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C^*-Algebras and Automorphism Groups

H. J. Borchers

Institut für Theoretische Physik der Universität Göttingen, Bunsenstrasse 9, D-3400 Göttingen, Federal Republic of Germany

Abstract. Let (A, G, α) be a C^*-dynamical system with G a topological group. Let π be a representation of A. We will show that there exists a quasi-equivalent representation $\hat{\pi}$ to π which is a covariant representation, if and only if the folium of π is invariant under the action of G and this action is strongly continuous.

I. Introduction and Notation

Symmetries are one of the most powerful concepts in physics. Many of the classification schemes of physical objects are based on symmetry-groups. Therefore it is no wonder that one finds a vast amount of literature on this subject. In earlier times most of these investigations focused on the classification of group representations. In the last $1^1/_2$ decade, however, the interest has changed more to the investigation of the interplay between the symmetry-group and the algebra of observables or the field-algebra. This subject now goes under the name of C^*-dynamical systems. Although this name is unsatisfactory from the physical point of view, because usually only the kinematical possibilities are studied, we use this name in this paper. The main tool of this subject is what has been called by Doplicher et al. [8] the covariance-algebra and which is now known as the crossed product between a C^*-algebra and a group. For a good survey on this subject see the book of Pedersen [11], where one also finds a list of references. One of the objects of this theory is to characterize the representations in which we have also a continuous representation of the symmetry-group implementing the automorphisms. This problem has been answered modulo problems of multiplicity by the author [3, 4].

Looking at this part of the theory of C^*-dynamical systems, one finds that there are two assumptions which are unsatisfactory. The first assumption is the continuity assumption, which says that the expressions $g \to \alpha_g(x)$ have to be continuous functions on the group with values in the C^*-algebra furnished with the norm topology. However, in quantum field theory or statistical mechanics one usually starts with a local net of von Neumann algebras, which means that the continuity assumption is violated. The standard argument out of this dilemma is usually that one postulates the existence of a sub-algebra fulfilling the continuity assumption. Since such a sub-algebra, if it exists, generally will not be dense in the norm topology, one cannot be sure whether one investigates the original object, in particular when different representations are involved. Existing theory shows that the norm topology on the dual space A^* of A, rather than the norm topology on the algebra A, is important for the representation of the group (see e.g. [3]). The assumption of the continuity of the action of G on A is therefore only a technical assumption used in order to have easy access to the problem of covariant representation.

Existing theory has a second defect, namely it can handle only locally compact groups and not arbitrary topological groups. The restriction to the case of locally compact groups was again dictated by mathematical convenience because the mathematical theory for such groups exists, but not for general topological groups. At the time covariance algebras were invented, it was enough to study locally compact groups, because all global symmetry-groups in physics were locally compact. During the past few years the emergence of gauge theories indicates we need to deal with topological groups which are not locally compact.

Because of these arguments, I feel it is necessary to study the problem of covariant representations again in a more general setting. This is done in this and a forthcoming paper. We assume that we are dealing with a C^*-dynamical system (A, G, α), this is a C^*-algebra A, a topological group G, and a mapping α of G into $\mathrm{Aut}(A)$. It is now no longer assumed that the action α_g on A is continuous in any topology on A. As mentioned before, what counts is the continuity property of the group action on the dual-space A^* of A. The action of G on A^* should be denoted by α'_g or α^*_g, but since there is usually no confusion possible, we denote the transposed and the double transposed of α_g again by α_g. We denote by A^*_c the set of functionals $\varphi \in A^*$ such that the function $g \to \alpha_g \varphi$ is a continuous function on G with values in A^* furnished with the norm topology.

Since the dual-space A^* of a C^*-algebra is at the same time the pre-dual of the enveloping von Neumann algebra A^{**}, and since our object of investigation is a subspace of A^*, it turns out that we deal exclusively with the dual pair (A^*, A^{**}). Therefore it is natural to forget about the original C^*-algebra A and work in the theory of von Neumann algebras. Thus we deal with (M_*, M), where M is a von Neumann algebra and M_* its pre-dual.

Let $M_{*,c}$ denote that part of M_* on which the group acts strongly continuous. We show in Sect. 2 that $M_{*,c}$ is generated by the positive elements in it. In Sect. 3 we use the theory of the natural cone based on Tomita's theory of modular Hilbert algebras to prove the main result on covariant representations. The short version of the proof is due to H. Araki and replaces my own version using covariance algebras. In the last section we remark on the structure of $M_{*,c}$ which we plan to elaborate in the future.

