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On the Energy Spectrum and Parameter Spaces of Classical CP^n Models★

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Abstract. It is known that a large class of smooth solutions of CP^n models can be constructed starting from holomorphic maps of an algebraic curve into complex projective spaces. Here we apply results from algebraic geometry to describe the energy spectrum and the parameter spaces for such models.

1. Introduction

Harmonic maps theory has been recently applied to a well known problem of mathematical physics, namely the CP^n models. These have been quite extensively studied in the physical literature not because of direct physical applications, which seem limited in number and perspective, but in that they exhibit interesting phenomena common to Yang-Mills theories both at the classical and at the quantum level [1, 2].

One of these coincidences has to do with the semi-classical domain of the theories, whereby the properties of classical solutions of the elliptic form of the field equations are examined. From the physical point of view, the interest in such "pseudoparticle" solutions of classical field equations was first pointed out by Polyakov in connection with the infrared problem in the quantum theory of Yang-Mills fields. The SU(2)-instanton solutions were first found by Belavin et al. [3] for Yang-Mills equations. Shortly afterwards, the same type of reasoning was applied by Belavin and Polyakov [4] to the standard SO(3)-invariant σ-model. This was subsequently generalized by Eichenherr [5] to SU$(n+1)$-invariant σ-models, which were christened CP^n-models by D'Adda et al. [2].

From the mathematical point of view, it was immediately clear that both these generalizations, as well as the original σ-model, could be considered as particular examples of harmonic problems. An extensive mathematical literature exists on this topic [6] from which we shall recall below some definitions and results. One of the basic outcomes of these developments is that classical solutions

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of CP^n-models are given by or can be constructed from holomorphic maps $\phi: C \to CP^n$ of a complex curve C into the complex projective space. Now, when C is compact, such maps are algebraic objects.

This fact seems to show another similarity with Yang-Mills fields, whose instantons are as well algebraic geometric objects on CP^3 [7]. Motivated by this further coincidence, we examine in this paper the problem of studying the energy spectrum and the space of moduli of CP^n classical solutions by algebraic geometrical tools, the main ones being deep results of the so-called Brill-Noether theory [17, 19–21, 23, 24]. As we shall see, this will lead to a rather complete classification of finite energy solutions subject to the standard boundary conditions and to a detailed description of their parameter spaces. Less complete, but still interesting results will be also derived for the CP^n models with generalized boundary conditions. Some of these results for instantons on S^2 have been already published as short letters [8, 9]. Here we shall concentrate on the mathematical aspects of the problem, while a more extended version of this paper, containing an introductory account both from the physical and the mathematical point of view, will be published elsewhere [10].

2. Results from Harmonic Maps Theory

In this section we shall briefly recall some of the results of harmonic maps theory, which we shall need in the following. For a general review on harmonic maps the reader is referred to the already quoted paper by Eells and Lemaire [6].

By a P^r model[1], we mean a field $\phi: C \to P^r$, where C is an algebraic curve (i.e. a real 2-dimensional orientable compact C^∞ surface with a fixed conformal class of metrics). Classical solutions are extremals of the energy integral

$$E(\phi) = \frac{i}{2} \int_C [h(\xi_z, \bar{\xi}_{\bar{z}}) + h(\bar{\xi}_z, \xi_{\bar{z}})] dz \Lambda d\bar{z}\,,$$

where h is the Fubini-Study metric on P^r and $\xi = \xi(z, \bar{z})$ is a local representation of ϕ. From the physical point of view one is interested in finite energy smooth extremals of the functional above.

One should further note that usually C is taken to be the Riemann sphere, which arises as the one-point compactification of the Euclidean 2-space R^2, thanks to the usual boundary condition that the field ϕ is a constant at infinity [4]. However one can envisage more general boundary and periodicity conditions [8, 10] which lead to the extension of the study of P^r models over an arbitrary algebraic curve C.

