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Cylindrically and Spherically Symmetric Monopoles in SU(3) Gauge Theory

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Abstract. We apply to the Atiyah-Ward ansätze a systematic procedure locating symmetric monopoles in SU(3) gauge theory broken to U(1) × U(1). In particular we recover the known spherically symmetric monopole as a limit of a cylindrically symmetric separated two monopole solution in SU(3). We also discuss the spherically symmetric monopole in SU(n). This latter is the only instance where we have properly shown the smoothness of the Higgs and gauge fields.

Introduction

Over the past year there has been a great deal of progress in the understanding of monopoles in gauge theories. It commenced with the discovery that the Atiyah-Ward construction [1] of self dual solutions is better suited to monopoles than to the instantons which motivated it. The doubly charged SU(2) monopole found by Ward [2] and independently by Forgacs et al. [3] was generalised by Prasad and Rossi and Forgacs et al. [4] to higher charges. As yet all these monopoles were located at a single point and had cylindrical symmetry. Ward [5] then produced the first true multimonopole, two charge one monopoles separated by a small distance. Corrigan and Goddard [6] generalised this to a $4n-1$ parameter family of SU(2) multimonopoles.

SU(3) is clearly the next place to look and, again, Ward [7] has found a one parameter family of cylindrically symmetric monopoles which have as spherically symmetric limit the solution for SU(3) broken to U(2) familiar from earlier work.

In this paper we shall look for such families when SU(3) is broken to U(1) × U(1).

Monopoles are finite energy solutions of the Bogomolny equation

$$D_i\phi = \pm\tfrac{1}{2}\varepsilon_{ijk}F_{jk} \qquad i,j,k=1,2,3\,, \tag{1.1}$$

where ϕ is the Higgs field, in the adjoint representation of SU(n), D_j is the covariant derivative $(\partial_i + iA_j)$ and F_{jk} the space part of the field strength tensor,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]\,. \qquad A_\mu = A_\mu^\dagger\,.$$

We impose as boundary conditions that $\phi \to \mathbb{C} - \frac{k}{r}$ as $r \to \infty$ for k and C constant matrices in SU(n).

Although we have no potential in (1.1) we preserve the boundary conditions that would be imposed by one of fourth order in ϕ. The matrix C then specifies the asymptotic direction of symmetry breaking either to U(1)×U(1) or to U(2). From (1.1) and the finite energy conditions we get

$$B_i = \tfrac{1}{2}\varepsilon_{ijk}F_{jk} \sim \frac{r_i}{r^3}k \quad \text{as} \quad r \to \infty .$$

So the matrix k specifies the monopole charge. It can be chosen to lie in the Cartan algebra of the gauge group seen as a vector space whose dimension is the rank of the group. Then the charge is specified, more or less, by the components of k in this vector space.

In this paper we shall use the Atiyah Ward method [1, 2]. We shall impose upon it the constraints of time independence and reality. We discuss the further conditions of spherical and cylindrical symmetries applying these to recover the known monopoles in SU(2) before going on to SU(3).

2. The Atiyah-Ward Construction [1, 8]

In the Atiyah-Ward construction static monopole solutions of an SU(n) gauge theory correspond to certain holomorphic, n-dimensional vector bundles over complex projective three space, $P_3(\mathbb{C})$. The transition matrix, g, must be a holomorphic function of coordinates ζ, μ, and ν, of $P_3(C)$ where $\mu = -it - z + \zeta(x - iy)$, and, $\nu = -it + z + \zeta^{-1}(x + iy)$. Here g is defined up to bundle equivalence: that is, up to multiplication on the left and right by SL(n, C) matrices holomorphic at $\zeta = \infty$ and at $\zeta = 0$, respectively. (This equivalence is not *simply* related to gauge equivalence, but does contain it.)

From such a transition matrix we can calculate the Higgs and gauge fields by factorising it

$$g = g_\infty g_0^{-1}$$

into matrices g_∞, holomorphic at $\zeta = \infty$, and g_0^{-1}, holomorphic at $\zeta = 0$. Then the fields are given by,

$$\phi = \tfrac{1}{2} g_0^{-1}\partial_z g_0 - \tfrac{1}{2} g_\infty^{-1}\partial_z g_\infty ,$$

$$A_z = -\frac{i}{2} g_0^{-1}\partial_z g_0 - \frac{i}{2} g_\infty^{-1}\partial_z g_\infty ,$$

$$A_x = -\frac{i}{2} g_0^{-1}\partial_x g_0 - \frac{i}{2} g_\infty^{-1}\partial_x g_\infty ,$$

$$A_y = \frac{i}{2} g_0^{-1}\partial_y g_0 - \frac{i}{2} g_\infty^{-1}\partial_y g_\infty .$$

For static, hermitian fields we require that g be bundle equivalent to a g which depends on ζ and $\gamma=\mu-\nu$, and that it be equivalent to a conjugate g defined by $g^{\dagger}\left(-\frac{1}{\zeta^*}\right)$.

For the case of SU(2) there is a canonical form for g:

$$g=\begin{bmatrix}\zeta^{\ell} & \varrho \\ 0 & \zeta^{-\ell}\end{bmatrix} \quad \text{for} \quad \ell\in\mathbb{Z}^{+},$$

and at least for the case of $U(1)\times U(1)$ it appears that a similar form may hold for SU(3):

$$g=\begin{bmatrix}\zeta^{\ell_1} & \varrho_{12} & \varrho_{13} \\ 0 & \zeta^{\ell_2} & \varrho_{23} \\ 0 & 0 & \zeta^{\ell_3}\end{bmatrix}, \quad \ell_i\in\mathbb{Z}, \quad \sum_{i=1}^{3}\ell_i=0.$$

This is not necessarily true in general.

By a bundle equivalence argument we may order the powers of ζ in a decreasing fashion down the diagonal.

Our design is to start from this form and whittle it away by the imposition of the constraints of time-independence, reality and symmetry, until we are left with those classes of monopoles we desire.

3. Time Independence and Reality

If g is a function of ζ, $\mu=x_{22}+\zeta x_{21}$ and $\nu=x_{11}+x_{12}/\zeta$ then it can only be time independent if its μ,ν dependence is entirely through $\gamma=\mu-\nu$ since $x_{11}=t-iz$, $x_{22}=t+iz$. However the normal form we have so far described may not be time independent. If we are to be able to remove the time by multiplication we must have

$$\varrho_{ij}=f_{ij}(\mu+\nu)\,\tilde{\varrho}_{ij}(\gamma). \tag{3.1}$$

But g satisfies the equations

$$(\partial_{i1}-\zeta\partial_{i2})g=0$$

and hence $\Box\,\varrho_{ij}=0$.

The above separation of variables in (3.1) then implies that the simplest form of $f_{ij}(\mu+\nu)$ is $\exp\{\alpha_{ij}(\mu+\nu)\}$ for α_{ij} constant.

We can remove this time dependence by bundle transformations to leave the general form:

$$g=\begin{bmatrix}\zeta^{\ell_1}e^{\alpha_1\gamma} & \tilde{\varrho}_{12}(\gamma,\zeta) & \tilde{\varrho}_{13}(\gamma,\zeta) \\ 0 & \zeta^{\ell_2}e^{\alpha_2\gamma} & \tilde{\varrho}_{23}(\gamma,\zeta) \\ 0 & 0 & \zeta^{\ell_3}e^{\alpha_3\gamma}\end{bmatrix}. \tag{3.2}$$

In the following sections we shall use this form and drop the tilda from $\tilde{\varrho}_{ij}$.

The values of the α_i determine the symmetry breaking and the scale of the Higgs field. If two of the α_i are equal we get breaking to U(2), otherwise to $U(1)\times U(1)$.

