n_1,\ldots,n_k\}}\sum_{\substack{X_1,\ldots,X_{k-1}:\\ \langle i,j\rangle\in\Gamma(X_1)\\ |\Gamma_2|=n_2,\ldots,|\Gamma_k|=n_k}}\!\!\cdots\sum\;\sum_{\eta}\int_0^1 ds_1,\ldots,ds_{k-1}$$
$$\cdot\left\{\prod_{j=2}^{k}(s_{\eta(j)},\ldots,s_{j-2})\chi_{X_{j-1}}(\Gamma_1,\ldots,\Gamma_{\eta(j)-1};\Gamma_{\eta(j)};\Gamma_{\eta(j)+1},\ldots,\Gamma_{j-1})\right.$$
$$\left.\cdot\exp[-(1/8)J|X_{j-1}|]\right\}. \tag{5.27}$$

The last step assumed that $\langle i,j\rangle\in\Gamma(X_1)$ and compensated by multiplying by k. We now claim that

$$\sum_{\substack{X_1,\ldots,X_{k-1}:\\ \langle i,j\rangle\in\Gamma(X_1)\\ |\Gamma_2|=n_2,\ldots,|\Gamma_k|=n_k}}\!\!\cdots\sum\left(\prod_{j=2}^{k}[s_{\eta(j)},\ldots,s_{j-2}]\chi_{X_{j-1}}(\Gamma_1,\ldots,\Gamma_{\eta(j)-1};\Gamma_{\eta(j)};\Gamma_{\eta(j)+1},\ldots,\Gamma_{j-1})\right.$$
$$\left.\cdot\exp[-(1/8)J|X_{j-1}|]\right)$$
$$\leqq\prod_{j=2}^{k}\{[s_{\eta(j)},\ldots,s_{j-2}]n_{\eta(j)}\}e^{-kJ/32}. \tag{5.28}$$

For $k=2$ this follows from the standard Peierls argument. Assume that (5.28) holds for $k\leqq m-1$

$$\sum_{X_{m-1}}(s_{\eta(m)},\ldots,s_{m-2})\chi_{X_{m-1}}(\Gamma_1,\ldots,\Gamma_{\eta(m)-1};\Gamma_{\eta(m)};\Gamma_{\eta(m)+1},\ldots,\Gamma_m)e^{-(J/8)|X_{m-1}|}$$
$$\leqq\sum_{\langle i,j\rangle\in\Gamma_{\eta(m)}}\sum_{\substack{X:\\ \langle i,j\rangle\in\Gamma(X)}}(s_{\eta(m)},\ldots,s_{m-2})e^{-(J/8)|X|}$$
$$\leqq(s_{\eta(m)},\ldots,s_{m-2})\times|\Gamma_{\eta(m)}|e^{-J/32}. \tag{5.29}$$

The first inequality follows because all terms in the first sum with $\Gamma(X_{m-1})\cap\Gamma_{\eta(m)}=\emptyset$ vanish. In the second inequality, the Peierls argument is used to bound the sum over X with $\langle i,j\rangle \in \Gamma(X)$. Inequality (5.29) implies (5.28) holds for $k=m$ since we use (5.29) to control the sum over X_m, and the induction hypothesis bounds the rest. By standard estimates (see [6])

$$\sum_{\eta} \int_0^1 ds_1, \ldots, ds_{k-1} \left[\prod_{j=1}^{k} (s_{\eta(j)}, \ldots, s_{j-2} n_{\eta(j)})\right] \leqq \exp\left(\sum_1^k n_j\right) = e^N. \tag{5.30}$$

Combining (5.26), (5.27), (5.28), and (5.30) yields

$$\sum_{\substack{Y : \langle i,j\rangle \in \tilde{\Gamma}(Y) \\ |\tilde{\Gamma}(Y)| = N}}' S(Y) \leqq 2^N \sum_{k=2}^{\infty} (k) e^{-(5/16)JN} \cdot e^N e^{-kJ/32} \leqq e^{-1/4JN}, \tag{5.31}$$

