Commun. Math. Phys. 86, 555–594 (1982)

Communications in Mathematical Physics

(Higgs)$_{2,3}$ Quantum Fields in a Finite Volume

II. An Upper Bound*

Tadeusz Bałaban

Department of Physics, Harvard University, Cambridge, MA 02138, USA

Abstract. This is the second part of the paper entitled, "(Higgs)$_{2,3}$ Quantum Fields in a Finite Volume." The proof of an upper bound for vacuum energy is completed with the exception of some technical estimates.

1. Introduction

This paper is a second part of the paper [1] and contains the second, more important part of the proof of the theorem formulated there. Let us recall the basic definitions and the theorem. We consider two spaces of field configurations on the torus

$$T_\varepsilon = \{x \in \varepsilon Z^d : -L_\mu \leqq x_\mu < L_\mu, \mu = 1, \ldots, d\}:$$

scalar fields and vector fields. Scalar field configurations are the functions $\phi : T_\varepsilon \to R^N$. Vector field configurations are the functions $A : T_\varepsilon \to R^d$ identified with the functions $A : T_\varepsilon^* \to R$ by the equality: $A_{\langle x, x+\varepsilon e_\mu \rangle} = A_\mu(x)$. Of course the periodic boundary conditions are understood here if the torus is identified with the subset of εZ^d. We consider the action

$$\begin{aligned} S^\varepsilon(A,\phi) = \tfrac{1}{2} \sum_{b \subset T_\varepsilon} \varepsilon^d |(D_A^\varepsilon \phi)(b)|^2 + \sum_{x \in T_\varepsilon} \varepsilon^d (\tfrac{1}{2} m_0^2 |\phi(x)|^2 \\ + \lambda |\phi(x)|^4) + \tfrac{1}{2} \sum_{b \subset T_\varepsilon} \varepsilon^d |(\partial^\varepsilon A)(b)|^2 + \tfrac{1}{2} \mu_0^2 \sum_{x \in T_\varepsilon} \varepsilon^d |A(x)|^2 - E, \end{aligned} \tag{1.1}$$

where $m_0^2 = m^2 + \delta m^2$, $m^2 > 0$ and δm^2 is the mass renormalization counterterm, $\mu_0^2 > 0$, $\lambda > 0$ and $E = E_0 + E_1$, E_0 is the normalization factor and E_1 is the renormalization counterterm of vacuum energy. The counterterms δm^2 and E_1 are defined with the help of perturbation expansions. The more detailed description of (1.1) is given in the first part [1, Chap. 1].

The partition function is defined as usual,

$$Z^\varepsilon = \int dA \int d\phi \exp(-S^\varepsilon(A,\phi)), \tag{1.2}$$

---

* Supported in part by the National Science Foundation under Grant No. PHY 79-16812

0010-3616/82/0086/0555/$08.00

and the integration is with respect to the natural Lebesgue measures on the field configurations. The fundamental result of the paper is

Theorem. *For the dimensions $d=2,3$ there exist the constants E_-, E_+ independent of ε, T_ε and such that*

$$\exp(-E_-|T_\varepsilon|) \leqq Z^\varepsilon \leqq \exp(E_+|T_\varepsilon|). \tag{1.3}$$

In the first part [1], we have proved the first inequality above, the lower bound. Now we will prove the upper bound. We will use the notations, methods and results of [1], and we will refer to this paper adding I before the numberings of chapters, theorems or formulas.

2. The Upper Bound

This chapter is a fundamental part of the paper. In Chap. I.3 we have analysed the forms of the actions and their expansions with respect to the vector fields. We will use these results here. The basic ideas of the proof of the upper bound are the same as described in this chapter although there are two essential differences. The first one is the form of the restrictions on the fields and their derivatives. Here, the restrictions will be introduced on the fields B, A, ψ, ϕ directly and not on the functions of these fields as in (I.3.27)–(I.3.29). These restrictions have a different form in accordance with the positivity properties which we will prove for the actions $S^{(k)}$. The second difference comes from the fact that "small" and "large" fields appear in each step of the procedure here. Each time we divide the lattice into two domains corresponding to these fields and we apply different procedures in these domains.

Similarly, as in Chap. I.3, we will describe the procedure for the first step very briefly, and next for the $k+1$ step.

A. First Renormalization Transformation

We have to calculate the integral $T^\varepsilon_{a,L}[T^\varepsilon_{a,L,A}[\exp(-S^\varepsilon)]]$. We rescale it from ε-lattice T_ε to 1-lattice T_1 and we get the integral (I.3.7). Omitting the constant factors we have

$$\begin{aligned}\int dA \int d\phi \exp\Big[&-\tfrac{1}{2}aL^{d-2} \sum_{y\in T_1'} |B(y)-(QA)(y)|^2 \\ &-\tfrac{1}{2}\sum_{b\subset T_1} |(\partial A)(b)|^2 - \tfrac{1}{2}\mu_0^2\varepsilon^2 \sum_{x\in T_1} |A(x)|^2 \\ &-\tfrac{1}{2}aL^{d-2}\sum_{y\in T_1'} |\psi(y)-(Q(A)\phi)(y)|^2 - \tfrac{1}{2}\sum_{b\subset T_1} |(D_A\phi)(b)|^2 \\ &-\tfrac{1}{2}m^2\varepsilon^2 \sum_{x\in T_1} |\phi(x)|^2 - \lambda\varepsilon^{4-d}\sum_{x\in T_1}|\phi(x)|^4 \\ &-\tfrac{1}{2}\delta m^2\varepsilon^2 \sum_{x\in T_1}|\phi(x)|^2 - E\Big].\end{aligned} \tag{2.1}$$

The expression in the above exponential function has several positive terms. Next we will prove that every action $S^{(k)}$ can be bounded from below by the positive terms of a similar type and the remaining terms are sufficiently small in comparison with them.

Now we will introduce the restrictions on the fields. Each restriction is connected with some positive term of the action, so the restrictions will be on the absolute values of the fields, their derivatives and the connections between the "old" and the "new" fields. If any of the following inequalities holds

$$\begin{gathered} |B(y)-(QA)(y)|>p(\varepsilon)\,, \qquad |(\partial A)(b)|>p(\varepsilon)\,, \\ |A(x)|>\frac{1}{\mu_0\varepsilon}\,p(\varepsilon)\,, \qquad |\psi(y)-(Q(A)\phi)(y)|>p(\varepsilon)\,, \\ |(D_A\phi)(b)|>p(\varepsilon)\,, \qquad |\phi(x)|>\frac{1}{(\lambda\varepsilon^{4-d})^{1/4}}\,p(\varepsilon)\,, \end{gathered} \tag{2.2}$$

then the corresponding factor in (2.1) satisfies the inequality

$$\exp(-(\ldots))<\exp(-c_0p(\varepsilon)^2)\,, \tag{2.3}$$

with some positive constant c_0, e.g. $c_0=\frac{1}{2}\min\{a,1\}$. The term on the right side above is very small. It follows from the definition of $p(\varepsilon)=b_0(1+\log\varepsilon^{-1})^p$, $p>2$, that $\exp(-c_0p(\varepsilon)^2)$ is smaller than the arbitrary power ε^K, so it can compensate the factor arising even from a very rough estimate of the action in a big neighbourhood of a point or a bond at which one of the inequalities (2.2) holds. This idea is basic for the procedure described below.

Let us denote by Λ^* the set of all bonds contained in Λ, i.e. with endpoints belonging to Λ, for arbitrary subset $\Lambda\subset T_1$. The following equality holds

$$\begin{aligned} 1= &\sum_{P_v\subset T_1'}\sum_{Q_v\subset T_1^*}\sum_{R_v\subset T_1}\prod_{y\in P_v}\chi(\{|B(y)-(QA)(y)|>p(\varepsilon)\}) \\ &\cdot\prod_{y\in P_v^c}\chi(\{|B(y)-(QA)(y)|\leqq p(\varepsilon)\})\prod_{b\in Q_v}\chi(\{|(\partial A)(b)|>p(\varepsilon)\}) \\ &\cdot\prod_{b\in Q_v^c}\chi(\{|(\partial A)(b)|\leqq p(\varepsilon)\})\prod_{x\in R_v}\chi\left(\left\{|A(x)|>\frac{1}{\mu_0\varepsilon}p(\varepsilon)\right\}\right) \\ &\cdot\prod_{x\in R_v^c}\chi\left(\left\{|A(x)|\leqq\frac{1}{\mu_0\varepsilon}p(\varepsilon)\right\}\right) \\ =: &\sum_{P_v\subset T_1'}\sum_{Q_v\subset T_1^*}\sum_{R_v\subset T_1}\chi^c_{P_v}\chi_{P_v^c}\chi^c_{Q_v}\chi_{Q_v^c}\chi^c_{R_v}\chi_{R_v^c}\,. \end{aligned} \tag{2.4}$$

An analogous equality for the scalar field is

$$\begin{aligned} 1= &\sum_{P_s\subset T_1'}\sum_{Q_s\subset T_1^*}\sum_{R_s\subset T_1}\prod_{y\in P_s}\chi(\{|\psi(y)-(Q(A)\phi)(y)|>p(\varepsilon)\}) \\ &\cdot\prod_{y\in P_s^c}\chi(\{|\psi(y)-(Q(A)\phi)(y)|\leqq p(\varepsilon)\})\prod_{b\in Q_s}\chi(\{|(D_A\phi)(b)|>p(\varepsilon)\}) \\ &\cdot\prod_{b\in Q_s^c}\chi(\{|(D_A\phi)(b)|\leqq p(\varepsilon)\})\prod_{x\in R_s}\chi\left(\left\{|\phi(x)|>\frac{1}{\lambda(\varepsilon)^{1/4}}p(\varepsilon)\right\}\right) \\ &\cdot\prod_{x\in R_s^c}\chi\left(\left\{|\phi(x)|\leqq\frac{1}{\lambda(\varepsilon)^{1/4}}p(\varepsilon)\right\}\right) \\ =: &\sum_{P_s\subset T_1'}\sum_{Q_s\subset T_1^*}\sum_{R_s\subset T_1}\chi^c_{P_s}\chi_{P_s^c}\chi^c_{Q_s}\chi_{Q_s^c}\chi^c_{R_s}\chi_{R_s^c}\,, \end{aligned} \tag{2.5}$$

where $\lambda(\varepsilon)=\lambda\varepsilon^{4-d}$.

Unifying these two expansions, i.e. multiplying (2.4) and (2.5), we get a joint expansion:

$$1=\sum_{P_v,\,P_s\subset T_1'}\ \sum_{Q_v,\,Q_s\subset T_1^*}\ \sum_{R_v,\,R_s\subset T_1} \chi^c_{P_v}\chi_{P_v^c}\chi^c_{P_s}\chi_{P_s^c}$$
$$\cdot\chi^c_{Q_v}\chi_{Q_v^c}\chi^c_{Q_s}\chi_{Q_s^c}\chi^c_{R_v}\chi_{R_v^c}\chi^c_{R_s}\chi_{R_s^c}\,. \tag{2.6}$$

We assign a division of the lattice T_1 into two domains with each term in the above sum on the right side. The first one contains points and bonds at which the fields or their derivatives are large, for the second domain they are small. We will use here the division of the lattice T_1 into large blocks described in the first part [1]. Let us define

Λ_0^c is the sum of all large blocks of T_1 distant from one of the sets $B(P_v)$, Q_v, R_v, $B(P_s)$, Q_s, R_s less than $r(\varepsilon)=R(1+\log\varepsilon^{-1})^r$. The numbers r, R satisfy $r>1$, $R>R_0$ (R_0 occurs in the formulation of Proposition I.2.1). (2.7)

Next let us define a sequence of sets $\Lambda_1, \Lambda_2, \ldots$ by an induction

Λ_{i+1}^c is the sum of all large blocks of T_1 with distances from the set Λ_i^c less or equal $r(\varepsilon)$. (2.8)

Of course all the fields are small on the set Λ_0 and on the neighbourhood of Λ_0 of the additional thickness $r(\varepsilon)$ also. In (2.6) some terms have a set Λ_0 in common, so we can represent this sum as the sum over all possible sets Λ_0, and next for each fixed Λ_0 we have a sum over all admissible sets $P_v, \ldots, R_s$, i.e. defining the set Λ_0^c by (2.7). In this last sum we can make a partial resummation. In the class of ordered 6-tuples $\{P_v, Q_v, R_v, P_s, Q_s, R_s\}$ the inclusion relation between the proper sets defines a natural partial order relation. Thus, there are minimal elements in the class. Let us denote by $\Lambda_{-1}^{(*,\prime)}$ the set of the points (the bonds, the blocks) in T_1 distant from Λ_0 less than $r(\varepsilon)$. It is easily seen that

$$\sum_{\{P_v,\ldots,R_s\}\,\text{admissible}} \chi^c_{P_v}\chi_{P_v^c}\cdot\ldots\cdot\chi^c_{R_s}\chi_{R_s^c}$$
$$=\sum_{\{P_v,\ldots,R_s\}\,\text{admissible, minimal}} \chi^c_{P_v}\chi^c_{P_s}\chi^c_{Q_v}\chi^c_{Q_s}\chi^c_{R_v}\chi^c_{R_s}\chi_{\Lambda_{-1}}, \tag{2.9}$$

where the last characteristic function denotes the product of the characteristic functions giving the corresponding restrictions on the vector and the scalar fields on the set Λ'_{-1}, Λ^*_{-1}, Λ_{-1}. Let us notice that the minimal elements are the elements for which the sets $P_v, \ldots, R_s$ are maximally "diluted." For example, this implies that each pair of elements belonging to the sum of these sets has a distance $>r(\varepsilon)$. Further we have

$$1=\sum_{\Lambda_0}\,[\text{the expression on the right side of (2.9)}]\,. \tag{2.10}$$

The above expansion is introduced under the integral (2.1) and we get a sum of terms.

Now let us make a first estimate of this integral, namely in each term of this some we remove the interaction terms from the set Λ_7^c. We use the fact that

$\delta m^2 = O(\varepsilon^{-1})$ for $d=3$ and $\delta m^2 = O(1+\log\varepsilon^{-1})$ for $d=2$. Hence

$$\lambda(\varepsilon)|\phi|^4 + \tfrac{1}{2}\delta m^2 \varepsilon^2 |\phi|^2 \geqq -O(\varepsilon^{\kappa_0}), \tag{2.11}$$

where $\kappa_0 = 1$ for $d=3$ and $\kappa_0 = 2-\alpha$ with arbitary $\alpha > 0$ for $d=2$, and

$$\lambda(\varepsilon)|\phi|^4 + \tfrac{1}{2}\delta m^2 \varepsilon^2 |\phi|^2 \geqq \tfrac{1}{2}\lambda(\varepsilon)|\phi|^4 \quad \text{for} \quad |\phi|^2 \geqq O(1)(1+\log\varepsilon^{-1}). \tag{2.12}$$

From these estimates we get the following one

$$\begin{aligned} &\chi^c_{R_s} \exp\left[-\sum_{x\in\Lambda_7^c} (\lambda(\varepsilon)|\phi(x)|^4 + \tfrac{1}{2}\delta m^2 \varepsilon^2 |\phi(x)|^2)\right] \\ &\leqq \exp(-\tfrac{1}{2}p(\varepsilon)^4 |R_s|) \exp(O(\varepsilon^{\kappa_0})|\Lambda_7^c|) \\ &\leqq \exp(-p(\varepsilon)^2 |R_s|) \exp(O(\varepsilon^{\kappa_0})|\Lambda_7^c|). \end{aligned} \tag{2.13}$$

Similarly, for the constant E_1 given by the perturbation expansion (I.1.13), we have

$$E_1 = E_1(\Lambda_7) + E_1'(\Lambda_7^c), \qquad E_1'(\Lambda_7^c) = O(\varepsilon^{\kappa_0})|\Lambda_7^c|. \tag{2.14}$$

The last estimate and the above mentioned estimates of δm^2 are connected with the properties of the perturbation expansion and they will be proved in the next paper. Let us denote

$$\begin{aligned} \zeta_{\Lambda_0} = &\sum_{\{P_v,\dots,R_s\}\,\text{admissible, minimal for}\,\Lambda_0} \chi^c_{P_v}\chi^c_{Q_v} \\ &\cdot \chi^c_{R_v}\chi^c_{P_s}\chi^c_{Q_s} \exp(-p(\varepsilon)^2|R_s|). \end{aligned} \tag{2.15}$$

Let us analyse more precisely the restrictions on the fields in the domains $\Lambda^{(*,')}_{-1}$. For each $y\in\Lambda'$, we have

$$\begin{aligned} |\psi(y)| &\leqq \left| L^{-d} \sum_{x\in B(y)} U(A(\Gamma_{y,x}))\phi(x)\right| \\ &+ p(\varepsilon) \leqq L^{-d} \sum_{x\in B(y)} |\phi(x)| \\ &+ p(\varepsilon) \leqq \frac{1}{\lambda(\varepsilon)^{1/4}} p(\varepsilon) + p(\varepsilon) \leqq \frac{2}{\lambda(\varepsilon)^{1/4}} p(\varepsilon). \end{aligned}$$

Further, for arbitrary $x\in B(y)$

$$\begin{aligned} &|U(A(\Gamma_{y,x}))\phi(x) - \phi(y)| \\ &= \left|\sum_{b\subset\Gamma_{y,x}} U(A(\Gamma_{y,b_-}))(D_A\phi)(b)\right| \\ &\leqq \sum_{b\subset\Gamma_{y,x}} |(D_A\phi)(b)| \leqq (L-1)dp(\varepsilon), \end{aligned}$$

hence

$$|(Q(A)\phi)(y) - \phi(y)| \leqq (L-1)dp(\varepsilon), \qquad |\psi(y) - \phi(y)| \leqq Ldp(\varepsilon),$$

and

$$|\psi(y) - U(A(\Gamma_{y,x}))\phi(x)| \leqq 2Ldp(\varepsilon).$$

Finally for arbitrary

$$\langle x, x'\rangle, \quad x \in B(y), \quad x' \in B(y'),$$

we have

$$|U(A(\Gamma_{y,x} \cup \langle x, x'\rangle \cup \Gamma_{x',y'}))\psi(y') - \psi(y)| \leqq |\psi(y') - \phi(y')|$$
$$+ \sum_{b \subset \Gamma_{y,x} \cup \langle x,x'\rangle \cup \Gamma_{x',y'}} |(D_A \phi)(b)| + |\psi(y) - \phi(y)| \leqq 4Ldp(\varepsilon).$$

In particular

$$|U(A(\langle y, y'\rangle))\psi(y') - \psi(y)| \leqq 3Ldp(\varepsilon).$$

It is worthwhile to notice here that this particular estimate implies the previous more general cases.

Let us gather the estimates for the scalar fields on Λ_{-1}:

$$\begin{aligned} |\psi(y) - U(A(\Gamma_{y,x}))\phi(x)| &\leqq 2Ldp(\varepsilon) \quad \text{for} \quad x \in B(y), \\ |\psi(y)| &\leqq \frac{2}{\lambda(\varepsilon)^{1/4}} p(\varepsilon) \quad \text{for} \quad y \in \Lambda'_{-1}, \\ |U(A(\langle y, y'\rangle))\psi(y') - \psi(y)| &\leqq 3Ldp(\varepsilon) \quad \text{for} \quad \langle y, y'\rangle \in \Lambda'^*_{-1}. \end{aligned} \tag{2.16}$$

The identical considerations can be done for vector fields and we get

$$\begin{aligned} |B(y) - A(x)| &\leqq 2Ldp(\varepsilon) \quad \text{for} \quad x \in B(y), \quad \text{and} \\ |B(y)| &\leqq \frac{2}{\mu_0 \varepsilon} p(\varepsilon) \quad \text{for} \quad y \in \Lambda'_{-1}, \\ |B(y) - B(y')| &\leqq 3Ldp(\varepsilon) \quad \text{for} \quad \langle y, y'\rangle \in \Lambda'^*_{-1}. \end{aligned} \tag{2.17}$$

The same estimates as (2.16), (2.17) will hold for the fields in each step with ε replaced by the corresponding $L^k \varepsilon$.

Now we will make a translation in the fields A analogous to the translation (I.3.10) in the proof of the lower bound, only now it is connected with a conditional integral, the conditioning in the set Λ_0^c. Thus we make a translation

$$A = A' + aL^{-2} C_{\Lambda_0}^{(0)} Q^* B, \tag{2.18}$$

where $C_{\Lambda_0}^{(0)}$ denotes the covariance with the Dirichlet boundary conditions outside Λ_0 introduced in Chap. I.2. Its properties were described in Proposition I.2.3.

In the third section of this chapter we will prove a general result from which it follows that the field A' is small on Λ_1, i.e. $|A'(x)| \leqq O(1)p(\varepsilon)$ for $x \in \Lambda_1$. Next we divide the field (2.18) into two components: one "small," with respect to which we will expand the action, and one "large" which will remain in all the expressions in the preceding form. To define this division let us introduce a function θ_1 equal to 1 on Λ_2 and changing smoothly from 1 to 0 on a slice of thickness $< M$ surrounding Λ_2. We have

$$\begin{aligned} A = &[(1-\theta_1)(A' + aL^{-2} C_{\Lambda_0}^{(0)} Q^* B) + \theta_1 a L^{-2} \zeta C^{(0)} Q^* B] \\ &+ \theta_1 (A' + aL^{-2} \delta C_{\Lambda_0}^{(0)} Q^* B + aL^{-2}(1-\zeta) C^{(0)} Q^* \Lambda'_0 B) = : \tilde{B}^{(1)} + A'', \end{aligned} \tag{2.19}$$

where $\zeta C^{(0)}Q^*$ denotes an operator with the kernel

$$(\zeta C^{(0)}Q^*)(x,y)=\zeta(x,y)L^{-d}\sum_{x'\in B(y)}C^{(0)}(x,x'),\qquad x\in T_1,\qquad y\in T_1',$$

and the function ζ is determined by the conditions:

$$\begin{aligned}\zeta(x,y)&=1\quad\text{if}\quad |x-y|\leqq\tfrac{1}{2}r(\varepsilon),\\ \zeta(x,y)&=0\quad\text{if}\quad |x-y|>\tfrac{1}{2}r(\varepsilon).\end{aligned}$$

A definition of the operator $(1-\zeta)C^{(0)}Q^*$ should be clear. A'' is a small field and we can expand the action with respect to this field. At first let us notice that the restrictions (2.16) on the scalar fields can be replaced by the same restrictions putting $\tilde{B}^{(1)}$ instead of A and $c_1p(\varepsilon)$ instead of $p(\varepsilon)$, with some constant c_1 independent of ε. Similarly the characteristic functions $\chi_{A_{-1},s}$ for the scalar fields can be estimated by the corresponding characteristic functions with $\tilde{B}^{(1)}$ and $c_1p(\varepsilon)$ instead of A and $p(\varepsilon)$. For example, we have

$$(D_A\phi)(b)=(D_{\tilde{B}^{(1)}}\phi)(b)+(U(A_b'')-1)U(B_b^{(1)})\phi(b_+),\tag{2.20}$$

hence

$$\begin{aligned}|(D_{\tilde{B}^{(1)}}\phi)(b)|&\leqq|(D_A\phi)(b)|+|U(A_b'')-1|\,|\phi(b_+)|\leqq p(\varepsilon)\\ &\quad+e(\varepsilon)O(1)p(\varepsilon)\frac{1}{\lambda(\varepsilon)^{1/4}}p(\varepsilon)\leqq c_1p(\varepsilon).\end{aligned}\tag{2.21}$$

The remaining restrictions can be considered in a similar way. The characteristic functions with the new restrictions will be denoted as previously.