It is known that smooth maps $\phi: C \to P^r$ fall into disjoint homotopy classes, labelled by their degree $\deg(\phi)$, or "topological charge" as it is called in the physical literature. Accordingly, we have to search for extremals of the energy functional in any given homotopy class. A well known result of Lichnerowicz's [11] applies here, telling us that in any given homotopy class holomorphic or anti-holomorphic maps (if any) give absolute minima of the functional E. This

1 Hereinafter we shall simply denote by P^r the complex projective r-dimensional space

follows from the fact that in any homotopy class (see e.g. [12])

$$E(\phi) \geqq |\deg(\phi)|,$$

equality holding if and only if ϕ is (anti)-holomorphic. These absolute minima will be called instanton solutions.

Of course, the inequality above does not imply that there are no other local extremals for E. It is known that

i) any classical solution $\phi : C \to P^1$ from a curve C of genus g with $|\deg(\phi)| \geqq g$ is (anti)-holomorphic [12]. In particular if C is the Riemann sphere P^1, then $g=0$ and any classical solution $\phi : P^1 \to P^1$ is (anti)-holomorphic.

ii) For the general case $\phi : C \to P^r$, besides instantons, there are as well non-(anti)-holomorphic classical solutions [13]. It turns out that for $|\deg(\phi)| \geqq r(g-1)/(r+1)$, these are saddle points for the energy functional [13].

We see then that in the general case, in any admissible homotopy class, there are solutions of minimum energy (instantons) and higher energy solutions, which will be called excitations. These were first found by Din and Zakrzewski [14] and Glaser and Stora [15] in the case $g=0$. Their existence was subsequently rigorously proved for any genus by Eells and Wood [13]. Even more, the last authors proved a general classification theorem, according to which there is a one-to-one correspondence between isotropic and full harmonic maps $\phi : C \to P^r$ and pairs (f, k), where $f : C \to P^r$ is a holomorphic map and k is an integer $0 \leqq k \leqq r$. We recall that by a full map it is meant a map whose image is not contained in any proper projective subspace of P^r. We refer to Eells and Wood [13] for the definition of the isotropy condition and for the proof of the theorem above. They further show that, when C is the Riemann sphere, any harmonic map is isotropic and hence can be constructed from a pair (f, k).

The most important point for our concern is to recall here how a holomorphic map generates solutions which are not holomorphic. By using homogeneous coordinates on P^r, any full holomorphic map $f : C \to P^r$ can be locally lifted to a vector-valued function $\mathbf{v} : C \to \mathbb{C}^{r+1}$; in other terms, $f(z)$ is the line through $\mathbf{0} \in \mathbb{C}^{r+1}$ and $\mathbf{v}(z) = (v_0(z), \ldots, v_r(z))$. Let now $\mathbf{v}^{(i)}(z) = \dfrac{d^i}{dz^i}\mathbf{v}(z)$ denote the i^{th} order derivative of $\mathbf{v}(z)$. For any i, $\mathbf{v}^{(i)}$ is still a vector-valued function $\mathbf{v}^{(i)} : C \to \mathbb{C}^{r+1}$ locally defined on $U \subset C$. From these data, one can construct a map which associates to $z \in U$ the linear span of the $\mathbf{v}^{(i)}$'s in $\mathbb{C}^{r+1}$. Let $\mathrm{Gr}(k+1, r+1)$ be the Grassmannian of $k+1$-planes in $\mathbb{C}^{r+1}$. The k^{th} *associated curve* of f is a map $f_k : C \to \mathrm{Gr}(k+1, r+1)$, which is locally defined by letting $f_k(z)$ be the $k+1$-plane in $\mathbb{C}^{r+1}$ spanned by $\mathbf{v}(z), \mathbf{v}^{(1)}(z), \ldots, \mathbf{v}^{(k)}(z)$. We refer to [16] in order to realize that f_k is well defined, in that it does not vanish identically and is independent of the choice of the local representation of f by $\mathbf{v}(z)$. Finally we note that $f_0 = f$ and that it is convenient to put f_{-1} equal to the zero map. Associated curves to anti-holomorphic maps can be obviously defined in the same way.

