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On Multimeron Solutions of the Yang-Mills Equation

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Abstract. We study a singular boundary value problem introduced by Glimm and Jaffe for the purpose of obtaining solutions of the Euclidean Yang-Mills equations with isolated singularities along an axis. Using comparison techniques, we prove existence, asymptotic behavior and also uniqueness in some special cases.

1. Introduction

In this note we study solutions of the singular boundary value problem

$$Lu \equiv \Delta u + \frac{1}{x_1^2}(u-u^3)=0 \quad \text{in} \quad R_+^2, \tag{1.1}$$

$$u(0,x_2)=(-1)^j, \qquad a_j<x_2<a_{j+1}, \quad j=0 \quad \text{to} \quad 2n, \tag{1.2}$$

where $R_+^2=\{x=(x_1x_2)\in R^2 : x_1>0\}$ and

$$-\infty=a_0<a_1<\ldots<a_{2n}<a_{2n+1}=+\infty. \tag{1.3}$$

The boundary value problem (1.1), (1.2) was introduced by Glimm and Jaffe [4] for the purpose of obtaining solutions of the Euclidean Yang-Mills equations in R^4 with isolated singularities along an axis. The existence of a solution was obtained heuristically in [4] and established in a rigorous way using variational methods in [6]. The solutions of the Yang-Mills equations that arise from (1.1), (1.2) are known as *multimeron* solutions and may be thought of as describing "pseudoparticles" located at the singular points (see [5]).

The two-meron solution ($n=1$) of (1.1), (1.2) is given explicitly by the formula

$$u(x)=\frac{(x-a_2e_2)}{|x-a_2e_2|}\cdot\frac{(x-a_1e_2)}{|x-a_1e_2|}, \tag{1.4}$$

(where e_1, e_2 is the standard basis of R^2). Here we exploit this fact and the invariance of Eq. (1.1) under change of scale $u(x)\to u(\lambda x)$ and inversions about a

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point in the x_2-axis: $u(x) \to v(y)$, $y - ae_2 = \dfrac{x - be_2}{|x - be_2|^2}$ to show that any solution of (1.1), (1.2) can be controlled by two-meron solutions. More precisely, we let φ_k be the explicit two meron solution with

$$\varphi_k = \begin{cases} +1 & \text{on} \quad (a_k, a_{k+1}) \\ -1 & \text{otherwise} \end{cases} \tag{1.5}$$

and define

$$\Phi = \max_{k \text{ even}} \varphi_k, \qquad \Psi = \min_{k \text{ odd}} - \varphi_k. \tag{1.6}$$

Then (see Sect. 3) Φ and Ψ are weak sub- and supersolutions of (1.1) respectively with $\Phi \leqq \Psi$ and Φ, Ψ satisfy (1.2)

Theorem 1.1. *There exist solutions* $u^{\pm}$ *of* (1.1), (1.2) *satisfying*

i) $\Phi \leqq u^- \leqq u^+ \leqq \Psi$.

ii) $u^{\pm}$ *are maximal and minimal solutions respectively*, i.e. *for any solution* u *of* (1.1), (1.2), $u^- \leqq u \leqq u^+$.

iii) *If either* u^+ *or* u^- *depends continuously on the parameters* a_i, *then* $u^+ = u^-$ *so there is uniqueness.*

Corollary 1.2. i) *The two-meron solution is unique.* ii) *If* u *is any solution of* (1.1), (1.2) *then near* $x = a_i e_2$

$$\left| u(x) - \frac{x_2 - a_i}{|x - a_2 e_2|} \right| = O(x_1^2). \tag{1.7}$$

Part i) of Corollary 1.2 follows from the fact that for $n = 1$, $u = u^+$ is the two-meron solution which depends continuously on its endpoints while part ii) follows from the explicit form of Φ and Ψ. In Sect. 4 we show that the asymptotic formula (1.7) remains true if u is only a local solution of (1.1), (1.2) with a "+, − singularity" at $x = a_i e_2$.

Using our methods it is very easy to obtain solutions to Eq. (1.1) with more general boundary conditions. As an example, we state

Theorem 1.3. *Let* O *be an arbitrary open set on the line* $x_1 = 0$. *Then there exists a solution* u *of* (1.1) *with* $u = -1$ *on* O *and* $u = +1$ *on the interior of* O^c, *the complement of* O.