for J sufficiently large. Note that

$$\sum_{\substack{(Z_1,\ldots,Z_m) \\ \tilde{\Gamma}(Z_s) \cap \tilde{\Gamma}(Y_1) \neq \emptyset \\ s=1,\ldots,m \\ \sum_s |\tilde{\Gamma}(Z_s)| = M}}' \prod_{s=1}^{m} S(Z_s) \leqq 2^M \cdot \sup_{\substack{\{n_1,\ldots,n_m\} \\ \sum n_k = M}} \sum_{\substack{(Z_1,\ldots,Z_m) \\ \tilde{\Gamma}(Z_s) \cap \tilde{\Gamma}(Y_1) \neq \emptyset \\ s=1,\ldots,m \\ |\tilde{\Gamma}(Z_1)| = n_1, \ldots, |\tilde{\Gamma}(Z_m)| = n_m}}' \prod_{s=1}^{m} S(Z_s). \tag{5.32}$$

The factor of 2^M bounds the number of possibilities for splitting the total number of bonds in $\sum |\tilde{\Gamma}(Z_s)|$ into subsets $n_1, \ldots, n_m$. Since each $\tilde{\Gamma}(Z_s)$ intersects $\tilde{\Gamma}(Y_1)$, the right hand side of (5.32) is bounded by

$$\begin{aligned} 2^M \cdot &\sup_{\substack{\{n_1,\ldots,n_m\} \\ \sum n_k = M}} \prod_{s=1}^{m} \left(\sum_{\langle i,j\rangle \in \tilde{\Gamma}(Y_1)} \sum_{\substack{Z_s : \langle i,j\rangle \in \tilde{\Gamma}(Z_s) \\ |\tilde{\Gamma}(Z_s)| = n_s}} S(Z_s) \right) \\ &\leqq 2^M \cdot \sup_{\substack{\{n_1,\ldots,n_m\} \\ \sum n_k = M}} \prod_{s=1}^{m} (|\tilde{\Gamma}(Y_1)| e^{-(1/4)Jn_s}) \\ &\leqq 2^M |\tilde{\Gamma}(Y_1)|^m \cdot e^{-1/4JM} \leqq |\tilde{\Gamma}(Y_1)|^m e^{-(3/16)JM}, \end{aligned} \tag{5.33}$$

for J sufficiently large. In the first inequality, we have used (5.26) to bound the sum over Z_s, and then bounded the number of terms in the sum over $\langle i,j\rangle$ by $|\tilde{\Gamma}(Y_1)|$. This completes the proof of Lemma 5.3.

In analogy with [2, 3, 9, 12] define cluster functions

$$\Psi(Y_1, \ldots, Y_n; Z_1, \ldots, Z_k) = \sum_{g \in G_c(Y_1, \ldots, Y_r; Z_1, \ldots, Z_k)} \left(\prod_{\ell \in g} A(\ell)\right) \prod_{j=1}^{k} S(Z_j). \tag{5.34}$$

Recall that $G_c(Y_1, \ldots, Y_r; Z_1, \ldots, Z_k)$ refers to the set of graphs in which each Z_j vertex is connected (directly or indirectly) to a Y_k vertex by a leg in g. One then has

Lemma 5.4. *There exists $J_0(\beta) > 0$ such that for all $J > J_0(\beta)$ one has*

$$\sum_{\substack{(Z_1,\ldots,Z_k) \\ \sum_1^k |\tilde{\Gamma}(Z_s)| = N}}' |\Psi(Y_1, \ldots, Y_r; Z_1, \ldots, Z_k)| \leqq k! \, e^{-(J/8)N} \exp\left(\sum_1^r |\tilde{\Gamma}(Y_j)|\right). \tag{5.35}$$

Proof. First note that $\Psi(\emptyset; Z_1, \ldots, Z_k) \equiv 0$ and $\Psi(Y_1, \ldots, Y_r; \emptyset) \equiv 1$. We then rewrite $\Psi(Y_1, \ldots, Y_r; Z_1, \ldots, Z_k)$ in terms of Ψ functions with less than $r+k$ vertices, via the manipulations detailed in [2, 3, 9, 12]. This leads to

$$\sum'_{\substack{(Z_1,\dots,Z_k)\\ \sum_1^k|\tilde\Gamma(Z_s)|=N}} |\Psi(Y_1,\dots,Y_r;Z_1,\dots,Z_k)| \leqq \sum'_{\substack{(Z_1,\dots,Z_k)\\ \sum|\tilde\Gamma(Z_s)|=N}} \prod_{s=1}^{k} [|A(Y_1,Z_s)|S(Z_s)]$$