Upon this general form we impose the constraint of reality. It suffices for this to use a single bundle transformation matrix $A(\zeta)$ and to write:

$$gA(\zeta)=A^{\dagger}\left(-\frac{1}{\zeta^*}\right)g^{\dagger}\left(-\frac{1}{\zeta^*}\right). \tag{3.3}$$

We note that under the combined conjugation and $\zeta\to-\frac{1}{\zeta^*}$ operations, γ is invariant.

For SU(2) the condition reads:

$$\begin{bmatrix} A_{11}\zeta^{\ell}e^{\gamma\alpha}+A_{21}\varrho & A_{12}\zeta^{\ell}e^{\gamma\alpha}+A_{22}\varrho \\ A_{21}\zeta^{-\ell}e^{-\gamma\alpha} & A_{22}\zeta^{-\ell}e^{-\gamma\alpha} \end{bmatrix}$$
$$=\begin{bmatrix} A_{11}^*\left(-\frac{1}{\zeta}\right)^{\ell}e^{\gamma\alpha}+A_{21}^*\varrho^* & A_{21}^*\left(-\frac{1}{\zeta}\right)^{-\ell}e^{-\gamma\alpha} \\ A_{12}^*\left(-\frac{1}{\zeta}\right)^{\ell}e^{\gamma\alpha}+A_{22}^*\varrho^* & A_{22}^*\left(-\frac{1}{\zeta}\right)^{-\ell}e^{-\gamma\alpha} \end{bmatrix}. \tag{3.4}$$

Because A_{22} is analytic in ζ and A_{22}^* in ζ^{-1}, the equation

$$A_{22}\zeta^{-l}=A_{22}^*(-1)^{\ell}\zeta^{\ell} \tag{3.5}$$

implies that up to a real constant $A_{22}\zeta^{-\ell}$ is a product of ℓ factors $(\gamma-\gamma_i)$ for $i=1,\dots,$ where $\gamma_i=c_i+a_{i/\zeta}-a_i^*\zeta$ with a_i and (real) c_i constants has the same reality properties as γ.

The off diagonal equations are equivalent and tell us that

$$\varrho=\frac{Fe^{\gamma\alpha}+fe^{-\gamma\alpha}}{\psi}, \quad \text{where} \quad \psi=\prod_{i=1}^{\ell}(\gamma-\gamma_i) \tag{3.6}$$

and F and f are arbitrary functions of ζ. We shall always use upper case letters for functions analytic at $\zeta=0$ and lower case for those analytic at $\zeta=\infty$. Hence $F=-A_{12}$, $f=(-1)^{\ell}A_{21}^*\left(-\frac{1}{\zeta^*}\right)$ up to a constant factor.

The remaining reality condition reduces to $A_{11}A_{22}-A_{21}A_{12}=1$ which is satisfied since $A\in\mathrm{SL}(2,C)$.

A precisely analogous procedure for SU(3) produces two polynomials in γ, ψ_1, ψ_3 of degrees ℓ_1 and $-\ell_3>0$ from conditions like (3.5) on the subdeterminant $A_{22}A_{33}-A_{32}A_{23}$ and on A_{33} respectively.

For the ϱ_{ij} we obtain:

$$\begin{aligned} \varrho_{12}&=\frac{Ge^{\gamma\alpha_1}+\zeta^{-\ell_3}ge^{\gamma\alpha_2}}{\psi_1}, \qquad \psi_1=\prod_{i=1}^{\ell_1}(\gamma-\gamma_{1i}),\\ \varrho_{23}&=\frac{fe^{\gamma\alpha_3}+\zeta^{-\ell_1}Fe^{\gamma\alpha_2}}{\psi_3}, \qquad \psi_3=\prod_{i=1}^{-\ell_3}(\gamma-\gamma_{3i}),\\ \varrho_{13}&=\frac{He^{\gamma\alpha_1}}{\psi_1}+\frac{he^{\gamma\alpha_3}}{\psi_3}+\frac{Fge^{\gamma\alpha_2}}{\psi_1\psi_3}. \end{aligned} \tag{3.7}$$

Note that a term appears in ϱ_{13} which mixes terms from ϱ_{12} and ϱ_{23}.

The only general demands we make on the ϱ_{ij} are that they be free of singularities, for all X, in some open annulus about $\zeta=0$ and that the splitting procedure should not introduce singularities into the field strength tensor.

We however will make further demands first.

4. Cylindrical and Spherical Symmetry

What do we mean by symmetry in a gauge theory? Normally by symmetry we mean that under a group transformation r acting on x, the points of space time, a symmetric object $S(x)$ obeys the relation:

$$rS(x)r^{-1}=S(r^{-1}xr). \tag{4.1}$$

The type of object S is defined according to the representation of the group which acts thereon.

In a gauge theory the gauge independent objects will satisfy such a relation but we ask of gauge dependent objects that they do so only up to a gauge transformation. So for the gauge potential A we have:

$$A(r^{-1}xr)=\theta rA(x)r^{-1}\theta^{-1}+\theta\, d\theta^{-1}, \tag{4.2}$$

where $\theta(x)$ has values in the gauge group.

To say the same thing in the Atiyah-Ward formalism:

$$g(r^{-1}xr)=a\left(\frac{1}{\zeta}\right)g(x)\,A(\zeta), \tag{4.3}$$

that is, the rotated and original g's give equivalent bundles. The relation between θ and a and A is not very simple.

For cylindrical and spherical symmetry we consider the general rotation [6] for which $X\to X'=rXr^{-1}$, as quaternions, and

$$r=\begin{pmatrix}\alpha & \beta\\ -\beta^* & \alpha^*\end{pmatrix}\in \mathrm{SU}(2).$$

From $X\pi=\omega$ and $X'\pi'=\omega'$ we see that $r\pi=\pi'$, $r\omega=\omega'$. So with the definitions of ζ', γ' as for ζ, γ

$$\begin{aligned}\zeta\to\zeta'&=\frac{\alpha\zeta+\beta}{\alpha^*-\beta^*\zeta},\\ \gamma\to\gamma'&=\gamma(\alpha+\beta/\zeta)^{-1}(\alpha^*-\beta^*\zeta)^{-1}.\end{aligned} \tag{4.4}$$

For rotations of the order of a small parameter ε equation (4.3) looks like

$$\frac{\Delta g}{\varepsilon}=gA-ag, \tag{4.5}$$

where we have replaced A and a by $1+\varepsilon A$ and $1-\varepsilon a$ where $\tau r A=\tau r a=0$ and Δg is the $O(\varepsilon)$ change in g due to the rotation.

We shall need for cylindrical symmetry a small rotation about the z-axis: $\alpha=1+i\varepsilon$, $\beta=0$ under which

$$\begin{aligned}\zeta&\to\zeta+2i\varepsilon\zeta\,,\\ \gamma&\to\gamma\,,\end{aligned}\tag{4.6'}$$

and for spherical symmetry this and the other two rotations:

$$\alpha=1\,,\qquad \beta=\varepsilon\quad \begin{aligned}\zeta&\to\zeta(1+\varepsilon(\zeta+\zeta^{-1}))\,,\\ \gamma&\to\gamma(1+\varepsilon(\zeta-\zeta^{-1}))\,,\end{aligned}\tag{4.7a'}$$

and

$$\alpha=1\,,\qquad \beta=i\varepsilon\quad \begin{aligned}\zeta&\to\zeta(1+i\varepsilon(\zeta^{-1}-\zeta))\,,\\ \gamma&\to\gamma(1-i\varepsilon(\zeta^{-1}+\zeta))\,.\end{aligned}\tag{4.7b'}$$

Under each of these Eq. (4.5) becomes:

$$2i\zeta\frac{\partial g}{\partial\zeta}=gA-ag\,,\tag{4.6}$$

$$\zeta\left(\zeta+\frac{1}{\zeta}\right)\frac{\partial g}{\partial\zeta}+\gamma\left(\zeta-\frac{1}{\zeta}\right)\frac{\partial g}{\partial\gamma}=gA'-a'g\,,\tag{4.7a}$$

$$-\zeta\left(\zeta-\frac{1}{\zeta}\right)\frac{\partial g}{\partial\zeta}-\gamma\left(\zeta+\frac{1}{\zeta}\right)\frac{\partial g}{\partial\gamma}=gA''-a''g\,.\tag{4.7b}$$

Given the general form of g we shall solve these equations to find the symmetric monopoles in SU(3). First we rehearse the procedure for SU(2) to obtain already known results.