We next state a result on the removability of certain isolated singularities.

Theorem 1.4. *Let* $\Omega_{\lambda,\mu} = \{x \in R_+^2 : 0 < x_1 < \mu, -\lambda < x_2 < \lambda\}$.

Suppose that $u(x)$ *is a* $C^2(\Omega_{\lambda,\mu}) \cap C^0(\bar{\Omega}_{\lambda,\mu} \setminus \{0\})$ *solution of* (1.1) *such that*

$$u(x) = +1 \ (\text{or } -1) \quad \text{on} \quad x_1 = 0, \quad 0 < |x_2| < \lambda. \tag{1.8}$$

Then the singularity at $x = 0$ *is removable.*

A consequence of Theorem 1.4 is that there is no solution of (1.1) with an odd number of singularities, i.e. *merons always come in pairs.* This justifies the form of the boundary condition (1.2) (see also Lemma 2.2).

The plan of the paper is as follows: In Sect. 2 we establish some technical lemmas which are needed in proving the main results. Section 3 contains the construction of the maximal and minimal solutions of (1.1), (1.2) given in Theorem 1.1 as well as some more general solutions of (1.1). Section 4 contains the proof of parts ii) and iii) of Theorem 1.1 as well as our local results on the nature of isolated singularities.

2. Technical Lemmas

In this section we derive some technical estimates which are needed in the following sections. Our first lemma is a special case of a result of Cheng and Yau [2]. But we provide here a self-contained proof.

Lemma 2.1. *Let u be a $C^2(R^2_+)$ solution of* (1.1). *Then*

$$-1 \leqq u(x) \leqq 1 \quad \textit{for all} \quad x \in R^2_+ . \tag{2.1}$$

Proof. Assume that $u(x_0) > 1$ for some point $x_0 \in R^2_+$. By a translation and scaling we may assume $x_0 = 1/2\, e_1$. We map R^2_+ onto the unit ball $B_1 = \{y \in R^2 : |y| \leqq 1)$ by the conformal inversion

$$y + e_1 = \frac{x + \frac{1}{2} e_1}{|x + \frac{1}{2} e_1|^2}, \tag{2.2}$$

and set $v(y) = u(x)$. Then v satisfies

$$\Delta v + \frac{4}{(1-|y|^2)^2}(v - v^3) = 0 \quad \text{in } B, \tag{2.3}$$

and

$$v(0) = u(x_0) > 1 .$$

Let $r = |y|$ and set $\bar{v}(r) = 1/2\pi \int\limits_0^{2\pi} v_+(r,\theta)\, d\theta$, $v_+ = \max(v, 0)$. Then (see Lemma 3.2)

$$\frac{1}{r}(r\bar{v}')' + \frac{4}{(1-r^2)^2}(\bar{v} - \bar{v}^3) \geqq 0 . \tag{2.4}$$

Integrating (2.4) we find

$$r\bar{v}'(r) \geqq \int\limits_0^2 \frac{4t}{(1-t^2)^2}(\bar{v}^3 - \bar{v})(t)\, dt . \tag{2.5}$$

Since $\bar{v}(0) > 1$, $(\bar{v}^3 - \bar{v}(0)) = \varepsilon > 0$ and (2.5) implies $\bar{v}'(r) > 0$ and so $r\bar{v}'(r) > \dfrac{2\varepsilon}{1-r^2} - 2\varepsilon$ which says $\bar{v} > c \ln \dfrac{1}{1-r^2}$ as $r \to 1$. To derive a contradiction we change variables by $\varrho = \ln \dfrac{1}{1-r^2}$, $\psi(\varrho) = \bar{v}(r)$. Then

$$\psi'' + (1 - e^{-\varrho})^{-1} \psi' \geqq (1 - e^{-\varrho})^{-1} (\psi^3 - \psi) \quad \text{on} \quad (0, \infty) . \tag{2.6}$$