Now let us expand the action with respect to the field A''. This expansion was described already in Chap. I.3. We use the formulas (I.3.14)–(I.3.16), with $\tilde{B}^{(1)}$ instead of $B^{(1)}$. Let us notice that now we have worse restrictions on the fields ψ, ϕ, with the additional factor $\lambda(\varepsilon)^{-1/4}$, so we have to expand to higher power than before to compensate for these factors, e.g. we have to take $\bar{n}\geqq 7$. After the expansion we get the fundamental quadratic form for the fields ψ, ϕ in the external field $\tilde{B}^{(1)}$ and the terms describing an interaction with the field A''. We remove this interaction from the set Λ_7^c, e.g. we estimate

$$|(D_{\tilde{B}^{(1)}}\phi)(b)\cdot F_1(-A_b'')\phi(b_-)|\leqq c_1p(\varepsilon)O(1)e(\varepsilon)p(\varepsilon)\lambda(\varepsilon)^{-1/4}p(\varepsilon)=O(\varepsilon^{\kappa_0}),\tag{2.22}$$

and similarly the other terms, thus the interaction is estimated by $O(\varepsilon^{\kappa_0})|\Lambda_7^c\cap\Lambda_2|$. From the definition (2.19) and the properties of the propagator $C_{\Lambda_0}^{(0)}$, it follows that for the interaction terms in the domain Λ_7 the field A'' can be replaced by A' and we get the additional term $O(\varepsilon^{\kappa})|\Lambda_7|$, $\kappa>d$.

We make a next transformation of the integral, namely we make the translation

$$\phi=\phi'+aL^{-2}C_{\Lambda_3}^{(0)}(\tilde{B}^{(1)})Q^*(\tilde{B}^{(1)})\psi\,.\tag{2.23}$$

This translation changes the interaction, and we get an expression almost identical to $V^{(0)}$ in Chap. I.3, only with the modified propagators for the scalar field and the summations with respect to variables in the vertices restricted to the set Λ_7. We will change slightly this expression. Let us introduce a configuration $B^{(1)}$ by the formula

$$B^{(1)}=aL^{-2}\zeta C^{(0)}Q^*B\,, \tag{2.24}$$

and let us notice that Lemma 2.3 implies also $(\partial_\mu B^{(1)})(x)=O(p(\varepsilon))$ for $x\in\Lambda_2$, $\mu=1,\ldots,d$. Hence the assumptions of Proposition I.2 are satisfied for $B^{(1)}$ on Λ_2, and we have

$$\begin{aligned}&aL^{-2}(C^{(0)}_{\Lambda_3}(\tilde{B}^{(1)})Q^*(\tilde{B}^{(1)})\psi)(x)\\&\quad=aL^{-2}(G_1(\Lambda_2,B^{(1)})Q^*(B^{(1)})\Lambda'_6\psi)(x)\\&\qquad+O(\varepsilon^\kappa)=:\psi^{(1)}(x)+O(\varepsilon^\kappa),\quad \kappa>d,\quad x\in\Lambda_7\,.\end{aligned} \tag{2.25}$$

We denote by $V^{(0)}(\Lambda_7,B^{(1)},\psi,A',\phi')$ the expression for the interaction obtained by applying (2.25) to the previous expression. We will prove also in the third section that the field ϕ' is small on the domain Λ_4.

Let us introduce the characteristic functions

$$\prod_{x\in\Lambda_5}\chi(\{|\phi'(x)|\leqq O(1)p(\varepsilon)\})\chi(\{|A'(x)|\leqq O(1)p(\varepsilon)\})\,,$$

and let us denote them by χ'. Further let us denote by χ_1 the characteristic functions giving the part of the restrictions (2.16), (2.17), which involves the new fields B, ψ only.

As an effect of the considerations of this point, we get the inequality

$$\begin{aligned}(2.1)\leqq&\sum_{\Lambda_0}\int \mathrm{d}A'\int \mathrm{d}\phi'\zeta_{\Lambda_0}\chi_{\Lambda_{-1}}\chi'\chi_1\\&\cdot\exp\Big[-\tfrac12 aL^{d-2}\sum_{y\in T'_1}|(\Lambda'^c_0B)(y)-(QA')(y)|^2\\&-\tfrac12\langle A',(-\Delta+\mu_0^2\varepsilon^2)A'\rangle+aL^{-2}\sum_{b\in st(\Lambda_0)}A'(b_+)\cdot(C^{(0)}_{\Lambda_0}Q^*B)(b_-)\\&-\tfrac12\langle B,\Delta^{(1),L}_{\Lambda_0}B\rangle-\tfrac12 aL^{d-2}\sum_{y\in T'_1}|(\Lambda'^c_3\psi)(y)-(Q(\tilde{B}^{(1)})\phi')(y)|^2\\&-\tfrac12\langle\phi',(-\Delta_{\tilde{B}^{(1)}}+m^2\varepsilon^2)\phi'\rangle\\&+aL^{-2}\sum_{b\in st(\Lambda_3)}\phi'(b_+)\cdot U(B^{(1)}_{-b})(C^{(0)}_{\Lambda_3}(B^{(1)})Q^*(B^{(1)})\psi)(b_-)\\&-\tfrac12\langle\psi,\Delta^{(1),L}_{\Lambda_3}(B^{(1)})\psi\rangle+V^{(0)}(\Lambda_7,B^{(1)},\psi,A',\phi')\\&+O(\varepsilon^{\kappa_0})|\Lambda^c_7|+O(\varepsilon^\kappa)|T_1|\Big],\end{aligned} \tag{2.26}$$

where the new quadratic form $\Delta^{(1),L}_{\Lambda_3}(B^{(1)})$ for the field ψ is defined by the general formula

$$\begin{aligned}\langle\psi,\Delta^{(k+1),L}_{\Lambda}(\Omega,\mathrm{A})\psi\rangle=&\,aL^{d-2}\sum_{y\in\Lambda'}|\psi(y)|^2\\&-a^2L^{-4}\langle\psi,Q(A)C^{(k)}_\Lambda(\Omega,A)Q^*(A)\psi\rangle\,,\qquad \Lambda\subset\Omega^{(k)}\,,\end{aligned} \tag{2.27}$$

and the similar formula holds for the vector field.

A next operation is a calculation of the integral in (2.26). This operation can not be done on the whole set T_1, as in Chap. I.3, but on some subsets of T_1 only, on which the fields A', ϕ' are small. We will integrate each term in the sum on the right side of (2.26) on the set Λ_5, and it will be a conditional integration with conditioning on Λ_5^c.

Let us recall this operation in a general case. Let Ω be a finite set, $\Lambda \subset \Omega$, and let A be a positive operator on a space of field configurations on Ω, A_Λ its restriction on Λ. Then we have

$$\begin{aligned}
&\int \prod_{x\in\Omega} d\phi(x) \exp(-\tfrac{1}{2}\langle\phi, A\phi\rangle) \exp(\langle f, \phi\rangle) F(\phi\restriction_\Lambda) G(\phi\restriction_{\Lambda^c}) \\
&\quad = \int \prod_{x\in\Omega} d\phi(x) \exp(-\tfrac{1}{2}\langle\phi, A\phi\rangle) \exp(\langle f, \phi\rangle) G(\phi\restriction_{\Lambda^c}) \\
&\quad\quad \cdot \int d\mu_{A_\Lambda^{-1}}(\phi') F(\phi' - A_\Lambda^{-1} A\phi\restriction_{\Lambda^c} + A_\Lambda^{-1} f), \qquad (2.28)
\end{aligned}$$

where $d\mu_{A_\Lambda^{-1}}$ is a probabilistic Gaussian measure with the covariance A_Λ^{-1}.

In our case A is given by the main quadratic form in the fields A', ϕ', f is obvious, $\Lambda = \Lambda_5$, $F = \chi' \exp(V^{(0)}(\Lambda_7))$, and the function G is the product of the characteristic functions $\zeta_{\Lambda_0}\chi_{\Lambda_{-1}\cap\Lambda_5^c}\chi_1$ and the remaining exponential functions, the rest of the characteristic functions are estimated by 1. The expression $(\phi' - A_\Lambda^{-1} A\phi\restriction_{\Lambda^c})(x)$ for vector fields has the form

$$A'(x) + \sum_{b\in st(\Lambda_5)} C_{\Lambda_5}^{(0)}(x, b_-) A'(b_+), \qquad x \in \Lambda_5, \qquad (2.29)$$

and for scalar fields

$$\phi'(x) + \sum_{b\in st(\Lambda_5)} C_{\Lambda_5}^{(0)}(B^{(1)}; x, b_-) U(B_b^{(1)}) \phi'(b_+), \qquad x \in \Lambda_5. \qquad (2.30)$$

Because the fields A', ϕ' are small on $\partial\Lambda_5 = \{x \in \Lambda_5^c : x = b_+ \text{ for some } b \in st(\Lambda_5)\}$, the second terms in (2.29), (2.30) can be estimated by $O(1)p(\varepsilon)$ and the characteristic functions χ' [the expression (2.29)]$\cdot\chi'$ [the expression (2.30)] can be estimated by the functions $\chi_{\Lambda_5}(A')\chi_{\Lambda_5}(\phi')$ defined in the same way as χ' but with a suitably larger constant $O(1)$. The expressions (2.29), (2.30) occur also in the interaction $V^{(0)}(\Lambda_7)$, but then $x \in \Lambda_7$, and the second terms are of the order $O(\varepsilon^\kappa)$. So the part of the interaction containing these terms can be estimated by $O(\varepsilon^\kappa)|\Lambda_7|$.

Thus in our case the integral with respect to the probabilistic measure in the last line of (2.28) is estimated by

$$\begin{aligned}
&\int d\mu_{C_{\Lambda_5}^{(0)}}(A') \int d\mu_{C_{\Lambda_5}^{(0)}(B^{(1)})}(\phi') \chi_{\Lambda_5}(A') \chi_{\Lambda_5}(\phi') \\
&\quad \cdot \exp(V^{(0)}(\Lambda_7, B^{(1)}, \psi, A', \phi') + O(\varepsilon^\kappa)|\Lambda_7|). \qquad (2.31)
\end{aligned}$$

To this integral we apply the lemma in [2]. Using the lemma and some results of the third paper, we can estimate the integral (2.31) by a cumulant expansion to a sufficiently high order $\bar{n}$ plus $O(\varepsilon^\kappa)|\Lambda_7|$, $\kappa > d$. Here there are stronger restrictions on $\bar{n}$ than in Chap. I.3 because the estimates on the expressions in vertices are weaker, thus $\bar{n}$ has to be >12 for $d=3$ and >4 for $d=2$.

As a result of the cumulant expansion we get part of the perturbation expansion in coupling constants, and we estimate the sum of terms of order higher than $\bar{n}$ by $O(\varepsilon^\kappa)|\Lambda_7|$. As a result we get an expression equal to $\mathscr{P}^{(1),L}(\Lambda_7, B^{(1)}, \psi)$, i.e.

to the expression obtained and analysed in Chap. I.3, the only differences being that the summations in the vertices are restricted to the set Λ_7 and the scalar field propagators are modified. Thus we have

$$(2.31) \leqq \exp(\mathscr{P}^{(1),L}(\Lambda_7, B^{(1)}, \psi) + O(\varepsilon^{\kappa})|\Lambda_7|). \tag{2.32}$$

It is worth mentioning here that the above inequality can be obtained in a simpler way, without using the lemma. This can be achieved by integration by parts as in [12] and then doing some elementary estimates.

As a result of all the operations which have been done up to now, we get the inequality

$$\begin{aligned}(2.1) \leqq \sum_{\Lambda_0} & \varrho''^{(1),L}(\Lambda_0, B, \theta_1 B^{(1)}, \psi) \\ & \cdot \exp(\mathscr{P}^{(1),L}(\Lambda_7, B^{(1)}, \psi) + O(\varepsilon^{\kappa_0})|\Lambda_7^c| + O(\varepsilon^{\kappa})|T_1|),\end{aligned} \tag{2.33}$$

with $\kappa_0 > 0$, $\kappa > d$, and $\varrho''^{(1),L}$ given by the formulas

$$\begin{aligned}\varrho''^{(1),L}(\Lambda_0, B, \theta_1 B^{(1)}, \psi) = & :\chi_1 \int dA \int d\phi \zeta_{\Lambda_0} \chi_{\Lambda_{-1} \cap \Lambda_5^c} \\ & \cdot \exp\Big[-\tfrac{1}{2} a L^{d-2} \sum_{y \in T_1'} |B(y) - (QA)(y)|^2 - \tfrac{1}{2}\langle A, (-\Delta + \mu_0^2 \varepsilon^2) A\rangle \\ & - \tfrac{1}{2} a L^{d-2} \sum_{y \in T_1'} |\psi(y) - (Q(\tilde{B}^{(1)})\phi)(y)|^2 - \tfrac{1}{2}\langle \phi, (-\Delta_{\tilde{B}^{(1)}} + m^2 \varepsilon^2)\phi\rangle\Big] \\ = & \chi_1 \int dA \restriction_{\Lambda_5^c} \int d\phi \restriction_{\Lambda_5^c} \zeta_{\Lambda_0} \chi_{\Lambda_{-1} \cap \Lambda_5^c} \cdot \exp\Big[-\tfrac{1}{2} a L^{d-2} \sum_{y \in \Lambda_5^c} |B(y) - (QA)(y)|^2 \\ & - \tfrac{1}{2}\langle A, (-\Delta^D_{\Lambda_5^c} + \mu_0^2 \varepsilon^2) A\rangle - \tfrac{1}{2} a L^{d-2} \sum_{y \in \Lambda_5^c} |\psi(y) - (Q(\tilde{B}^{(1)})\phi)(y)|^2 \\ & - \tfrac{1}{2}\langle \phi, (-\Delta^D_{\tilde{B}^{(1)}, \Lambda_5^c} + m^2 \varepsilon^2)\phi\rangle + \tfrac{1}{2} \sum_{b, b' \in st(\Lambda_5)} A(b_+) \cdot C^{(0)}_{\Lambda_5}(b_-, b'_-) A(b'_+) \\ & + a L^{-2} \sum_{b \in st(\Lambda_5)} A(b_+) \cdot (C^{(0)}_{\Lambda_5} Q^* B)(b_-) \\ & - \tfrac{1}{2}\langle B, \Delta^{(1),L}_{\Lambda_5} B\rangle + \tfrac{1}{2} \sum_{b, b' \in st(\Lambda_5)} \phi(b_+) \cdot U(B^{(1)}_{-b}) C^{(0)}_{\Lambda_5}(B^{(1)}; b_-, b'_-) \\ & \cdot U(B^{(1)}_{b'})\phi(b'_+) + a L^{-2} \sum_{b \in st(\Lambda_5)} \phi(b_+) \cdot U(B^{(1)}_{-b})(C^{(0)}_{\Lambda_5}(B^{(1)}) Q^*(B^{(1)})\psi)(b_-) \\ & - \tfrac{1}{2}\langle \psi, \Delta^{(1),L}_{\Lambda_5}(B^{(1)})\psi\rangle\Big] \int dA \restriction_{\Lambda_5} \exp(-\tfrac{1}{2}\langle A, (C^{(0)}_{\Lambda_5})^{-1} A\rangle) \\ & \cdot \int d\phi \restriction_{\Lambda_5} \exp(-\tfrac{1}{2}\langle \phi, (C^{(0)}_{\Lambda_5}(B^{(1)}))^{-1}\phi\rangle).\end{aligned} \tag{2.34}$$

Because the first representation on the right side above was obtained by doing the translations in the fields A', ϕ' inverse to (2.18), (2.23), we have

$$\tilde{B}^{(1)} = (1 - \theta_1) A + \theta_1 B^{(1)}, \tag{2.35}$$

and the characteristic functions $\chi_{\Lambda_{-1} \cap \Lambda_5^c}$ give the restrictions on the fields B, ψ, A, ϕ on the set $\Lambda_{-1} \cap \Lambda_5^c$. The functions χ_1 and ζ_{Λ_0} have the same meaning as before.

The integrals in the last line of (2.34), after proper rescaling and multiplication by

$$\left(\frac{a(L\varepsilon)^{d-2}}{2\pi}\right)^{\frac{d}{2}|\Lambda_5|}, \quad \left(\frac{a(L\varepsilon)^{d-2}}{2\pi}\right)^{\frac{N}{2}|\Lambda_5|},$$

are equal to the factors $Z^{(0),\varepsilon}$, $Z^{(0),\varepsilon}(B^{(1)})$ calculated on the set Λ_5 instead of T_ε. We will denote them by $Z^{(0),\varepsilon}_{\Lambda_5}$, $Z^{(0),\varepsilon}_{\Lambda_5}(B^{(1)})$. A product of these factors coming from each step of the procedure will be used in the sequel, instead of the factors Z^ε_k, $Z^\varepsilon_k(A)$ used in Chap. I.3.

The second representation in (2.34) was obtained by integration with respect to $A\restriction_{\Lambda_5}$, $\phi\restriction_{\Lambda_5}$. We will transform it further in order to get the same representation as in Chap. I.3, when the fields B, ψ are restricted to the set Λ_6. We have

$$aL^{-2}\sum_{b\in st(\Lambda_5)} A(b_+)\cdot(C^{(0)}_{\Lambda_5}Q^*B)(b_-)=aL^{-2}\sum_{b\in st(\Lambda_5)} A(b_+) \cdot(C^{(0)}_{\Lambda_5}Q^*\Lambda'^c_6B)(b_-)+O(\varepsilon^\kappa)|\partial\Lambda_5|, \tag{2.36}$$

$$\begin{aligned}\tfrac{1}{2}\langle B,\Delta^{(1),L}_{\Lambda_5}B\rangle &= \tfrac{1}{2}\langle \Lambda'^c_6B,\Delta^{(1),L}_{\Lambda_5}\Lambda'^c_6B\rangle \\ &\quad -a^2L^{-4}\langle(\Lambda'_5\cap\Lambda'^c_6)B, QC^{(0)}_{\Lambda_5}Q^*(\Lambda'_6\cap\Lambda'^c_7)B\rangle \\ &\quad +\tfrac{1}{2}\langle\Lambda'_6B,\Delta^{(1),L}\Lambda'_6B\rangle+O(\varepsilon^\kappa)|\Lambda'_5|,\end{aligned} \tag{2.37}$$

for arbitrary κ.

Similar representations and estimates hold for the scalar field expressions, but there are essential changes also. In this case it is convenient to make all the expressions, except the basic quadratic form, independent of the field $B^{(1)}\restriction_{\Lambda_7}$, because then they are unchanged in the next step of the procedure. This is achieved by imposing proper boundary conditions on the fundamental Laplace difference operator. We have from Proposition I.2.1.

$$\begin{aligned}&\tfrac{1}{2}\sum_{b,b'\in st(\Lambda_5)}\phi(b_+)\cdot U(B^{(1)}_{-b})C^{(0)}_{\Lambda_5}(B^{(1)};b_-,b'_-)U(B^{(1)}_{b'})\phi(b'_+) \\ &\quad =\tfrac{1}{2}\sum_{b,b'\in st(\Lambda_5)}\phi(b_+)\cdot U(B^{(1)}_{-b})C^{(0)}_{\Lambda_5}(\Lambda^c_6,B^{(1)};b_-,b'_-)U(B^{(1)}_{b'})\phi(b'_+) \\ &\quad\quad +O(\varepsilon^\kappa)|\partial\Lambda_5|,\end{aligned} \tag{2.38}$$

$$\begin{aligned}&aL^{-2}\sum_{b\in st(\Lambda_5)}\phi(b_+)\cdot U(B^{(1)}_{-b})(C^{(0)}_{\Lambda_5}(B^{(1)})Q^*(B^{(1)})\psi)(b_-) \\ &\quad =aL^{-2}\sum_{b\in st(\Lambda_5)}\phi(b_+)\cdot U(B^{(1)}_{-b})(C^{(0)}_{\Lambda_5}(\Lambda^c_6,B^{(1)})Q^*(B^{(1)})\psi)(b_-) \\ &\quad\quad +O(\varepsilon^\kappa)|\partial\Lambda_5|,\end{aligned} \tag{2.39}$$

$$\begin{aligned}&\tfrac{1}{2}\langle\psi,\Delta^{(1),L}_{\Lambda_5}(B^{(1)})\psi\rangle \\ &\quad =\tfrac{1}{2}\langle\Lambda'^c_6\psi,\Delta^{(1),L}_{\Lambda_5}(\Lambda^c_7,B^{(1)})\Lambda'^c_6\psi\rangle \\ &\quad\quad -a^2L^{-4}\langle\Lambda'^c_6\psi,Q(B^{(1)})C^{(0)}_{\Lambda_5}(\Lambda^c_7,B^{(1)})Q^*(B^{(1)})(\Lambda'_6\cap\Lambda'^c_7)\psi\rangle \\ &\quad\quad +\tfrac{1}{2}\langle\Lambda'_6\psi,\Delta^{(1),L}(\Lambda_2,B^{(1)})\Lambda'_6\psi\rangle+O(\varepsilon^\kappa)|\Lambda_5|.\end{aligned} \tag{2.40}$$

Let us denote by $\varrho^{(1),L}(A_0,B,\theta_1B^{(1)},\psi)$ a density given by the second representation in (2.34), with the expressions on the left sides of (2.36)–(2.40) replaced by the expressions on the right sides, obviously without the errors. Also let us denote by $\varrho'^{(1),L}(\Lambda_0, B, \theta_1 B^{(1)}, \psi)$ a density obtained from $\varrho^{(1),L}(\Lambda_0, B, \theta_1 B^{(1)}, \psi)$ by removing the basic quadratic forms on Λ'_6 from (2.37), (2.40). Then we have

$$\begin{aligned}\varrho^{(1),L}(\Lambda_0, B, \theta_1 B^{(1)}, \psi) = \varrho'^{(1),L}(\Lambda_0, B, \theta_1 B^{(1)}, \psi)\\ \cdot\exp[-\tfrac{1}{2}\langle\Lambda'_6 B, \Delta^{(1),L}\Lambda'_6 B\rangle\\ -\tfrac{1}{2}\langle\Lambda'_6\psi, \Delta^{(1),L}(\Lambda_2, B^{(1)})\Lambda'_6\psi\rangle],\end{aligned} \tag{2.41}$$

$$\varrho'^{(1),L}(\Lambda_0, B, \theta_1 B^{(1)}, \psi) = \varrho'^{(1),L}(\Lambda_0, \Lambda_7'^{c} B, \theta_1 B^{(1)}, \Lambda_7'^{c}\psi), \tag{2.42}$$

and $\varrho'^{(1),L}$ depends on $\Lambda_7 B^{(1)}$ only through the factor $Z^{(0)}_{\Lambda_5}(B^{(1)})$.