II. Continuity of the Map $\varphi \to |\varphi|$

Let M be a von Neumann algebra and M_* its pre-dual. For $\varphi \in M_*$ denote by $S_l(\varphi)$ and $S_r(\varphi)$ the left and right support of φ. If $x \in M$, then $x\varphi$, respectively, φx denotes the functional $y \to \varphi(xy)$, respectively, $y \to \varphi(yx)$. Tomita [13] has shown that to every $\varphi \in M_*$ a positive functional exists, denoted by $|\varphi|$, and a partial isometry $V \in M$ such that $\varphi = V|\varphi|$ holds. Here V can be chosen to fulfill the relations $VV^* = S_r(\varphi)$ and $V^*V = S_l(\varphi)$. If these relations are fulfilled, then V and $|\varphi|$ are uniquely defined by φ. Later Effros [9] has shown that the map $\varphi \to |\varphi|$ is continuous, if the initial space is furnished with the norm and the final space is furnished with the weak topology $\sigma(M_*, M)$. Here we want to give an estimate for the convergence in the norm topology.

II.1. Theorem. *Let M be a von Neumann algebra and M_* its pre-dual. Assume $\varphi, \psi \in M_*$ and $|\varphi|$ and $|\psi|$ are absolute values in M_*^+, then we get the estimate:*

$$\| |\varphi| - |\psi| \| + \| |\varphi^*| - |\psi^*| \| \leqq 2\{\|\varphi - \psi\| + \sqrt{\|\varphi - \psi\|} \cdot [2(\|\varphi\| + \|\psi\|)]^{1/2}\}.$$

Proof. We first restrict our attention to real elements. The general case will then be reduced to the special case by the 2×2 matrix method.

Let $\psi = \varphi^*$, $\psi = \psi^*$, and $\Delta = \varphi - \psi$. Let $\psi = \varphi^+ - \varphi^-$, $\psi = \psi^+ - \psi^-$, $\Delta = \Delta^+ - \Delta^-$ be their canonical decomposition into the positive and negative parts. Moreover let e^+, e^- be two projections with $e^+ + e^- = 1$, $e^+ e^- = 0$, $e^+ \varphi^+ = \varphi^+ e^+ = \varphi^+$, $e^+ \varphi^- = \varphi^- e^+ = 0$, $e^- \varphi^+ = \varphi^+ e^- = 0$, $e^- \varphi^- = \varphi^- e^- = \varphi^-$. This means e^+ majorizes the support of φ^+ and is majorized by $1-$ (support of φ^-). In the same manner define f^+ and f^- fulfilling the same relations with ψ^+ and ψ^-. From $f^+ + f^- = 1$ follows:

$$\varphi^+ - \psi^+ = f^+ \varphi^+ f^+ + f^- \varphi^+ f^- + f^+ \varphi^+ f^- + f^- \varphi^+ f^+ - \psi^+, \qquad (1^+)$$

and

$$\varphi^- - \psi^- = f^+ \varphi^- f^+ + f^- \varphi^- f^- + f^+ \varphi^- f^- + f^- \varphi^- f^+ - \psi^-. \qquad (1^-)$$

Multiplying the defining equation for Δ on both sides with f^+, respectively, with f^-, we obtain:

$$f^+ \varphi^+ f^+ - \psi^+ = f^+ \varphi^- f^+ + f^+ \Delta f^+, \qquad (2^+)$$

$$f^- \varphi^+ f^- - f^- \Delta f^- = f^- \varphi^- f^- - \psi^-, \qquad (2^-)$$

Inserting Eqs. (2) into Eqs. (1), one finds:

$$\varphi^+ - \psi^+ = f^+ \varphi^- f^+ + f^- \varphi^+ f^- + f^+ \varphi^+ f^- + f^- \varphi^+ f^+ + f^+ \Delta f^+,$$

and

$$\varphi^- - \psi^- = f^+ \varphi^- f^+ + f^- \varphi^+ f^- + f^+ \varphi^- f^- + f^- \varphi^- f^+ - f^- \Delta f^-,$$

and by adding them:

$$\begin{aligned} |\varphi| - |\psi| = {} & f^+ \Delta f^+ - f^- \Delta f^- + 2(f^+ \varphi^- f^+ + f^- \varphi^+ f^-) \\ & + f^+ \varphi^+ f^- + f^- \varphi^+ f^+ + f^+ \varphi^- f^- + f^- \varphi^- f^+. \end{aligned} \qquad (3)$$