For $\varrho \geqq R$ large enough, ψ is a subsolution of the equation

$$\begin{gathered}\varphi'' + a\varphi' = \tfrac{1}{2}\varphi^3, \qquad a > 0, \\ \varphi(R) = \psi(R), \qquad \varphi'(R) = \psi'(R),\end{gathered} \tag{2.7}$$

and so $\psi \geqq \varphi$. As before $\varphi' > 0$. Multiplying (2.7) by $2\varphi'$ and integrating gives

$$\varphi'^2(\varrho) + 2a \int_R^\varrho \varphi'^2(t)\,dt = \frac{\varphi^4(\varrho)}{4} + C_1, \tag{2.8}$$

so $\varphi'^2(\varrho) \leqq \dfrac{\varphi^4(\varrho)}{4} + C_1$. Using this again in (2.7) gives

$$\varphi'^2(\varrho) \geqq c_2^2 \varphi^4(\varrho) \quad \text{for} \quad \varrho > \varrho_1 \quad \text{large}. \tag{2.9}$$

Integrating (2.9) gives $-\dfrac{1}{\varphi(\varrho)} + \dfrac{1}{\varphi(\varrho_1)} \geqq c_2(\varrho - \varrho_1)$ which gives the desired contradiction as $\varrho \to \infty$.

The next lemma determines the boundary values of a solution of (1.1). It will not be used in the sequel.

Lemma 2.2. *Let x_0 be a boundary point of R_+^2, and let $\Omega_R = \{x \in R_+^2 : |x - x_0| < R\}$. If $u(x)$ is a $C^2(\Omega_R) \cap C^0(\Omega_R \cup \{x_0\})$ solution of* (1.1), *then $u(x_0)$ is* 0, $+1$, *or* -1.

Proof. Set $v^\lambda(x) = u(x_0 + \lambda(x - x_0))$. Then $|v^\lambda| \leqq 1$ and v^λ satisfies (1.1) in $\Omega_{R/\lambda}$. As $\lambda \to 0$, $v^\lambda \to u(x_0)$ in $C^{2+\alpha}(K)$ for any compact subset K of R_+^2 by elliptic regularity theory. Hence $0 \equiv \dfrac{u^3(x_0) - u(x_0)}{x_1^2}$ so that

$$u(x_0) = 0, \ +1, \text{ or } -1.$$

Next we recall the maximum principle for Eq. (1.1).

Lemma 2.3. *Let Ω be a bounded open set in R_+^2. Suppose that $u(x)$, $v(x)$ are two $C^2(\Omega)$ solutions of* (1.1) *such that $u(x) \geqq v(x)$. Then $u(x) > v(x)$ in Ω, unless $u(x) \equiv v(x)$.*

Proof. Let $h(x) = u(x) - v(x)$ and suppose $h(x_0) = 0$ for $x_0 \in \Omega$. Then

$$\Delta h + c(x) h(x) = 0,$$

where

$$c(x) = \frac{1}{x_1^2}(1 - u^2 - uv - v^2).$$

Let $B \subset \Omega$ be a small ball about x_0 and set $M = \sup\limits_B c(x)$. Then (since $h \geqq 0$ in Ω)

$$\Delta h - (M - c(x))h = -Mh \leqq 0,$$

so the lemma follows via the maximum principle.

Corollary 2.4. *Let $u(x)$ be a $C^2(R_+^2)$ solution of* (1.1) *in R_+^2. Then $u(x)$ cannot attain the values ± 1 in the interior of R_+^2.*

The following lemma plays the role of the Hopf boundary point lemma and is central to the analysis of Sect. 4.

Lemma 2.4. *Let* $\Omega_{\lambda,\mu}=\{x\in R^2: 0<x_1<\mu, |x_2|<\lambda\}$.