5. Symmetric Monopoles in SU(2)

Equation (4.6) in SU(2) reads:

$$\begin{aligned}2i\zeta\begin{bmatrix}\ell\zeta^{\ell-1}e^{\gamma\alpha} & \partial\varrho/\partial\zeta\\ 0 & -\ell\zeta^{-\ell-1}e^{-\gamma\alpha}\end{bmatrix}&=\begin{bmatrix}\zeta^{\ell}e^{\gamma\alpha} & \varrho\\ 0 & \zeta^{-\ell}e^{-\gamma\alpha}\end{bmatrix}\begin{bmatrix}A_{11} & A_{12}\\ A_{21} & -A_{11}\end{bmatrix}\\ &\quad-\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & -a_{11}\end{bmatrix}\begin{bmatrix}\zeta^{\ell}e^{\gamma\alpha} & \varrho\\ 0 & \zeta^{-\ell}e^{\gamma\alpha}\end{bmatrix}.\end{aligned}\tag{5.1}$$

Of these four equations the simplest is

$$\zeta^{-\ell}e^{-\gamma\alpha}A_{21}=a_{21}\zeta^{\ell}e^{\gamma\alpha}\,.\tag{5.2}$$

We are to preserve time independence. This and the analyticities of A_{21} and a_{21} require that they be functions of $\gamma\zeta$ and ζ, and $\gamma\zeta^{-1}$ and ζ^{-1}, respectively. Then they cannot cancel the exponentials in (5.2) and, unless $\alpha=0$, we must have $A_{21}\equiv 0$, $a_{21}\equiv 0$. But α sets the scale of the Higgs field and cannot be zero.

The diagonal entries in (5.1) give

$$A_{11}=i(c+\ell), \quad a_{11}=-i(-c+\ell).$$

By use of the same argument that we applied in (5.2) we may disconnect the $e^{\gamma\alpha}$ and $e^{-\gamma\alpha}$ parts of the remaining equation to give, from the general form (3.6) of ϱ,

$$2i\zeta\frac{\partial}{\partial\zeta}\left(\frac{F}{\psi}\right)=\zeta^{\ell}A_{12}-2ic\frac{F}{\psi},$$

$$2i\zeta\frac{\partial}{\partial\zeta}\left(\frac{f}{\psi}\right)=-2ic\frac{f}{\psi}-a_{12}\zeta^{-\ell},$$

which we rewrite as:

$$2i\zeta\frac{\partial}{\partial\zeta}\left(\frac{F\zeta^{c}}{\psi}\right)=\zeta^{\ell}A_{12}, \tag{5.3}$$

$$2i\zeta\frac{\partial}{\partial\zeta}\left(\frac{f\zeta^{c}}{\psi}\right)=-\zeta^{-\ell}a_{12}. \tag{5.4}$$

Now F is a function of ζ and $\gamma\zeta$. Therefore if ψ divides it, being a polynomial of order ℓ in γ, it will push out a factor ζ^{ℓ} from F. But if ψ divides F it must also divide f to prevent ϱ having singularities other than at 0 and ∞. Since f is a function of ϱ^{-1} and γ/ζ this pushes out a factor $\zeta^{-\ell}$. Then ϱ has the form

$$\varrho=\zeta^{\ell}F'(\zeta,\gamma\zeta)+\zeta^{-\ell}f'(\zeta^{-1},\gamma\zeta^{-1}),$$

which is precisely that removable by a bundle equivalence. So we may assume some part of ψ does not divide F. Since f/ψ must have the same x-dependent poles, the same part does not divide f.

However, A_{12} may only have poles at $\zeta=\infty$. Therefore (5.3) implies $A_{12}\equiv 0$. Similarly $a_{12}\equiv 0$. Hence the arguments of the derivatives in (5.3), (5.4) are functions of γ alone.

$$\frac{F\zeta^{c}}{\psi}=\tilde{F}(\gamma), \quad \frac{f\zeta^{c}}{\psi}=\tilde{f}(\gamma).$$

But now $F(\gamma\zeta,\zeta)$ and $f(\gamma/\zeta,1/\zeta)$ must both have factors of ζ to cancel the power ζ^{c} which is only possible if $c\leqq 0$, and $c\geqq 0$. So $c=0$. But further since F and f can only be functions of γ if they are also functions of ζ, independence of the latter implies that of the former. So F and f are constants, F_0 and f_0.

So the general form for cylindrical symmetry is:

$$\varrho_c=(F_0e^{\gamma\alpha}+f_0e^{-\gamma\alpha})\Big/\prod_{i=1}^{\ell}(\gamma-c_i), \tag{5.5}$$

where the c_i's are constants, since the general ζ-dependence of the γ_i is clearly incompatible with (5.3) and (5.4). This agrees with Ward's [2] and Prasad and Rossi's [4] results.

To consider spherical symmetry, let us first add and subtract Eqs. (4.7a) and (4.7b) to obtain the simpler equations,

$$\frac{\partial g}{\partial \zeta}-\frac{\gamma}{\zeta}\cdot\frac{\partial g}{\partial \gamma}=gB-bg\,, \tag{5.6a}$$

$$\zeta^2\frac{\partial g}{\partial \zeta}+\gamma\zeta\cdot\frac{\partial g}{\partial \gamma}=g\mathbb{C}-cg\,, \tag{5.6b}$$

and impose these upon the general form (5.5), ϱ_c.

Again the lower off-diagonal entry gives B_{21}, b_{21}, $\mathbb{C}_{21}$, and c_{21} all identically zero.

The diagonal entries in (5.6a) give

$$B_{11}=c\,,\qquad b_{11}=c-\frac{1}{\zeta}(\ell-\alpha\gamma)\,.$$

Then as before the upper off-diagonal entry leaves us with two equations:

$$-\frac{\gamma F_0}{\zeta}\frac{\partial}{\partial\gamma}\left(\frac{1}{\psi}\right)=\zeta^\ell B_{12}+\left(\frac{\ell}{\zeta}-2c\right)\frac{F_0}{\psi}\,,$$

$$\frac{\alpha f_0\gamma}{\zeta\psi}-\frac{\gamma f_0}{\zeta}\frac{\partial}{\partial\gamma}\left(\frac{1}{\psi}\right)=-\zeta^{-\ell}b_{12}+\left(\frac{1}{\zeta}(\ell-\gamma\alpha)-2c\right)\frac{f_0}{\psi}\,.$$

Since $\ell>0$ we may extract the ζ^{-1} dependence from the first of these which gives

$$-\gamma\frac{\partial}{\partial\gamma}\left(\frac{1}{\psi}\right)=\frac{\ell}{\psi}\,.$$

Therefore, up to a multiplicative constant, $\psi=\gamma^\ell$. The part remaining is then:

$$\zeta^\ell B_{12}=\frac{2cF_0}{\gamma^\ell}\,.$$

Here B_{12} is allowed no such singularities and so $B_{12}\equiv 0$ and $c=0$.