In order to get an estimate of the norm of $(|\varphi| - |\psi|)$, we take the sum of the norms of each term of the right hand side of (3). For the first two terms we get:

$$\begin{aligned} \|f^+ \Delta f^+\| + \|f^- \Delta f^-\| & \leqq \|f^+ \Delta^+ f^+\| + \|f^+ \Delta^- f^+\| + \|f^- \Delta^+ f^-\| + \|f^- \Delta^- f^-\| \\ & = \Delta^+(f^+) + \Delta^+(f^-) + \Delta^-(f^+) + \Delta^-(f^-) \\ & = \Delta^+(1) + \Delta^-(1) = \|\Delta\|. \end{aligned} \qquad (4)$$

For treating the last four terms of (3) we need the following

II.2. Lemma. *Let M be a von Neumann algebra and let $\omega \in M_*$ be a positive linear functional. Let e_1, e_2 be two orthogonal projections in M and define $\omega_{i,k} = e_i \omega e_k$. Then we get the estimates:*

$$\|\omega_{1,2}\|^2 \leqq \|\omega_{1,1}\| \, \|\omega_{2,2}\| .$$

Proof. Since ω is positive it follows that $\omega_{1,2}^* = \omega_{2,1}$. Let $\omega_{1,2} = V|\omega_{1,2}|$ be the polar decomposition of $\omega_{1,2}$, then $V^*V \leqq e_2$ follows and $VV^* \leqq e_1$. For $\lambda \in \mathbb{C}$ define $x = VV^* + \lambda V + \bar{\lambda} V^* + |\lambda|^2 V^*V$. Since V is a partial isometry it follows by easy computation that $x^2 = (1 + |\lambda|^2)x$ (using the orthogonality of e_1 and e_2). Since x is selfadjoint, it follows from this that x is positive. Hence we obtain again by orthogonality:

$$\begin{aligned} 0 &\leqq (e_1 + e_2)\omega(e_1 + e_2)(x) \\ &= \omega_{1,1}(VV^*) + \bar{\lambda}\omega_{1,2}(V^*) + \lambda\omega_{2,1}(V) + |\lambda|^2 \omega_{2,2}(V^*V) . \end{aligned}$$

Since $\omega_{1,2}(V^*) = \|\omega_{1,2}\|$ by definition of V,

$$\|\omega_{1,2}\|^2 \leqq \omega_{1,1}(VV^* | \omega_{2,2}(V^*V| \leqq \|\omega_{1,1}\| \, \|\omega_{2,2}\|$$

follows.

Continuation of the Proof of the Theorem. Using this last lemma we obtain from (3) and (4):

$$\begin{aligned} \| |\varphi| - |\psi| \| \leqq \|\Delta\| + 2(&\|f^+\varphi^- f^+\| + \|f^-\varphi^+ f^-\| \\ &+ \{\|f^+\varphi^+ f^+\| \, \|f^-\varphi^+ f^-\|\}^{1/2} + \{\|f^-\varphi^- f^-\| \, \|f^+\varphi^- f^+\}^{1/2}), \end{aligned}$$

and hence by Schwarz' inequality:

$$\begin{aligned} \| |\varphi| - |\psi| \| &\leqq \|\Delta\| + 2\{\varphi^+(f^+) + \varphi^+(f^-) + \varphi^-(f^+) + \varphi^-(f^-)\}^{1/2} \\ &\qquad \cdot \{2(\varphi^+(f^-) + \varphi^-(f^+)\}^{1/2} \\ &= \|\Delta\| + 2\sqrt{2}\sqrt{\|\varphi\|}\{\varphi^+(f^-) + \varphi^-(f^+)\}^{1/2} . \end{aligned}$$

Interchanging the role of φ and ψ, we find also:

$$\| |\varphi| - |\psi| \| \leqq \|\Delta\| + 2\sqrt{2}\sqrt{\|\psi\|}\{\psi^+(e^-) + \psi^-(e^+)\}^{1/2} .$$

Taking the average of these two equations and using Schwarz' inequality again, we obtain:

$$\begin{aligned} \| |\varphi| - |\psi| \| \leqq \|\Delta\| + &\sqrt{2}\{\|\varphi\| + \|\psi\|\}^{1/2} \\ &\cdot \{\varphi^+(f^-) + \varphi^-(f^+) + \psi^+ e^-) + \psi^-(e^+)\}^{1/2} . \end{aligned} \tag{5}$$