Suppose u, v *are solutions of* (1.1) *in* $\Omega_{\lambda,\mu}$, $\mu<2\lambda$. *Then*

i) *If* $u\geqq v$ *in* $\Omega_{\lambda,\mu}$

$$u-v\geqq \frac{3}{20}\,\frac{m(\mu)}{\mu^2}x_1^2 \quad \text{in} \quad \Omega_{\frac{\lambda}{2\sqrt{5}},\mu}, \tag{2.10}$$

(where $m(\mu)=\inf(u-v)$ *on* $x_1=\mu$, $|x_2|<\lambda$*).*

ii) *If in addition to* i) $u, v\in C^0(\bar{\Omega}_{\lambda,\mu})$, $u(0,x_2)=v(0,x_2)=1$ *then*

$$u-v\leqq cx_1^2 \quad \text{in} \quad \Omega_{\lambda/2,\mu/2}. \tag{2.11}$$

Proof. i) Let $w=u-v$. Then

$$\Delta w+\frac{w}{x_1^2}(1-(u^2+uv+v^2))=0. \tag{2.12}$$

In particular, since $|u|\leqq 1$ $|v|\leqq 1$

$$\Delta w-\frac{2}{x_1^2}w\leqq 0 \quad \text{in} \quad \Omega_{\lambda,\mu}. \tag{2.13}$$

Set

$$\begin{aligned} h(x)&=ax_1^2+bx_1^4-cx_1^2x_2^2,\\ a&=m(\mu)\left(\frac{1}{\mu^2}-\frac{1}{5\lambda^2}\right),\\ b&=\frac{m(\mu)}{5\lambda^2\mu^2},\\ c&=\frac{m(\mu)}{\lambda^2\mu^2}. \end{aligned} \tag{2.14}$$

Then using (2.14) $h=0$ on $x_1=0$, $h=x_1^2((a-c)\lambda^2+bx_1^2)\leqq x_1^2((a-c\lambda^2)+b\mu^2)=0$ on $|x_2|=\lambda$, $0<x_1<\mu$ and $h=\mu^2(a+b\mu^2-cx_2^2)\leqq\mu^2(a+b\mu^2)=m(\mu)\leqq w$ on $x_1=\mu|x_2|<\lambda$. Finally

$$\Delta h-\frac{2}{x_1^2}h=(10b-2c)x_1^2\equiv 0.$$

Applying the maximum principle, $w\geqq h$ in $\Omega_{\lambda,\mu}$. In particular for $|x_2|\leqq\dfrac{\lambda}{2\sqrt{5}}$, $0<x_1<\mu$

$$h\geqq 3/20\frac{m(\mu)}{\mu^2}x_1^2.$$

ii) We observe that it suffices to prove ii) in the special case $u\equiv 1$ for supposing this case

$$u-v=(u-1)+(1-v)\leqq 1-v\leqq cx_1^2 \quad \text{in} \quad \Omega_{\lambda/2,\mu/2}.$$

Let $w=1-v\geqq 0$ and take $\lambda=\mu$. Then

$$\Delta w=\frac{2w-3w^2+w^3}{x_1^2}\geqq\frac{2w-3w^2}{x_1^2} \tag{2.15}$$

$$\text{or } Lw\equiv\Delta w-\frac{(2w-3w^2)}{x_1^2}\leqq 0.$$

Set $h(x)=ax_1^2-bx_1^4+cx_2^4$, where

$$\begin{aligned} &1\geqq a\mu^2>50M(\mu),\\ &b=\frac{1}{2}\frac{a}{\mu^2}, \qquad c=\frac{M(\mu)}{\mu^4},\\ &M(\mu)=\sup_{\Omega_{\mu,\mu}} w. \end{aligned} \tag{2.16}$$

Then

$$\begin{aligned} Lh= & -(10b-3a^2)x_1^2-3bx_1^4(2a-bx_1^2)\\ & +(12cx_2^2+6acx_2^4)-2c\frac{x_2^4}{x_1^2}-6bcx^2y^4\leqq 0 \end{aligned}$$

in $\Omega_{\mu,\mu}$ for μ small enough. Moreover, $h\geqq w$ on $\partial\Omega_{\mu,\mu}$ as is easily seen from 2.16. Hence by the maximum principle $h\geqq w$ on $\Omega_{\mu,\mu}$. In particular, $w(x_1,0)\leqq\frac{a}{2}x_1^2$, $0<x_1<\mu$. Since "0 is an arbitrary point" on $x_1=0$ the lemma follows.