$$+k!\sum_{|\Omega|=1}^{k-1}\frac{1}{|\Omega|!}\sum_{M=|\Omega|}^{N-1}\sum'_{\substack{(Z'_1,\dots,Z'_{|\Omega|})\\ \sum|\Gamma(Z'_s)|=M}}\left|\prod_{s\in\Omega}A(Y_1,Z_s)\right|\left(\prod_{s\in\Omega}S(Z'_s)\right)$$

$$\cdot\frac{1}{(k-|\Omega|)!}\sum'_{\substack{(Z''_1,\dots,Z''_{k-|\Omega|})\\ \sum|\tilde\Gamma(Z''_s)|=N-M}}|\Psi(Y_2,\dots,Y_r,Z'_1,\dots,Z'_{|\Omega|};Z''_1,\dots,Z''_{k-|\Omega|})|$$

$$+\sum_{\substack{(Z_1,\dots,Z_k)\\ \sum|\tilde\Gamma(Z_s)|=N}}|\Psi(Y_2,\dots,Y_r;Z_1,\dots,Z_k)|. \tag{5.36}$$

Given a graph $g\in G_c$, $\Omega\equiv\{s:\ell(Y_1,Z_s)\in g\}$. The first term is handled by noting that $A(Y_1,Z_s)$ vanishes unless $\tilde\Gamma(Z_s)\cap\tilde\Gamma(Y_1)\neq\emptyset$, and then applying Lemma 5.3. The third term is handled by the induction hypothesis and the second combines the induction hypothesis and Lemma 5.3. This bounds (5.36) by

$$k!\,e^{-(J/8)N}\exp\left(\sum_2^r|\tilde\Gamma(Y_j)|\right)\{1+(1/k!)|\tilde\Gamma(Y_1)|^k e^{-(1/8)JN}$$

$$+\sum_{|\Omega|=1}^{k-1}\frac{1}{|\Omega|!}e^M e^{(1/8)JM}|\tilde\Gamma(Y_1)|^{|\Omega|}e^{-(1/4)JM}. \tag{5.37}$$

If J is sufficiently large $\exp((1/8)J+1-(1/4)JM)\leqq 1$, and the quantity in curly brackets is bounded by $\exp(|\tilde\Gamma(Y_1)|)$, which completes the induction.

Finally, we use Lemma 5.4 to prove Lemma 5.2.

$$\sum'_{\substack{(Y_1,\dots,Y_m)\\ \langle i,j\rangle\in\bigcup_1^m\tilde\Gamma(Y_k)}}\left|\sum_{g\in G_c}\left(\prod_{\ell}A(\ell)\right)\prod_{k=1}^m S(Y_k)\right| \leqq m\cdot\sum'_{\substack{(Y_1,\dots,Y_m)\\ \langle i,j\rangle\in\Gamma(Y_1)}}\left|\sum_{g\in G_c}\left(\prod_{\ell}A(\ell)\right)\prod_{k=1}^m S(Y_k)\right|, \tag{5.38}$$

where we assumed $\langle i,j\rangle\in\tilde\Gamma(Y_1)$, and compensated by introducing a factor of m. Using (5.34) this may be bounded by

$$m\cdot\sum'_{\substack{Y_1:\\ \langle i,j\rangle\in\tilde\Gamma(Y_1)}}\left(\sum_{N=m}^{\infty}\sum'_{\substack{(Y_2,\dots,Y_m)\\ \sum|\tilde\Gamma(Y_s)|=N}}|\Psi(Y_1;Y_2,\dots,Y_m)|\right)S(Y_1)$$

$$\leqq\sum_{\substack{Y_1:\\ \langle i,j\rangle\in\tilde\Gamma(Y_1)}}\left(\sum_{N=m}^{\infty}m(m-1)!\,e^{-(J/8)N}e^{|\tilde\Gamma(Y_1)|}S(Y_1)\right), \tag{5.39}$$