Now let $f_k(z)^\perp$ be the orthogonal complement of $f_k(z)$ with respect to the standard hermitian metric of $\mathbb{C}^{r+1}$. The (anti-holomorphic) map $\tilde{f} : C \to P^r$ given by $\tilde{f}(z) = f_{r-1}(z)^\perp$ is called the *polar curve* of f. [Recall that

$$C \xrightarrow{f_{r-1}} \mathrm{Gr}(r, r+1) \xrightarrow{\perp} \mathrm{Gr}(1, r+1) = \mathbb{P}^r.]$$

Now for any k, $0 \leqq k \leqq r$, we can define a (nonholomorphic) map $\psi_k : C \to P^r$ by

$$\psi_k(z) = (f_{k-1}(z) \oplus \tilde{f}_{r-k-1}(z))^{\perp}.$$

To explain this definition, we note that $f_{k-1}(z)$ is a k-plane in $\mathbb{C}^{r+1}$, $\tilde{f}_{r-k-1}(z)$ is an $(r-k)$-plane in $\mathbb{C}^{r+1}$. Their direct sum is an r-plane in $\mathbb{C}^{r+1}$ whose orthogonal complement is a line in $\mathbb{C}^{r+1}$, i.e. a point in P^r.

It can be proved that $\psi_k(z)$ is harmonic and isotropic [13]. Finally recall that the energy and the degree of the solutions ψ_k constructed above are given by [13]

$$E(\psi_k) = \deg(f_k) + \deg(f_{k-1}),$$

and

$$\deg(\psi_k) = \deg(f_k) - \deg(f_{k-1}).$$

3. Energy Spectrum of P^r Models

The results recalled in Sect. 2 show that a certain subclass of classical solutions of P^r models, with generalized boundary conditions, can be represented as holomorphic maps into P^r of a certain Riemann surface C. In any case, holomorphic maps play a central rôle, either because they represent instanton solutions or because they provide building blocks from which more general solutions can be constructed. Moreover, when $C = P^1$, these more general solutions exhaust the whole class of finite energy classical solutions.

Since holomorphic maps from an algebraic curve into projective complex spaces are themselves algebraic objects, in this section we shall apply known results from algebraic geometry to study the energy spectrum of P^r models. To compute $E(\psi_k)$ and $\deg(\psi_k)$, we need to know the degrees of the curves associated to f. Let $d_k = \deg(f_k)$; obviously $d_0 = \deg(f) = d$ and $d_{-1} = \deg(f_{-1}) = 0$. The higher d_k's are given by the Plücker formulas

$$d_{k+1} - 2d_k + d_{k-1} = 2g - 2 - \beta_k,$$

where β_k is the ramification index of f_k (for more details, see [16]). Since f_r is a constant map, d_r must vanish. Summing the recurrence relations above, one easily finds that

$$\sum_{j=0}^{r-1} (r-j)\beta_j = (r+1)d + r(r+1)(g-1).$$

Plücker formulas can be explicitly solved. If the conditions $d_0 = d$ and $d_r = 0$ are imposed, we find

$$d_k = (k+1)d + k(k+1)(g-1) - \sum_{j=0}^{k-1} (k-j)\beta_j, \qquad (0 \leqq k \leqq r).$$

We have then proved the following

Proposition 3.1. *Let ψ_k be a full isotropic map ($0 \leqq k \leqq r$), associated to a holomorphic map $f : C \to P^r$ of $\deg(f) = d$. Then the energy and the degree of ψ_k are given by*

$$E(\psi_k) = \deg(\psi_k) + 2\left\{kd + k(k+1)(g-1) - \sum_{j=0}^{k-2} (k-j-1)\beta_j\right\},$$

$$\deg(\psi_k) = d + 2k(g-1) - \sum_{j=0}^{k-1} \beta_j.$$

Remark. The formula above for the degree can be also found in Eells and Wood [13].