To estimate the expression in the last bracket we remark:

$$\begin{aligned} \varphi^+(f^-) + \varphi^-(f^+) + \psi^+(e^-) + \psi^-(e^+) &= (\varphi^+ - \varphi^- - \psi^+ + \psi^-)(e^+ - f^+) \\ &= \Delta(e^+ - f^+) \leqq \|\Delta\| . \end{aligned}$$

If φ and ψ are both positive or negative, then one can choose $e^+=f^+=1$, respectively, $e^+=f^+=0$. This means that in this case the expression vanishes. Therefore it should be possible to find an estimate taking this into account. But we work with this simple estimate and obtain:

$$\||\varphi|-|\psi|\| \leqq \|\Delta\| + \sqrt{2}\{\|\varphi\|+\|\psi\|\}^{1/2}\sqrt{\|\Delta\|}\,. \tag{6}$$

This proves the theorem for the case where φ and ψ are real functionals.

For generalizing this result to arbitrary functionals we start with:

II.3. Lemma. *Let M_2 be the two by two matrices with values in M; that is $x=(x_{i,k})$ with $x_{i,k}\in M$. For $\varphi_{i,k}\in M_*$ define the functional $\Phi=(\psi_{i,k})$ by $\Phi(x)=\Sigma\varphi_{i,k}(x_{i,k})$. Then we have for $\Phi=\begin{pmatrix}0, & \varphi^*\\ \varphi, & 0\end{pmatrix}$*

$$\Phi^+=\frac{1}{2}\begin{pmatrix}|\varphi|, & \varphi^*\\ \varphi, & |\varphi^*|\end{pmatrix}, \qquad \Phi^-=\frac{1}{2}\begin{pmatrix}|\varphi|, & -\varphi^*\\ -\varphi, & |\varphi^*|\end{pmatrix},$$

and hence

$$|\Phi|=\begin{pmatrix}|\varphi|, & 0\\ 0, & |\varphi^*|\end{pmatrix}.$$

Proof. Let $\varphi=V|\varphi|$ be the polar decomposition of φ. Then we get

$$\varphi^*=V^*V|\varphi|V^*=V^*|\varphi^*|\,.$$

This allows us to write

$$\Phi=\begin{pmatrix}0, & V\\ V^*, & 0\end{pmatrix}\begin{pmatrix}|\varphi|, & 0\\ 0, & |\varphi^*|\end{pmatrix}.$$

Now the matrix $U=\begin{pmatrix}0, & V\\ V^*, & 0\end{pmatrix}$ satisfies $U=U^*$ and $(U^2)^2=U^2$. Furthermore we have $U^2=S\begin{pmatrix}|\varphi|, & 0\\ 0, & |\varphi^*|\end{pmatrix}$. In order to show that this is the polar decomposition of Φ, it is sufficient to show that $\Phi(U^*)=\|\Phi\|$ holds. From $\Phi(U^*)=\varphi^*(V)+\varphi(V^*)$ $=2\|\varphi\|$ it follows that $\|\Phi\|\geqq 2\|\varphi\|$. On the other hand $\|x\|\leqq 1$ implies $\|x_{i,k}\|\leqq 1$, and hence from

$$\|\Phi\|=\sup\{\varphi(x_{2,1})+\varphi^*(x_{1,2});\ \|x\|=1\}$$

it follows that $\|\Phi\|\leqq\|\varphi\|+\|\varphi^*\|=2\|\varphi\|$. This gives $\|\Phi\|=2\|\varphi\|$ and hence $|\Phi|=\begin{pmatrix}|\varphi|, & 0\\ 0, & |\varphi^*|\end{pmatrix}$. From $\Phi^+=\frac{1}{2}\{\Phi+|\Phi|\}$, $\Phi^-=\frac{1}{2}\{|\Phi|-\Phi\}$ the lemma follows.