The last step is a rescaling of the obtained expression from L-lattice to $L\varepsilon$-lattice. In the final expression we must include all the numerical factors. We do not write this expression here because it will be written in a general case in an inductive hypothesis.

B. Inductive Description of the Action after k Steps

Now we will write the expressions and their estimates we get after k steps of the renormalization procedure. We have

$$\begin{aligned}Z^{\varepsilon} \leqq \int dA \int d\phi \sum_{\Lambda_0^{(k-1)}\subset T_1^{(k-1)},\,\text{admissible}}\\ \dots \sum_{\Lambda_0^{(0)}\subset T_1,\,\text{admissible}} \varrho^{(k),L^k\varepsilon}(\Lambda_0^{(0)}, \dots, \Lambda_0^{(k-1)}, A, \theta_k A^{(k),\varepsilon}, \phi)\\ \cdot\exp\Bigg(\mathscr{P}^{(k),L^k\varepsilon}(\Lambda_7^{(k-1)\prime}, \theta_k A^{(k),\varepsilon}, \phi) - E_0\\ + \sum_{j=0}^{k-1} O((L^j\varepsilon)^{\kappa_0})|\Lambda_7^{(j-1)\prime}\cap\Lambda_7^{(j)c}|\Bigg)\\ \cdot\exp\left(\sum_{j=0}^{k-1} O((L^j\varepsilon)^{\kappa})|T_1^{(j)}|\right).\end{aligned} \tag{2.43}$$

Here the word "admissible" means that the sets $\Lambda_0^{(j)}$ have to satisfy all the conditions resulting from the construction. The sets are unions of big blocks, the set $\Lambda_0^{(0)c}$ is either empty or has at least one point whose distance from $\Lambda_0^{(0)}$ is bigger than $r(\varepsilon)$. In general $\Lambda_0^{(j+1)}\subset\Lambda_7^{(j)\prime}$, and either $\Lambda_0^{(j+1)}$ is a maximal set composed of big blocks and satisfying this inclusion, or the set $\Lambda_0^{(j+1)c}\cap\Lambda_7^{(j)\prime}$ has at least one point whose distance from $\Lambda_0^{(j+1)}$ is bigger than $r(L^{j+1}\varepsilon)$, and so on. Of course the sets $\Lambda_i^{(j)}$ are defined in the same way as Λ_i, $r(\varepsilon)$ is replaced by $r(L^j\varepsilon)$ only. Finally let us notice that $\Lambda_i^{(j)}$ does not mean the prime operation applied j times to a set Λ_i, for different j these are independent sets. We will use the same notations for the sets in different scales. Now we will give a detailed description of the expressions in (2.43). At first we have the following formula for $A^{(k),\varepsilon}$

$$A^{(k),\varepsilon} = a_k(L^k\varepsilon)^{-2}\zeta^{(k)}G_k^{\varepsilon}Q_k^{*}A\,, \tag{2.44}$$

where the function $\zeta^{(k)}(x,y)$ is defined for $x\in T_\eta$, $y\in T_1^{(k)}$, is "smooth" with respect to x in the sense that $|(\partial_x^\eta\zeta^{(k)})(b,y)|\leqq 1$, $\operatorname{supp}\zeta^{(k)}(\cdot,y)$ is contained in the set $\{x\in T_\eta : |x-y| < r(L^k\varepsilon) - 2M\}$ and $\zeta^{(k)}(x,y)=1$ if $|x-y|\leqq\frac{1}{2}r(L^k\varepsilon)$. The function θ_k is defined on T_η, is equal to 1 on $B^{k-1}(\Lambda_2^{(k-1)})$ and varies "smoothly" from 1 to 0 on a slice of thickness $<M$ surrounding $B^{k-1}(\Lambda_2^{(k-1)})$. These functions are rescaled in (2.44). The density $\varrho^{(k),L^k\varepsilon}$ is defined inductively by the formulas generalizing (2.34)–(2.42). At first we define

$$\begin{aligned}
&\varrho''^{(k),L^k\varepsilon}(\Lambda_0^{(0)},\ldots,\Lambda_0^{(k-1)},A,\theta_k A^{(k),\varepsilon},\phi)\\
&=\chi_k T_{a,L}^{L^{k-1}\varepsilon}[T_{a,L,\tilde{A}'}^{L^{k-1}\varepsilon}[\zeta_{\Lambda_0^{(k-1)}}\chi_{\Lambda_{-1}^{(k-1)}\cap\Lambda_5^{(k-1)c}}\chi_{k-1,\Lambda_5^{(k-1)c}}\\
&\quad\cdot\varrho'^{(k-1),L^{k-1}\varepsilon}(\Lambda_0^{(0)},\ldots,\Lambda_0^{(k-2)},A',\tilde{A}',\phi')\\
&\quad\cdot\exp[-\tfrac{1}{2}\langle\Lambda_6^{(k-2)\prime}A',\Delta^{(k-1),L^{k-1}\varepsilon}\Lambda_6^{(k-2)\prime}A'\rangle\\
&\quad-\tfrac{1}{2}\langle(\Lambda_6^{(k-2)\prime}\cap\Lambda_3^{(k-1)c})\phi',\Delta^{(k-1),L^{k-1}\varepsilon}(B^{k-1}(\Lambda_2^{(k-2)\prime}\cap\Lambda_5^{(k-1)c}),\tilde{A}')\\
&\quad\cdot(\Lambda_6^{(k-2)\prime}\cap\Lambda_3^{(k-1)c})\phi'\rangle-\langle(\Lambda_6^{(k-2)\prime}\cap\Lambda_3^{(k-1)c})\phi',\\
&\quad\Delta^{(k-1),L^{k-1}\varepsilon}(B^{k-1}(\Lambda_2^{(k-2)\prime}\cap\Lambda_5^{(k-1)c}),\tilde{A}')\\
&\quad\cdot(\Lambda_3^{(k-1)}\cap\Lambda_4^{(k-1)c})\phi'\rangle\\
&\quad-\tfrac{1}{2}\langle\Lambda_3^{(k-1)}\phi',\Delta^{(k-1),L^{k-1}\varepsilon}(B^{k-1}(\Lambda_2^{(k-1)}),A^{(k),\varepsilon})\Lambda_3^{(k-1)}\phi'\rangle]]], \qquad (2.45)
\end{aligned}$$

where $\tilde{A}=(1-\theta_k)\theta_{k-1}A^{(k-1),\varepsilon}+\theta_k A^{(k),\varepsilon}$, and the characteristic functions χ_k, $\zeta_{\Lambda_0^{(k-1)}}$, etc. are defined analogously to the corresponding functions in Sect. A, with ε replaced by $L^{k-1}\varepsilon$ and Λ_0 by $\Lambda_0^{(k-1)}$. Another representation is obtained by calculation of a conditional integral in (2.45) with the conditioning on $\Lambda_5^{(k-1)c}$:

$$\begin{aligned}
&\varrho''^{(k),L^k\varepsilon}(\Lambda_0^{(0)},\ldots,\Lambda_0^{(k-1)},A,\theta_k A^{(k),\varepsilon},\phi)\\
&=\chi_k\int dA'\restriction_{\Lambda_5^{(k-1)c}}\int d\phi'\restriction_{\Lambda_5^{(k-1)c}}t_{a,L}^{L^{k-1}\varepsilon}(\Lambda_5^{(k-1)c};A,A')t_{a,L,\tilde{A}'}^{L^{k-1}\varepsilon}(\Lambda_5^{(k-1)c};\phi,\phi')\\
&\quad\cdot\zeta_{\Lambda_0^{(k-1)}}\chi_{\Lambda_{-1}^{(k-1)}\cap\Lambda_5^{(k-1)c}}\chi_{k-1,\Lambda_5^{(k-1)c}}\varrho'^{(k-1),L^{k-1}\varepsilon}(\Lambda_0^{(0)},\ldots,\Lambda_0^{(k-2)},A',\tilde{A}',\phi')\\
&\quad\cdot\exp[-\tfrac{1}{2}\langle(\Lambda_6^{(k-2)\prime}\cap\Lambda_5^{(k-1)c})A',\Delta^{(k-1),L^{k-1}\varepsilon}(\Lambda_6^{(k-2)\prime}\cap\Lambda_5^{(k-1)c})A'\rangle\\
&\quad+\tfrac{1}{2}\langle(\Lambda_6^{(k-2)\prime}\cap\Lambda_5^{(k-1)c})A',\Delta^{(k-1),L^{k-1}\varepsilon}C_{\Lambda_5^{(k-1)}}^{(k-1),L^{k-1}\varepsilon}\Delta^{(k-1),L^{k-1}\varepsilon}(\Lambda_6^{(k-2)\prime}\cap\Lambda_5^{(k-1)c})A'\rangle\\
&\quad-a(L^k\varepsilon)^{-2}\langle(\Lambda_6^{(k-2)\prime}\cap\Lambda_5^{(k-1)c})A',\Delta^{(k-1),L^{k-1}\varepsilon}C_{\Lambda_5^{(k-1)}}^{(k-1),L^{k-1}\varepsilon}Q^*A\rangle-\tfrac{1}{2}\langle A,\Delta_{\Lambda_5^{(k-1)}}^{(k),L^k\varepsilon}A\rangle\\
&\quad-\tfrac{1}{2}\langle(\Lambda_6^{(k-2)\prime}\cap\Lambda_3^{(k-1)c})\phi',\Delta^{(k-1),L^{k-1}\varepsilon}(B^{k-1}(\Lambda_2^{(k-2)\prime}\cap\Lambda_5^{(k-1)c}),\tilde{A}')\\
&\quad\cdot(\Lambda_6^{(k-2)\prime}\cap\Lambda_3^{(k-1)c})\phi'\rangle\\
&\quad-\langle(\Lambda_6^{(k-2)\prime}\cap\Lambda_3^{(k-1)c})\phi',\Delta^{(k-1),L^{k-1}\varepsilon}(B^{k-1}(\Lambda_2^{(k-2)\prime}\cap\Lambda_5^{(k-1)c}),\tilde{A}')\\
&\quad\cdot(\Lambda_3^{(k-1)}\cap\Lambda_4^{(k-1)c})\phi'\rangle\\
&\quad-\tfrac{1}{2}\langle(\Lambda_3^{(k-1)}\cap\Lambda_5^{(k-1)c})\phi',\Delta^{(k-1),L^{k-1}\varepsilon}(B^{k-1}(\Lambda_2^{(k-1)}),A^{(k),\varepsilon})\\
&\quad\cdot(\Lambda_3^{(k-1)}\cap\Lambda_5^{(k-1)c})\phi'\rangle\\
&\quad+\tfrac{1}{2}\langle(\Lambda_3^{(k-1)}\cap\Lambda_5^{(k-1)c})\phi',\Delta^{(k-1),L^{k-1}\varepsilon}(B^{k-1}(\Lambda_2^{(k-1)}),A^{(k),\varepsilon})\\
&\quad\cdot C_{\Lambda_5^{(k-1)}}^{(k-1),L^{k-1}\varepsilon}(B^{k-1}(\Lambda_2^{(k-1)}),A^{(k),\varepsilon})\\
&\quad\cdot\Delta^{(k-1),L^{k-1}\varepsilon}(B^{k-1}(\Lambda_2^{(k-1)}),A^{(k),\varepsilon})(\Lambda_3^{(k-1)}\cap\Lambda_5^{(k-1)c})\phi'\rangle\\
&\quad-a(L^k\varepsilon)^{-2}\langle(\Lambda_3^{(k-1)}\cap\Lambda_5^{(k-1)c})\phi',\Delta^{(k-1),L^{k-1}\varepsilon}(B^{k-1}(\Lambda_2^{(k-1)}),A^{(k),\varepsilon})\\
&\quad\cdot C_{\Lambda_5^{(k-1)}}^{(k-1),L^{k-1}\varepsilon}(B^{k-1}(\Lambda_2^{(k-1)}),A^{(k),\varepsilon})Q^*(A^{(k),\varepsilon})\phi\rangle\\
&\quad-\tfrac{1}{2}\langle\phi,\Delta_{\Lambda_5^{(k-1)}}^{(k),L^k\varepsilon}(B^{k-1}(\Lambda_2^{(k-1)}),A^{(k),\varepsilon})\phi\rangle]\\
&\quad\cdot Z_{\Lambda_5^{(k-1)}}^{(k-1),L^{k-1}\varepsilon}Z_{\Lambda_5^{(k-1)}}^{(k-1),L^{k-1}\varepsilon}(B^{k-1}(\Lambda_2^{(k-1)}),A^{(k),\varepsilon}). \qquad (2.46)
\end{aligned}$$

This representation is changed further in a way generalizing (2.36)–(2.40). The aim is to get a representation proportional to the basic Gaussian density after k renormalization transformations, i.e

$$\exp[-\tfrac{1}{2}\langle A, \Delta^{(k), L^k\varepsilon} A\rangle - \tfrac{1}{2}\langle \phi, \Delta^{(k), L^k\varepsilon}(B^{k-1}(\Lambda_2^{(k-1)}), A^{(k),\varepsilon})\phi\rangle],$$

if the fields A, ϕ are restricted to the domain $\Lambda_6^{(k-1)\prime}$. We use Proposition I.2.1 and restrictions on the fields, and we estimate unnecessary terms by $O((L^{k-1}\varepsilon)^\kappa)|\Lambda_6^{(k-1)}|$ with $\kappa > d$. More exactly, we have

$$\begin{aligned}
&\tfrac{1}{2}\langle \phi, \Delta^{(k), L^k\varepsilon}_{\Lambda_5^{(k-1)}}(B^{k-1}(\Lambda_2^{(k-1)}), A^{(k),\varepsilon})\phi\rangle \\
&\quad = \tfrac{1}{2}\langle \Lambda_6^{(k-1)\prime c}\phi, \Delta^{(k), L^k\varepsilon}_{\Lambda_5^{(k-1)}}(B^{k-1}(\Lambda_2^{(k-1)} \cap \Lambda_7^{(k-1)c}), A^{(k),\varepsilon})\Lambda_6^{(k-1)\prime c}\phi\rangle \\
&\quad + \langle \Lambda_6^{(k-1)\prime c}\phi, \Delta^{(k), L^k\varepsilon}_{\Lambda_5^{(k-1)}}(B^{k-1}(\Lambda_2^{(k-1)} \cap \Lambda_7^{(k-1)c}), A^{(k),\varepsilon})(\Lambda_6^{(k-1)\prime} \cap \Lambda_7^{(k-1)\prime c})\phi\rangle \\
&\quad + \tfrac{1}{2}\langle \Lambda_6^{(k-1)\prime}\phi, \Delta^{(k), L^k\varepsilon}(B^{k-1}(\Lambda_2^{(k-1}(\Lambda_2^{(k-1)}), A^{(k),\varepsilon})\Lambda_6^{(k-1)\prime}\phi\rangle \\
&\quad + O((L^{k-1}\varepsilon)^\kappa)|\Lambda_6^{(k-1)}|,
\end{aligned} \tag{2.47}$$

and similarly for the form $\langle A, \Delta^{(k), L^k\varepsilon}_{\Lambda_5^{(k-1)}} A\rangle$, except that Neumann boundary conditions are not introduced. The third and ninth terms in the exponential in (2.46) are linear forms in the fields A, ϕ. We restrict them to $\Lambda_5^{(k-1)\prime} \cap \Lambda_6^{(k-1)\prime c}$ and in the ninth term we restrict the domains of the definitions of the operators by introducing Neumann boundary conditions on the boundary of $B^{k-1}(\Lambda_2^{(k-1)} \cap \Lambda_7^{(k-1)c})$. This is done also for seventh and eighth terms. Terms omitted are described more precisely later, in (2.109) particularly. This way we get a density which we define as $\varrho^{(k), L^k\varepsilon}(\Lambda_0^{(0)}, \ldots, \Lambda_0^{(k-1)}, A, A^{(k),\varepsilon}, \phi)$. It has the following property:

$$\begin{aligned}
&\varrho''^{(k), L^k}(\Lambda_0^{(0)}, \ldots, \Lambda_0^{(k-1)}, A, \theta_k A^{(k),\varepsilon}, \phi) \\
&\quad = \varrho^{(k), L^k\varepsilon}(\Lambda_0^{(0)}, \ldots, \Lambda_0^{(k-1)}, A, \theta_k A^{(k),\varepsilon}, \phi) \\
&\qquad \cdot \exp(O((L^{k-1}\varepsilon)^\kappa)|\Lambda_5^{(k-1)}|),
\end{aligned} \tag{2.48}$$

and if we define a density $\varrho'^{(k), L^k\varepsilon}$ by the equality

$$\begin{aligned}
&\varrho^{(k), L^k\varepsilon}(\Lambda_0^{(0)}, \ldots, \Lambda_0^{(k-1)}, A, \theta_k A^{(k),\varepsilon}, \phi) \\
&\quad = \varrho'^{(k), L^k\varepsilon}(\Lambda_0^{(0)}, \ldots, \Lambda_0^{(k-1)}, A, \theta_k A^{(k),\varepsilon}, \phi) \\
&\qquad \cdot \exp[-\tfrac{1}{2}\langle \Lambda_6^{(k-1)\prime} A, \Delta^{(k), L^k\varepsilon}\Lambda_6^{(k-1)\prime} A\rangle \\
&\qquad - \tfrac{1}{2}\langle \Lambda_6^{(k-1)\prime}\phi, \Delta^{(k), L^k\varepsilon}(B^k(\Lambda_2^{(k-1)\prime}), A^{(k),\varepsilon})\Lambda_6^{(k-1)\prime}\phi\rangle],
\end{aligned} \tag{2.49}$$

then we have

$$\begin{aligned}
&\varrho'^{(k), L^k\varepsilon}(\Lambda_0^{(0)}, \ldots, \Lambda_0^{(k-1)}, A, \theta_k A^{(k),\varepsilon}, \phi) \\
&\quad = \varrho'^{(k), L^k\varepsilon}(\Lambda_0^{(0)}, \ldots, \Lambda_0^{(k-1)}, \Lambda_7^{(k-1)\prime c} A, \theta_k A^{(k),\varepsilon}, \Lambda_7^{(k-1)\prime c}\phi).
\end{aligned} \tag{2.50}$$

Here we treat $\theta_k A^{(k)}$ as an independent field configuration appearing only as an external vector field in the expressions with scalar fields. Further, $\varrho^{(k), L^k\varepsilon}$ is defined in such a way that $\varrho'^{(k), L^k\varepsilon}$ depends on the configuration $B^k(\Lambda_7^{(k-1)\prime})A^{(k),\varepsilon}$ through the factors $Z^{(j), L^j\varepsilon}_{\Lambda_5^{(j)}}(B^j(\Lambda_2^{(j)}), A^\varepsilon)$ only.

The inductive definitions (2.46)–(2.50) allow us to write an explicit formula for

$$\varrho^{(k), L^k\varepsilon}(\Lambda_0^{(0)}, \ldots, \Lambda_0^{(k-1)}, A, \theta_k A^{(k),\varepsilon}, \phi).$$

We do not do it because we will not need this formula, but let us write a formula for a whole external vector field in its scalar field part:

$$\tilde{A}^{\varepsilon} = (1-\theta_1)A_0 + \sum_{j=1}^{k-1} (1-\theta_{j+1})\theta_j A^{(j),\varepsilon} + \theta_k A^{(k),\varepsilon}, \tag{2.51}$$

where $A^{(j),\varepsilon}$ is defined by the formula (2.44) with j and A_j instead of k and A. Obviously we have $B^{k-2}(\Lambda_2^{(k-2)})\tilde{A}^{\varepsilon} = \tilde{A}'$ [we write here $A_{k-1} = A'$, $\phi_{k-1} = \phi'$ in (2.45), (2.46)]. Also let us write a formula for the $Z^{(j)}$-factors. In the previous paper they were composed into a factor Z_k, but now it is inconvenient because they depend on different sets $\Lambda_5^{(j)}$. We have

$$\begin{aligned} &Z^{(j), L^j\varepsilon}_{\Lambda_5^{(j)}}(B^j(\Lambda_2^{(j)}), \tilde{A}^{\varepsilon}) \\ &= \left(\frac{a(L^{j+1}\varepsilon)^{d-2}}{2\pi}\right)^{\frac{N}{2}|\Lambda_5^{(j)}|} \int d\phi \restriction_{\Lambda_5^{(j)}} \exp(-\tfrac{1}{2}\langle \phi, (C^{(j), L^j\varepsilon}_{\Lambda_5^{(j)}}(B^j(\Lambda_2^{(j)}), \tilde{A}^{\varepsilon}))^{-1}\phi\rangle) \end{aligned} \tag{2.52}$$

for $j = 0, 1, \ldots, k-1$. In each step we expand these factors with respect to a new small field, hence the configuration $\tilde{A}^{\varepsilon}$ is changed after each step.

The expression $\mathscr{P}^{(k), L^k\varepsilon}(\Lambda_7^{(k-1)\prime}, \theta_k A^{(k)\varepsilon}, \phi)$ is the same as the corresponding expression in Chap. I.3, except that there are different propagators and summations in vertices are restricted to the set $B^{k-1}(\Lambda_7^{(k-1)})$ with the help of the characteristic function, or some function g_k. A detailed description of $\mathscr{P}^{(k), L^k\varepsilon}(\Lambda_7^{(k-1)\prime})$ will be given in a paper on renormalization of perturbation expansions.

C. A Renormalization Transformation in a General Case ($k+1$ *Step*)

We apply the renormalization transformation $T^{L^k\varepsilon}_{a,L}[T^{L^k_\varepsilon}_{a,L,\theta_k A^{(k)},\varepsilon}[\cdot]]$ to the expression under the integral on the right side of (2.43). We get an expression dependent on the new fields B, ψ and its integral over these fields is $\geqq Z^{\varepsilon}$.