by Lemma 5.4. This may be rewritten as

$$m!e^{-Jm/8}(1-e^{-J/8})^{-1}\times\sum_{n=1}^{\infty}\sum_{\substack{Y_1:\\ \langle i,j\rangle\in\tilde{\Gamma}(Y_1)\\ |\tilde{\Gamma}(Y_1)|=n}}' e^{|\tilde{\Gamma}(Y_1)|}S(Y_1)$$

$$\leqq m!e^{-Jm/8}(1-e^{-J/8})^{-1}\sum_{m=1}^{\infty} e^{n}\cdot e^{-(1/4)Jn}$$

$$\leqq m!e^{-Jm/8}, \tag{5.40}$$

provided J is sufficiently large. In the next to last inequality, we have applied (5.26). This completes the proof of Lemma 5.2.

Appendix. The Dirichlet Covariance

Let Ξ be a set of periodic lines, separated by a distance $(2m+2)$ in the x and y directions, which divides $\mathbb{R}^2$ into $(2m+2)\times(2m+2)$ squares Δ_j, one of which, Δ_0, is centered at the origin of the dual lattice (and hence contains Λ^*). For any point $x\in\Lambda$, define a set of points, $\{x_j\}$, invariant under reflection in any line belonging to Ξ and with $x=x_0$. One then has, following [10], a representation for the Dirichlet covariance in terms of the free covariance,

$$C_D(i,0)-C_D(i,x)=\sum_{j=0}^{\infty}(-1)^{\varepsilon_j}[C_F(i-0_j)-C_F(i-x_j)], \tag{A.1}$$

where ε_j is the number of reflections in lines in Ξ necessary to obtain x_j from x. We now prove Lemma 4.1. From (4.2) one has

$$\begin{aligned} a_{i^*}-a_{j^*}&=\varepsilon\sum_{\ell^*\in\gamma_L^*}[C_D(i^*,\ell^*-e_y)-C_D(i^*,\ell^*)-C_D(j^*,\ell^*-e_y)+C_D(j^*,\ell^*)]\\ &=\varepsilon\sum_{k=0}^{\infty}(-1)^{\varepsilon_k}\sum_{\ell^*\in\gamma_L^*}\{C_F(i^*-(\ell^*-e_y)_k)-C_F(i^*-\ell_k^*)\\ &\quad -C_F(j^*-(\ell^*-e_y)_k)+C_F(j^*-\ell_k^*)\}, \end{aligned}\tag{A.2}$$

where we used (A.1) in the second equality.

To bound the terms with $k\neq 0$, we note that

$$F(i^*)=\sum_{j=1}^{\infty}(-1)^{\varepsilon_j}\sum_{\ell^*\in\gamma_L^*}\{C_F(i^*-(\ell^*-e_y)_k)-C_F(i^*-\ell_k^*)\} \tag{A.3}$$

is a solution of Laplace's equation throughout Δ_0. Thus by the maximum-minimum principle it must attain both its maximum and minimum values on $\partial\Delta_0$, the boundary of Δ_0. However we know that for any site $i^*\in\partial\Delta_0$, $\ell^*\in\Delta_0$,

$$\sum_{\ell^*\in\gamma_L^*}C_D(i^*,(\ell^*-e_y))-C_D(i^*,\ell^*)=0=F(i^*)+\sum_{\ell^*\in\gamma_L^*}C_F(i^*-(\ell^*-e_y))-C_F(i^*-\ell^*), \tag{A.4}$$

so that

$$|F(i^*)|\leqq|x|\max_{\substack{j^*\in\partial\Delta_0\\ \ell^*\in\gamma_L^*}}\frac{K}{|j^*-\ell^*|} \tag{A.5}$$

for some constant K, and any $i^* \in \Lambda_0$ using standard estimates on the free covariance. Thus if we choose Λ_0 larger than $9|x|^2$ we find

$$|F(i^*)| \leqq K. \tag{A.6}$$

Applying the same estimate to those terms in (A.3) containing j^* rather than i^* allows us to bound all terms in (A.3) with $k \neq 0$ by $2\varepsilon K$.