Among the $r+1$ full isotropic solutions generated by $\tilde{\tilde{f}}$, ψ_0 and ψ_r are respectively holomorphic and antiholomorphic. In fact $\psi_0 = \tilde{\tilde{f}}$ is the polar of the polar curve of f and it is not difficult to show that actually $\tilde{\tilde{f}} = f$, while $\psi_r = \tilde{f}$, being the polar of f, is antiholomorphic. From the formulas above, we have that $E(\psi_0) = \deg(f) = \deg(\psi_0)$ and $E(\psi_r) = \deg(f_{r-1}) = -\deg(\psi_r)$, so that the energy coincides with (minus) the degree for (anti)-holomorphic maps, as we already know. Note that ψ_0 and ψ_r are not homotopic, because $\deg(\psi_0) \neq \deg(\psi_r)$. Also, $\tilde{f} = \psi_r$ is not in general homotopic to $\overline{f(z)}$ (i.e. to the anti instanton solution associated to f), since $\deg(\psi_r) \neq -\deg(f)$ and obviously $E(\psi_r) \neq E(f)$. However, if f is such that $\sum_{j=0}^{r-1} \beta_j = 2d + 2r(g-1)$, ψ_r is homotopic to $\overline{f(z)}$ and $E(\psi_r) = E(f)$.

Let us now come to discuss the energy spectrum. We know that instanton solutions fall into (disjoint) homotopy classes. These are classified by their degree or, in physical terms, by their topological charge. For any (admissible) value of the topological charge, there is only one possible value for the energy, i.e. $E(f) = |\deg(f)|$. So the energy spectrum of these solutions is in principle known, provided one knows which values of the topological charge are admissible. We shall limit ourselves to the holomorphic case, the antiholomorphic one being obtained by a reversal of orientation. From the Brill-Noether theorem [17], we have that for a general curve[2], it must be

$$d \geqq \frac{s}{s+1} g + s,$$

where s is the dimension of the least linear subspace of P^r containing the image of f. Thus, for a general curve, there are no instantons when $d < (g/2) + 1$.

What one would really like to know is the spectrum of the energy of isotropic solutions at a given degree, that is the energy spectrum of the excited states of a given instanton solution.

We can answer this question only partially; nevertheless, a number of results can be proved and a qualitative description of the energy spectrum of the excited states can be given. From Proposition 3.1, it is apparent that $E(\psi_k) - \deg(\psi_k)$ is even, but we do not know if any even number is actually attained. We can however prove that there are infinitely many excitations at a given degree (larger than a suitable limit) with arbitrarily high energies. The proof will be given in two steps. First we state a result which is in itself interesting, since it holds for P^r model over $P^1 (r \geqq 2)$. Besides, it extends Theorem 8.3 of Eells and Wood [13] which states the existence of at least one full isotropic solution in each homotopy class.

Proposition 3.2. *For any $|d| \geqq r-2$, there exist full classical solutions $\phi: P^1 \to P^r$ $(r \geqq 2)$ with* $\deg(\phi) = d$ *and arbitrarily high energy.*

Proof. We may limit ourselves to consider solutions of the form ψ_1, with $\deg(\psi_0) = d' \geqq r$. For $r > 2$, we consider the maps ψ_0 obtained by projecting the Veronese map, locally given by $z \to (1, z, z^2, \ldots, z^{d'})$, onto P^r in such a way that ψ_0 can be

2 Cf. the next section for a precise explanation of the meaning of "general"

locally represented by $z \to (1, z^{i_1}, z^{i_2}, \dots, z^{i_{r-1}}, z^{d'})$ with $0 < i_1 < \dots < i_{r-1} < d'$. These maps and their associated curves are ramified at $z = 0, \infty$. According to [16], their ramification indices are given by $\beta_l = i_{l+1} - i_l + i_{r-l} - i_{r-l-1} - 2$, where we put $i_l = 0$ for $l \leqq 0$, and $i_r = d'$. Hence $\sum_{j=0}^{k-1} \beta_j = d' + (i_k - i_{r-k}) - 2k$ and $\deg(\psi_k) = i_{r-k} - i_k$. As for ψ_1, we have that any value

$$-(d'-2) \leqq \deg(\psi_1) \leqq -(r-2); \quad (r-2) \leqq \deg(\psi_1) \leqq (d'-2),$$