Remarks 2.5. i) Lemmas 2.3 and 2.4 i) remain true if u is a supersolution, v is a subsolution if (1.1) and $|u|\leqq 1$, $|v|\leqq 1$.

ii) Lemma 2.4 ii) has been observed earlier in [1].

We end this section by deriving a bound of an explicit solution of (1.1). The function

$$\varphi(x)=\frac{x_2}{|x|} \tag{2.17}$$

is a solution of (1.1) in R_+^2, and has two isolated singularities located at $x=0$ and $x=\infty$. We displace the two singularities at points ae_2 and be_2, $a,b\in R$, by using the conformal transformation

$$y-ae_2=\frac{x-\dfrac{1}{a-b}e_2}{\left|x-\dfrac{1}{a-b}e_2\right|^2}, \tag{2.18}$$

and defining $\psi^{a,b}(y)=\varphi(x)$. This gives

$$\psi^{a,b}(y)=\frac{a-b}{|a-b|}\,\frac{(y-ae_2)\cdot(y-be_2)}{|y-ae_2|\,|y-be_2|}. \tag{2.19}$$

This is the "two-meron" solution mentioned in the introduction. We set $\psi^{\varepsilon}(y)=\psi^{\varepsilon,-\varepsilon}(y)$, i.e.

$$\psi^{\varepsilon}(y)=\frac{(y-\varepsilon e_2)\cdot(y+\varepsilon e_2)}{|y-\varepsilon e_2|\,|y+\varepsilon e_2|}. \tag{2.20}$$

If $\varepsilon>0$ is sufficiently small, then the following bounds hold for small enough y_1:

$$\frac{c_1}{\varepsilon^2} y_1^2 \leqq 1+\psi^\varepsilon(y) \leqq \frac{c_2}{\varepsilon^2} y_1^2 \quad \text{for} \quad y_2 \in(-\varepsilon, \varepsilon), \tag{2.21a}$$

$$c_3 \varepsilon^2 y_1^2 \leqq 1-\psi^3(y) \leqq c_4 \varepsilon^2 y_1^2 \quad \text{for} \quad |y_2|>\varepsilon, \tag{2.21b}$$

$$c_5 \frac{\varepsilon^2}{|y|^2} \leqq 1-\psi^\varepsilon(y) \leqq c_6 \frac{\varepsilon^2}{|y|^2} \quad \text{for} \quad |y| \quad \text{large}. \tag{2.21c}$$

Here $c_1, \ldots, c_6$ are strictly positive constants. Estimates (2.21) are obtained by straightforward computation. They give the rate of convergence of $\psi^\varepsilon(y)$ to ± 1, and are consistent with Lemma 2.4.

3. Existence of Multimeron Solutions

In this section we prove the existence of a solution of the boundary value problem (1.1), (1.3) and an extension to the case when there is an infinite number of isolated singularities accumulating at a point. Our proof is via the classical Perron's method of sub- and supersolutions.

Definition 3.1. A function $u \in H^1(\Omega) \cap L^\infty(\Omega)$ is said to be an L-subsolution (supersolution) if for all sufficiently small balls $B \Subset \Omega$

$$\int \left\{\nabla u \cdot \nabla \zeta - \frac{\zeta u}{x_1^2} + \frac{\zeta u^3}{x_1^2}\right\} dx \leqq 0 \quad (\geqq 0) \tag{3.1}$$

for all $\zeta \in H_0^1(B)$, $\zeta \geqq 0$.

In the following, by sufficiently small balls $B \Subset \Omega$, we will mean that $-\Delta - \dfrac{1}{x_1^2}$ is strictly coercive on $H_1^0(B)$. This will be the case, for example, if the first Dirichlet eigenvalue $\lambda_1(B)$ satisfies

$$\frac{\lambda_1(B)}{2} \geqq \sup_B \frac{1}{x_1^2}. \tag{3.2}$$

Assuming this is the case, it follows (see [3, 7]) that the Dirichlet problem $Lu=0$ is uniquely solvable in B for *arbitrary* continuous boundary data.

The following properties of weak sub- and supersolutions are well-known [7].