Proof of the Theorem. Put $\Phi=\begin{pmatrix}0, & \varphi^*\\ \varphi, & 0\end{pmatrix}$, $\Psi=\begin{pmatrix}0, & \psi^*\\ \psi, & 0\end{pmatrix}$, and $\underline{\Delta}=\begin{pmatrix}0, & \varphi^*-\psi^*\\ \varphi-\psi, & 0\end{pmatrix}$. Then we get

$$\|\Phi\|=2\|\varphi\|\,,\qquad \|\underline{\Psi}\|=2\|\psi\|\,,\quad\text{and}\quad \|\underline{\Delta}\|=2\|\varphi-\psi\|=2\|\Delta\|\,.$$

From Lemma II.3 we have $|\Phi|=\begin{pmatrix}|\varphi|, & 0\\ 0, & |\varphi^*|\end{pmatrix}$, $|\Psi|=\begin{pmatrix}|\psi|, & 0\\ 0, & |\psi^*|\end{pmatrix}$. So we obtain from Eq. (6):

$$\||\varphi|-|\psi|\|+\||\varphi^*|-|\psi^*|\|\leqq 2\|\Delta\|+2\sqrt{2}\{\|\varphi\|+\|\psi\|\}^{1/2}\cdot\sqrt{\|\Delta\|}\,.$$

This proves the theorem.

Next we want to apply Theorem II.1 to our problem of continuous covariant representations. First we need some notation.

II.4. Definition. Let G be a topological group and $\{M,G,\alpha\}$ a W^*-dynamical system. Then we denote by $M_{*,c}=\{\varphi\in M_*,\ \alpha_g\varphi$ is continuous in the norm topology at $g=1\}$. (This means $\|\alpha_g\varphi-\varphi\|\to 0$ for $g\to 1$.)

II.5. Proposition. *Let $\{M,G,\alpha\}$ be a W^*-dynamical system (G a topological group). Then $M_{*,c}$ is a norm-closed linear subspace of M_* with the additional properties:*

a) *$M_{*,c}$ is invariant under α_g for all $g\in G$,*

b) *$\varphi\in M_{*,c}\Rightarrow|\varphi|$ and $\varphi^*\in M_{*,c}$,*

c) *$M_{*,c}$ is generated by its positive part $M^+_{*,c}$.*

Proof. It is clear from the definition of $M_{*,c}$ that it is a linear space. Let now φ_i be a norm convergent sequence with $\varphi_i\in M_{*,c}$ and with limit φ. Let $\varepsilon>0$. Then there exists n_0 such that $\|\varphi-\varphi_{n_0}\|<\frac{\varepsilon}{3}$. Since $\varphi_{n_0}\in M_{*,c}$ there exists a neighbourhood U of the identity in G with $\|\varphi_{n_0}-\alpha_g\varphi_{n_0}\|\leqq\frac{\varepsilon}{3}$ for $g\in U$. Hence we get for $g\in U$

$$\|\varphi-\alpha_g\varphi\|\leqq\|\varphi-\varphi_{n_0}\|+\|\varphi_{n_0}-\alpha_g\varphi_{n_0}\|+\|\alpha_g(\varphi_{n_0}-\varphi)\|<\varepsilon\,.$$

This shows $M_{*,c}$ is norm-closed. Next from

$$\|\alpha_h\alpha_g\varphi-\alpha_g\varphi\|=\|\alpha_g(\alpha_{g^{-1}hg}\varphi-\varphi)\|=\|\alpha_{g^{-1}hg}\varphi-\varphi\|$$

it follows that $M_{*,c}$ is invariant under α_g.

Since the norm is invariant under the involution it follows that $M_{*,c}$ is invariant under involution. Since the map $\varphi\to|\varphi|$ is continuous in the norm topology by Theorem II.1, it follows that with $\varphi\in M_{*,c}$, $|\varphi|$ also belongs to $M_{*,c}$.

III. Quasi-Covariant Representations

In this and the following section we will work with C^*-algebras. Everything which is said here is true in the context of von Neumann algebras if one replaces the concept of representation by that of normal representation.

Let A be a C^*-algebra, then we denote by $S(A)$ the set of states. If (π,H) is a representation of A (which is always assumed to be nondegenerate), then we denote by F_π the folium of π. These are the normal states of $\pi(A)$. Here F_π is convex and norm-closed and invariant under the map $\omega\to x\omega x^*/\omega(xx^*)$ for all $x\in A$ with $\omega(xx^*)\neq 0$. Two representations π_1,π_2 of A are called quasi-equivalent if $\pi_1(A)$ and $\pi_2(A)$ are normal faithful representations of each other. It is well known that π_1 and π_2 are quasi-equivalent to each other if and only if $F_{\pi_1}=F_{\pi_2}$.