Now concentrate on the case $k=0$. Since i^* and j^* are nearest neighbors, if $i^* = (x_1, x_2)$, j^* is either $(x_1 \pm 1, x_2)$ or $(x_1, x_2 \pm 1)$. Assume that it is (x_1+1, x_2). All other cases are estimated in analogous fashion. If $\ell^* \in \gamma_L^*$ it has coordinates $(m, 0)$, $0 \leqq m \leqq |x|$. (Assume that the origin of our coordinate system is at the leftmost point in γ_L^*.) Then, using standard estimates on the free covariance,

$$\begin{aligned}
&\left|\sum_{\ell^* \in \gamma_L^*} C_F(i^*, \ell^* - e_y) - C_F(i^*, \ell^*) - C_F(j^*, \ell^* - e_y) + C_F(j^*, \ell^*)\right| \\
&\leqq \sum_{m=0}^{|x|} \frac{|K \cdot x_2|}{1 + x_1^2 + x_2^2 + m^2} \leqq K'
\end{aligned} \tag{A.7}$$

for some constants K, K'. Combining this with estimates on the terms in (A.2) with $k \neq 0$, we see that

$$|a_{i^*} - a_{j^*}| \leqq K'' \cdot \varepsilon \tag{A.8}$$

as claimed.

From the definitions of a_{i^*} and $\delta_{\langle i^*, j^* \rangle, \gamma}$, one has

$$\begin{aligned}
&\sum_{\langle i, j^* \rangle} (a_{i^*} - a_{j^*} + \varepsilon \delta_{\langle i^*, j^* \rangle, \gamma})^2 \\
&= \sum_{\langle i^*, j^* \rangle} [(a_{i^*} - a_{j^*})^2 + 2\varepsilon (a_{i^*} - a_{j^*}) \delta_{\langle i^*, j^* \rangle, \gamma} + \varepsilon^2 \delta^2_{\langle i^*, j^* \rangle, \gamma}] \\
&= \langle a_{i^*}, (-\Delta_D) a_{j^*} \rangle - 2\varepsilon \sum_{i^*} a_{i^*} (\partial_2 \chi)(i^*) + \varepsilon^2 |\gamma| \\
&= -\varepsilon \langle C_D \partial_2 \chi, \partial_2 \chi \rangle + \varepsilon^2 |\gamma|,
\end{aligned} \tag{A.9}$$

where $\langle \cdot, \cdot \rangle$ is the usual inner product on $\ell_2(\Lambda^*)$, and Δ_D is the lattice Laplacian with Dirichlet boundary conditions outside Λ^*.

Now examine

$$\begin{aligned}
\langle C_D \partial_2 \chi, \partial_2 \chi \rangle &= \sum_{i^*, j^*} C_D(i^*, j^*)(\partial_2 \chi)(i^*)(\partial_2 \chi)(j^*) \\
&= \sum_{k=0}^{\infty} (-1)^{\varepsilon_k} \sum_{i^*, j^*} C_F(i^*, j_k^*)(\partial_2 \chi)(i^*)(\partial_2 \chi)(j^*).
\end{aligned} \tag{A.10}$$

The terms with $k \neq 0$ are treated in a manner analagous to that used earlier. The $k=0$ term may be rewritten as

$$\begin{aligned}
\langle C_F \partial_2 \chi, \partial_2 \chi \rangle &= -\langle \partial_2^2 C_F \chi, \chi \rangle \\
&= -\langle (\partial_2^2 + \partial_2^2) C_F \chi, \chi \rangle + \langle \partial_1^2 C_F \chi, \chi \rangle \\
&= \langle \chi, \chi \rangle - \langle C_F \partial_1 \chi, \partial_1 \chi \rangle \\
&= |\gamma| - 2(C_F(0^*) - C_F(x^*)),
\end{aligned} \tag{A.11}$$

where we used summation by parts, the fact that $-\Delta C_F\chi = -(\partial_1^2+\partial_2^2)C_F\chi=\chi$, and 0^* and x^* are the points at which $(\partial_1\chi)$ is nonvanishing. Standard estimates on the free covariance then imply

$$|-\langle C_F\partial_2\chi, \partial_2\chi\rangle + |\gamma|| \leqq K\ln(1+|x|). \tag{A.12}$$

Acknowledgements. It is a pleasure to acknowledge Professor A. Jaffe for his encouragement and support of this work, and Dr. J. Imbrie for several useful discussions.

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Communicated by A. Jaffe

Received February 18, 1983; in revised form April 7, 1983