Lemma 3.1. *Let u be a $C^0(B)$ weak L-subsolution and v a $C^0(B)$ weak L-supersolution with $u \leqq v$ on ∂B. Then $u \leqq v$ in B.*

Lemma 3.2. *Let u, v be weak L-supersolutions (subsolutions). Then* $\min(u, v)(\max(u, v))$ *is a weak L-supersolution (subsolution).*

Lemma 3.3. *Let v be a weak L-subsolution in Ω and $B \Subset \Omega$. Let u denote the solution of $Lu=0$ in B with $u=v$ on ∂B. Then*

$$\tilde{v}(x)=\begin{cases} u(x) & \text{in} \quad B \\ v(x) & \text{in} \quad \Omega-\bar{B}\end{cases}$$

is a weak L-subsolution in Ω.

Let $I_j=\{x=(x_1,x_2): x_1=0, a_j<x_2<a_{j+1}\}$, $j=0,\dots,2n$, where the a_j's are defined in (1.3). We seek a solution of (1.1) which is $+1$ on I_j for j even, and -1 on I_j for j odd. Let φ_j be the two meron solution defined by (1.5) and let Φ, Ψ be given by (1.6). Note that Φ, Ψ satisfy the boundary conditions (1.2).

Since $\pm\varphi_j$ are solutions of (1.1) it follows from Lemma 3.2 that Φ, Ψ are weak L sub- and supersolutions respectively. Using the explicit expression for φ_j (or the arguments in Sect. 4) it is easily seen that $\varphi_j \leqq -\varphi_k$ with j even, k odd. Hence $\Phi \leqq \Psi$. Let F^+ (respectively F^-) denote the class of weak L subsolutions (supersolutions) $v(x)$ on R^2_+ such that $\Phi \leqq v(x) \leqq \Psi$. Note that $\Phi \in F^+$, $\Psi \in F^-$. We are now set up to use Perron's method to construct $u^\pm(x)$, the maximal and minimal solutions.

Theorem 3.4. *The functions*

$$u^+(x)=\sup_{v\in F^+} v(x),\qquad u^-(x)=\inf_{v\in F^-} v(x) \tag{3.3}$$

are solutions of (1.1), (1.2) *with* $\Phi \leqq u^- \leqq u^+ \leqq \Psi$.

We do not give the details, which are standard [3]. Instead, we provide an alternative construction of $u^\pm$ well known to afficionados of bifurcation theory.

To construct $u^-(u^+)$ we define a sequence $\{u_k\}$ as follows:

$$\begin{gathered} u_0=\Phi(\Psi), \\ Lu \equiv \Delta u_{k+1} - \frac{1}{x_1^2}u_{k+1}^3 = \frac{-u_k}{x_1^2} \quad \text{in} \quad R^2_+, \\ u_{k+1}=\Phi \qquad \text{on} \quad x_1=0. \end{gathered} \tag{3.4}$$

The boundary value problem can be *uniquely* solved using the Perron process described earlier. This is done inductively: given u_k, $\Phi \leqq u_k \leqq \Psi$, we observe that

$$L\Phi = -\frac{\Phi}{x_1^2} \geqq -\frac{u_k}{x_1^2},$$

$$L\Psi = -\frac{\Psi}{x_1^2} \leqq -\frac{u_k}{x_1^2},$$

so Φ, Ψ are weak "$L+\frac{u_k}{x_1^2}$" sub- and supersolutions. We can therefore, uniquely solve for u_{k+1}. By the maximum principle $\Phi \leqq u_1 \leqq \dots \leqq u_k \leqq u_{k+1} \leqq \dots \leqq \Psi$ (respectively $\Psi \geqq u_1 \geqq u_2 \geqq \dots \geqq u_k \geqq u_{k+1} \geqq \dots \geqq \Phi$). Clearly

$$u^-(x)=\lim_{k\to\infty} u_k(x), \quad u_0=\Phi, \qquad u^+(x)=\lim_{k\to\infty} u_k(x), \quad u_0=\Psi. \tag{3.5}$$

We end the section with the generalization of the boundary value problem (1.1), (1.2) mentioned in the introduction.