III.1. Definition. Let (A, G, α) be a C^*-dynamical system with G a topological group. Then

a) A representation π on H is called covariant if there exists a unitary continuous representation $U(g)$ of G on H with

$$U(g)\pi(x)U^*(g) = \pi(\alpha_g x).$$

b) A representation π is called quasi-covariant if there is a representation π_1 which is quasi-equivalent to π and which is at the same time a covariant representation.

We now want to generalize a result which is known for locally compact groups and the additional assumption that α_g acts strongly continuous on A.

III.2. Theorem. *Let (A, G, α) be a C^*-dynamical system with G a topological group, and let π be a representation of A. Then π is quasi-covariant if and only if*

1. *F_π is invariant under the action of α_g,*
2. *α_g acts strongly continuous on F_π; this means for $\varepsilon > 0$ and $\omega \in F_\pi$ there exists a neighbourhood $N \subset G$ of the identity such that*

$$\|\alpha_g \omega - \omega\| < \varepsilon \quad \textit{for} \quad g \in N.$$

Proof. This theorem will be proved with the help of Tomita's theory of modular Hilbert algebras [14] (see e.g. Takesaki [19]) for a representation of this subject. In particular the theory of the so-called natural cone and standard representation of positive functionals is needed. This theory of the natural cone has been developed by Araki [1, 2], Connes [6, 7], and Haagerup [10]. In the papers of Araki and Connes one finds the case where the von Neumann algebra M has a separating state. The general case which uses weights instead of states is treated by Haagerup. Since we have in mind that our von Neumann algebra is the double dual of a C^*-algebra we need the general case, since, except for special situations the double dual of a C^*-algebra will not have separating states. For an introduction into the theory of the natural self-dual cone, see e.g. the textbook of Bratteli and Robinson [5, Vol. I, Sect. 2.5.4] for the case where M has a separating normal state, and Sect. 2.7.3 for the general situation.

We now apply this theory to the von Neumann algebra $\pi(A)''$. It is well-known that a central projection E in A^{**} exists (the enveloping von Neumann algebra) such that EA^{**} and $\pi(A)''$ has the characterization

$$\{\pi(A)''\}_* = \{\varphi \in A^*;\ \operatorname{supp}\varphi \leqq E\},$$

where one can choose either the right or the left support of φ since E belongs to the center.

Let w be a faithful, normal, semi-finite weight on $\pi(A)''$ and $(\hat{\pi}_1, H)$ its GNS representation. Let π_1 be the restriction of $\hat{\pi}_1$ to A, then π_1 and π are quasi-equivalent. Let P be the natural cone in the Hilbert space H. (The natural cone is often denoted by $P^\natural$.) Then $\xi \in P \to \omega_\xi(x) = (\xi, x\xi)$ gives a homeomorphic bijection from P to F_π. If α is any automorphism of $\pi(A)''$, then there exists a unitary operator U_α acting on H such that $U_\alpha \xi(\omega) = \xi(\alpha^* \omega)$ holds for $\omega \in F_\pi$.

Since $\omega \to \xi(\omega)$ is a homeomorphism, it follows that the representation $U(g)\xi(\omega) = \xi(\alpha_g^* \omega)$ is continuous if Condition 2 of the theorem is fulfilled. Hence π_1 is a covariant representation and consequently π is quasi-covariant.

The converse of the statement is well-known and can be found in [3].

IV. Some Remarks on the Structure of $M_{*,c}$

We end this paper with some additional remarks on the structure of $M_{*,c}$. From Sect. II we know the following properties:

(i) $\varphi \in M_{*,c}$ implies $\varphi^* \in M_{*,c}$.

(ii) $\varphi = \varphi^* \in M_{*,c}$ and $\varphi = \varphi^+ - \varphi^-$, the canonical decomposition of φ, then φ^+ and $\varphi^- \in M_{*,c}$. Hence $M_{*,c}$ is linearly generated by its positive elements.

(iii) $\varphi \in M_{*,c}$ implies $|\varphi| \in M_{*,c}$.

In order to obtain more properties of $M_{*,c}$, we look at the standard representation of M and the natural cone P. We denote by P_c the representatives $\{\xi_\omega;\ \omega \in M_{*,c}^+\}$, and by H_c the smallest sub-Hilbert-space containing P_c.