can be obtained, by a suitable choice of i_{r-1} and i_1. This shows that we have at least one isotropic solution of degree $|d| \geqq r-2$ generated by holomorphic maps of any degree $d' \geqq r$. The case $r = 2$ needs further consideration, since if ψ_0 is taken as above, $\deg(\psi_1) = 0$ in any case. We then consider solutions of the form ψ_1 generated by holomorphic maps $\psi_0 : P^1 \to P^2$ given locally by $z \to (1, (z+1)^{d'-i}, z^{d'})$, with $1 \leqq i \leqq d' - 1$. In this case $\beta_0 = i - 1$, so that $\deg(\psi_1) = d' - i - 1$. By considering these maps together with those generated by $\bar{\psi}_0$, one has that also in this case any value of $\deg(\psi_1)$ within the limits given above is possible. As for the energy, we have in any case that $E(\psi_1) = \deg(\psi_1) + 2d'$. Since d' can be arbitrarily high, E can be arbitrarily large in any homotopy class.

Remark. Incidentally, we note that any value of the energy $E = \deg(\psi_1) + 2m$, with $m \geqq r$ is admissible in the case of P^r models over P^1. In fact there exists a solution of the form ψ_1 with that energy and degree.

The results of Proposition 3.2 can be somewhat extended to P^r models over a general curve C, by considering maps ψ_k generated by quite special holomorphic maps. In fact we do not know very much about the ramification properties of holomorphic maps of C into P^r, while more can be said about maps which arise by composition as follows:

$$C \xrightarrow{h} P^1 \xrightarrow{f} P^r.$$

Here h is a branched covering of P^1, which has degree $n = \deg(h) \geqq [(g+1)/2] + 1$, where [ ] stands for the integral part.

Let $\psi'_0 = f \circ h$. It is not difficult to compute the ramification indices β'_k of ψ'_0 and its associated curves; one has

$$\beta'_k = n\beta_k + 2(n + g - 1),$$

where β_k are the ramification indices of f and its associated curves. Accordingly, the energy and the degree of the isotropic maps ψ'_k generated by ψ'_0 are

$$E(\psi'_k) = \deg(\psi'_k) + n\{E(\psi_k) - \deg(\psi_k)\} + (g-1)\{k^2(1-n) - 2k + 1 - n\} - k(k-1)n,$$

$$\deg(\psi'_k) = n\deg(\psi_k) - 2kng.$$

We have now the following:

Proposition 3.3. *Let $d' = nd - 2kng$, with $n \geqq [(g+1)/2] + 1, d \geqq r, 0 < k < r$. There exist full isotropic solutions $\phi : C \to P^r$ of degree d' and arbitrarily high energy.*

Proof. If $\deg(\phi) = d'$, there exists in the homotopy class of ϕ at least one solution of the form ψ'_k generated by $\psi'_0 = f \circ h$. From Proposition 3.2, we see that one can choose f such that $E(\psi_k)$ is arbitrarily high. The result then follows from the formula above for the energy $E(\psi'_k)$.

Remark. Albeit far less complete than in the case of P^r models over P^1, the proposition above leads to conjecture that the energy spectrum of a general P^r model is qualitatively similar to that of Proposition 3.2, showing arbitrarily high energy excitations of any given instanton solution. Our proof, however, holds only for certain degrees. This is because we restricted ourselves to consider suitable composite maps, which let the proof be technically easy. To prove a proposition analogous to 3.2, one would need to classify the possible ramification behaviours of holomorphic maps $\phi: C \to P^r$ and their associated curves, for a general curve C. Very little seems to be known in this direction.

4. Parameter Spaces for P^r Classical Solutions

The next question we shall ask is, roughly speaking, "how many" classical solutions of a given degree and satisfying certain boundary conditions do we expect to exist. In view of the preceding remarks, this question is naturally translated into the problem of classifying holomorphic maps of a Riemann surface into a projective space. Fortunately enough, a number of recent and deep results in algebraic geometry, constituting the so-called Brill-Noether theory, tell us a good deal about this problem[3].