Theorem 3.5. *Let O be an arbitrary open set on the line $x_1=0$, then there exists a solution $u(x)$ of* (1.1) *satisfying*

$$\begin{aligned} u&=-1 \quad \text{on } O, \\ u&=+1 \quad \text{on interior } O^c. \end{aligned} \tag{3.6}$$

The proof of Theorem 3.5 is similar to the proof of Theorem 3.4 and will not be repeated here. We only observe that on the line, an arbitrary open set is the countable union of open intervals and that the construction of Φ, Ψ [see (1.6)] involves locally only a finite number of functions. As with Theorem 3.4 the boundary value problem (1.1), (3.6) has a maximal and minimal solution $\Phi \leqq u^- \leqq u^+ \leqq \Psi$.

In the next section we show that any solution of (1.1), (1.2) (or (1.1), (3.6)) lies between u^+ and u^-. It seems quite likely that $u^+ = u^-$ but this remains to be shown.

4. Uniqueness and Local Results

We begin with a removable singularities theorem.

Theorem 4.1. *Let $\Omega_R=\{x\in R^2_+ : |x|<R\}$ and suppose that u is a $C^2(\Omega_R)\cap C^0(\bar{\Omega}_R\backslash\{0\})$ solution of* (1.1) *such that*

$$u(x)=+1 \quad (or\ -1) \quad on \quad x_1=0,\ 0<|x_2|<R.$$

Then the singularity at $x=0$ is removable.

Proof. We first show $u\geqq 0$ in a neighborhood of 0 minus "a narrow cusp" centered on $x_1=0$. To be precise consider the family of two meron solutions $\psi^{a,b}$ given by (2.19) with $a>b\geqq 0$. We fix a so small that $\psi^{a,0}<u$ in a full neighborhood of $|x|=R$, $x_1\geqq 0$. For b near a $\psi^{a,b}<u$ in Ω_R by (2.21) and Lemmas 2.3 and 2.4. We claim that $\psi^{a,b}<u$ in Ω_R for all $a>b>0$. For if not, let $b_0=\inf\{b\in(0,a):\psi^{a,t}<u$ in Ω_R, $0\leqq t\leqq b\}$. Then applying Lemma 2.4, $u-\psi^{a,b_0}\geqq cx_1^2$ in a neighborhood of $x_1=0$. To see that we can decrease b_0 further, note that the two meron $\frac{x_2}{|x|}$ (conformally equivalent to $\psi^{a,b}$) tends to 1 more slowly than $1-cx_1^2$ in a suitable neighborhood of 0. Therefore if we choose $b'<b^0$, b^0-b' sufficiently small we can make $\psi^{a,b'}<u$ near $x_1=0$ and so $\psi^{a,b'}<u$ in Ω_R, a contradiction. Therefore $b=0$ so $u\geqq\psi^{a,0}$. By the same argument we find also $u\geqq\psi^{0,-a}$. Therefore $u\geqq\max(\psi^{a,0},\psi^{0,-a})$ which precise our claim.

To prove the theorem, assume for contradiction that $m=\liminf_{x\to 0} u(x)<1$. Let x_k be a sequence tending to 0 with $m=\lim u(x_k)$ and $\lim\frac{x_k}{|x_k|}=y$. Define

$$v^k(x)=u(\lambda_k x)\lambda_k=|x_k|\,|x|<\frac{R_1}{\lambda_k}. \tag{4.1}$$

As $k\to\infty$, v^k converges to a solution $v\geqq 0$ of (1.1) in R^2 uniformly in $C^{2+\alpha}$ on compacta. Moreover, $v=+1$ on $x_1=0$, $|x_2|>0$ [this is easily seen from

$u \geqq \max(\psi^{a,0}, \psi^{0,-a})]$ and v has an interior min at y, since $v^k\left(\frac{x_k}{|x_k|}\right) = u(x_k) \to m$. But $\Delta v = \frac{v^3 - v}{x_1^2} \leqq 0$, a contradiction.