IV.1. Lemma. *With the previous notations we have the following properties:*

(i) *P_c is a cone.*

(ii) *H_c is invariant under the canonical involution $\mathscr{I}$.*

(iii) *Let H_c^r denote the vectors which are real in H_c. Then P_c is a selfdual cone in H_c^r and H_c is generated by P_c algebraically.*

(iv) *Denote by e_c the projection onto H_c. Then for every $\xi \in P$ it follows that $e_c\xi \in P_c$.*

Proof. (i) Let $\xi_1, \xi_2 \in P_c$. Then it follows from Theorem III.1 that the functional $x \to (\xi_1, x\xi_2)$ belongs to $M_{*,c}$. Hence the functional generated by $\xi_1 + \xi_2$ is in $M_{*,c}$, which implies that P_c is a cone.

(ii) This follows from the fact that P_c is pointwise invariant under the involution $\mathscr{I}$, and that $\mathscr{I}$ is a continuous operator.

(iii) Assume $\xi_i, \eta_i \in P_c$, $i = 0, \dots, 3$.

Then the functional $x \to \left(\sum_0^3 (i)^k \xi_k, x \sum_0^3 (i)^L \eta_L \right)$ belongs to $M_{*,c}$. Since $M_{*,c}$ is norm-closed, it follows that $x \to (\xi, x\eta) \in M_{*,c}$ for all $\xi, \eta \in H_c$. This in turn implies that $P_c = P \cap H_c$ is a closed cone. Now let $\eta \in H_c$ with $\mathscr{I}\eta = \eta$, then $(\eta, \cdot\, \eta) \in M_{*,c}^+$, and hence a vector $\xi \in P_c$ exists with $(\eta, \cdot\, \eta) = (\xi, \cdot\, \xi)$ and consequently a partial isometry $W' \in M'$ with $\eta = W^*\xi$ and $\xi = W'^*\eta$. From $\mathscr{I}\xi = \xi$ and $\mathscr{I}\eta = \eta$ it follows with $W = \mathscr{I} W' \mathscr{I} \in M$ also that $\eta = W\xi$, $\xi = W^*\eta$ holds. Without loss of generality we may assume that W^*W is the support of ξ. Now from $W\xi = \mathscr{I} W \mathscr{I} \xi$, $W^2\xi = W \mathscr{I} W \mathscr{I} \xi \in P$ follows and for $x \in M$

$$
\begin{aligned}
(W^2\xi, xW^2\xi) &= (W\mathscr{I}W\mathscr{I}\xi, xW\mathscr{I}W\mathscr{I}\xi) \\
&= (W\xi, xW\mathscr{I}W^*W\mathscr{I}\xi) = (W\xi, xW\xi) \\
&= (\mathscr{I}W\mathscr{I}\xi, x\mathscr{I}W\mathscr{I}\xi) = (\xi, x\xi).
\end{aligned}
$$

Hence by the uniqueness of the representing vector $W^2\xi = (\mathscr{I}W\mathscr{I})^2\xi = \xi$ follows. From the minimality of W it follows that W^2 is the support projection of ξ or

$W = W^*$. Since $(\eta, \cdot\xi) = W(\xi, \cdot\xi)$ it follows that $(\eta, \cdot\xi)$ is selfadjoint and this formula gives the polar decomposition. From this follows $\xi^+ = W^+\xi \in P_c$ and $\xi^- = W^-\xi \in P_c$ and $\eta = \xi^+ - \xi^-$, $\xi = \xi^+ + \xi^+$. Hence $P_c^+ - P_c^+ = H_c^r$. Finally for $\xi_1, \xi_2 \in P_c$, $(\xi_1, \xi_2) \geqq 0$ follows, since P_c is a subcone of P. If $\eta \in H_c^r$, then from the previous result $\eta = \xi^+ - \xi^-$ follows, with $\xi^+, \xi^- \in P_c$ and $(\xi^+, \xi^-) = 0$. Hence if $(\eta, \xi) \geqq 0$ for all $\xi \in P_c$ $(\eta, \xi^-) = -\|\xi^-\|^2 \geqq 0$ follows, which implies $\xi^- = 0$ and consequently $\eta \in P_c$.

(iv) Let $\xi \in P$. Then $(\xi, \xi_c) \geqq 0$ follows for all $\xi_c \in P_c$, and hence $(e_c\xi, \xi_c) = (\xi, \xi_c) \geqq 0$, which implies $e_c\xi \in P_c$.

These remarks show that the space $M_{*,c}$ has some interesting structure. Consequences of this will be treated in a forthcoming paper.

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