We first recall that isomorphism classes of complex structures on a compact orientable surface S_g of genus g are parametrized by an irreducible quasi-projective [18] variety M_g, whose (complex) dimension is $3g-3$, when $g \geqq 2$, and 0 or 1, when $g=0,1$, respectively [16]. For any point $p \in M_g$, we have a complex structure turning S_g into a curve C, different points corresponding to inequivalent structures. The variety M_g is classically called the *moduli space* for genus g curves; it is singular when $g \geqq 2$ and its singularities all arise from curves with a nontrivial automorphism group. From a "physical" point of view, one may think of M_g as the variety of inequivalent conformal structures on S_g. When we say that a certain property is satisfied by a *general* curve of genus g, we mean that there is a proper algebraic subvariety Z of M_g such that every curve corresponding to a point of M_g not in Z satisfies the property.

Next we shall briefly recall how holomorphic maps can be described in algebraic geometrical terms. Let $f: C \to P^r$ be a non-constant holomorphic map. If H stands for the hyperplane line bundle of P^r, f determines a line bundle $L = f^*(H)$ on C of degree $d = \deg(f)$, plus $r+1$ distinguished sections of L, gotten by pulling back the homogeneous coordinates on P^r; these sections never vanish simultaneously. Conversely, given a line bundle L of degree $d > 0$ on C and $r+1$ sections with no common zeros $s_0, \ldots, s_r$ of L, we can construct a non-constant map from C to P^r by setting $f(p) = [s_0(p), \ldots, s_r(p)]$. The sections $s_0, \ldots, s_r$ span an $(s+1)$-dimensional vector subspace V of $H^0(C, L)$, i.e. a *linear series* of degree d and

3 A detailed introduction to the subject will be provided by the forthcoming book [25]. An introductory review of the results needed in this section is also contained in [10]

dimension s on C (a g_d^s for short) with $1 \leqq s \leqq r$. Clearly, V is base-point-free, that is, no point of C is a common zero of all the elements of V. Thus, to give a non-constant map of degree d from C to P^r is the same as giving a degree d line bundle on C, a base-point-free g_d^s $V \subseteqq H^0(C, L)$ $(1 \leqq s \leqq r)$ and $r+1$ spanning vectors for V, up to homothethy. Note that full maps are characterized by $s=r$.

In the following we shall denote by $G_d^s(C)$ the space of all g_d^s's on C (cf. [22] for more details); in addition, we let $\tilde{G}_d^s(C)$ be the open subset consisting of base-point-free series. We shall also denote by $B_d^{s,r}(C)$ the holomorphic bundle over $\tilde{G}_d^s(C)$ whose fibre over $V \in \tilde{G}_d^s(C)$ consists of all sets of $r+1$ spanning vectors for V, up to homothethies. Clearly, $B_d^{s,r}(C)$ parametrizes instanton solutions $f: C \to P^r$ such that the span $f(C)$ has dimension s. The closure $\overline{B_d^{s,r}(C)}$ of $B_d^{s,r}(C)$ in the space of all degree d instanton solutions is obtained by relaxing the condition on the $r+1$ vectors to be chosen in V to the mere requirement that they do not have common zereos.

With these preliminaries in mind, we now describe $B_d^{s,r}(C)$ using the main results of the Brill-Noether theory. The existence theorem for special divisors [19–21] implies that $G_d^s(C)$ [and hence $B_d^{s,r}(C)$] is non-empty whenever $\varrho(s) = g - (s+1)(g-d+s) \geqq 0$. From now on we assume that C is *general*. To compute the dimension of $B_d^{s,r}(C)$, recall that [17] when $\varrho(s) < 0$, $G_d^s(C)$ is empty; hence $B_d^{s,r}(C)$ is also empty when $\varrho(s) < 0$. When $\varrho(s) = 0$, $G_d^s(C)$ is a discrete set containing [19–21]

$$n = \frac{s! \dots 0!}{(g-d+2s)! \dots (g-d+s)!} g!$$

points. Accordingly, $B_d^{s,r}(C)$ is the disjoint union of n copies of the homogeneous space $\mathrm{PGl}(r)/\Gamma$, where Γ is the group of all linear transformations fixing $s+1$ independent vectors. Finally, when $\varrho(s) > 0$, $G_d^s(C)$, and hence $\tilde{G}_d^s(C)$, is a smooth connected complex manifold of dimension $\varrho(s)$ [23, 24]; thus $B_d^{s,r}(C)$ is a smooth connected complex manifold of dimension $\varrho(s) + \dim(\mathrm{PGl}(r)/\Gamma)$ $= \varrho(s) + (r+1)(s+1) - 1$.