Now we will estimate this expression from above. At first we rescale it from the $L^k\varepsilon$-lattice $T^{(k)}_{L^k\varepsilon}$ to 1-lattice $T_1^{(k)}$. Omitting the numerical factors, we get the integral

$$\begin{aligned} &\sum_{\Lambda_0^{(0)}, \ldots, \Lambda_0^{(k-1)}} \int dA \int d\phi \exp\Big[-\tfrac{1}{2}aL^{d-2} \sum_{y \in T^{(k)\prime}} |B(y) - (QA)(y)|^2 \\ &\quad -\tfrac{1}{2}aL^{d-2} \sum_{y\in T^{(k)\prime}} |\psi(y) - (Q(\theta_k A^{(k)})\phi)(y)|^2\Big] \\ &\quad \cdot \varrho'^{(k)}(\Lambda_0^{(0)}, \ldots, \Lambda_0^{(k-1)}, \Lambda_7^{(k-1)\prime c}A, \theta_k A^{(k)}, \Lambda_7^{(k-1)\prime c}\phi) \\ &\quad \cdot \exp[-\tfrac{1}{2}\langle \Lambda_6^{(k-1)\prime}A, \Delta^{(k)}\Lambda_6^{(k-1)\prime}A\rangle \\ &\quad -\tfrac{1}{2}\langle \Lambda_6^{(k-1)\prime}\phi, \Delta^{(k)}(B^k(\Lambda_2^{(k-1)\prime}), A^{(k)})\Lambda_6^{(k-1)\prime}\phi\rangle] \\ &\quad \cdot \exp \mathscr{P}^{(k)}(\Lambda_7^{(k-1)\prime}, \theta_k A^{(k)}, \phi). \end{aligned} \tag{2.53}$$

Let us recall the formula for $A^{(k)}$ after the rescaling

$$A^{(k)} = a_k \zeta^{(k)} G_k Q_k^* A, \tag{2.54}$$

and the restrictions on the fields A, ϕ given by the characteristic functions χ_k:

$$
\begin{gathered}
|(\partial A)(b)| \leqq c_1 p(L^{k-1}\varepsilon), \qquad |A(x)| \leqq \frac{c_1}{\mu_0 L^{k-1}\varepsilon} p(L^{k-1}\varepsilon), \\
|(D_{\bar{A}^{(k)}}\phi)(b)| \leqq c_1 p(L^{k-1}\varepsilon), \\
|\phi(x)| \leqq \frac{c_1}{\lambda(L^{k-1}\varepsilon)^{1/4}} p(L^{k-1}\varepsilon) \quad \text{for} \quad x \in \Lambda_{-1}^{(k-1)\prime}, \\
b \subset \Lambda_{-1}^{(k-1)\prime}, \qquad \bar{A}_b^{(k)} = L^{-k} \sum_{\langle x, x'\rangle \subset b} A_{\langle x, x'\rangle}^{(k)}.
\end{gathered}
\tag{2.55}
$$

Now we introduce the new restrictions on the fields B, ψ, A, ϕ considered on the set $\Lambda_7^{(k-1)\prime}$. The restrictions are identical to these considered in the first step, Sect. A, and are given by (2.4)–(2.6), with the replacements of ε by $L^k\varepsilon$, T_1 by $\Lambda_7^{(k-1)\prime}$ and the field A by $\bar{A}^{(k)}$ in the expressions for scalar fields. We will not rewrite these definitions, except the definition of $\Lambda_0^{(k)}$: $\Lambda_0^{(k)}$ is the sum of large blocks of the lattice $T_1^{(k)}$ contained in $\Lambda_7^{(k-1)\prime}$, distant from the set

$$
B(P_v^{(k)}) \cup Q_v^{(k)} \cup R_v^{(k)} \cup B(P_s^{(k)}) \cup Q_s^{(k)} \cup R_s^{(k)}
$$

more than $r(L^k\varepsilon)$. Replacing Λ_i by $\Lambda_i^{(k)}$, we have the identities (2.9), (2.10) also. We introduce (2.10) under the integral (2.53) and we make a first estimate of it, namely we remove the interaction from the set $B^k(\Lambda_7^{(k-1)\prime} \cap \Lambda_7^{(k)c})$ leaving the expression

$$
-\lambda(Lj\varepsilon) \sum_{x \in B^k(\Lambda_7^{(k-1)\prime} \cap \Lambda_7^{(k)c})} \eta^d |\phi^{(k)}(x)|^4
$$

only, $\eta = L^{-k}$, where of course

$$
\phi^{(k)} = a_k G_k(B^k(\Lambda_2^{(k-1)\prime}), A^{(k)}) Q_k^*(A^{(k)}) \Lambda_6^{(k-1)\prime}\phi\,. \tag{2.56}
$$

Such a possibility is assured by the following theorem.

Proposition 2.1. *Under the conditions* (2.55), *we have*

$$
\begin{aligned}
&\mathscr{P}^{(k)}(\Lambda_7^{(k-1)\prime}, \theta_k A^{(k)}, \phi) \\
&\quad = -\lambda(L^k\varepsilon) \sum_{x \in B^k(\Lambda_7^{(k-1)\prime} \cap \Lambda_7^{(k)c})} \eta^d |\phi^{(k)}(x)|^4 \\
&\qquad + \mathscr{P}^{(k)}(\Lambda_7^{(k)}, \theta_k A^{(k)}, \phi) + O((L^k\varepsilon)^{\kappa_0}) |\Lambda_7^{(k-1)\prime} \cap \Lambda_7^{(k)c}|\,.
\end{aligned}
\tag{2.57}
$$

Similar conclusions hold for other expressions which will be included in the action in the later stages of the procedure.

This theorem is a corollary of the analysis of the perturbation expansions.

Now we want to use the first term on the right side of (2.57) to produce small "convergence factors" as in (2.13). To do it we will prove the lemmas which were used in Sect. A already. They will be based on the following

Proposition 2.2. *Let Ω and A satisfy the assumptions of Proposition I.2.1, then for $e(L^k\varepsilon)$ sufficiently small there exist positive constants δ_0, c_0, R_0 independent of A, k, Ω and depending on d, a, M, such that*

$$
|(D_A^\eta G_k(\Omega, A) Q_k^*(A))(b, y)| \leqq c_0 \exp(-\delta_0 \operatorname{dist}(b, y))\,, \tag{2.58}
$$

for $b \subset \Omega$, $\operatorname{dist}(b, \Omega^c) \geqq R_0$, $y \in \Omega^{(k)}$. *The identical inequality holds for* $G_k(\Omega, A)Q_k^*(A)$, *and for* $D_A^\eta \delta G_k(\Omega, \Omega_0, A)Q_k^*(A)$, $\delta G_k(\Omega, \Omega_0, A)Q_k^*(A)$ *with the additional factor* $\exp(-\delta_0(\operatorname{dist}(b, \Omega^c) + \operatorname{dist}(y, \Omega^c)))$.

This proposition is a simple corollary of Proposition I.2.1. At first we will apply it to prove the necessary properties of $A^{(k)}$; especially we would like to prove that this configuration satisfies the assumption made on vector field configurations in Propositions I.2.1. It will follow from

Lemma 2.3. *Under the restrictions* (2.55), *we have*

$$A^{(k)}(x) = A(y) + O(p(L^k\varepsilon)) = (Q_k^* A)(x) + O(p(L^k\varepsilon)), \quad x \in B^k(y), \quad y \in \Lambda_2^{(k-1)'}, \tag{2.59}$$

$$(\partial_\mu^\eta A^{(k)})(x) = O(p(L^k\varepsilon)), \quad x \in B^k(\Lambda_2^{(k-1)'}). \tag{2.60}$$

Let us notice that the conclusions of Proposition 2.2 hold for $G_k Q_k^*$. We have

$$\begin{aligned} A^{(k)}(x) &= a_k(\zeta^{(k)} G_k Q_k^*(A - A(y)))(x) \\ &\quad - a_k((1-\zeta^{(k)}) G_k Q_k^* 1)(x) A(y) + a_k(G_k Q_k^* 1)(x) A(y), \\ &\quad x \in B^k(y), \quad y \in \Lambda_2^{(k-1)'}. \end{aligned} \tag{2.61}$$

Using Proposition 2.2 and the restrictions (2.55) we can estimate the first two terms in (2.61) by $O(1)p(L^k\varepsilon)$. The third term can be calculated in the following way

$$\begin{aligned} G_k Q_k^* 1 &= a_k^{-1} G_k a_k P_k Q_k^* 1 = a_k^{-1} G_k(-\Delta^\eta + \mu_0^2 (L^k\varepsilon)^2 + a_k P_k) 1 \\ &\quad - \frac{\mu_0^2 (L^k\varepsilon)^2}{a_k} G_k Q_k^* 1 . \end{aligned} \tag{2.62}$$

Hence

$$(a_k G_k Q_k^* 1)(x) = 1 - \frac{\mu_0^2 (L^k\varepsilon)^2}{a_k + \mu_0^2 (L^k\varepsilon)^2}, \tag{2.63}$$

and, again using (2.55), we have (2.59). Furthermore, because $G_k Q_k^* 1$ is a constant, from (2.61) we have

$$\begin{aligned} (\partial_\mu^\eta A^{(k)})(x) &= a_k(\partial_\mu^\eta \zeta^{(k)} G_k Q_k^*(A - A(y)))(x) \\ &\quad - a_k(\partial_\mu^\eta (1-\zeta^{(k)}) G_k Q_k^* 1)(x) A(y), \quad x \in B^k(y), \quad y \in \Lambda_2^{(k-1)'}, \end{aligned} \tag{2.64}$$

and the restrictions (2.55), Proposition 2.2 and the properties of the function $\zeta^{(k)}$ imply (2.60). Thus Lemma 2.3 is proved.

The inequality (2.60) implies that the configuration $A^{(k)}$ considered on the set $B^k(\Lambda_2^{(k-1)'})$ satisfies the assumption of Proposition I.2.1. on a vector field configuration. The inequalities of Lemma 2.3 will be applied also when a scale is changed.

Now we can prove the corresponding inequalities for the scalar field configurations.

Lemma 2.4. *Under the restrictions* (2.55) *we have*

$$\begin{aligned}\phi^{(k)}(x) &= U(A^{(k)}(\Gamma^{(k)}_{x,y}))\phi(y) + O(p(L^k\varepsilon)) \\ &= (Q^*_k(A^{(k)})\phi)(x) + O(p(L^k\varepsilon)), \quad \textit{for} \quad x \in B^k(y), \quad y \in \Lambda_7^{(k-1)\prime}, \end{aligned} \tag{2.65}$$

$$(D^\eta_{A^{(k)}}\phi^{(k)})(b) = O(p(L^k\varepsilon)) \quad \textit{for} \quad b \subset B^k(\Lambda_7^{(k-1)\prime}). \tag{2.66}$$

Let us define $\square_1$, $\square_2$ as the sums of large blocks contained in $\Lambda_7^{(k-1)\prime}$ and distant from the point y less than $2r(L^k\varepsilon)$, $4r(L^k\varepsilon)$ respectively, and let us denote $\square = B^k(\square_2)$. Of course $\square \subset B^k(\Lambda_2^{(k-1)\prime})$. Using Proposition 2.2 and the restrictions (2.55) we get

$$\phi^{(k)}(x) = (a_k G_k(\square, A^{(k)}) Q^*_k(A^{(k)}) \square_1 \phi)(x) + O((L^k\varepsilon)^\kappa), \qquad x \in B^k(y), \tag{2.67}$$

and the same equality for the covariant derivative of $\phi^{(k)}$. From the property (2.60) we have the inequality

$$|A^{(k)}(x) - A^{(k)}(y)| \leqq O(p(L^k\varepsilon) r(L^k\varepsilon)).$$

Let us denote by A_0 a constant configuration equal to $A^{(k)}(y)$ at each point, thus $A^{(k)} - A_0 = O(p(L^k\varepsilon) r(L^k\varepsilon))$. Using the expansion formula (I.3.44) and Proposition I.2.2. we have

$$\begin{aligned}&(a_k G_k(\square, A^{(k)}) Q^*_k(A^{(k)}) \square_1 \phi)(x) \\ &\quad = (a_k G_k(\square, A_0) Q^*_k(A_0) \square_1 \phi)(x) + (a_k G_k(\square, A_0) F_{2,k}(A^{(k)} - A_0, A_0) \square_1 \phi)(x) \\ &\qquad + (a_k G_k(\square, A_0) V_k(A^{(k)} - A_0, A_0) G_k(\square, A^{(k)}) Q^*_k(A^{(k)}) \square_1 \phi)(x) \\ &\quad = (a_k G_k(\square, A_0) Q^*_k(A_0) \square_1 \phi)(x) + O((L^k\varepsilon)^{\kappa_0}), \qquad \kappa_0 > 0, \end{aligned} \tag{2.68}$$

and similarly for the derivative. Now the constant field A_0 can be "gauged out" from the last expression above. We use a gauge transformation defined on $\square$ by the formula $\phi_0(x) = U(A_0(\Gamma_{x,y}))\phi'_0(x)$, $x \in \square$, where the contour $\Gamma_{x,y} = -\Gamma_{y,x}$ is defined as in (I.2.1.), but now for the points x from $\square$ instead of the block $B(y)$. Let us consider how the operator $+\Delta^{\eta,N}_{A_0,\square} + m^2(L^k\varepsilon)^2 + a_k P_k(A_0)\square$ transforms itself under this gauge transformation. We consider the terms determining the corresponding quadratic form:

$$\begin{aligned}(D^\eta_{A_0}\phi_0)(b) &= \eta^{-1}(U(A_{0,b}) U(A_0(\Gamma_{b_+,y}))\phi'_0(b_+) - U(A_0(\Gamma_{b_-,y}))\phi'_0(b_-) \\ &= U(A_0(\Gamma_{b_-,y}))\eta^{-1}(U(A_0(\Gamma_{y,b_-} \cup b \cup \Gamma_{b_+,y}))\phi'_0(b_+) - \phi'_0(b_-)), \end{aligned} \tag{2.69}$$

but the contour $\Gamma_{y,b_-} \cup b \cup \Gamma_{b_+,y}$ is closed and bounds some surface $\sum \subset \square$, so using Stokes' theorem we have

$$A_0(\Gamma_{y,b_-} \cup b \cup \Gamma_{b_+,y}) = A_0(\partial \textstyle\sum) = \partial^\eta A_0(\textstyle\sum) = 0,$$

hence

$$\begin{aligned}(D^\eta_{A_0}\phi_0)(b) &= U(A_0(\Gamma_{b_-,y}))(\partial^\eta \phi'_0)(b), \quad \text{and} \\ |(D^\eta_{A_0}\phi_0)(b)|^2 &= |(\partial^\eta \phi'_0)(b)|^2 \quad \text{for} \quad b \subset \square. \end{aligned} \tag{2.70}$$

Furthermore for $y' \in \square_2$,

$$\begin{aligned}(Q_k(A_0)\phi_0)(y') &= \sum_{x' \in B^k(y')} \eta^d U(A_0(\Gamma^{(k)}_{y',x'}))\, U(A_0(\Gamma_{x',y}))\phi'_0(x') \\ &= U(A_0(\Gamma_{y',y})) \sum_{x' \in B^k(y')} \eta^d U(A_0(\Gamma_{y,y'} \cup \Gamma^{(k)}_{y',x'} \cup \Gamma_{x',y}))\phi'_0(x'), \end{aligned} \tag{2.71}$$

and because $A_0(\Gamma_{y,y'} \cup \Gamma^{(k)}_{y',x'} \cup \Gamma_{x',y}) = 0$ again, so we have

$$\begin{aligned}&(Q_k(A_0)\phi_0)(y') = U(A_0(\Gamma_{y',y}))\,(Q_k\phi'_0)(y') \quad \text{and} \\ &|(Q_k(A_0)\phi_0)(y')|^2 = |(Q_k\phi'_0)(y')|^2 . \end{aligned} \tag{2.72}$$

Hence the operator $+\Delta^{\eta,N}_{A_0,\square} + m^2(L^k\varepsilon)^2 + a_k P_k(A_0)\square$ is transformed into the operator $-\Delta^{\eta,N}_{0,\square} + m^2(L^k\varepsilon)^2 + a_k P_k\square$, and we have

$$G_k(\square, A_0; x, x') = U(A_0(\Gamma_{x,y}))G_k(\square, 0; x, x')\,U(A_0(\Gamma_{y,x'})) . \tag{2.73}$$

Together with the gauge transformation of the propagators we make the corresponding transformation of the field ϕ, i.e. $\phi(y') = U(A_0(\Gamma_{y',y}))\phi'(y')$. Then the last expression in (2.68) transforms itself as follows

$$(a_k G_k(\square, A_0)Q^*_k(A_0)\square_1\phi)(x) = U(A_0(\Gamma_{x,y}))\,(a_k G_k(\square,0)Q^*_k\square_1\phi')(x) . \tag{2.74}$$

Now let us consider the restrictions (2.55) on the field ϕ. The estimates of the covariant derivatives give us

$$|U(A_0(\langle y', y''\rangle))\phi(y'') - \phi(y')| \leqq O(1)p(L^k\varepsilon) .$$

After the gauge transformation we finally get

$$\begin{aligned}&|U(A_0(\langle y', y''\rangle))\,U(A_0(\Gamma_{y'',y}))\phi'(y'') - U(A_0(\Gamma_{y',y}))\phi'(y')| \\ &= |\phi'(y'') - \phi'(y')| \leqq O(1)p(L^k\varepsilon) .\end{aligned}$$

Now we can apply the same reasoning to the configuration $a_k G_k(\square, 0)Q^*_k\square_1\phi'$ as to $A^{(k)}$ in the proof of Lemma 2.3, especially we have

$$\begin{aligned}a_k G_k(\square, 0)Q^*_k 1 &= G_k(\square, 0)\,(-\Delta^{\eta,N}_{0,\square} + m^2(L^k\varepsilon) + a_k P_k\square)1 \\ &\quad - m^2(L^k\varepsilon)^2 G_k(\square, 0)Q^*_k 1 = 1 - m^2(L^k\varepsilon)^2 G_k(\square, 0)Q^*_k 1 , \end{aligned} \tag{2.75}$$

so the same conclusion holds and we get

$$(a_k G_k(\square, 0)Q^*_k\square_1\phi')(x) = \phi'(y) + O(p(L^k\varepsilon)), \qquad x \in B^k(y) . \tag{2.76}$$

Combining the equalities (2.67), (2.68), (2.74), (2.76) and taking into account the equalities $\phi'(y) = \phi(y)$,

$$U(A_0(\Gamma_{x,y}))\phi(y) = U(A^{(k)}(\Gamma^{(k)}_{x,y}))\,\phi(y) + O((L^k\varepsilon)^{\kappa_0})$$

we finally get (2.65). It was mentioned several times that the corresponding equalities hold for the covariant derivatives of $\phi^{(k)}$ and we have

$$\begin{aligned}(D^\eta_{A^{(k)}}\phi^{(k)})(b) &= U(A_0(\Gamma_{b_-,y}))\,(a_k\partial^\eta G_k(\square, 0)Q^*_k\square_1\phi')(b) \\ &\quad + O((L^k\varepsilon)^{\kappa_0}), \qquad b \subset \square . \end{aligned} \tag{2.77}$$

Using again the similar considerations as in the proof of Lemma 2.3 we get (2.66). This ends the proof of Lemma 2.4.

Now we can use (2.65) to estimate the first term on the right side of (2.57):

$$-\lambda(L^k\varepsilon)\sum_{x\in B^k(\Lambda_7^{(k-1)'}\cap\Lambda_7^{(k)c})}\eta^d|\phi^{(k)}(x)|^4=-\lambda(L^k\varepsilon)\sum_{y\in\Lambda_7^{(k-1)'}\cap\Lambda_7^{(k)c}}|\phi(y)|^4$$
$$+O((L^k\varepsilon)^{\kappa_0})|\Lambda_7^{(k-1)'}\cap\Lambda_7^{(k)c}|. \tag{2.78}$$

Of course there holds

$$\chi^c_{R_s^{(k)}}\exp\left(-\lambda(L^k\varepsilon)\sum_{y\in\Lambda_7^{(k-1)'}\cap\Lambda_7^{(k)c}}|\phi(y)|^4\right)\leqq\exp(-p(L^k\varepsilon)^4|R_s^{(k)}|). \tag{2.79}$$

Let us introduce the functions $\zeta_{\Lambda_0^{(k)}}$ by the formula (2.15) with the obvious modifications.

The next operation is a translation in the vector fields and an expansion of the action with respect to a proper small field. We make the translation

$$A=A'+aL^{-2}C^{(k)}_{\Lambda_0^{(k)}}Q^*B. \tag{2.80}$$

We would like to show that the field A' is small on $\Lambda_1^{(k)}$. The restrictions on the fields A, ϕ introduced by the characteristic functions $\chi_{\Lambda_{-1}^{(k)}}$ imply the corresponding restrictions (2.16), (2.17), with ε replaced by $L^k\varepsilon$, on the fields B, ψ. For the second term on the right side of (2.80) we have

Lemma 2.5.

$$aL^{-2}(C^{(k)}_{\Lambda_0^{(k)}}Q^*B)(x)=B(y)+O(p(L^k\varepsilon))$$
$$=(Q^*B)(x)+O(p(L^k\varepsilon)),\qquad x\in B(y),\qquad y\in\Lambda_1^{(k)'}. \tag{2.81}$$

To prove it let us notice that

$$\begin{aligned}&aL^{-2}(C^{(k)}_{\Lambda_0^{(k)}}Q^*B)(x)\\&=aL^{-2}(C^{(k)}Q^*1)(x)B(y)-aL^{-2}(C^{(k)}Q^*\Lambda_0^{(k)'c})(x)B(y)\\&\quad+aL^{-2}(C^{(k)}Q^*\Lambda_0^{(k)'}(B-B(y)))(x)+aL^{-2}(\delta C^{(k)}_{\Lambda_0^{(k)}}Q^*B)(x)\\&=aL^{-2}(C^{(k)}Q^*1)(x)B(y)+O(p(L^k\varepsilon)),\end{aligned} \tag{2.82}$$

where Proposition I.2.3 and the restrictions on the field B were used. We have to calculate $aL^{-2}C^{(k)}Q^*1=aL^{-2}C^{(k)}1$, where the two units are in different scales. We proceed the same way as in (2.62):

$$aL^{-2}C^{(k)}1=C^{(k)}aL^{-2}P1=C^{(k)}(\Delta^{(k)}+aL^{-2}P)1$$
$$-C^{(k)}\Delta^{(k)}1=1-C^{(k)}\Delta^{(k)}1. \tag{2.83}$$

We have to calculate $\Delta^{(k)}1$. From the equalities (I.2.21) and (2.63) we get

$$\Delta^{(k)}1=a_k1-a_k^2Q_kG_kQ_k^*1=a_k1-a_k\left(1-\frac{\mu_0^2(L^k\varepsilon)^2}{a_k+\mu_0^2(L^k\varepsilon)^2}\right)$$
$$=\frac{a_k\mu_0^2(L^k\varepsilon)^2}{a_k+\mu_0^2(L^k\varepsilon)^2}, \tag{2.84}$$

and hence

$$aL^{-2}C^{(k)}1=\left(1+\frac{a^{-1}L^2a_k\mu_0^2(L^k\varepsilon)^2}{a_k+\mu_0^2(L^k\varepsilon)^2}\right)^{-1}=1+O(\mu_0^2(L^k\varepsilon)^2). \tag{2.85}$$

This together with (2.82) proves (2.81).