The other equation now simplifies to

$$2\alpha f_0(\gamma/\zeta)=-b_{12}(\gamma/\zeta)^\ell\,,$$

which implies that $\ell=1$ and b_{12} is constant.

Equation (5.6b) leads to the same result. Hence there is a unique spherically symmetric monopole in SU(2) given by

$$\varrho_s=F_0\frac{e^{\gamma\alpha}-e^{-\gamma\alpha}}{\gamma}\,, \tag{5.7}$$

where the choice $f_0=-F_0$ renders the correct singularities. Again this agrees with Ward [2].

These results have been known for some time but we derive them to illustrate the method we shall now apply to SU(3).

6. Cylindrically Symmetric Monopoles in SU(3)

An important difference between SU(2) and SU(3) is that whereas in the former the upper triangularity of $gA-ag$ requires that of A and a this is only the case in the latter when α_1, α_2, and α_3 are all unequal (3.2). This is because we get, for instance,

$$\zeta^{\ell_3}e^{\gamma\alpha_3}A_{31}-a_{31}\zeta^{\ell_1}e^{\gamma\alpha_1}=0\,, \tag{6.1}$$

and only for $\alpha_3 \neq \alpha_1$ can we argue that A_{31}, a_{31} vanish identically. In that case the other two lower off-diagonal entries are of the same form as (6.1) for the other pairwise choices of the α_i. In this paper we shall only consider this case but should bear in mind that the others are equally important and ought not necessarily to be limits of the following results as pairs of α_i coincide.

Looking first at small rotations about the z-axis the diagonal terms in (4.6) are:

$$2i\ell_j=A_{jj}-a_{jj}\,,\qquad j=1,\ldots,3\quad(\text{No sum}),$$

from which we write:

$$A_{jj}=i(\ell_j+c_j)\,,$$
$$a_{jj}=-i(\ell_j-c_j)\,,\quad \text{for}\quad c_j \text{ constant}\,,$$

and where

$$\sum_{j=1}^{3} c_j=0\,.$$

Disconnecting the terms in the $e^{\gamma\alpha_i}$ in the upper off-diagonal entries gives us, from the general forms (3.7) of the ϱ_{ij}, three sets of equations:

$$\left.\begin{aligned}
&2i\zeta\frac{\partial}{\partial\zeta}\left(\frac{G\zeta^{n_{12}}}{\psi_1}\right)\zeta^{-n_{12}}=A_{12}\zeta^{\ell_1}\,,\\
&2i\zeta\frac{\partial}{\partial\zeta}\left(\frac{g\zeta^{n_{12}-\ell_3}}{\psi_1}\right)\zeta^{-n_{12}}=-a_{12}\zeta^{\ell_2}\,,
\end{aligned}\right\} \tag{6.2}$$

$$\left.\begin{aligned}
&2i\zeta\frac{\partial}{\partial\zeta}\left(\frac{f\zeta^{n_{23}}}{\psi_3}\right)\zeta^{-n_{23}}=-a_{23}\zeta^{\ell_3}\,,\\
&2i\zeta\frac{\partial}{\partial\zeta}\left(\frac{F\zeta^{-\ell_1+n_{23}}}{\psi_3}\right)\zeta^{-n_{23}}=A_{23}\zeta^{\ell_2}\,,
\end{aligned}\right\} \tag{6.3}$$

$$\left.\begin{aligned}
&2i\zeta\frac{\partial}{\partial\zeta}\left(\frac{H\zeta^{n_{13}}}{\psi_1}\right)\zeta^{-n_{13}}=-a_{13}\zeta^{\ell_3}+A_{23}\frac{G}{\psi_1}\,,\\
&2i\zeta\frac{\partial}{\partial\zeta}\left(\frac{h\zeta^{n_{13}}}{\psi_3}\right)\zeta^{-n_{13}}=A_{13}\zeta^{\ell_1}-a_{12}\frac{f}{\psi_3}\,,\\
&2i\zeta\frac{\partial}{\partial\zeta}\left(\frac{gF\zeta^{n_{13}}}{\psi_1\psi_3}\right)\zeta^{-n_{13}}=A_{23}\frac{\zeta^{-\ell_3}g}{\psi_1}-a_{12}\frac{\zeta^{-\ell_1}F}{\psi_3}\,,
\end{aligned}\right\} \tag{6.4}$$

where

$$n_{ij}=\tfrac{1}{2}(c_i-c_j-\ell_i-\ell_j), \qquad n_{13}-n_{12}-n_{23}=\ell_2 .$$

By a bundle equivalence argument similar to the SU(2) case we can remove any of the arbitrary functions which divide by ψ_1 or ψ_3. So either because we can replace G and g by zero or by the argument from poles in G/ψ_1 and g/ψ_1 we have $A_{12}\equiv 0$, $a_{12}\equiv 0$ in (6.2). Similarly $a_{13}\equiv 0$, $A_{23}\equiv 0$ in (6.3). Therefore $a_{13}\equiv 0$, $A_{13}\equiv 0$ in (6.4). So all the arguments of the derivatives in (6.2)–(6.4) are either zero or ζ-independent. Therefore either $\varrho_{ij}\equiv 0$ or $\varrho_{ij}=\zeta^{-n_{ij}}\hat{\varrho}_{ij}(\gamma)$, and the γ_{1i} and γ_{3i} in ψ_1 and ψ_3 are real constants c_{1i} and c_{3i}.

Suppose $\varrho_{12}\neq 0$. Then

$$\frac{G}{\psi_1}=\zeta^{-n_{12}}\frac{\hat{G}(\gamma,M_{12})}{\phi_1(\gamma)}, \tag{6.5}$$

and

$$\frac{g}{\psi_1}\zeta^{-\ell_3}=\zeta^{-n_{12}}\frac{\hat{g}(\gamma,m_{12})}{\phi_1(\gamma)}, \tag{6.6}$$

where $\phi_1(\gamma)$ is of order $p_1\leqq l_1$ and gives the (common) poles which do not divide G and g. Here $\hat{G}(\gamma,M_{12})$ and $\hat{g}(\gamma,m_{12})$ are polynomials in γ alone of degrees M_{12} and m_{12}, respectively. Then the ζ-dependence of G implies

$$\frac{G(\gamma\zeta,\zeta)}{\psi_1}=\zeta^{\ell_1-p_1}\frac{G'(\gamma\zeta,\zeta)}{\phi_1(\gamma)}=\zeta^{-n_{12}}\frac{\hat{G}(\gamma,M_{12})}{\phi_1(\gamma)}. \tag{6.7}$$

Therefore $0\leqq M_{12}\leqq -n_{12}+p_1-\ell_1$.

Also

$$\frac{g(\gamma/\zeta,1/\zeta)}{\psi_1}=\zeta^{-\ell_1+p_1}\frac{g'(\gamma/\zeta,1/\zeta)}{\phi_1(\gamma)}=\zeta^{\ell_3-n_{12}}\frac{\hat{g}(\gamma,m_{12})}{\phi_1(\gamma)}. \tag{6.8}$$

Therefore $0\leqq m_{12}\leqq n_{12}+p_1+\ell_2$.

Similarly if $\varrho_{23}\neq 0$, we obtain

$$\frac{f}{\psi_3}=\zeta^{-n_{23}}\frac{\hat{f}(\gamma,m_{23})}{\phi_3(\gamma)}, \tag{6.9}$$

$$\frac{F\zeta^{-\ell_1}}{\psi_3}=\zeta^{-n_{23}}\frac{\hat{F}(\gamma,M_{23})}{\phi_3(\gamma)}, \tag{6.10}$$

where ϕ_3 has degree $-p_3\leqq -\ell_3$ and,

$$0\leqq m_{23}\leqq n_{23}+\ell_3-p_3, \tag{6.11}$$

$$0\leqq M_{23}\leqq -n_{23}-\ell_2-p_3. \tag{6.12}$$

Equations (6.7) and (6.8) place the following constraints on n_{12} and p_1: $\ell_3\leqq n_{12}\leqq 0$ and $\ell_1\geqq p_1\geqq\frac{1}{2}(\ell_1-\ell_2)$. Similarly from (6.11) and (6.12): $0\leqq n_{23}\leqq\ell_1$ and $\ell_3\leqq p_3\leqq\frac{1}{2}(\ell_3-\ell_2)$.