Turning to the full space $B_d^r(C)$ of instanton solutions $f: C \to P^r$, for a general curve C this is the disjoint union

$$B_d^r(C) = B_d^{1,r}(C) \cup B_d^{2,r}(C) \cup \dots \cup B_d^{r,r}(C).$$

In particular, we see that $B_d^r(C)$ has irreducible components of varying dimensions. However, as we observed, the closure of each component of $B_d^{s,r}(C)$ intersects $B_d^{t,r}(C)$ for every $t < s$; thus, with the sole exception of the case when $\varrho(1) = 0$, $B_d^r(C)$ is connected.

Note that all the maps parametrized by a fibre of $B_d^{s,r}(C)$ can be obtained one from the other by the action of $\mathrm{PGl}(r)$ on P^r itself. They are all homotopic and have the same energy. However, they cannot be obtained one from the other by an action of the internal symmetry group of the model, which leaves the Lagrangian invariant and trivially sends solutions into solutions. As we know, this group is the isometry group $\mathrm{SU}(r+1)/\mathbb{Z}_{r+1}$ of the Fubini-Study metric of P^r. Accordingly, we would like better to parametrize the orbits of the internal symmetry group in the parameter space $B_d^{s,r}(C)$, i.e. to parametrize the instanton solutions up to a

$\mathrm{SU}(r+1)/\mathbb{Z}_{r+1}$ gauge transformation. To simplify matters, we shall do this only for full solutions, i.e. for $B_d^{r,r}(C)$. From the discussion above, we have the following

Theorem 4.1. A) *The space of full holomorphic P^r-instantons of degree d (up to a global* $\mathrm{SU}(r+1)/\mathbb{Z}_{r+1}$ *gauge transformation) is the bundle*

$$N_d^r(C) \xrightarrow{\mathrm{PGl}(r)/\mathrm{SU}(r+1)} \tilde{G}_d^r(C),$$

where $N_d^r(C)$ is the quotient bundle $B_d^r/\mathrm{SU}(r+1)$.

B) *Let $\varrho = g-(r+1)(g-d+r)$. For a general curve C, we then have*

i) *if $\varrho<0$, then $N_d^r(C)=\emptyset$;*

ii) *if $\varrho=0$, $N_d^r(C)$ is the disjoint union of*

$$\frac{r!\dots 0!}{(g-d+2r)!\dots(g-d+r)!}g!$$

copies of $\mathrm{PGL}(r)/\mathrm{SU}(r+1)$;

iii) *if $\varrho>0$, $N_d^r(C)$ is a smooth connected manifold of real dimension*

$$\dim N_d^r(C) = (r+1)(2d-r+1)-2rg-1\,.$$

Remark. Note that there are special curves for which there are "more" instantons than stated in B), while A) holds in any case. This depends on the structure of $G_d^r(C)$. The case of P^r models over P^1 has been also discussed in [8].

Besides instantons, for $r>1$ there are other maps at which the energy functional of P^r models is stationary. From Sect. 2, we know that all these maps give saddle points for the energy, that is there are perturbations which lower their energy. Hence, at a given admissible degree, we have minimum energy solutions (i.e. instantons) and possibly higher energy unstable solutions, which are homotopic to the instantons and may be thought as their "excitations."

Our knowledge about these excitations for a general P^r model over a curve C is far from being complete. Indeed we do know something about those, among them, which are full and isotropic. Recall that, according to Eells and Wood [13], from any instanton ψ_0 we can generate $r-1$ isotropic solutions $\psi_k(0<k<r)$ which are neither holomorphic nor antiholomorphic. It is clear that each ψ_k will depend on the same parameters as ψ_0. However, in general, the ψ_k's will not have the same degree of ψ_0 and hence cannot be considered as excited states of ψ_0 itself.