From the above lemma and the restrictions on the fields B, A it follows that the field A' is small on $\Lambda_1^{(k)}$:

$$\begin{aligned}|A'(x)|&=|A(x)-aL^{-2}(C^{(k)}_{\Lambda_0^{(k)}}Q^*B)(x)|\\&\leqq|A(x)-(Q^*B)(x)|+O(1)p(L^k\varepsilon)\leqq O(1)p(L^k\varepsilon).\end{aligned} \tag{2.86}$$

Now let us investigate a result of the translation (2.80) in the integral (2.53). For the quadratic form in the fields B, A standing in the exponential function under the integral, we have

$$\begin{aligned}&\tfrac{1}{2}aL^{d-2}\sum_{y\in T_1^{(k)\prime}}|B(y)-(QA)(y)|^2+\tfrac{1}{2}\langle\Lambda_6^{(k-1)\prime}A,\Delta^{(k)}\Lambda_6^{(k-1)\prime}A\rangle\\&=\tfrac{1}{2}aL^{d-2}\sum_{y\in T_1^{(k)\prime}}|(\Lambda_0^{(k)\prime c}B)(y)-(QA')(y)|^2\\&\quad+\tfrac{1}{2}\langle\Lambda_6^{(k-1)\prime}A',\Delta^{(k)}\Lambda_6^{(k-1)\prime}A'\rangle\\&\quad+aL^{-2}\langle(\Lambda_6^{(k-1)\prime}\cap\Lambda_0^{(k)c})A',\Delta^{(k)}C^{(k)}_{\Lambda_0^{(k)}}Q^*B\rangle\\&\quad+\tfrac{1}{2}\langle B,\Delta^{(k+1),L}_{\Lambda_0^{(k)\prime}}B\rangle.\end{aligned} \tag{2.87}$$

It is easily seen that the quadratic form in the field A' restricted e.g. to the set $\Lambda_5^{(k)}$ has the form $\frac{1}{2}\langle\Lambda_5^{(k)}A',(C^{(k)}_{\Lambda_5^{(k)}})^{-1}\Lambda_5^{(k)}A'\rangle$. The remaining terms of the action depend on the field $\Lambda_0^{(k)}A$ only through the field $A^{(k)}$. The result of the translation (2.80) on the configuration $A^{(k)}$ will be represented in different ways depending on the sets on which the configuration is considered. Thus we have $A^{(k)}=(1-\theta_{k+1})A^{(k)}+\theta_{k+1}A^{(k)}$ and

$$(1-\theta_{k+1})\theta_kA^{(k)}=a_k(1-\theta_{k+1})\theta_k\zeta^{(k)}G_kQ_k^*(A'+aL^{-2}C^{(k)}_{\Lambda_0^{(k)}}Q^*B)), \tag{2.88}$$

$$\begin{aligned}\theta_{k+1}A^{(k)}&=a_k\theta_{k+1}\zeta^{(k)}G_kQ_k^*A'\\&\quad+a_kaL^{-2}\theta_{k+1}\zeta^{(k)}G_kQ_k^*\delta C^{(k)}_{\Lambda_0^{(k)}}Q^*B\\&\quad-a_kaL^{-2}\theta_{k+1}(1-\zeta^{(k)})G_kQ_k^*C^{(k)}Q^*\Lambda_0^{(k)\prime}B\\&\quad+a_{k+1}L^{-2}\theta_{k+1}(1-\zeta^{(k+1)})G^\eta_{k+1}Q^*_{k+1}\Lambda_0^{(k)\prime}B\\&\quad+a_{k+1}L^{-2}\theta_{k+1}\zeta^{(k+1)}G^\eta_{k+1}Q^*_{k+1}B\\&=:\theta_{k+1}A'^{(k)}+\tilde{B}^{(k)}+\theta_{k+1}B^{(k+1),\eta},\end{aligned} \tag{2.89}$$

where by the definition $\tilde{B}^{(k)}$ is the sum of the second, third and fourth terms on the right side of (2.89). From the restrictions introduced by the functions $\zeta^{(k)}$, $\zeta^{(k+1)}$, the restrictions on the field B and the Propositions I.2.1–I.2.3, 2.2 we get

$$\tilde{B}^{(k)}(x)=O((L^k\varepsilon)^\kappa)\quad\text{for every }\kappa, \tag{2.90}$$

and similar estimates for the derivative. From (2.86) it follows that the field $\theta_{k+1}A'^{(k)}$ is smooth and small:

$$|\theta_{k+1}A'^{(k)}(x)|\leqq O(1)p(L^k\varepsilon), \tag{2.91}$$

similarly for the derivative.

Now we will expand the expression under the integral (2.53) with respect to the small field $A'^{(k)}+\tilde{B}^{(k)}$. Let us stress here once more that in the density $\varrho'^{(k)}$ only the factors $Z^{(j)}$ depend on this field, so we will consider the expansions of these factors. They are given by the formulas (2.52) rescaled from the $L^j\varepsilon$-lattice to the $L^j\eta$-lattice, thus the field $\tilde{A}^{(k)}$ is defined on the η-lattice, $\eta=L^{-k}$. To understand better the properties of the expression in (2.52), let us rescale further from the $L^j\eta$-lattice to the 1-lattice and let us write it as a determinant. Omitting a numerical factor we have

$$[\det(C^{(j)}_{\Lambda_5^{(j)}}(B^j(\Lambda_2^{(j)}),\tilde{A}^{L^{-j}}))^{-1}]^{-1/2}, \tag{2.92}$$

where $\Lambda_i^{(j)}$ are subsets of the 1-lattice $T_1^{(j)}$ and the configuration $\tilde{A}^{(k)L^{-j}}$ is defined on $T_{L^{-j}}$. The expression (2.92) depends on the configuration $B^j(\Lambda_2^{(j)})\tilde{A}^{(k),L^{-j}}$. This configuration, after the translation (2.80) and using the formulas (2.51), (2.88), (2.89), is given by

$$\begin{aligned} &B^j(\Lambda_2^{(j)})\tilde{A}^{(k),L^{-j}} \\ &\quad= B^j(\Lambda_2^{(j)})(1-\theta_{j+2})\theta_{j+1}A^{(j+1),L^{-j}}+\sum_{l=j+2}^{k-1}(1-\theta_{l+1})\theta_l A^{(l),L^{-j}} \\ &\qquad+(1-\theta_{k+1})\theta_k A^{(k),L^{-j}}+\theta_{k+1}B^{(k+1),L^{-j}}+(\theta_{k+1}A'^{(k),L^{-j}}+\tilde{B}^{(k),L^{-j}}). \end{aligned} \tag{2.93}$$

We will need to apply Propositions I.2.1–I.2.3 to operators with external vector field equal to (2.93), so we have to verify the assumption of regularity (I.2.23) for this field. From (2.90) and (2.91) it follows that

$$\begin{aligned} &|(\partial_\mu^{L^{-j}}(\theta_{k+1}A'^{(k),L^{-j}}+\tilde{B}^{(k),L^{-j}}))(x)| \\ &\quad\leqq O(1)(L^j\eta)^{d/2}p(L^k\varepsilon)\leqq O(1)p(L^j\varepsilon), \end{aligned} \tag{2.94}$$

so the field $\tilde{A}:=\theta_{k+1}A'^{(k),L^{-j}}+\tilde{B}^{(k),L^{-j}}$ satisfies this assumption. Let us consider the remaining part of the expression (2.93) and let us denote it by $\tilde{B}$. The derivative $(\partial_\mu^{L^{-j}}\tilde{B})(x)$ of this configuration is equal to one of the derivatives $(\partial_\mu^{L^{-j}}A^{(l),L^{-j}})(x)$ or to $(\partial_\mu^{L^{-j}}B^{(k+1),L^{-j}})(x)$ if the point x does not lie in a slice of thickness M surrounding one of the sets $B^l(\Lambda_2^{(l)})$. For examples, if x belongs to $B^l(\Lambda_2^{(l-1)\prime}\cap\Lambda_2^{(l)c})$ with the exception of the slice, then Lemma 2.3 implies

$$\begin{aligned} |(\partial_\mu^{L^{-j}}\tilde{B})(x)| &= |(\partial_\mu^{L^{-j}}A^{(l),L^{-j}})(x)| \\ &\leqq O(1)(L^{j-l})^{d/2}p(L^l\varepsilon)\leqq O(1)p(L^j\varepsilon). \end{aligned} \tag{2.95}$$

If x belongs to this slice, then the above inequality implies

$$\begin{aligned} &(\partial_\mu^{L^{-j}}\tilde{B})(x) \\ &\quad=(\partial_\mu^{L^{-j}}(1-\theta_{l+1})A^{(l),L^{-j}})(x)+(\partial_\mu^{L^{-j}}\theta_{l+1}A^{(l+1),L^{-j}})(x) \\ &\quad=(\partial_\mu^{L^{-j}}\theta_{l+1})(x)(A^{(l+1),L^{-j}}(x)-A^{(l),L^{-j}}(x))+O(p(L^j\varepsilon)), \end{aligned} \tag{2.96}$$

and applying Lemma 2.3, we have

$$A^{(l+1),L^{-j}}(x)-A^{(l),L^{-j}}(x)=A_{l+1}(y')-A_l(y)+(L^{j-l})^{\frac{d-2}{2}}O(p(L^l\varepsilon)), \tag{2.97}$$

where $x\in B^{l+1}(y')$, $x\in B^l(y)$, thus $y\in B(y')$ and the restrictions on the fields A_l, A_{l+1} on the set $\Lambda_0^{(l)}$ imply

$$A_{l+1}(y')-A_l(y)=(L^{j-l})^{\frac{d-2}{2}}O(p(L^l\varepsilon))$$

again. These inequalities give us finally

$$|(\partial_\mu^{L^{-j}}\tilde{B})(x)|\leqq O(1)p(L^j\varepsilon), \tag{2.98}$$

and it means that the regularity assumption is satisfied for the configurations $\tilde{B}$ and (2.93).

Now we will expand the operator $(C^{(j)}_{\Lambda_5^{(j)}}(B^j(\Lambda_2^{(j)}),\tilde{B}+\tilde{A}))^{-1}$ with respect to the field $\tilde{A}$. We use the formulas (I.3.15), (I.3.44) for the expansions of $Q_j(A+B)$, $G_j(\Omega,A+B)$ and we get

$$\begin{aligned}
&\Delta^{(j)}(\Omega,A+B)+aL^{-2}P_{A+B}\\
&=a_jI-a_j^2Q_j(A+B)G_j(\Omega,A+B)Q_j^*(A+B)+aL^{-2}P_{A+B}\\
&=\Delta^{(j)}(\Omega,B)+aL^{-2}P_B+aL^{-2}Q^*(B)F_2(A,B)+aL^{-2}F_2^*(A,B)Q(B)\\
&\quad+aL^{-2}F_2^*(A,B)F_2(A,B)-a_j^2F_{2,j}(A,B)G_j(\Omega,A+B)Q_j^*(A+B)\\
&\quad-a_j^2Q_j(A+B)G_j(\Omega,A+B)F_{2,j}^*(A,B)\\
&\quad-a_j^2F_{2,j}(A,B)G_j(\Omega,A+B)F_{2,j}^*(A,B)\\
&\quad-a_j^2Q_j(B)G_j(\Omega,B)V_j(A,B)G_j(\Omega,A+B)Q_j^*(B)\\
&=:\Delta^{(j)}(\Omega,B)+aL^{-2}P_B-W^{(j)}(A,B).
\end{aligned} \tag{2.99}$$

The operator $W^{(j)}$ is built with the help of the propagator $G_j(B^j(\Lambda_2^{(j)}),\tilde{A}+\tilde{B})$, so (2.99) does not give a full expansion in A, which will be obtained later.

The formula (2.99) implies

$$\begin{aligned}
&(C^{(j)}_{\Lambda_5^{(j)}}(B^j(\Lambda_2^{(j)}),\tilde{B}+\tilde{A}))^{-1}\\
&\quad=(C^{(j)}_{\Lambda_5^{(j)}}(B^j(\Lambda_2^{(j)}),\tilde{B}))^{-1/2}(I-(C^{(j)}_{\Lambda_5^{(j)}}(\dots))^{1/2}W^{(j)}\\
&\quad\cdot(C^{(j)}_{\Lambda_5^{(j)}}(\dots))^{1/2})(C^{(j)}_{\Lambda_5^{(j)}}(\dots))^{-1/2},
\end{aligned} \tag{2.100}$$

hence

$$\begin{aligned}
&[\det(C^{(j)}_{\Lambda_5^{(j)}}(B^j(\Lambda_2^{(j)}),\tilde{B}+\tilde{A}))^{-1}]^{-1/2}\\
&\quad=[\det(C^{(j)}_{\Lambda_5^{(j)}}(B^j(\Lambda_2^{(j)}),\tilde{B}))^{-1}]^{-1/2}\\
&\quad\cdot[\det(I-(C^{(j)}_{\Lambda_5^{(j)}}(\dots))^{1/2}W^{(j)}(C^{(j)}_{\Lambda_5^{(j)}}(\dots))^{1/2})]^{-1/2}.
\end{aligned} \tag{2.101}$$

The first determinant on the right side above, multiplied by the numerical factors we have omitted in (2.92) define the factor $Z^{(j)}$ with the field $\tilde{A}^{(k),L^{-j}}$ replaced by $\tilde{B}$. Thus it is of the form required by the inductive assumption for the density $\varrho^{(k+1)}$.

We have proved that the configurations (2.93) and $\tilde{B}$ are regular on $B(\Lambda_2^{(j)})$ in the sense of Proposition I.2.1, hence we can apply Proposition I.2.3 and we get

$$\begin{aligned}
&\frac{\gamma_0}{\gamma_1}I\leqq I-(C^{(j)}_{\Lambda_5^{(j)}}(\dots))^{1/2}W^{(j)}(C^{(j)}_{\Lambda_5^{(j)}}(\dots))^{1/2}\\
&\quad=(C^{(j)}_{\Lambda_5^{(j)}}(\dots))^{1/2}(C^{(j)}_{\Lambda_5^{(j)}}(B^j(\Lambda_2^{(j)}),\tilde{B}+\tilde{A}))^{-1}(C^{(j)}_{\Lambda_5^{(j)}}(\dots))^{1/2}\leqq\frac{\gamma_1}{\gamma_0}I.
\end{aligned} \tag{2.102}$$

The inequality

$$-\tfrac{1}{2}\log(1-\lambda)\leqq\tfrac{1}{2}\lambda+\tfrac{1}{4}\lambda^2+\tfrac{1}{6}\lambda^3+\ldots+\frac{1}{2n}\lambda^n+O(1)\lambda^{n+1}$$

holding for n odd and λ satisfying $\frac{\gamma_0}{\gamma_1}\leqq 1-\lambda\leqq\frac{\gamma_1}{\gamma_0}$ implies the inequality

$$\begin{aligned}&[\det(I-(C^{(j)}_{\Lambda_5^{(j)}}(\ldots))^{1/2}W^{(j)}(C^{(j)}_{\Lambda_5^{(j)}}(\ldots))^{1/2})]^{-1/2}\\ &\leqq\exp\left[\sum_{l=1}^{\bar{n}}\frac{1}{2l}\operatorname{Tr}(C^{(j)}_{\Lambda_5^{(j)}}(B^j(\Lambda_2^{(j)}),\tilde{B})W^{(j)})^l\right.\\ &\qquad\left.+O(1)\operatorname{Tr}(C^{(j)}_{\Lambda_5^{(j)}}(B^j(\Lambda_2^{(j)}),\tilde{B})W^{(j)})^{\bar{n}+1}\right].\end{aligned}\tag{2.103}$$

The possibility of dealing with bounded operators only, and especially the inequality (2.102), is an essential reason why we are treating the factors $Z^{(j)}$ separately instead of composing them into the factor Z_k. A simple analysis of the operator $W^{(j)}$ shows that the last term on the right side of (2.103) can be estimated by

$$\begin{aligned}&O((L^j\varepsilon)^\kappa)|B^{k-j}(\Lambda_2^{(k)})|\\ &\quad=O((L^k\varepsilon)^\kappa)|\Lambda_2^{(k)}|L^{(j-k)\kappa_0},\quad\text{with}\quad\kappa>d,\quad\kappa_0>0.\end{aligned}$$

In a similar way we can estimate the terms from the sum in (2.103) containing the R-vertices, the field $\tilde{B}^{(k),L^{-j}}$ or the terms of an order in the coupling constants larger than $\bar{n}$. These estimates are quite elementary because the terms are represented by one-loop graphs and easy applications of Propositions I.2.1–I.2.3 are sufficient here. The expression we get contains still the propagators $G_j(B^j(\Lambda_2^{(j)}),\tilde{A}+\tilde{B})$. We expand them iterating (I.3.44) to sufficiently high orders, and estimate again the terms of an order $>\bar{n}$ as above. Finally we get the following expression

$$\sum_{n=1}^{\bar{n}}\frac{1}{n!}\left(\frac{d^n}{d\tau^n}\log Z^{(j)}_{\Lambda_5^{(j)}}(B^j(\Lambda_2^{(j)}),\tau\theta_{k+1}A'^{(k),L^{-j}}+\tilde{B})\right)\Bigg|_{\tau=0}.\tag{2.104}$$

We will transform it further.

The summations in the vertices in (2.104) are restricted to $\operatorname{supp}\theta_{k+1}$ and the distance of this set to the set $B^k(\Lambda_0^{(k)c})$ measured on the lattice $T_{L^{-j}}$ is $>L^{k-j}r(L^k\varepsilon)>r(L^j\varepsilon)$. Using Proposition I.2.1, we can replace the propagator $G_j(B^j(\Lambda_2^{(j)}),\tilde{B})$ by $G_j(B^k(\Lambda_0^{(k)}),\tilde{B})$ and the terms containing the difference of these propagators can be estimated by

$$O((L^j\varepsilon)^\kappa)|B^{k-j}(\Lambda_2^{(k)})|=O((L^k\varepsilon)^\kappa)|\Lambda_2^{(k)}|L^{(j-k)\kappa_0}$$

with arbitrary $\kappa>d$, $\kappa_0=\kappa-d$.

Similarly using Proposition I.2.3 the propagator $C^{(j)}_{\Lambda_5^{(j)}}(B^j(\Lambda_2^{(j)}),\tilde{B})$ can be replaced by $C^{(j)}(B^k(\Lambda_0^{(k)}),\tilde{B})$. Thus we estimate (2.104) by the same expression but without the subscript $\Lambda_5^{(j)}$ and with $B^j(\Lambda_2^{(j)})$ replaced by $B^k(\Lambda_0^{(k)})$, plus an error

$$O((L^k\varepsilon)^\kappa)|\Lambda_2^{(k)}|L^{(j-k)\kappa_0}.$$

Finally we rescale the expressions from 1-lattice to η-lattice. Now we can use the formula (I.2.40) for a composition of $Z^{(j)}$ factors:

$$\prod_{j=0}^{k-1} Z^{(j),L^j\eta}(B^k(\Lambda_0^{(k)}),\cdot)=Z_k(B^k(\Lambda_0^{(k)}),\cdot). \tag{2.105}$$

This formula implies that the sum of the expressions over $j=0,1,\ldots,k-1$ is equal to

$$\sum_{n=1}^{\bar n}\frac{1}{n!}\left(\frac{d^n}{d\tau^n}\log Z_k(B^k(\Lambda_0^{(k)}),\tau\theta_{k+1}A'^{(k)}+\tilde B)\right)\bigg|_{\tau=0}. \tag{2.106}$$

This expression is by the definition equal to the polynomial $Q^{(k)}(B^k(\Lambda_0^{(k)});\ \tilde B,\ \theta_{k+1}A'^{(k)})$ defined in Chap. I.3, with the basic set $\Omega=B^k(\Lambda_0^{(k)})$ instead of $\Omega=T_\eta$, thus we have the propagator $G_k(B^k(\Lambda_0^{(k)}),\tilde B)$ instead of $G_k(B^{(k+1),\eta})$. Again let us notice that the propagator $G_k(B^k(\Lambda_0^{(k)}),\tilde B)$ depends on the configuration

$$B^k(\Lambda_0^{(k)})\tilde B=B^k(\Lambda_0^{(k)})(1-\theta_{k+1})A^{(k)}+\theta_{k+1}B^{(k+1),\eta}$$

and this configuration satisfies the regularity assumption of Proposition I.2.1 on the basis of the estimate (2.98). The terms containing the vertices localized in $B^k(\Lambda_7^{(k)c})$ can be estimated by

$$O((L^k\varepsilon)^{\kappa_0})|\Lambda_2^{(k)}\cap\Lambda_7^{(k)c}|,$$

as it follows from the extension of Proposition 2.1 to the polynomial $Q^{(k)}$. We choose the new localization with the help of a function g_k, where g_k is a smooth function with $\operatorname{supp} g_k\subset\Lambda_7^{(k)}$ and $g_k=1$ on $\Lambda_8^{(k)}$.

Now the summations in the vertices are restricted to $\Lambda_7^{(k)}$, and we can make the final transformation, namely we replace the propagators $G_k(B^k(\Lambda_0^{(k)}),\tilde B)$ by $G_k(B^k(\Lambda_2^{(k)}),\ B^{(k+1),\eta})$, with an error $O((L^k\varepsilon)^\kappa)|\Lambda_7^{(k)}|$.