We identify the following cases:

Case (i). ψ_1 divides G, ψ_3 divides f.

Then $\varrho_{12} \equiv 0$, $\varrho_{23} \equiv$ and ϱ_{13} loses its $e^{\gamma\alpha_2}$ term.

$$\varrho_{13} = \frac{He^{\gamma\alpha_1}}{\psi_1} + \frac{he^{\gamma\alpha_3}}{\psi_3} = \zeta^{-n_{13}}\hat{\varrho}(\gamma).$$

The analyticities of $H(\gamma\zeta, \zeta)$ and $h(\gamma/\zeta, 1/\zeta)$ then require $0 \leqq n_{13} \leqq 0$, since we can only extract positive powers from the former and negative ones from the latter. This in turn requires H and h to be constants since any γ-dependence must be accompanied by ζ-dependence. But now ψ_1 and ψ_3 must have the same (constant) poles in γ or ϱ_{13} will not be analytic away from 0 and ∞. Therefore $\ell_1 + \ell_3 = 0$, $\ell_2 = 0$ and ϱ_{13} looks exactly like the ϱ_c of SU(2), (5.5) but with α and $-\alpha$ replaced by α_2 and α_3:

$$g = \begin{bmatrix} \zeta^{\ell} e^{\gamma\alpha_1} & 0 & (H_0 e^{\gamma\alpha_1} + h_0 e^{\gamma\alpha_3}) \Big/ \prod_{i=1}^{\ell} (\gamma - c_i) \\ 0 & e^{\gamma\alpha_2} & 0 \\ 0 & 0 & \zeta^{-\ell} e^{\gamma\alpha_3} \end{bmatrix}.$$

For non-singularity the c_i must all be different and of the form:

$$c_j = \frac{c_0 + 2\pi i n_j}{\alpha_1 - \alpha_3}.$$

As in the SU(2) case we expect the non-singularity of the splitting procedure, at each x, to constrain the c_j to ℓ particular values.

Case (ii). ψ_3 divides f.

From (6.5) and (6.6)

$$\varrho_{12} = \zeta^{-n_{12}} \frac{\hat{G}(\gamma, M_{12}) e^{\gamma\alpha_{12}} + \hat{g}(\gamma, m_{12}) e^{\gamma\alpha_2}}{\phi_1(\gamma)},$$

where (6.7) and (6.8) hold. Thus ϱ_{13} will vanish or not accordingly as ψ_1 divides H, and ψ_3 divides h or not, since F vanishes by ψ_3 dividing f. If it does not then it will have the same form as in case (i), and for the same reasons. Then ℓ_2 is zero. Here $\phi_1(\gamma)$ is a selection of p_1 factors from $\psi_1 = \prod_{i=1}^{\ell} (\gamma - c_i)$. Satisfying the singularity constraints on ϱ_{12} will leave free parameters in $\phi_1(\gamma)$ only in the case where $\varrho_{13} \equiv 0$.

Case (iii). ψ_1 divides G.

The same holds with respect to ϱ_{13} and ϱ_{23} here as did for ϱ_{13} and ϱ_{12} in case (ii).

Case (iv). ψ_1 does not divide G, ψ_3 does not divide F.

Then ϱ_{13} has the general form: since $n_{13} = \ell_2 + n_{12} + n_{23}$,

$$\varrho_{13} = \zeta^{-n_{13}} \left(\frac{He^{\gamma\alpha_1}}{\psi_1} + \frac{he^{\gamma\alpha_3}}{\psi_3} + \frac{\hat{g}(\gamma, m_{12}) \hat{F}(\gamma, M_{23})}{\phi_1 \phi_3} e^{\gamma\alpha_2} \right).$$

(a) If ψ_1 divides H and ψ_3 divides h, then ϕ_1 divides $\hat{F}$ and ϕ_3 divides $\hat{g}$. Therefore

$$0 \leqq p_1 \leqq M_{23} \leqq -n_{23} - \ell_2 - p_3,$$
$$0 \leqq -p_3 \leqq m_{12} \leqq n_{12} + p_1 + \ell_2.$$

These imply $n_{12} \geqq n_{23}$. But $n_{12} \leqq 0$ and $n_{23} \geqq 0$ so they must both vanish. Inequalities (6.7) and (6.11) then give $\ell_1 = p_1$, $\ell_3 = p_3$ and so,

$$\varrho_{12} = \frac{\hat{G}_0 e^{\gamma\alpha_1} + \hat{g}(\gamma, -\ell_3) e^{\gamma\alpha_2}}{\psi_1(\gamma)},$$
$$\varrho_{23} = \frac{\hat{f}_0 e^{\gamma\alpha_3} + \hat{F}(\gamma, \ell_1) e^{\gamma\alpha_2}}{\psi_3(\gamma)},$$
$$\varrho_{13} = 0.$$

(b) ψ_1 does not divide H, ψ_3 does not divide h.

Then as in case (i), $n_{13} = 0$ and H, h are constants. All we can say in general, since the $e^{\gamma\alpha_2}$ term in ϱ_{13} is determined from those in ϱ_{12} and ϱ_{23}, is that enough cancellation must occur between ϕ_1 and $\hat{F}$, and ϕ_3 and $\hat{g}$ to ensure that the number of poles in $\frac{\hat{g}\hat{F}}{\phi_1\phi_2}$ away from zero and infinity balances the distinct poles in ψ_1, ψ_3.

(c) ψ_1 divides H or ψ_3 divides h.

In this case n_{13} is no longer zero and the usual remarks apply as regards singularities.

7. Spherically Symmetric Monopoles in SU(3)

Not all the cases listed under Sect. 7 have spherically symmetric monopoles amongst their cylindrical ones.

The equations of spherical symmetry are:

$$\frac{\partial g}{\partial \zeta} - \frac{\gamma}{\zeta} \cdot \frac{\partial g}{\partial \zeta} = gB - bg, \tag{7.1}$$

$$\zeta^2 \frac{\partial g}{\partial \zeta} + \gamma\zeta \frac{\partial g}{\partial \zeta} = g\mathbb{C} - cg. \tag{7.2}$$

From the diagonal terms we obtain:

$$B_{ii} = k_i, \qquad b_{ii} = k_i - \frac{1}{\zeta}(\ell_i - \gamma\alpha_i).$$
$$\mathbb{C}_{ii} = k_i' + \zeta(\ell_i + \gamma\alpha_i), \qquad c_{ii} = k_i'.$$

For case (i) the analysis is similar to that in the SU(2) case (Sect. 5) yielding $\ell = 1$ and $\varrho_{13} = \frac{1}{\gamma}(e^{\gamma\alpha_1} - e^{\gamma\alpha_3})$.