Corollary 4.2. *Let $u(x)$ be a $C^2(R_+^2)$ solution of* (1.1) *which is continuous up to $x_1 = 0$ except possibly at the points $a_1 e_2, \ldots, a_N e_2$, $a_1 < a_2 < \ldots < a_N$. Assume that $u(0, x_2) = +1$ (or -1) for $x_2 \neq a_1, \ldots, a_N$ and $u(x) \to +1\,(-1)$ as $|x| \to \infty$. Then $u(x) \equiv 1$ (or -1).*

Proof. By Theorem 4.1, u is continuous up to $x_1 = 0$. Hence using the argument of the first part of Theorem 4.1, $u \geqq \psi^{a,b}$ for all $a > b$. Since $\psi^{a,b} \to +1$ uniformly on compacta as $a \to +\infty$, $b \to -\infty$ the corollary follows.

Next we characterize "$+$, $-$" singularities.

Theorem 4.3. *Let $\Omega_R = \{x \in R_+^2 : |x| < R\}$ and suppose u is a $C^2(\Omega_R) \cap C^0(\bar{\Omega}_R \setminus \{0\})$ solution of* (1.1) *satisfying*

$$u(0, x_2) = \begin{cases} +1 & 0 < x_2 < R \\ -1 & -R < x_2 < 0 . \end{cases}$$

Then as $x \to 0$

$$u(x) - \varphi(x) = O(x_1^2), \tag{4.2}$$

where $\varphi(x) = \frac{x_2}{|x|}$.

Proof. The proof is very similar to the first part of the proof of Theorem 4.1. Fix $a > 0$ so small that $\psi^{a,0} < u$ in a full neighborhood of $|x| = R$, $x_1 > 0$. Arguing as before, it then follows that $\psi^{a,0} \leqq u$. Similarly, $u \leqq -\psi^{0,-a}$ so that for a small,

$$\psi^{a,0} \leqq u \leqq -\psi^{0,-a} \quad \text{in } \Omega_R . \tag{4.3}$$

Using (4.3) and Lemma 2.4 and the properties of the two-merons $\psi^{a,b}$ (which are conformally equivalent to φ) (4.2) follows.

We will now apply the same "continuity method" to show that any solution that can be continuously connected to $+1$ (-1) lies above (below) any other solution.

Definition 4.4. Let u be a solution (supersolution) of (1.1), (1.2). We say that *u can be continuously connected to* $+1$ if there is a family u^b of solutions (supersolutions) of (1.1) $b = (b_0, b_1, \ldots, b_{2n}, b_{2n+1})$. $(b_0 = -\infty, b_{2n+1} = +\infty)$ continuous in the parameter b satisfying

(i) $a_j < b_j < b_{j+1} < a_{j+1}$, $j = 1, 3, 5, \ldots, 2n-1$,
(ii) $u^b(0, x_2) = (-1)^j\, b_j < x_2 < b_{j+1}$, $j = 0$ to $2n$,
(iii) $u^b = u$ if $b = a = (a_0, a_1, \ldots, a_{2n}, a_{2n+1})$.

In a similar way, we say that a solution (subsolution) u of (1.1) *can be continuously connected to* -1 if the continuous family of solutions (subsolutions) u^b satisfies (ii), (iii) and (i)′ $b_j < a_j < a_{j+1} < b_{j+1}$, $j = 1, 3, 5, \ldots, 2n-1$.

Remark 4.5. By the construction of Φ, Ψ [see (1.6)] it is easily seen that Φ, Ψ can be continuously connected to $+1$, -1 respectively. Thus, for $n=1$, $u^+ = \psi^{a_2,a_1}$ can be continuously connected to both $+1$ and -1.

Theorem 4.6. *For any solution u of* (1.1), (1.2), $\Phi \leqq u \leqq \Psi$.

The proof of Theorem 4.6 is essentially the same as the first argument of the proof of Theorem 4.1. For example, to show $u \leqq \Psi$, we "shrink the – intervals" until $\Psi^b > u$. This is possible by the explicit form of the two merons. We then "open up the – intervals" decreasing Ψ^b. By our previous argument, we must reach Ψ. The case $\Phi \leqq u$ is analogous.

Corollary 4.7. *For any solution u of* (1.1), (1.2), $u^- \leqq u \leqq u^+$.

Theorem 4.8. *Let u^0 be a solution of* (1.1), (1.2) *which can be continuously connected to both $+1$ and -1. Then $u^0 = u^+ = u^-$, so there is uniqueness.*

The proof is very similar and will not be given.

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