Gathering together all the expansions and the estimates of the factors $Z^{(j)}$ we get

$$\begin{aligned}\prod_{j=0}^{k-1} Z_{\Lambda^{(j)}}^{(j),L^j\eta}(B^j(\Lambda_2^{(j)}),A+B)\leqq{}&\prod_{j=0}^{k-1} Z_{\Lambda_5^{(j)}}^{(j),L^j\eta}(B^j(\Lambda_2^{(j)}),\tilde B)\\ &\cdot\exp(Q^{(k)}(B^k(\Lambda_2^{(k)});B^{(k+1),\eta},g_kA'^{(k)})\\ &+O((L^k\varepsilon)^{\kappa_0})|\Lambda_2^{(k)}\cap\Lambda_7^{(k)c}|+O((L^k\varepsilon))^\kappa|\Lambda_7^{(k)}|).\end{aligned} \tag{2.107}$$

Let us consider the expansions of the remaining expressions under the integral in (2.53), i.e. the basic quadratic form in the ψ, ϕ fields and the polynomial $\mathscr{P}^{(k)}$. These expansions were analysed in Chap. I.3 and are given by the formulas (I.3.15), (I.3.16), (I.3.44), and (I.3.45). Proposition I.3.1 can also be applied in the present situation and we estimate the terms with R-vertices, the terms containing the configuration $\tilde B^{(k)}$ and the terms of order $>\bar n$ by $O((L^k\varepsilon)^\kappa)|\Lambda_2^{(k)}|$. We estimate the terms localized in $B^k(\Lambda_7^{(k)c})$ by

$$O((L^k\varepsilon)^{\kappa_0})|\Lambda_2^{(k)}\cap\Lambda_7^{(k)c}|.$$

The interaction terms we get after these expansions and estimates contain the propagator $G_k(B^{k-1}(\Lambda_2^{(k-1)}),\tilde B)$. Our final transformation is a change of this propagator by the propagator $G_k(B^k(\Lambda_2^{(k)}),\ B^{(k+1),\eta})$. Here we use a property already used in the inequality (2.107) and summarized in

Proposition 2.6. *If in the interaction terms localized in $B^k(\Lambda_7^{(k)})$ we replace the propagator $G_k(B^{k-1}(\Lambda_2^{(k-1)}), \tilde{B})$ by $G_k(B^k(\Lambda_2^{(k)}), B^{(k+1),\eta})$, then all the terms containing at least one difference of these propagators can be estimated by $O((L^k\varepsilon)^\kappa)|\Lambda_7^{(k)}|$.*

Now we will transform the basic quadratic form in the scalar fields and next we will do a translation in the field ϕ. We will localize this form in the set $\Lambda_6^{(k-1)'} \cap \Lambda_3^{(k)c}$, $\Lambda_3^{(k)}$, and we will change the operators of the forms using Proposition I.2.2 and the restrictions on the fields ψ, ϕ. We have

$$\begin{aligned}
&\tfrac{1}{2}aL^{d-2}\sum_{y\in T_1^{(k)'}}|\psi(y)-(Q(\theta_k\tilde{B})\phi)(y)|^2\\
&\quad+\tfrac{1}{2}\langle(\Lambda_6^{(k-1)'}\cap\Lambda_3^{(k)c})\phi, \Delta^{(k)}(B^k(\Lambda_2^{(k-1)'}\cap\Lambda_5^{(k)c}), \tilde{B})(\Lambda_6^{(k-1)'}\cap\Lambda_3^{(k)c})\phi\rangle\\
&\quad+\langle(\Lambda_6^{(k-1)'}\cap\Lambda_3^{(k)c})\phi, \Delta^{(k)}(B^k(\Lambda_2^{(k-1)'}\cap\Lambda_5^{(k)c}), \tilde{B})(\Lambda_3^{(k)}\cap\Lambda_4^{(k)c})\phi\rangle\\
&\quad+\langle\Lambda_3^{(k)}\phi, \Delta^{(k)}(B^k(\Lambda_2^{(k)}), B^{(k+1),\eta})\Lambda_3^{(k)}\phi\rangle\\
&\quad+O((L^k\varepsilon)^\kappa)|\Lambda_3^{(k)}|\,. \qquad (2.108)
\end{aligned}$$

More exactly the difference between the quadratic forms is a quadratic form $\frac{1}{2}\langle\Lambda_6^{(k-1)'}\phi, \mathrm{H}_k\Lambda_6^{(k-1)'}\phi\rangle$ and for the matrix elements $h_k(x,x')$ of the operator H_k of this form, the following inequality holds

$$\begin{aligned}
|h_k(x,x')| &\leqq O(1)\exp(-\delta_1 r(L^k\varepsilon))\exp(-\delta_0|x-x'|)\\
&\leqq O((L^k\varepsilon)^\kappa)\exp(-\delta_0|x-x'|) \qquad (2.109)
\end{aligned}$$

for x, $x'\in\Lambda_6^{(k-1)'}$ and arbitrary κ. This remark holds for the other expressions of this type, e.g. for the expression connected with (2.47), and will be used in the next chapter, where these formulas will be applied in the reversed order.

Now we make a translation of the form

$$\phi=\phi'+aL^{-2}C^{(k)}_{\Lambda_4^{(k)}}(B^k(\Lambda_2^{(k)}), B^{(k+1),\eta})Q^*(B^{(k+1),\eta})\psi\,. \qquad (2.110)$$

This translation changes the first and the fourth terms on the right side of (2.108) in an obvious way. We get an expression analogous to (2.87), so we will not write it here. After the translation (2.110) we make some changes in the interaction terms. If x in

$$(aL^{-2}C^{(k)}_{\Lambda_4^{(k)}}(B^k(\Lambda_2^{(k)}), B^{(k+1),\eta})\,Q^*(B^{(k+1),\eta})\psi)(x)$$

is a vertex variable, then $x\in\Lambda_7^{(k)}$, and we replace this configuration by

$$(aL^{-2}C^{(k)}(B^k(\Lambda_2^{(k)}), B^{(k+1),\eta})Q^*(B^{(k+1),\eta})\Lambda_6^{(k)'}\psi)(x)\,.$$

If the configuration appears in the expression

$$\begin{aligned}
&(a_kG_k(B^k(\Lambda_2^{(k)}), B^{(k+1),\eta})Q_k^*(B^{(k+1),\eta})\Lambda_6^{(k)}aL^{-2}\\
&\quad\cdot C^{(k)}_{\Lambda_4^{(k)}}(B^k(\Lambda_2^{(k)}), B^{(k+1),\eta})Q^*(B^{(k+1),\eta})\psi)(x)\,, \qquad (2.111)
\end{aligned}$$

then $x\dot{\in}B^k(\Lambda_7^{(k)})$, and we replace $C^{(k)}_{\Lambda_4^{(k)}}$ by $C^{(k)}$, $\Lambda_6^{(k)}$ by 1, and using the recursive equation (I.2.41) we get

$$\begin{aligned}
(2.111)=&(a_{k+1}L^{-2}G^\eta_{k+1}(B^k(\Lambda_2^{(k)}), B^{(k+1),\eta})Q^*_{k+1}(B^{(k+1),\eta})\Lambda_6^{(k)'}\psi)(x)\\
&+O((L^k\varepsilon)^\kappa)\,. \qquad (2.112)
\end{aligned}$$

After all the transformations of this and previous points, we get the interaction

$$V^{(k)}(\Lambda_7^{(k)}, B^{(k+1),\eta}, \psi, g_k A'^{(k)}, \phi'),$$

i.e. the same interaction which appeared in Chap. I.3, except that the summations in the vertices are restricted either to the set $B^k(\Lambda_7^{(k)})$, or by the function g_k, and the propagators are defined by the operator $-\Delta^{\eta,N}_{B^{(k+1),\eta}B^k(\Lambda_2^{(k)})}$. This interaction has the same properties as $V^{(k)}$ in Chap. I.3, especially Proposition I.3.2 holds for it.

Let us consider the restrictions satisfied by the field ϕ'. We want to prove that this field is small on $\Lambda_5^{(k)}$. We have the following analog of Lemma 2.5.

Lemma 2.7. *The following estimate holds*

$$\begin{aligned} &aL^{-2}(C^{(k)}_{\Lambda_4^{(k)}}(B^k(\Lambda_2^{(k)}), B^{(k+1),\eta})Q^*(B^{(k+1),\eta})\psi)(x) \\ &\quad = (Q^*(B^{(k+1),\eta})\psi)(x) + O(p(L^k\varepsilon)), \qquad x \in \Lambda_5^{(k)}. \end{aligned} \tag{2.113}$$

A proof of this lemma is based on the ideas which were described before in the proofs of Lemmas 2.4 and 2.5, so we omit it here.

Lemma 2.7 and the restrictions on the fields ψ, ϕ imply

$$|\phi'(x)| \leqq O(1)p(L^k\varepsilon), \qquad x \in \Lambda_5^{(k)}. \tag{2.114}$$

We introduce the characteristic functions $\chi(A')$, $\chi(\phi')$ giving the restrictions (2.86), (2.114) on the set $\Lambda_5^{(k)}$, and we estimate all the remaining characteristic functions for the fields on $\Lambda_5^{(k)}$ by 1, with the exception of the functions χ_{k+1}.

After all these transformations, the conditional integration with respect to A', ϕ' with the conditioning on $\Lambda_5^{(k)c}$, and the translations inverse to (2.80), (2.110), we get the inequality

$$\begin{aligned} &[\text{the integral (2.53)}] \\ &\leqq \sum_{\Lambda_0^{(0)}, \dots, \Lambda_0^{(k-1)}, \Lambda_0^{(k)}} \int dA \int d\phi \chi_{k+1} \zeta_{\Lambda_0^{(k)}} \chi_{\Lambda_{-1}^{(k)} \cap \Lambda_5^{(k)c}} \chi_{k, \Lambda_5^{(k)c}} \\ &\cdot \varrho'^{(k)}(\Lambda_0^{(0)}, \dots, \Lambda_0^{(k-1)}, \Lambda_7^{(k-1)\prime c}A, \tilde{B}, \Lambda_7^{(k-1)\prime c}\phi) \\ &\cdot \exp\Bigg[-\tfrac{1}{2}aL^{d-2} \sum_{y \in T^{(k)\prime}} |B(y) - (QA)(y)|^2 \\ &-\tfrac{1}{2}\langle \Lambda_6^{(k-1)\prime}A, \Delta^{(k)}\Lambda_6^{(k-1)\prime}A\rangle \\ &-\tfrac{1}{2}aL^{d-2} \sum_{y \in T^{(k)\prime}} |\psi(y) - (Q(\tilde{B})\phi)(y)|^2 - \tfrac{1}{2}\langle (\Lambda_6^{(k-1)\prime} \cap \Lambda_3^{(k)c})\phi, \\ &\Delta^{(k)}(B^k(\Lambda_2^{(k-1)\prime} \cap \Lambda_5^{(k)c}), \tilde{B})(\Lambda_6^{(k-1)\prime} \cap \Lambda_3^{(k)c})\phi\rangle \\ &-\langle (\Lambda_6^{(k-1)\prime} \cap \Lambda_3^{(k)c})\phi, \Delta^{(k)}(B^k(\Lambda_2^{(k-1)\prime} \cap \Lambda_5^{(k)c}), \tilde{B}) \\ &\cdot (\Lambda_3^{(k)} \cap \Lambda_4^{(k)c})\phi\rangle - \tfrac{1}{2}\langle \Lambda_3^{(k)}\phi, \Delta^{(k)}(B^k(\Lambda_2^{(k)}), B^{(k+1),\eta}) \\ &\cdot \Lambda_3^{(k)}\phi\rangle \Bigg] \int d\mu_{C^{(k)}_{\Lambda_5^{(k)}}}(A') \int d\mu_{C^{(k)}_{\Lambda_5^{(k)}}(B^k(\Lambda_2^{(k)}), B^{(k+1),\eta})}(\phi') \\ &\cdot \chi(A')\chi(\phi') \exp(V^{(k)}(\Lambda_7^{(k)}, B^{(k+1),\eta}, \psi, A'^{(k)}, \phi')) \\ &\cdot \exp(O((L^k\varepsilon)^{\kappa_0})|\Lambda_7^{(k-1)\prime} \cap \Lambda_7^{(k)c}| + O((L^k\varepsilon)^{\kappa})|T_1^{(k)}|), \end{aligned} \tag{2.115}$$

where

$$\tilde{B}=(1-\theta_{k+1})\theta_k A^{(k)}+\theta_{k+1}B^{(k+1),\eta}.$$

The terms $A_\Lambda^{-1}A$ coming from the formula (2.28) have been removed as in the discussion preceding (2.31).

Let us consider the integrals in (2.115). The first integral on the right side of (2.115) is equal to the density

$$\varrho''^{(k+1),L}(\Lambda_0^{(0)},\ldots,\Lambda_0^{(k-1)},\Lambda_0^{(k)},B,\theta_{k+1}B^{(k+1),\eta},\psi)$$

without the constants, i.e. the normalization constants and the constants coming from rescalings.

We transform it into the density $\varrho^{(k+1),L}$ using the same arguments as in Sect. B. For the second integral we have almost the same situation as for the integral (I.3.56) of Chap. I.3, the only difference is that the integration is over the fields defined on $\Lambda_5^{(k)}$ instead of $T_1^{(k)}$. The conclusion is the same and for this integral we have the cumulant expansion (I.3.59). Proceeding as previously, i.e. estimating the terms of order higher than $\bar{n}$, replacing the covariances with Dirichlet boundary conditions by the "free" ones and composing some of them with the help of recursive equations (I.2.42), we get the expression

$$\mathscr{P}^{(k+1),L}(\Lambda_7^{(k)},B^{(k+1),\eta},\psi).$$

The last operation is a rescaling of the obtained expressions from the L-lattice to $L^{k+1}\varepsilon$-lattice. Gathering together all the estimates, we get the inequality (2.43) but with $k+1$ instead of k.

D. The Final Step

The procedure is continued until $k=K$, where K is such that $L^K\varepsilon\leqq\varepsilon_0$, $L^{K+1}\varepsilon>\varepsilon_0$. Then we estimate

$$\mathscr{P}^{(K),L^K\varepsilon}(\Lambda_7^{(K-1)},B^{(K),\varepsilon},\psi)\leqq O((L^K\varepsilon)^{\kappa_0})|\Lambda_7^{(K)}|. \tag{2.116}$$

Now it is sufficient to prove the estimate

$$\begin{aligned}&\int dB\int d\psi\sum_{\Lambda_0^{(0)},\ldots,\Lambda_0^{(K-1)}}\varrho^{(K),L^K\varepsilon}(\Lambda_0^{(0)},\ldots,\Lambda_0^{(K-1)},B,\theta_K B^{(K),\varepsilon},\psi)\exp(-E_0)\\&\cdot\exp\left(O(1)\varepsilon^{\kappa_0}|\Lambda_7^{(0)c}|+\sum_{j=1}^{K-1}O(1)(L^j\varepsilon)^{\kappa_0}|\Lambda_7^{(j-1)\prime}\cap\Lambda_7^{(j)c}|\right.\\&\left.+O(1)(L^{K-1}\varepsilon)^{\kappa_0}|\Lambda_7^{(K-1)}|\right)\leqq\exp(O(1)|T_\varepsilon|),\end{aligned} \tag{2.117}$$

with the constant $O(1)$ independent of ε, because for the last sum in the exponent on the right side of (2.43), we have

$$\sum_{j=0}^{K-1}O((L^j\varepsilon)^\kappa)|T_1^{(j)}|=\sum_{j=0}^{K-1}O(1)(L^j\varepsilon)^{\kappa_0}|T_\varepsilon|\leqq O(1)|T_\varepsilon|. \tag{2.118}$$

The inequality (2.117) will be proved in the next chapter.

3. Analysis of the Density $\varrho^{(K), L^K\varepsilon}$

A. A Preliminary Transformation of the Density $\varrho^{(K)}$

We will make operations inverse to those which were done in the proof of the upper bound, i.e. we will replace the expressions obtained as the results of conditional integrations and further transformations by the corresponding integrals. Rescaling the density $\varrho^{(K), L^K\varepsilon}$ to the unit lattice, using the restrictions on the fields B, ψ arising from the characteristic functions χ_K, and making the transformations inverse to (2.47), we get the inequality

$$\begin{aligned}
&\varrho^{(K)}(\Lambda_0^{(0)}, \ldots, \Lambda_0^{(K-1)}, B, \theta_K B^{(K)}, \psi) \\
&\leqq \varrho''^{(K)}(\Lambda_0^{(0)}, \ldots, \Lambda_0^{(K-1)}, B, \theta_K B^{(K)}, \psi) \\
&\quad \cdot \exp(O((L^K\varepsilon)^\kappa)|\Lambda_5^{(K-1)\prime}|) .
\end{aligned} \tag{3.1}$$

The density $\varrho''^{(K)}$ is represented by the formula (2.45).

Let us formulate the inequalities we get in such a way after $K-k$ steps as an inductive hypothesis. At first let us recall, or introduce, some definitions and notations. The fields with respect to which the integration is done in the $k+1$ renormalization transformation are denoted by A_k, ϕ_k. For simplicity we denote B, ψ by A_K, ϕ_K also. Finally in the expressions with scalar fields we have the following external vector field

$$\tilde{A}^\varepsilon = (1-\theta_1)A_0 + \sum_{k=1}^{K-1} (1-\theta_{k+1})\theta_k A^{(k),\varepsilon} + \theta_K A^{(K),\varepsilon}, \tag{3.2}$$

where the fields $A^{(k),\varepsilon}$ are defined by

$$A^{(k),\varepsilon} = a_k (L^k\varepsilon)^{-2} \zeta^{(k)} G_k^\varepsilon Q_k^* A_k . \tag{3.3}$$

Also let us denote

$$\tilde{A}^{(k),\varepsilon} = \sum_{l=k}^{K-1} (1-\theta_{l+1})\theta_l A^{(l),\varepsilon} + \theta_K A^{(K),\varepsilon} . \tag{3.4}$$

Now we can formulate the inductive hypothesis:

$$\begin{aligned}
&\varrho^{(K), L^K\varepsilon}(\Lambda_0^{(0)}, \ldots, \Lambda_0^{(K-1)}, A_K, \tilde{A}^{(K),\varepsilon}, \phi_K) \\
&\leqq T_{a,L}^{L^{K-1}\varepsilon} T_{a,L,\tilde{A}^{(K-1),\varepsilon}}^{L^{K-1}\varepsilon} [\chi_K \zeta_{\Lambda_0^{(K-1)}} \chi_{\Lambda_{-1}^{(K-1)} \cap \Lambda_5^{(K-1)c}} \\
&\quad \cdot \chi_{K-1, \Lambda_5^{(K-1)c}} T_{a,L}^{L^{K-2}\varepsilon} T_{a,L,\tilde{A}^{(K-2),\varepsilon}}^{L^{K-2}\varepsilon} [\zeta_{\Lambda_0^{(K-2)}} \\
&\quad \cdot \chi_{\Lambda_{-1}^{(K-2)} \cap \Lambda_5^{(K-2)c}} \chi_{K-2, \Lambda_5^{(K-2)c}} \ldots T_{a,L}^{L^k\varepsilon} T_{a,L,\tilde{A}^{(k),\varepsilon}}^{L^k\varepsilon} \\
&\quad \cdot [\zeta_{\Lambda_0^{(k)}} \chi_{\Lambda_{-1}^{(k)} \cap \Lambda_5^{(k)c}} \chi_{k, \Lambda_5^{(k)c}} \varrho'^{(k), L^k\varepsilon}(\Lambda_0^{(0)}, \ldots, \Lambda_0^{(k-1)}, \\
&\quad A_k, \tilde{A}^{(k),\varepsilon}, \phi_k) \exp[-\tfrac{1}{2}\langle \Lambda_6^{(k-1)\prime} A_k, \Delta^{(k), L^k\varepsilon} \Lambda_6^{(k-1)\prime} A_k \rangle \\
&\quad -\tfrac{1}{2}\langle (\Lambda_6^{(k-1)\prime} \cap \Lambda_3^{(k)c})\phi_k, \Delta^{(k), L^k\varepsilon}(B^k(\Lambda_2^{(k-1)\prime} \cap \Lambda_5^{(k)c}), \tilde{A}^{(k),\varepsilon}) \\
&\quad \cdot (\Lambda_6^{(k-1)\prime} \cap \Lambda_3^{(k)c})\phi_k \rangle - \langle (\Lambda_6^{(k-1)\prime} \cap \Lambda_3^{(k)c})\phi_k, \Delta^{(k), L^k\varepsilon}, \\
&\quad (B^k(\Lambda_2^{(k-1)\prime} \cap \Lambda_5^{(k)c}), \tilde{A}^{(k),\varepsilon}) (\Lambda_3^{(k)} \cap \Lambda_4^{(k)c})\phi_k \rangle \\
&\quad -\tfrac{1}{2}\langle \Lambda_3^{(k)}\phi_k, \Delta^{(k), L^k\varepsilon}(B^k(\Lambda_2^{(k)}), \tilde{A}^{(k+1),\varepsilon})\Lambda_3^{(k)}\phi_k \rangle]] \ldots]] \\
&\quad \cdot \exp\left(\sum_{l=k+1}^{K} O(1)(L^l\varepsilon)^\kappa |T_\varepsilon| \right).
\end{aligned} \tag{3.5}$$

The expression on the right side seems to be rather complicated, however it has a simple structure. Here $T_1^{(k)}$ can be represented as a sum

$$T_1^{(k)} = \Lambda_5^{(k)c} \cup \bigcup_{l=k+1}^{K-1} B^{l-k}(\Lambda_5^{(l-1)'} \cap \Lambda_5^{(l)c}) \cup B^{K-1-k}(\Lambda_5^{(K-1)}), \tag{3.6}$$

and to each set $\Lambda_5^{(l-1)'} \cap \Lambda_5^{(l)c}$ there corresponds a set of characteristic functions and a sequence of partial renormalization transformations localized in this set. In this point we will use only the functions $\chi_{l,\Lambda_5^{(l)c}}$ giving the suitable restrictions on the fields A_l, ϕ_l considered on the set $\Lambda_5^{(l-1)'} \cap \Lambda_5^{(l)c}$. All the partial renormalization transformations defined on the sets $B^{l-j}(\Lambda_{-1}^{(l-1)} \cap \Lambda_5^{(l)c})$, $k \leqq j < l$, can be composed according to formula (I.2.12) of Chap. I.2. We have to notice that the vector fields, with respect to which the integration is done when the composition is formed, do not occur in the correspondingly localized configuration $\tilde{A}^{(k),\varepsilon}$. Let us omit from the right side of (3.36) the composed transformations which do not depend on A_k, $\phi_k \upharpoonright \Lambda_5^{(k)}$. The remaining terms, after rescaling from $L^k\varepsilon$-lattice to 1-lattice and removing numerical factors, can be written as follows:

$$\begin{aligned}
&\chi_K \chi_{K-1,\Lambda_5^{(K-1)c}} \cdots \chi_{k,\Lambda^{(k)c}} \int dA_k \upharpoonright_{\Lambda_5^{(k)}} \int d\phi_k \upharpoonright_{\Lambda_5^{(k)}} \\
&\cdot \varrho'^{(k)}(\Lambda_0^{(0)}, \ldots, \Lambda_0^{(k-1)}, \Lambda_7^{(k-1)'c} A_k, \tilde{A}^{(k)}, \Lambda_7^{(k-1)'c} \phi_k) \\
&\cdot \exp\Bigg[- \sum_{l=k+1}^{K} \tfrac{1}{2} a_{l-k} (L^{l-k})^{d-2} \sum_{x_l \in \Lambda_5^{(l-1)'} \cap \Lambda_5^{(l)c}} |A_l(x_l) - (Q_{l-k} A_k)(x_l)|^2 \\
&- \sum_{l=k+1}^{K} \tfrac{1}{2} a_{l-k} (L^{l-k})^{d-2} \sum_{x_l \in \Lambda_5^{(l-1)'} \cap \Lambda_5^{(l)c}} |\phi_l(x_l) - (Q_{l-k}(\tilde{A}^{(k+1)})\phi_k)(x_l)|^2 \\
&- \tfrac{1}{2} \langle \Lambda_6^{(k-1)'} A_k, \Delta^{(k)} \Lambda_6^{(k-1)'} A_k \rangle \\
&- \tfrac{1}{2} \langle \Lambda_6^{(k-1)'} \phi_k, \Delta^{(k)}(B^k(\Lambda_2^{(k-1)'}), \tilde{A}^{(k)}) \Lambda_6^{(k-1)'} \phi_k \rangle \\
&+ \tfrac{1}{2} \langle \Lambda_6^{(k-1)'} \phi_k, H_k \Lambda_6^{(k-1)'} \phi_k \rangle \Bigg].
\end{aligned} \tag{3.7}$$

Also we have applied formula (2.108) together with the remark following it to the expression in the exponent in (3.5). According to the remark, the matrix elements $h_k(x, x')$ of the operator H_k satisfy the estimates

$$|h_k(x, x')| \leqq O(1) \exp(-\delta_1 r(L^k\varepsilon)) \exp(-\delta_0 |x - x'|), \; x, x' \in \Lambda_6^{(k-1)'}. \tag{3.8}$$