In any of the cases where $\varrho_{13}\equiv 0$ we have the equation:

$$0=\frac{\partial\varrho_{13}}{\partial\gamma}-\frac{\gamma}{\zeta}\cdot\frac{\partial\varrho_{13}}{\partial\gamma}=B_{13}\zeta^{\ell_1}e^{\gamma\alpha_1}+B_{23}\varrho_{12}-b_{12}\varrho_{23}-b_{13}\zeta^{\ell_3}e^{\gamma\alpha_3},$$

and similarly for $\mathbb{C}$ and c. If ϱ_{12}, ϱ_{23} are non-vanishing this gives, $B_{23}=\zeta^{\ell_1}\psi_1 B'_{23}$ and $b_{12}=\zeta^{\ell_3}\psi_3 b'_{12}$ and so also, from the $e^{\alpha_2\gamma}$ term: $B'_{23}\hat{g}\zeta^{\ell_1-n_{12}}=-b'_{12}\hat{F}\zeta^{\ell_3-n_{23}}\equiv 0$ since the left-hand side has powers of ζ greater than zero and the right-hand side less than zero. Hence only B_{12} and b_{23} are not zero. The equation in ϱ_{12} then becomes:

$$\frac{\partial\varrho_{12}}{\partial\gamma}-\frac{\gamma}{\zeta}\cdot\frac{\partial\varrho_{12}}{\partial\gamma}=B_{12}\zeta^{\ell_1}e^{\gamma\alpha_1}+\left((k_2-k_1)+\frac{1}{\zeta}(\ell_1-\gamma\alpha_1)\right)\varrho_{12},$$

in which the $e^{\alpha_2\gamma}$ term is:

$$\frac{\partial}{\partial\gamma}(\tilde{g}/\phi_1)=(\alpha_2-\alpha_1)\tilde{g}/\zeta_1+\frac{\ell_1-n_{12}}{\gamma}\cdot\tilde{g}/\phi_1+\frac{k_2-k_1}{\gamma}\zeta.$$

The ζ dependence implies $k_2=k_1$. But this leaves us with an equation which for $\alpha_1 \neq \alpha_2$ has no finite polynomial solution, and we have no spherical monopoles here.

So suppose $\varrho_{23}\equiv 0$ as well. In this case B_{12} and b_{12} only remain. Then the equations for ϱ_{12} we have are:

$$\begin{aligned}\frac{\partial\varrho_{12}}{\partial\zeta}-\frac{\gamma}{\zeta}\cdot\frac{\partial\varrho_{12}}{\partial\gamma}&=B_{12}\zeta^{\ell_1}e^{\gamma\alpha_1}+\left((k_2-k_1)+\frac{1}{\zeta}(\ell_1-\gamma\alpha_1)\right)\varrho_{12}\\&\quad-b_{12}\zeta^{\ell_2}e^{\gamma\alpha_2},\\ \zeta_2\frac{\partial\varrho_{12}}{\partial\zeta}+\gamma\zeta\frac{\partial\varrho_{12}}{\partial\gamma}&=\mathbb{C}_{12}\zeta^{\ell_1}e^{\gamma\alpha_1}+(k'_2-k'_1+\zeta(\ell_2+\gamma\alpha_2))\varrho_{12}\\&\quad-c_{12}\zeta^{\ell_2}e^{\gamma\alpha_2}.\end{aligned}\tag{7.3}$$

The $e^{\gamma\alpha_1}$ term in the first equation gives,

$$-\gamma\zeta^{-n_{12}-1}\frac{\partial}{\partial\gamma}\left(\frac{\hat{G}}{\phi_1}\gamma^{\ell_1+n_{12}}\right)\gamma^{-\ell_1-n_{12}}=B_{12}\zeta^{\ell_1}+(k_2-k_1)\zeta^{-n_{12}}\frac{\tilde{G}}{\phi_1}.$$

If ϕ_1 had a pole in γ other than at $\gamma=0$ then it would be made second order by differentiation and could not be compensated on the right hand side. So up to a multiplicative constant $\phi_1=\gamma^{p_1}$. Similarly $\phi_3=\gamma^{-p_3}$. Also, the ζ-dependence requires that $k_2=k_1$. Now,

$$B_{12}=-\gamma\zeta^{-n_{12}-\ell_1-1}\frac{\partial}{\partial\gamma}(\tilde{G}/\gamma^{p_1-\ell_1-n_{12}})\gamma^{-\ell_1-n_{12}}.$$

Since $\tilde{G}$ can have no factors of γ we can only avoid poles in B_{12} if $p_1-\ell_1-n_{12}=0$ and $\tilde{G}$ is constant. Likewise $\tilde{g}$ is constant and $p_1+n_{12}+\ell_2=0$, from the $e^{\gamma\alpha_2}$ term in the second of Eq. (7.3).

The remaining equations are:

$$\hat{g}(p_1-\ell_1-n_{12})-\gamma\frac{\partial\hat{g}}{\partial\gamma}=(\alpha_2-\alpha_1)\gamma\hat{g}-b_{12}\zeta^{n_{12}+\ell_2+1}\gamma^{p_1},$$

$$\hat{G}(-p_1-\ell_2-n_{12})+\gamma\frac{\partial\hat{G}}{\partial\gamma}=(\alpha_2-\alpha_1)\gamma\hat{G}+\mathbb{C}_{12}\zeta^{\ell_1+n_{12}-1}\gamma^{p_1}.$$

Then, since $\hat{g}$ and $\hat{G}$ have no factors of γ,

$$p_1=\ell_1+n_{12}=-\ell_2-n_{12}.$$

But the ζ-dependence requires that $n_{12}+\ell_2+1\geqq 0$ and $\ell_1+n_{12}-1\leqq 0$ which can be written $1-p_1\geqq 0$; therefore $p_1=1$ and $\hat{G}$, $\hat{g}$ are constants. This leaves us with the following for g:

$$g=\begin{pmatrix}\zeta^{\ell+2}e^{\gamma\alpha_1} & \zeta^{\ell+1}\cdot\dfrac{g_0e^{\gamma\alpha_2}+G_0e^{\gamma\alpha_1}}{\gamma} & 0\\ 0 & \zeta^{\ell}e^{\gamma\alpha_2} & 0\\ 0 & 0 & \zeta^{-2\ell-2}e^{\gamma\alpha_3}\end{pmatrix}. \tag{7.4}$$

Similarly in the case $\varrho_{12}\equiv 0$ we obtain:

$$g=\begin{pmatrix}\zeta^{2\ell+2}e^{\gamma\alpha} & 0 & 0\\ 0 & \zeta^{-\ell}e^{\gamma\alpha_2} & \zeta^{-\ell-1}\dfrac{f_0e^{\gamma\alpha_3}+F_0e^{\gamma\alpha_2}}{\gamma}\\ 0 & 0 & \zeta^{-\ell-2}e^{\gamma\alpha_3}\end{pmatrix}. \tag{7.5}$$

These appear to be other simple SU(2) embeddings.

A more interesting case occurs when $\varrho_{13}\neq 0$. This time from the ϱ_{13} equations we obtain $\phi_1=\gamma^{p_1}$, $\phi_3=\gamma^{-p_3}$ as before. The other equations reduce to the following shape, for example,

$$\zeta^{-n_{12}-1}\left((-n_{12}-\ell_1)\frac{\hat{G}}{\phi_1}-\gamma\frac{\partial}{\partial\gamma}\left(\frac{\hat{G}}{\phi_1}\right)\right)$$

$$=B_{12}\zeta^{\ell_1}+(k_2-k_1)\zeta^{-n_{12}}\frac{\hat{G}}{\phi_1}.$$

The non-singularity of B_{12} requires $k_2=k_1$ and that $[(p_1-\ell_1-n_{12})\hat{G}-\gamma\hat{G}']\gamma^{-p_1}$ have no poles at $\gamma=0$. This requires that $p_1-\ell_1-n_{12}=0$ and either $\hat{G}'=0$ or $p_1=1$. But the companion equation to the above is:

$$\zeta^{-n_{12}-1}\left((\alpha_1-\alpha_2)\hat{g}-\frac{\partial}{\partial\gamma}(\hat{g})\right)\gamma^{1-p_1}=-b_{12}\zeta^{\ell_2},$$

and this clearly requires $p_1=1$. A similar argument for the other three sets of equations produces the following:

$$\begin{aligned}n_{12}-\ell_1&=-1,\\ n_{12}-\ell_2&=1,\\ -n_{23}-\ell_2&=-1,\\ -n_{23}-\ell_3&=1.\end{aligned}$$