As for the parameter spaces of full isotropic solutions, we have that they coincide with the parameter spaces of the holomorphic maps from which isotropic maps are generated. In principle, the question is answered by Proposition 4.1, with minor modifications concerning the notion of "effective" parameters. However, such information is of little use, since one would like to know the parameter space of solutions with a given degree and energy. It should be clear by now that we cannot answer this question in full generality, because we do not know which energy values are admissible for excitations of a given degree. Nor do we know how many isotropic solutions generated by different holomorphic maps have the same energy and degree. Once again, to solve these questions, one needs to study in full detail the ramification properties of holomorphic maps of C into P^r. Finally, if C is not P^1 or a torus, there may be excitations which are not isotropic. About these last solutions nothing is known.

7. Concluding Remarks

Let us summarise the results obtained above. There are two classes of solutions of P^r-models subjected to generalized boundary conditions, namely

i) (anti) *instantons*, i.e. minima of the energy,

ii) *unstable excitations*, i.e. saddle points of the energy.

All the instanton solutions can be in principle constructed as (anti)holomorphic maps $f: C \to P^r$, where C is a Riemann surface which depends on the chosen conditions. Moreover, which is more important, much is known about the space of their parameters so that one might deal with such solutions, without needing an explicit form for their functional dependence.

As for the unstable solutions, our knowledge is far less complete. However,

a) for the standard nonlinear σ-model with the usual boundary conditions, we know that there are no unstable solutions at all. So (anti)instantons represent the general classical solution,

b) for P^r models with the usual boundary conditions (i.e. $C=P^1$), all the unstable solutions are given by isotropic maps. Those, together with instantons give the general classical solution for P^r models over P^1,

c) for the general P^r model, isotropic maps represent a subclass of unstable solutions.

All the classical solutions above fall into disjoint homotopy classes, labelled by their degree d.

When standard boundary conditions are considered, for any d we have the lowest energy state of the field, given by an instanton or an anti-instanton (according to the sign of d) if $d<r$, these solutions cannot be full. Besides, if $r>1$, there are infinite excitations of the same degree, with an energy spectrum extending to infinity. However, only certain values of the energy are allowed for excitations of a given degree.

The pattern is qualitatively the same in the case of generalized boundary conditions. We have again instantons and excitations in any admissible homotopy class, with an energy spectrum bounded from below, but possibly extending to infinity. From the quantitative point of view, however, our results are far from being complete.

As an example of application, we shall briefly discuss the P^2 model over P^1, referring to [9] for a more detailed treatment. Given a d-instanton ψ_0 of degree d, we can construct two other solutions ψ_1 and ψ_2, of which $\psi_2=\tilde{\psi}_0$ is the anti-instanton polar to ψ_0. So, basically, we have only one kind of classical solutions which are not holomorphic or anti-holomorphic, namely those of the form $\bar{\psi}_1=\overline{(\psi_0 \oplus \tilde{\psi}_0)}^{\perp}$ which may be considered in some sense as "composite" states of the instanton ψ_0 and its polar anti-instanton. The topological and energetic properties of this composite state can be easily investigated.

First of all, note that the energy and the degree of ψ_0 and of $\bar{\psi}_1$ depend on the degree and on the ramification of ψ_0. From the results of Sect. 3 one finds

$$\begin{aligned} E(\psi_0)&=d; & \deg(\psi_0)&=d,\\ E(\tilde{\psi}_0)&=d+m; & \deg(\tilde{\psi}_0)&=-(d+m),\\ E(\bar{\psi}_1)&=2d+m; & \deg(\bar{\psi}_1)&=-m, \end{aligned}$$

where $m=d-2-\beta_0$.

The formulas above are quite suggestive. If $\bar{\psi}_1$ is considered as a product of some suitable interaction of the d-instanton ψ_0 with the $d+m$-anti-instanton $\tilde{\psi}_0$, we see that the instanton number, as well as the energy is conserved. The d instantons are annihilated by d anti-instantons, and the leftover is an excited state of an m anti-instanton, the energy gap with respect to the lowest energy state compatible with instanton number being exactly that implied by energy conservation.

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