Further we apply formula (2.49) and we replace $\varrho'^{(k)}$ exp[(the proper quadratic forms)] by $\varrho^{(k)}$. Next we complete and transform the quadratic forms in $\varrho^{(k)}$ to the forms appearing in formula (2.46) for the density $\varrho''^{(k)}$. The differences can be written again as the forms satisfying the property (3.8). Now the problem is that we cannot estimate these forms because we have no restrictions on the fields $A_k \upharpoonright_{\Lambda_5^{(k)}}$, $\phi_k \upharpoonright_{\Lambda_5^{(k)}}$. We have restrictions on the fields A_k, ϕ_k on the set $\Lambda_{-1}^{(k-1)'} \cap \Lambda_5^{(k)c}$ given by the functions $\chi_{k,\Lambda_5^{(k)c}}$, so we estimate only part of the quadratic forms involving these fields by

$$O((L^k\varepsilon)^\kappa) |\Lambda_5^{(k-1)'} \cap \Lambda_5^{(k)c}|.$$

Let us consider formula (2.46) for the density $\varrho''^{(k)}$, and let us take these expressions which depend on the fields $A_k \upharpoonright_{\Lambda_5^{(k)}}$, $\phi_k \upharpoonright_{\Lambda_5^{(k)}}$. Then the integral (3.7), after removing all the terms independent of these fields, is of the form

$$\begin{aligned}
&\int dA_k \upharpoonright_{\Lambda_5^{(k)}} \int d\phi_k \upharpoonright_{\Lambda_5^{(k)}} \exp\Bigg[-\tfrac{1}{2} \sum_{l=k+1}^{K} a_{l-k}(L^{l-k})^{d-2} \\
&\cdot \sum_{x_l \in \Lambda_5^{(l-1)'} \cap \Lambda_5^{(l)c}} |A_l(x_l) - (Q_{l-k}A_k)(x_l)|^2 \\
&-\tfrac{1}{2} \sum_{l=k+1}^{K} a_{l-k}(L^{l-k})^{d-2} \sum_{x_l \in \Lambda_5^{(l-1)'} \cap \Lambda_5^{(l)c}} |\phi_l(x_l) \\
&-(Q_{l-k}(\tilde{A}^{(k+1)})\phi_k)(x_l)|^2 - \tfrac{1}{2}\langle \Lambda_5^{(k)} A_k, \Delta^{(k)}_{\Lambda_5^{(k-1)}} \Lambda_5^{(k)} A_k \rangle \\
&-\tfrac{1}{2}\langle \Lambda_5^{(k)}\phi_k, \Delta^{(k)}_{\Lambda_5^{(k-1)}}(B^k(\Lambda_2^{(k-1)'}), \tilde{A}^{(k)})\Lambda_5^{(k)}\phi_k \rangle \\
&-\langle (\Lambda_5^{(k-1)'} \cap \Lambda_5^{(k)c}) \mathrm{A}_k, \Delta^{(k)}_{\Lambda_5^{(k-1)}} \Lambda_5^{(k)} \mathrm{A}_k \rangle \\
&-\langle (\Lambda_5^{(k-1)'} \cap \Lambda_5^{(k)c})\phi_k, \Delta^{(k)}_{\Lambda_5^{(k-1)}}(B^k(\Lambda_2^{(k-1)'}), \tilde{A}^{(k)})\Lambda_5^{(k)}\phi_k \rangle \\
&+\langle f_k', \Lambda_5^{(k)} A_k \rangle + \langle f_k'', \Lambda_5^{(k)}\phi_k \rangle \\
&+\tfrac{1}{2}\langle \Lambda_5^{(k)} A_k, H_k' \Lambda_5^{(k)} A_k \rangle + \tfrac{1}{2}\langle \Lambda_5^{(k)}\phi_k, H_k'' \Lambda_5^{(k)}\phi_k \rangle \\
&+\langle F_k', \Lambda_5^{(k)} A_k \rangle + \langle F_k'', \Lambda_5^{(k)}\phi_k \rangle \Bigg],
\end{aligned} \tag{3.9}$$

where the forms H_k', H_k'' are composed of terms associated with the changes in (2.47) and the discussion following, and the term H_k introduced in (3.7). They satisfy the estimate (3.8). The functions f_k', f_k'' are defined by the third and ninth terms in the exponential in (2.46) and we want to have these terms in a final formula. The functions F_k', F_k'' are defined by these terms with opposite signs and by the terms coming from quadratic forms H_k', H_k'', with only one variable localized to $\Lambda_5^{(k)}$. They have the property

$$|F_k'(x)|, \quad |F_k''(x)| \leqq O(1)\exp(-\delta_1 r(L^k\varepsilon)). \tag{3.10}$$

We will estimate this integral by the same integral without the last four terms in the exponent, i.e. the terms with H_k', H_k'', F_k', F_k''. Since (3.9) is a Gaussian integral, we can easily calculate it and estimate the obtained expression. We need the estimates of the quadratic forms in (3.9). Let us denote the quadratic forms in A_k, ϕ_k, connected with the first four terms in the exponential under the integral (3.9), by $\langle A_k, G_k' A_k\rangle$, $\langle \phi_k, G_k''\phi_k\rangle$ correspondingly. We will give the estimates from below for these forms. It is sufficient to get very weak estimates because we have the strong estimates (3.8), (3.10). To get them it suffices to use the mass terms in the fundamental operators $-\Delta^{\eta} + \mu_0^2(L^k\varepsilon)^2$ and

$$-\Delta^{\eta, N}_{\tilde{A}^{(k)}, B^k(\Lambda_2^{(k-1)'})} + m^2(L^k\varepsilon)^2 .$$

In the next section of this chapter we will formulate a much stronger estimate, which as a corollary gives

$$G_k' \geqq \gamma_1 \mu_0^2 (L^k\varepsilon)^2 I \upharpoonright_{\Lambda_5^{(k)}}, \qquad G_k'' \geqq \gamma_1 m^2 (L^k\varepsilon)^2 I \upharpoonright_{\Lambda_5^{(k)}} . \tag{3.11}$$

The property (3.8) implies in turn the following estimates for the norms of H'_k, H''_k in $L^2(\Lambda_5^{(k)})$:

$$\|H'_k\|, \quad \|H''_k\| \leqq O(1)\exp(-\delta_1 r(L^k\varepsilon)) \leqq O(1)(L^k\varepsilon)^\kappa \tag{3.12}$$

for every κ.

Hence for $L^k\varepsilon$ sufficiently small, the operators $G'_k - H'_k$ and $G''_k - H''_k$ are positive and satisfy the inequalities (3.11) with $\frac{1}{2}\gamma_1$ instead of γ_1. Let us introduce some new notations:

$$\Phi' = \sum_{l=k+1}^{K} a_{l-k}(L^{l-k})^{-2} B^{l-k}(\Lambda_5^{(l-1)'} \cap \Lambda_5^{(l)c}) Q^*_{l-k} A_l, \tag{3.13}$$

$$\Phi'' = \sum_{l=k+1}^{K} a_{l-k}(L^{l-k})^{-2} B^{l-k}(\Lambda_5^{(l-1)'} \cap \Lambda_5^{(l)c}) Q^*_{l-k}(\tilde{A}^{(k+1)}) \phi_l. \tag{3.14}$$

These configurations are defined on $\Lambda_5^{(k)}$. Because the fields A_l, ϕ_l considered on the subset $\Lambda_{-1}^{(l-1)'} \cap \Lambda_5^{(l)c}$ of the L^{l-k}-lattice satisfy the inequalities

$$\begin{aligned} |A_l(x_l)| &\leqq O(1) L^{-(l-k)\frac{d}{2}} \frac{1}{\mu_0 L^k \varepsilon} p(L^k\varepsilon), \\ |\phi_l(x_l)| &\leqq O(1) L^{-(l-k)\frac{d}{4}} \frac{1}{\lambda(L^k\varepsilon)^{1/4}} p(L^k\varepsilon), \end{aligned} \tag{3.15}$$

so we have

$$\begin{aligned} |\Phi'(x)| &\leqq O(1) \frac{1}{\mu_0 L^k \varepsilon} p(L^k\varepsilon) \leqq O(1)(L^k\varepsilon)^{-2}, \\ |\Phi''(x)| &\leqq O(1) \frac{1}{\lambda(L^k\varepsilon)^{1/4}} p(L^k\varepsilon) \leqq O(1)(L^k\varepsilon)^{-2} \end{aligned} \tag{3.16}$$

for $x \in \Lambda_5^{(k)}$. Of course the above estimates are very rough, especially the second.

After these preliminary remarks we can estimate the integral (3.9). At first we have

$$\begin{aligned} (3.9) = {} & (2\pi)^{\frac{d}{2}|\Lambda^{(k)}|} [\det(G'_k - H'_k)]^{-1/2} (2\pi)^{\frac{N}{2}|\Lambda_5^{(k)}|} [\det(G''_k - H''_k)]^{-1/2} \\ & \cdot \exp\Bigg[-\frac{1}{2} \sum_{l=k+1}^{K} a_{l-k}(L^{l-k})^{d-2} \sum_{x_l \in \Lambda_5^{(l-1)'} \cap \Lambda_5^{(l)c}} (|A_l(x_l)|^2 + |\phi_l(x_l)|^2) \\ & + \tfrac{1}{2}\langle (\Phi' - \Lambda_5^{(k)} \Delta^{(k)}_{\Lambda_5^{(k-1)}} (\Lambda_5^{(k-1)'} \cap \Lambda_5^{(k)c}) A_k + f'_k + F'_k), \\ & \cdot (G'_k - H'_k)^{-1} (\Phi' - \Lambda_5^{(k)} \Delta^{(k)}_{\Lambda_5^{(k-1)}} (\Lambda_5^{(k-1)'} \cap \Lambda_5^{(k)c}) A_k + f'_k + F'_k) \rangle \\ & + \tfrac{1}{2}\langle (\Phi'' - \Lambda_5^{(k)} \Delta^{(k)}_{\Lambda_5^{(k-1)}} (B^k(\Lambda_2^{(k-1)'}), \tilde{A}^{(k)}) \\ & \cdot (\Lambda_5^{(k-1)'} \cap \Lambda_5^{(k)c}) \phi_k + f''_k + F''_k), (G''_k - H''_k)^{-1} (\Phi'' - \ldots) \rangle \Bigg]. \end{aligned} \tag{3.17}$$

Now we will estimate the above expressions by the analogous expressions with $H'_k = H''_k = 0$ and $F'_k = F''_k = 0$. Let us consider the determinants in (3.17). We have

$$[\det(G'_k - H'_k)]^{-1/2} = (\det G'_k)^{-1/2} [\det(I - G'^{-1/2}_k H'_k G'^{-1/2}_k)]^{-1/2}, \tag{3.18}$$

and the operator under the second determinant satisfies

$$\|G_k'^{-1/2}H_k'G_k'^{-1/2}\| \leqq \gamma_1^{-1}\mu_0^2(L^k\varepsilon)^{-2}O(1)\exp(-\delta_1 r(L^k\varepsilon)) \leqq O(1)(L^k\varepsilon)^\kappa, \ \kappa \text{ arbitrary}. \tag{3.19}$$

Hence this determinant can be estimated as follows

$$\begin{aligned}&[\det(I-G_k'^{-1/2}H_k'G_k'^{-1/2})]^{-1/2}\\ &\leqq \exp[\tfrac{1}{2}\mathrm{Tr}(G_k'^{-1/2}H_k'G_k'^{-1/2})+O(1)\,\mathrm{Tr}(G_k'^{-1/2}H_k'G_k'^{-1/2})^2]\\ &\leqq \exp[\tfrac{1}{2}\|G_k'^{-1/2}H_k'G_k'^{-1/2}\|\\ &\quad\cdot \mathrm{Tr}(I\restriction_{\Lambda_5^{(k)}})+O(1)\|G_k'^{-1/2}H_k'G_k'^{-1/2}\|^2\,\mathrm{Tr}(I\restriction_{\Lambda_5^{(k)}})]\\ &\leqq \exp O((L^k\varepsilon)^\kappa)|\Lambda_5^{(k)}|,\end{aligned} \tag{3.20}$$

where an analog of the inequality (2.103) was used. The determinant $[\det(G_k''-H_k'')]^{-1/2}$ has the identical properties, so we have the required estimates for both determinants in (3.17). Now we will estimate the last two terms in the exponent in (3.17). We have $(G_k'-H_k')^{-1}=G_k'^{-1}+G_k'^{-1}H_k'(G_k'-H_k')^{-1}$ and the norm of the second operator can be estimated as in (3.19). From (3.16), (3.10) and the restrictions on the fields A_k considered on $\Lambda_5^{(k-1)\prime}\cap\Lambda_5^{(k)c}$, it follows that the norm of the configuration

$$\Phi'-\Lambda_5^{(k)}\Delta^{(k)}_{\Lambda_5^{(k-1)}}(\Lambda_5^{(k-1)\prime}\cap\Lambda_5^{(k)c})A_k+f_k'+F_k'$$

can be estimated by $O((L^k\varepsilon)^{-2})|\Lambda_5^{(k)}|^{1/2}$. Hence in the terms considered, we can replace the operators $(G_k'-H_k')^{-1}$ and $(G_k''-H_k'')^{-1}$ by $G_k'^{-1}$ and $G_k''^{-1}$, and we can estimate the rest by $O((L^k\varepsilon)^\kappa)|\Lambda_5^{(k)}|$. Finally the terms of the form $\langle F_k', G_k'^{-1}(\Phi'-\ldots)\rangle$ and other terms containing one of the functions F_k', F_k'' can be estimated by

$$O(1)\exp(-\tfrac{1}{2}\delta_1 r(L^k\varepsilon))\gamma_1^{-1}\mu_0^{-2}(L^k\varepsilon)^{-2}O((L^k\varepsilon)^{-2})|\Lambda_5^{(k)}| \leqq O((L^k\varepsilon)^\kappa)|\Lambda_5^{(k)}|.$$

Thus the integral (3.9) can be estimated by the same integral with $H_k'=H_k''=0$, $F_k'=F_k''=0$ and multiplied by the factor $\exp O((L^k\varepsilon)^\kappa)|\Lambda_5^{(k)}|$. Gathering together all the inequalities obtained until now, we estimate the integral (3.7) by an integral in which the density $\varrho'^{(k)}$ and the exponential function, except the part defining the renormalization transformation, is replaced by $\varrho''^{(k)}(\Lambda_0^{(0)},\ldots,\Lambda_0^{(k-1)}, A_k, \tilde{A}^{(k)}, \phi_k)$ multiplied by the factor $\exp O((L^k\varepsilon)^\kappa)|T_1^{(k)}|$. For the density $\varrho''^{(k)}$ we have the formula (2.46). Completing it by the expressions and integrals, which were omitted when we have passed from the right side of (3.5) to (3.7), and rescaling from the 1-lattice to the $L^k\varepsilon$-lattice, we get the expression on the right side of (3.5) again, but with $k-1$ instead of k.

Thus the inductive proof of inequality (3.5) is finished for arbitrary $k=0,1,\ldots,K$. The most interesting case for us is $k=0$.

B. The Basic Estimate Giving the Small Factors

In this section we will estimate the integral of the density $\varrho^{(K),L^k\varepsilon}$. This estimate is the most important one in this Chapter. In particular, it will give the convergence factors $\exp(-c_0 p(L^k\varepsilon)^2)$ for all the characteristic functions in $\zeta_{\Lambda_0^{(k)}}$, i.e. for all the points, bonds and blocks at which the corresponding functions of fields are large.

Let us notice that we can integrate at first with respect to scalar fields, treating the configuration $\tilde{A}^{\varepsilon}$ as a fixed external field, and next we can integrate with respect to vector fields. Thus the integral over the fields A_K, ϕ_K can be estimated in the following way, using inequality (3.5) for $k=0$

$$\begin{aligned}
&\int dA_K \int d\phi_K \varrho^{(K),L^K\varepsilon}(\Lambda_0^{(0)},\ldots,\Lambda_0^{(K-1)},A_K,\tilde{A}^{(K),\varepsilon},\phi_K)\\
&\quad\leqq \int dA_K \chi_{K,v} T_{a,L}^{L^{K-1}\varepsilon}[\chi_{K-1,\Lambda_5^{(K-1)c},v}\chi_{\Lambda_{-1}^{(K-1)}\cap\Lambda_5^{(K-1)c},v}\\
&\quad\quad\cdot\ldots\cdot T_{a,L}^{\varepsilon}[\chi_{\Lambda_{-1}^{(0)}\cap\Lambda_5^{(0)c},v}\exp(-\tfrac{1}{2}\langle A_0,(-\Delta^{\varepsilon}+\mu_0^2)A_0\rangle)\{\int d\phi_K\chi_{K,s}\\
&\quad\quad\cdot T_{a,L,\tilde{A}^{(K-1),\varepsilon}}^{L^{K-1}\varepsilon}[\zeta_{\Lambda_0^{(K-1)}}\chi_{K-1,\Lambda_5^{(K-1)c},s}\chi_{\Lambda_{-1}^{(K-1)}\cap\Lambda_5^{(K-1)c},s}\\
&\quad\quad\cdot\ldots\cdot T_{a,L,\tilde{A}^{\varepsilon}}^{\varepsilon}[\zeta_{\Lambda_0^{(0)}}\chi_{\Lambda_{-1}^{(0)}\cap\Lambda_5^{(0)c},s}\exp(-\tfrac{1}{2}\langle\phi_0,(-\Delta_{\tilde{A}^{\varepsilon}}^{\varepsilon}+m^2)\phi_0\rangle)]\\
&\quad\quad\ldots]\}]\ldots]\exp\left(\sum_{k=1}^{K}O(1)(L^k\varepsilon)^{\kappa_0}|T_{\varepsilon}|\right).
\end{aligned}\tag{3.21}$$

In the sequel we will omit the last factor because of inequality (2.118). We will estimate at first the internal integral over the scalar fields. Let us consider the expression in the curly bracket $\{\ldots\}$. We will transform it in a similar way as in Sect. A, i.e. we compose the renormalization transformations localized in the sets $\Lambda_5^{(k-1)\prime}\cap\Lambda_5^{(k)c}\subset T_{L^k\varepsilon}^{(k)}$, integrating over the fields $\phi_1,\ldots,\phi_{k-1}$ localized suitably in the sets $B^{k-1}(\Lambda_5^{(k-1)\prime}\cap\Lambda_5^{(k)c})$, $\ldots$, $B^1(\Lambda_5^{(k-1)\prime}\cap\Lambda_5^{(k)c})$. After these compositions the expression transforms into the following form

$$\begin{aligned}
\{\ldots\}=&\int d\phi_K T_{a,L,\tilde{A}^{(K-1),\varepsilon}}^{L^{K-1}\varepsilon}(\Lambda_5^{(K-1)c})\\
&\cdot\ldots\cdot T_{a,L,\tilde{A}^{(1),\varepsilon}}^{L\varepsilon}(\Lambda_5^{(1)c})\,T_{a,L,\tilde{A}^{\varepsilon}}^{\varepsilon}(\Lambda_5^{(0)c})\\
&\cdot[\chi_{K,s}\zeta_{\Lambda_0^{(K-1)}}\chi_{K-1,\Lambda_5^{(K-1)c},s}\chi_{\Lambda_{-1}^{(K-1)}\cap\Lambda_5^{(K-1)c},s}\\
&\cdot\ldots\cdot\zeta_{\Lambda_0^{(0)}}\chi_{\Lambda_{-1}^{(0)}\cap\Lambda_5^{(0)c},s}\\
&\cdot\left(\prod_{k=1}^{K}T_{a_k,L^k,\tilde{A}^{\varepsilon}}^{\varepsilon}(B^k(\Lambda_5^{(k-1)\prime}\cap\Lambda_5^{(k)c}))\right)\\
&\cdot[\exp(-\tfrac{1}{2}\langle\phi_0,(-\Delta_{\tilde{A}^{\varepsilon}}^{\varepsilon}+m^2)\phi_0\rangle)]].
\end{aligned}\tag{3.22}$$

Now let us consider the expression standing on the right of the characteristic functions. Its properties are essential for the whole analysis. Let us write it explicitly, omitting the constants in the definition of renormalization transformations:

$$\begin{aligned}
&\int d\phi_0 \upharpoonright_{\Lambda_5^{(0)}}\exp\left[-\frac{1}{2}\sum_{k=1}^{K}a_k(L^k\varepsilon)^{d-2}\sum_{x_k\in\Lambda_5^{(k-1)\prime}\cap\Lambda_5^{(k)c}}|\phi_k(x_k)-(Q_k(\tilde{A}^{\varepsilon})\phi_0)(x_k)|^2\right.\\
&\quad\left.-\tfrac{1}{2}\langle\phi_0,(-\Delta_{\tilde{A}^{\varepsilon}}^{\varepsilon}+m^2)\phi_0\rangle\right],
\end{aligned}\tag{3.23}$$

where $\Lambda_5^{(K)}=\emptyset$.

It is convenient to introduce the following new notations. We will consider a configuration Φ defined on the sum of sets $\Lambda_5^{(0)c}\cup\bigcup_{k=1}^{K}(\Lambda_5^{(k-1)\prime}\cap\Lambda_5^{(k)c})$ by the formula

$$\Phi=\Lambda_5^{(0)c}\phi_0+\sum_{k=1}^{K}(\Lambda_5^{(k-1)\prime}\cap\Lambda_5^{(k)c})\phi_k,\tag{3.24}$$

and we will write the integral (3.23) in the form

$$(3.23)=Z(\tilde{A}^{\varepsilon})\exp(-\tfrac{1}{2}\langle\Phi,\Delta(\tilde{A}^{\varepsilon})\Phi\rangle). \tag{3.25}$$

The properties of the quadratic form in the above exponent are fundamental for further analysis. We will prove that it has suitable positivity properties, such that the exponential function in (3.25) together with the functions $\zeta_{\Lambda_0^{(k)}}$ give the convergence factors. These properties are formulated in the following theorem.

Proposition 3.1. *There exists a constant $\gamma_0>0$ dependent on the space dimension d and the constant a only, and independent of ε and a choice of the sets $\Lambda_5^{(0)},\ldots,\Lambda_5^{(K-1)}$, such that for arbitrary configurations $\tilde{A}^{\varepsilon}$, Φ defined by the formulas (3.2), (3.3), (3.24), and satisfying the restrictions given by the characteristic functions in (3.21), the following inequality holds*

$$\begin{aligned}\langle\Phi,\Delta(\tilde{A}^{\varepsilon})\Phi\rangle \geqq \gamma_0 \sum_{k=0}^{K}\sum_{\langle x,x'\rangle\subset\Lambda_5^{(k-1)\prime}\cap\Lambda_5^{(k)c}} &(L^k\varepsilon)^{d-2}|U(\tilde{A}^{\varepsilon}(\langle x,x'\rangle))\phi_k(x')-\phi_k(x)|^2\\ +\gamma_0\sum_{k=0}^{K}\sum_{x\in\Lambda_5^{(k-1)\prime}\cap\Lambda_5^{(k)c}} &(L^k\varepsilon)^d m^2|\phi_k(x)|^2\\ -\sum_{k=1}^{K} O((L^k\varepsilon)^{\kappa_0})|(\Lambda_5^{(k-1)\prime}&\cap\Lambda_5^{(k)c})_1|,\end{aligned} \tag{3.26}$$

with $\kappa_0>0$. The last symbol in the above inequality denotes the measure of a set rescaled to the unit lattice, i.e. the number of points in the set. We assume $m^2\leqq O(1)$ also. If $\tilde{A}^{\varepsilon}=0$, then the inequality holds without the last sum on the right side and without any restrictions on the configuration Φ.