Since $\ell_1+\ell_2+\ell_3=0$ also we must have $\ell_1=2$, $\ell_2=0$, $\ell_3=-2$. Then (we can only satisfy the singularity conditions when ϱ_{13} has all those $e^{\gamma\alpha_i}$ terms) we obtain for g:

$$g=\begin{pmatrix} \zeta^2 e^{\gamma\alpha_1} & \zeta\tilde{\varrho}_{12} & \varrho_{13} \\ 0 & e^{\gamma\alpha_2} & \zeta^{-1}\tilde{\varrho}_{23} \\ 0 & 0 & \zeta^{-2}e^{\gamma\alpha_3} \end{pmatrix}, \tag{7.6}$$

where

$$\tilde{\varrho}_{12}=\frac{g_0}{\gamma}(e^{\gamma\alpha_2}-e^{\gamma\alpha_1}),$$

$$\tilde{\varrho}_{23}=\frac{F_0}{\gamma}(e^{\gamma\alpha_2}-e^{\gamma\alpha_3}),$$

$$\varrho_{13}=-\frac{F_0 g_0 \alpha_{32}\alpha_{21}}{\gamma^2}\left(\frac{e^{\gamma\alpha_1}}{\alpha_{21}\alpha_{31}}+\frac{e^{\gamma\alpha_2}}{\alpha_{32}\alpha_{12}}+\frac{e^{\gamma\alpha_3}}{\alpha_{32}\alpha_{31}}\right),$$

and $\alpha_{ij}=\alpha_i-\alpha_j$.

Remarks. It is known that the so-called maximal embedding of SU(2) in SU(3) has a spherically symmetric monopole [10] whose behaviour is determined by the functions

$$\frac{1}{r^2}Q_1(r)=\frac{1}{r^2}\left(\frac{e^{\alpha_1 r}}{\alpha_{12}\alpha_{13}}+\frac{e^{\alpha_2 r}}{\alpha_{32}\alpha_{12}}+\frac{e^{\alpha_3 r}}{\alpha_{31}\alpha_{32}}\right),$$

$$\frac{1}{r^2}Q_2(r)=\frac{1}{r^2}\left(\frac{e^{-\alpha_1 r}}{\alpha_{12}\alpha_{13}}+\frac{e^{-\alpha_2 r}}{\alpha_{32}\alpha_{12}}+\frac{e^{-\alpha_3 r}}{\alpha_{31}\alpha_{32}}\right),$$

where the α_i are associated with the direction of symmetry breaking. For the Higgs field along the z-axis is given by

$$\phi\Big|_{r=z}=H_1\frac{\partial}{\partial r}\left(\ln\frac{Q_1(r)}{r^2}\right)\Big|_{r=z}+H_2\frac{\partial}{\partial r}\left(\ln\frac{Q_2(r)}{r^2}\right)\Big|_{r=z}, \tag{7.7}$$

where $H_1=\begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix}$ and $H_2=\begin{pmatrix}0&0&0\\0&1&0\\0&0&-1\end{pmatrix}$ span the Cartan algebra of SU(3).

If we assume $\alpha_1>\alpha_2>\alpha_3$, then the asymptotic behaviour of ϕ is:

$$\phi\to\begin{pmatrix}\alpha_1&0&0\\0&\alpha_2&0\\0&0&\alpha_3\end{pmatrix}-\frac{2}{r}\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}.$$

That the patching-matrix (7.6) gives this monopole is strongly suggested by the following two interesting observations,

$$\varrho_{13}=-F_0 f_0 \alpha_{23}\alpha_{21}Q_1(\gamma)/\gamma^2,$$

and

$$\begin{vmatrix} \zeta\varrho_{12} & \varrho_{13} \\ e^{\gamma\alpha_2} & \zeta^{-1}\varrho_{23} \end{vmatrix} = \varrho_{12}\frac{\partial\varrho_{23}}{\partial\gamma} - \frac{\partial\varrho_{12}}{\partial\gamma}\varrho_{23} = -F_0 f_0 \alpha_{31}\alpha_{21}\alpha_{23}Q_2(\gamma)/\gamma^2 .$$

This immediately generalises the pattern in SU(2), where

$$\phi|_{r=z} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{\partial}{\partial r}\left(\ln\frac{\sinh r}{r}\right)\Bigg|_{r=z},$$

since here $\varrho = \dfrac{\sinh\gamma}{\gamma}$.

The form of (7.6) suggests a generalization to SU(n). For we can write the whole of the patching matrix g as an integral in a new complex variable η, around a contour enclosing the α_i in the following way:

$$g_{ij} = \frac{1}{2\pi i}\oint_C d\eta\, e^{\gamma\eta}(\zeta^{\frac{1}{2}(\ell_i+\ell_j)}\gamma^{i-j}) \Big/ \prod_{i\leq k\leq j}(\eta-\alpha_k). \tag{7.8}$$

Then we conjecture the following generalization to SU(n). That,

$$\ell_1 = n-1,\ \ell_2 = n-3, \ldots, \ell_{n-1} = 3-n,\ \ell_n = 1-n$$

and that g_{ij} be defined by (7.8) for $1\leq i, j\leq n$. The extension of (7.7) to SU(n) is known to be [10]

$$\phi|_{r=z} = \sum_{m=1}^{n-1} H_m \frac{\partial}{\partial r}\left(\ln\frac{Q_m^{(r)}}{r^{p_m}}\right)\Bigg|_{r=z},$$

where $p_m = m(n-m)$, and the $\gamma^{-p_m}Q_m(\gamma)$ are precisely the determinants of the upper right-hand submatrices of g. They are non-singular and non-vanishing

$$D_m = \frac{Q_m(z)}{z^{p_m}} = \mathrm{Det}\begin{vmatrix} g_{1\ n+1-m}\cdots g_{1n} \\ g_{m\ n+1-m}\cdots g_{m\ n} \end{vmatrix} \qquad D_0 = D_n = 1 .$$

The H_m are a basis of the Cartan subalgebra of SU(n):

$$H_m = \mathrm{diag}(0, 0, \ldots, \underset{n-m}{1}, \underset{n-m+1}{-1}, 0, \ldots, 0).$$

In fact it is not difficult to do the splitting procedure on the z-axis. For when $x_{12} = x_{21} = 0$, γ loses its ζ dependence and is, up to a scale, equal to $z = r$. Then we split g by solving the following equations:

$$\begin{pmatrix} \zeta^{n-1}e^{\alpha_1 z} & \zeta^{n-2}\tilde{\varrho}_{12}(z) & \ldots & \zeta\tilde{\varrho}_{1n-1}(z) & \tilde{\varrho}_{1n}(z) \\ 0 & \zeta^{n-3}e^{\alpha_1 z} & & \tilde{\varrho}_{2n-1}(z) & \zeta^{-1}\tilde{\varrho}_{2n}(z) \\ \vdots & 0 & & & \vdots \\ 0 & \vdots & \ldots & 0 & \zeta^{-n+1}e^{\alpha_n z} \end{pmatrix} k(\zeta) = h(1/\zeta).$$

We choose $k(\zeta=0)=1_n$, in which case

$$h(\zeta=\infty)=\begin{pmatrix} & & 0 & & & D_{n-1}D_n^{-1} \\ & & & & -D_{n-2}D_{n-1}^{-1} & \\ & & & \cdot^{\cdot^{\cdot}} & & \\ & & D_1D_2^{-1}(-1)^n & & & \\ & & & & 0 & \\ D_0D_1^{-1}(-1)^{n+1} & & & & & \end{pmatrix}(-1)^n .$$

Since this is time independent and the Higgs field comes from $A_{11}+A_{22}$ we have

$$\phi|_{r=z}=h^{-1}(\zeta=\infty)\frac{\partial}{\partial z}h(\zeta=\infty),$$

$$\phi|_{r=z}=\sum_{m=1}^{n-1}H_m\frac{\partial}{\partial r}(\ln D_m(r))\bigg|_{r=z},$$

as desired. To calculate the $A_i(x)$ we need to know the patching matrix in a neighbourhood of the z-axis. Because it is spherically symmetric we may use the fact that under a small rotation: $g\to agA$ and hence

$$h(\zeta=\infty)\to a(\zeta=\infty)\,h(\zeta=\infty)$$
$$k(\zeta=0)\to A^{-1}(\zeta=0)\,k(\zeta=0).$$

From these we may calculate $A_i(x)$ on the z-axis and since a and A are non-singular we preserve the non-singularity of the splitting. Then by spherical symmetry we conclude that the splitting is everywhere non-singular.

Further this splitting will work for SU(3) in the limit $\alpha_1=\alpha_2$ to give the U(2) breaking of SU(3) starting from

$$g=\begin{pmatrix}\zeta^2e^\gamma & \zeta e^\gamma & \varrho_{13}(\alpha_1=\alpha_2)\\ 0 & e^\gamma & \varrho_{23}(\alpha_1=\alpha_2)\\ 0 & 0 & \zeta^{-2}e^{-2\gamma}\end{pmatrix},\qquad \alpha_1=\alpha_2=1,$$

which is bundle equivalent to that found by Ward [7, 12].

Again for SU(3) we can find a family of cylindrically symmetric monopoles which contain the above spherically symmetric one. The family is:

$$\begin{aligned}\varrho_{12}&=\zeta\cdot\frac{e^{\gamma\alpha_2}-e^{\gamma\alpha_1+\alpha_{21}c_1}}{\alpha_{21}(\gamma-c_1)},\\ \varrho_{23}&=\zeta^{-1}\cdot\frac{e^{\gamma\alpha_2}-e^{\gamma\alpha_3+c_2\alpha_{23}}}{\alpha_{23}(\gamma-c_2)},\\ \varrho_{13}&=\frac{e^{\alpha_2\gamma}+H_0e^{\gamma\alpha_1}+h_0e^{\gamma\alpha_3}}{\alpha_{21}\alpha_{23}(\gamma-c_1)(\gamma-c_2)},\end{aligned}\tag{7.9}$$

where

$$H_0 = e^{(c_1+c_2)\alpha_2} \cdot \frac{e^{\alpha_{32}c_1} - e^{\alpha_{32}c_2}}{e^{\alpha_3 c_2 + \alpha_1 c_1} - e^{\alpha_1 c_2 + \alpha_3 c_1}},$$

$$h_0 = e^{(c_1+c_2)\alpha_2} \cdot \frac{e^{\alpha_{12}c_2} - e^{\alpha_{12}c_1}}{e^{\alpha_3 c_2 + \alpha_1 c_1} - e^{\alpha_1 c_2 + \alpha_3 c_1}},$$

where c_1 and c_2 are x- and ζ-independent. By a translation along the z-axis we could choose $c_1 + c_2 = 0$.

There are a number of interesting limits in (7.9). Firstly $c_1 = c_2$ yields the spherically symmetric monopole of (7.6) translated along the z-axis.

Secondly, if we assume $\alpha_1 > \alpha_2 > \alpha_3$ and let $c_1 = 0$, $c_2 \to -\infty$ then in the neighbourhood $1 - \varepsilon < |\zeta| < 1 + \varepsilon$ and in the neighbourhood of the origin in $\mathbb{R}^3$, $\varrho_{23} \to 0$, $\varrho_{13} \to 0$ and

$$g \to \begin{pmatrix} \zeta^2 e^{\gamma\alpha_1} & \zeta \cdot \dfrac{e^{\alpha_2\gamma} - e^{\alpha_1\gamma}}{\alpha_{21}\gamma} & 0 \\ 0 & e^{\gamma\alpha_2} & 0 \\ 0 & 0 & \zeta^{-2} e^{\gamma\alpha_2} \end{pmatrix}.$$

This is simply an embedding of the SU(2) monopole in SU(3). Likewise the limit $c_2 = 0$, $c_1 \to \infty$ yields $\varrho_{13} \to 0$, $\varrho_{12} \to 0$, and we have an SU(2) embedding in the ϱ_{23} position. This strongly suggests that (7.9) represents two separated SU(2) monopoles, of separation $c_1 - c_2$, embedded in SU(3). In SU(2) such a multimonopole has no symmetry but here with the greater gauge freedom of SU(3) it appears to have such. If we allow these two monopoles to coalesce we obtain the spherically symmetric monopole of charge 2 in the breaking of SU(3) to U(1) × U(1) or to U(2) if $\alpha_1 = \alpha_2$. In the limit $\alpha_1 = \alpha_2$ (7.9) is gauge equivalent to Ward's family of cylindrically symmetric, U(2) broken monopoles [7, 12].

Conclusions

We have presented a systematic way of finding cylindrically and spherically symmetric monopoles in SU(3) gauge theory, where SU(3) is broken to U(1) × U(1). In particular we have written down an Atiyah-Ward patching matrix which appears to represent a separated two monopole solution which is cylindrically symmetric and reduces to the charge two spherically symmetric monopole, in the U(1) × U(1) breaking, as the separation vanishes. By taking another limit we recover the U(2) breaking with a patching function equivalent to Ward's.

Further we have found the patching matrix for the spherically symmetric charge $n-1$ monopoles of SU(n) gauge theory and would conjecture that this arises from a cylindrically symmetric solution of $n-1$ SU(2) monopoles separated along the axis of symmetry.

However we have only shown the non-singularity of the splitting procedure, to recover the $A_\mu(x)$, in the spherically symmetric cases. This remains to be done in the general case but is difficult if attempted in the conventional fashion.

We have not fully analysed all the cylindrically symmetric monopoles nor have we repeated the above methods for the case where two of the α_i are equal. This case is more involved because the bundle transition matrices A and a etc., are no longer upper triangular. Nevertheless it ought to be examined.

Finally we have assumed throughout that the general form of g in SU(3) can be chosen to be upper triangular.

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Note added. Ward's deformed imbedding of the charge 1 SU(2) monopole in SU(3) has the following form [11]:

$$\ell_1=1, \quad \ell_2=0, \quad \ell_3=-1,$$

$$\varrho_{12}=m\zeta\cdot\frac{ae^{\alpha_1\gamma}-a^{-1}e^{\alpha_2\gamma}}{\gamma-m^2}, \qquad \varrho_{23}=\frac{m}{\zeta}\cdot\frac{e^{\alpha_3\gamma}-e^{\alpha_2\gamma}}{\gamma},$$

$$\varrho_{13}=\frac{-a^2\gamma e^{\alpha_1\gamma}+m^2e^{\alpha_2\gamma}+(\gamma-m^2)e^{\alpha_3\gamma}}{a\gamma(\gamma-m^2)},$$

$$a^2=e^{(\alpha_2-\alpha_1)m^2}, \quad \text{in a real parameter}.$$

It belongs to case (iv) (b) with the choices:

$$\psi_1=\gamma-m^2,$$

$$\psi_3=\gamma,$$

$$g=-\frac{m}{a}, \qquad G=ma\zeta,$$

$$f=\frac{m}{\zeta}, \qquad F=-m,$$

$$h=\frac{1}{a}, \qquad H=-a.$$

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