Remark. The above inequality can be strengthened by adding on the right side of it all the expressions of the form

$$\gamma_0(L^{k-1}\varepsilon)^{d-2}|U(\tilde{A}^{\varepsilon}(\langle x,x'\rangle))\phi_k(x')-\phi_{k-1}(x)|^2$$

for the neighbouring points $x\in\Lambda_5^{(k-2)\prime}\cap\Lambda_5^{(k-1)c}$, $x'\in\Lambda_5^{(k-1)\prime}\cap\Lambda_5^{(k)c}$, i.e. such that the intersection of the blocks $B^{k-1}(x)$, $B^k(x')$ is of "dimension" $d-1$. We will not use this generalization in the future.

The proposition follows easily from the corresponding inequalities for the actions $\Delta^{(k),L^k\varepsilon}$. We estimate the integral (3.23) by a similar integral with the covariant derivatives $-\frac{1}{2}\varepsilon^d|(D^{\varepsilon}_{\tilde{A}^{\varepsilon}}\phi_0)(b)|^2$ replaced by 0 for all the bonds b connecting the set $B^k(\Lambda_5^{(k)c})$ with $B^k(\Lambda_5^{(k)})$ for some $k=0,1,\ldots,K-1$. This inequality holds for an arbitrary configuration Φ. If we denote the integral on the right side of the obtained inequality by $Z'(\tilde{A}^{\varepsilon})\exp(-\frac{1}{2}\langle\Phi,\Delta'(\tilde{A}^{\varepsilon})\Phi\rangle)$, then we have

$$\langle\Phi,\Delta(\tilde{A}^{\varepsilon})\Phi\rangle\geqq\langle\Phi,\Delta'(\tilde{A}^{\varepsilon})\Phi\rangle, \tag{3.27}$$

because the inequality for the integrals holds for all Φ. Thus we have separated the expressions in the corresponding sets by Neumann boundary conditions. It is worth mentioning that just in this place we have omitted the additional terms described in the Remark. The form $\langle\Phi,\Delta'(\tilde{A}^{\varepsilon})\Phi\rangle$ is given by the formula

$$\langle\Phi,\Delta'(\tilde{A}^{\varepsilon})\Phi\rangle=\sum_{k=0}^{K}\langle\phi_k,\Delta^{(k),L^k\varepsilon}(B^k(\Lambda_5^{(k-1)\prime}\cap\Lambda_5^{(k)c}),\tilde{A}^{\varepsilon})\phi_k\rangle. \tag{3.28}$$

The term for $k=0$ already has the form required by the right side of (3.26), so we need the inequalities for the remaining terms. Let us rescale the k^{th} term from the $L^k\varepsilon$-lattice to the 1-lattice, $\phi_k(x)=(L^k\varepsilon)^{-\frac{d-2}{2}}\phi'_k((L^k\varepsilon)^{-1}x)$, and let us denote for simplicity $\Lambda_k=(\Lambda_5^{(k-1)\prime}\cap\Lambda_5^{(k)c})_1$.

Now inequality (3.26) of the proposition follows from

$$\begin{aligned}&\langle\phi'_k,\Delta^{(k)}(B^k(\Lambda_k),\tilde{A}^\eta)\phi'_k\rangle\\&\geqq\gamma_0\Big(\sum_{\langle x,x'\rangle\subset\Lambda_k}|U(\tilde{A}^\eta(\langle x,x'\rangle))\phi'_k(x')-\phi'_k(x)|^2\\&\quad+\sum_{x\in\Lambda_k}m^2(L^k\varepsilon)^2|\phi'_k(x)|^2\Big)-O((L^k\varepsilon)^{\kappa_0})|\Lambda_k|,\end{aligned}\tag{3.29}$$

with a constant γ_0 independent of k,Λ_k and for $\phi'_k,\tilde{A}^\eta$ satisfying suitable restrictions.

This inequality will be proved together with the properties of the covariances (formulated in Propositions I.2.1 and I.2.3).

Now let us come back to the expression (3.22). The part of it standing on the right of the characteristic functions is equal to $Z(\tilde{A}^\varepsilon)\exp(-\frac{1}{2}\langle\Phi,\Delta(\tilde{A}^\varepsilon)\Phi\rangle)$, where all the constants coming from the renormalization transformations are included in $Z(\tilde{A}^\varepsilon)$. We use Proposition 3.1 and we obtain

$$\begin{aligned}Z(\tilde{A}^\varepsilon)\exp(-\tfrac{1}{2}\langle\Phi,\Delta(\tilde{A}^\varepsilon)\Phi\rangle)&\leqq Z(\tilde{A}^\varepsilon)\exp(-\tfrac{1}{4}\langle\Phi,\Delta(\tilde{A}^\varepsilon)\Phi\rangle)\\&\quad\cdot\exp(-\tfrac{1}{4}(\text{the right side of (3.26)}))\,.\end{aligned}\tag{3.30}$$

Now we estimate the characteristic functions by 1, except the functions $\zeta_{\Lambda_0^{(k)}}$. These contain the functions $\chi^c_{Q_s^{(k)}}$, which give the restrictions of the form $(L^k\varepsilon)^{d-2}|U(\tilde{A}^\varepsilon(\langle x,x'\rangle))\phi_k(x')-\phi_k(x)|^2>p(L^k\varepsilon)^2$ for all the bonds $\langle x,x'\rangle\in Q_s^{(k)}$. The bonds of this set are contained in $\Lambda_7^{(k-1)\prime}\cap\Lambda_0^{(k)c}\subset\Lambda_5^{(k-1)\prime}\cap\Lambda_5^{(k)c}$, hence

$$\begin{aligned}&\chi^c_{Q_s^{(k)}}\exp(-\tfrac{1}{4}(k^{\text{th}}\text{ term of the right side of (3.26)}))\\&\leqq\exp(-\tfrac{1}{4}\gamma_0p(L^k\varepsilon)^2|Q_s^{(k)}|+O((L^k\varepsilon)^{\kappa_0})|\Lambda_k|)\,.\end{aligned}\tag{3.31}$$

Further, because there are no characteristic functions on the right side of (3.22) except the functions remaining in $\zeta_{\Lambda_0^{(k)}}$, so we can integrate with some exceptions, with respect to the fields $\phi_K\restriction_{\Lambda_5^{(K-1)\prime c}},\phi_{K-1}\restriction_{\Lambda_5^{(K-2)\prime c}},\dots,\phi_1\restriction_{\Lambda_5^{(0)\prime c}}$, using the normalization properties of the renormalization transformations

$$\int d\phi_{k+l}(y)\,t^{L^k\varepsilon}_{a_l,L^l,A}(\phi_{k+l}(y),\phi_k\restriction_{B^l(y)})=1\,.\tag{3.32}$$

The exceptions are when we integrate over $\phi_{k+1}(y)$ with the points $y\in P_s^{(k)}$. In this case the characteristic functions $\chi^c_{P_s^{(k)}}$ give the restrictions $(L^k\varepsilon)^{d-2}\cdot|\phi_{k+1}(y)-(Q(\tilde{A}^\varepsilon)\phi_k)(y)|^2>p(L^k\varepsilon)^2$, and instead of the integral (3.32) we have

$$\begin{aligned}&\int d\phi_{k+1}(y)\chi(\{(L^k\varepsilon)^{d-2}|\phi_{k+1}(y)-(Q(\tilde{A}^\varepsilon)\phi_k)(y)|^2\\&>p(L^k\varepsilon)^2\})\left(\frac{a(L^k\varepsilon)^{d-2}}{2\pi}\right)^{1/2N}\exp[-\tfrac{1}{2}a(L^k\varepsilon)^{d-2}\\&\cdot|\phi_{k+1}(y)-(Q(\tilde{A}^\varepsilon)\phi_k)(y)|^2]\leqq2^{1/2N}\exp(-\tfrac{1}{4}ap(L^k\varepsilon)^2)\\&\leqq\exp(-\tfrac{1}{8}ap(L^k\varepsilon)^2)\,.\end{aligned}\tag{3.33}$$

Thus, defining the functions

$$\zeta'_{\Lambda_0^{(k)}} = \sum_{\{P_v^{(k)}, \dots, R_s^{(k)}\} \text{ admissible, minimal}} \chi^c_{P_v^{(k)}} \cdot \chi^c_{Q_v^{(k)}} \chi^c_{R_v^{(k)}} \exp(-\tfrac{1}{8} a p (L^k \varepsilon)^2 |P_s^{(k)}|) \cdot \exp(-\tfrac{1}{4}\gamma_0 p (L^k \varepsilon)^2 |Q_s^{(k)}|) \exp(-p(L^k\varepsilon)^2 |R_s^{(k)}|), \tag{3.34}$$

we get the inequality

$$(\text{the right side of (3.22)}) \leqq \int d\Phi Z(\tilde{A}^\varepsilon) \cdot \exp(-\tfrac{1}{4}\langle \Phi, \Delta(\tilde{A}^\varepsilon)\Phi\rangle) \prod_{k=0}^{K-1} \zeta'_{\Lambda_0^{(k)}} \cdot \exp \sum_{k=1}^{K} O((L^k\varepsilon)^{\kappa_0})|\Lambda_k|. \tag{3.35}$$

The functions $\zeta'_{\Lambda_0^{(k)}}$ depend on the vector fields only. In the integral over Φ we make the transformation $\Phi = \sqrt{2}\Phi'$ and we get

$$\int d\Phi' Z(\tilde{A}^\varepsilon) \exp(-\tfrac{1}{2}\langle \Phi', \Delta(\tilde{A}^\varepsilon)\Phi'\rangle) \exp \tfrac{1}{2}\log 2 \sum_{k=0}^{K} |\Lambda_k|. \tag{3.36}$$

We apply the formula (3.23) to the underintegral expression. Next we use (3.32) again and it follows that the integral (3.36) is equal to

$$\int d\phi_0 \exp(-\tfrac{1}{2}\langle \phi_0, (-\Delta^\varepsilon_{A^\varepsilon} + m^2)\phi_0\rangle). \tag{3.37}$$

We apply the "diamagnetic inequality" of paper [I.5] to this integral, and we estimate it by

$$\int d\phi \exp(-\tfrac{1}{2}\langle \phi, (-\Delta^\varepsilon + m^2)\phi\rangle) = \exp(E_{0,s}). \tag{3.38}$$

Gathering together the equalities and the estimates, we get

$$(\text{the expression } \{\dots\} \text{ on the left side of (3.22)}) \leqq \prod_{k=0}^{K-1} \zeta'_{\Lambda_0^{(k)}} \exp(E_{0,s}) \exp\left(O(1) \sum_{k=0}^{K} |\Lambda_k|\right). \tag{3.39}$$

Using the above inequality we estimate the right side of (3.21) by an expression in which the curly bracket $\{\dots\}$ is replaced by the right side of (3.39). The expression we get can be estimated in a way similar to (3.22). Now it is even simpler because we do not need the characteristic functions, as it was noticed in Proposition 3.1. We get the inequality

$$\int dB \int d\psi \varrho^{(K), L^K\varepsilon}(\Lambda_0^{(0)}, \dots, \Lambda_0^{(K-1)}, B, B^{(K),\varepsilon}, \psi) \leqq \prod_{k=0}^{K-1} \zeta''_{\Lambda_0^{(k)}} \exp(E_0) \exp\left(O(1) \sum_{k=1}^{K} |\Lambda_k|\right) \exp(O(1)|T_\varepsilon|), \tag{3.40}$$

where

$$\zeta''_{\Lambda_0^{(k)}} = \sum_{\{P_v^{(k)}, \ldots, R_s^{(k)}\}\text{ admissible, minimal}} \cdot \exp(-\tfrac{1}{8}ap(L^k\varepsilon)^2|P_v^{(k)}|)\exp(-\tfrac{1}{4}\gamma_0 p(L^k\varepsilon)^2|Q_v^{(k)}|) \cdot \exp(-\tfrac{1}{4}\gamma_0 p(L^k\varepsilon)^2|R_v^{(k)}|)\exp(-\tfrac{1}{8}ap(L^k\varepsilon)^2|P_s^{(k)}|) \cdot \exp(-\tfrac{1}{4}\gamma_0 p(L^k\varepsilon)^2|Q_s^{(k)}|)\exp(-p(L^k\varepsilon)^2|R_s^{(k)}|). \tag{3.41}$$

To prove the inequality (2.117), which is the fundamental inequality necessary to complete the proof of the upper bound, it is sufficient to show that

$$\sum_{\Lambda_0^{(0)}, \ldots, \Lambda_0^{(K-1)}} \prod_{k=0}^{K-1} \zeta''_{\Lambda_0^{(k)}} \cdot \exp\left(\sum_{k=0}^{K} O(1)(L^k\varepsilon)^{\kappa_0}|\Lambda_7^{(k-1)'} \cap \Lambda_7^{(k)c}|\right) \cdot \exp\left(\sum_{k=0}^{K} O(1)|\Lambda_5^{(k-1)'} \cap \Lambda_5^{(k)c}|\right) \leqq \exp(O(1)|T_\varepsilon|). \tag{3.42}$$

C. The Combinatorial Estimate

In this section we will prove the inequality (3.42). The proof is purely combinatoric and model-independent. At first we introduce the quantities which we will use later to express all the other quantities.

Let us consider a regular partition of T_1 into a lattice of cubes, each cube is a sum of large blocks and a length of its side is bigger $r(\varepsilon)$, and less $2r(\varepsilon)$. Let us define $\mathscr{C}_0$ as the set of the cubes having common points with $\Lambda_0^{(0)c}$. Thus

$$\Lambda_0^{(0)c} \subset \bigcup_{\square \in \mathscr{C}_0} \square = \cup \mathscr{C}_0 . \tag{3.43}$$

Now if to every element of the set $P_v^{(0)} \cup \ldots \cup R_s^{(0)}$ we assign a cube $\square$ having a common point with this element and $3^d - 1$ cubes neighbouring with $\square$, then the sum of all these cubes contains the set $\cup \mathscr{C}_0$. It is so because a distance of each large block contained in $\Lambda_0^{(0)c}$ from the set $P_v^{(0)} \cup \ldots \cup R_s^{(0)}$ is $\leqq r(\varepsilon)$. Thus we have

$$|\mathscr{C}_0| \leqq 3^d(|P_v^{(0)}| + \ldots + |R_s^{(0)}|). \tag{3.44}$$

Here and in the sequel the symbol $|\cdot|$ means the number of elements in a given set.

In a similar way we divide $T_1^{(k)}$ into a regular lattice of cubes consisting of large blocks and having sides of length bigger than $r(L^k\varepsilon)$ and less than $2r(L^k\varepsilon)$. We define $\mathscr{C}_k$ as the set of these cubes, which have the common points with $\Lambda_7^{(k-1)'} \cap \Lambda_0^{(k)c}$. Because each large block of the lattice $T_1^{(k)}$ contained in $\Lambda_0^{(k)c}$ and having common points with $\Lambda_7^{(k-1)'}$ has the distance from the set $P_v^{(k)} \cup \ldots \cup R_s^{(k)}$ less than or equal to $r(L^k\varepsilon)$, so

$$|\mathscr{C}_k| \leqq 3^d(|P_v^{(k)}| + \ldots + |R_s^{(k)}|). \tag{3.45}$$

The sets $\mathscr{C}_k$, more exactly their numbers of elements $|\mathscr{C}_k|$, will be just these basic quantities and we will express the other quantities with their help.

Let us define $c_0 = \min\{\frac{1}{8}a3^{-d}, \frac{1}{4}\gamma_0 3^{-d}, 3^{-d}\}$. From (3.41) we have

$$\begin{aligned}\zeta''_{\Lambda_0^{(k)}} &\leqq \sum_{\{P_v^{(k)},\dots,R_s^{(k)}\}\text{ admissible, minimal}} \exp(-c_0 p(L^k\varepsilon)^2|\mathscr{C}_k|)\\ &\leqq \sum_{\text{all the subsets of }\Lambda_0^{(k)c}} \exp(-c_0 p(L^k\varepsilon)^2|\mathscr{C}_k|)\\ &= 2^{6|\Lambda_0^{(k)c}|}\exp(-c_0 p(L^k\varepsilon)^2|\mathscr{C}_k|),\ k=0,1,\dots,K-1.\end{aligned} \tag{3.46}$$

After easy transformations we get

$$\begin{aligned}(\text{the left side of (3.42)}) \leqq &\sup_{\{\Lambda_0^{(0)},\dots,\Lambda^{(K-1)}\}\text{ admissible}}\\ &\cdot\exp\left(-\sum_{k=0}^{K-1} c_0 p(L^k\varepsilon)^2|\mathscr{C}_k|\right)\\ &\cdot\exp\left(\sum_{k=0}^{K-1} O(1)(1+\log(L^k\varepsilon)^{-1})|\Lambda_0^{(k)c}|\right)\\ &\cdot\exp(O(1)|T_1^{(K)}|).\end{aligned} \tag{3.47}$$

In the second exponent we have gathered all the expressions dependent on $\Lambda_0^{(k)c}$. The third exponent has the required form and can be omitted in further considerations. Now we will express $|\Lambda_0^{(k)c}|$ by the help of $|\mathscr{C}_k|$. We will construct a sequence $\mathscr{D}_0, \mathscr{D}_1, \dots, \mathscr{D}_{K-1}$ of families of cubes with the properties that $\Lambda_0^{(k)c}$ is contained in the sum of cubes of the family $\mathscr{D}_k$. We take $\mathscr{D}_0 = \mathscr{C}_0$. Of course

$$\Lambda_0^{(0)c} \subset \bigcup_{\square\in\mathscr{D}_0}\square \quad\text{and}\quad |\Lambda_0^{(0)c}| \leqq (2r(\varepsilon))^d|\mathscr{C}_0|. \tag{3.48}$$

To each cube from $\mathscr{D}_0$ we add a "corridor" consisting of large blocks and of thickness $>9r(\varepsilon)$, but $<10r(\varepsilon)$. We get a cube with a side of length $<22r(\varepsilon)$ and we apply the operation $'$ to it, i.e. we take the set of small blocks. After rescaling we get a cube of the lattice $T_1^{(1)}$ with a side of length $<L^{-1}22r(\varepsilon)$. We add $\mathscr{C}_1$ to the obtained set of cubes and we denote the sum by $\mathscr{D}_1$. From the definition of $\Lambda_0^{(1)}$ we have

$$\Lambda_0^{(1)c} \subset \bigcup_{\square\in\mathscr{D}_1}\square \quad\text{and}\quad |\Lambda_0^{(1)c}| \leqq (L^{-1}22r(\varepsilon))^d|\mathscr{C}_0| + (2r(L\varepsilon))^d|\mathscr{C}_1|. \tag{3.49}$$

We continue this procedure and we get a sequence of families of cubes $\mathscr{D}_0, \mathscr{D}_1, \mathscr{D}_2, \dots, \mathscr{D}_{K-1}$ with the following properties

$$\begin{aligned}\Lambda_0^{(k)c} \subset \bigcup_{\square\in\mathscr{D}_k}\square \quad\text{and}\quad |\Lambda_0^{(k)c}| &\leqq (L^{-k}22r(\varepsilon) + L^{-(k-1)}20r(L\varepsilon)\\ &+ \dots + L^{-1}20r(L^{k-1}\varepsilon))^d|\mathscr{C}_0| + (L^{-(k-1)}22r(L\varepsilon)\\ &+ \dots + L^{-1}20r(L^{k-1}\varepsilon))^d|\mathscr{C}_1| + \dots + (L^{-1}22r(L^{k-1}\varepsilon))^d|\mathscr{C}_{k-1}|\\ &+ (2r(L^k\varepsilon))^d|\mathscr{C}_k|.\end{aligned} \tag{3.50}$$

The expression on the right sides of the above inequalities can be simplified. Thus $r(L^{k-j}\varepsilon) \leqq (1+j\log L)^r r(L^k\varepsilon)$ and the factor standing at $|\mathscr{C}_l|$ can be estimated by

$$(22r(L^k\varepsilon))^d(L^{-(k-1)}(1+(k-1)\log L)^r + \ldots + L^{-1}(1+\log L)^r + 1)$$
$$\leqq 22^d \sum_{j=0}^{\infty} L^{-j}(1+j\log L)^r (r(L^k\varepsilon))^d = O(1)(r(L^k\varepsilon))^d, \tag{3.51}$$

hence

$$|\Lambda_0^{(k)c}| \leqq O(1) r(L^k\varepsilon)^d (|\mathscr{C}_0| + \ldots + |\mathscr{C}_k|). \tag{3.52}$$

We can estimate the sum in the second exponent on the right side of (3.47) using the above inequality:

$$\sum_{k=0}^{K-1} O(1)(1+\log(L^k\varepsilon)^{-1})|\Lambda_0^{(k)c}|$$
$$\leqq \sum_{k=0}^{K-1} O(1) r(L^k\varepsilon)^d (1+\log(L^k\varepsilon)^{-1}) \sum_{j=0}^{k} |\mathscr{C}_j|$$
$$\leqq \sum_{k=0}^{K-1} O(1) r(L^k\varepsilon)^{d+1} |\mathscr{C}_k|, \tag{3.53}$$

because $K \leqq (\log L)^{-1} \log\varepsilon^{-1}$, $K-k \leqq (\log L)^{-1}\log(L^k\varepsilon)^{-1}$ and $r \geqq 2$. Now the sum on the right side of (3.43) together with the sum in the first exponent on the right side of (3.47) give

$$-\sum_{k=0}^{K-1} (c_0 p(L^k\varepsilon)^2 - O(1) r(L^k\varepsilon)^{d+1})|\mathscr{C}_k| \leqq 0 \tag{3.54}$$

if $2p \geqq (d+1)r$ and ε_0 is sufficiently small. Thus from (3.47) inequality (3.42) follows and this ends the proof of inequality (2.117).

References

1. Bałaban, T.: $(\text{Higgs})_{2,3}$ quantum fields in a finite volume. I. A lower bound. Commun. Math. Phys. **85**, 603 (1982) and references in this paper
2. Bałaban, T.: Ultraviolet Stability for a model of interacting scalar and vector fields. Aarhus Universitet, Matematisk Institut. Preprint Series 1981/82, No. 19
3. Balaban, T.: Ultraviolet stability for a model of interacting scalar and vector fields. I. A lower bound. II. An upper bound. Harvard preprints HUTMP 82/B116 and B117

Communicated by A. Jaffe

Received March 26, 1982; in revised form June 18, 1982