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On the Positivity of the Effective Action in a Theory of Random Surfaces

E. Onofri*

Theory Division, CERN, CH-1211 Geneva 23, Switzerland

Abstract. It is shown that the functional $S[\eta]=\frac{1}{24\pi}\int(\frac{1}{2}|\nabla\eta|^2+2\eta)d\mu_0$, defined on C^∞ functions on the two-dimensional sphere, satisfies the inequality $S[\eta]\geqq 0$ if η is subject to the constraint $\int(e^\eta-1)d\mu_0=0$. The minimum $S[\eta]=0$ is attained at the solutions of the Euler–Lagrange equations. The proof is based on a sharper version of Moser–Trudinger's inequality (due to Aubin) which holds under the additional constraint $\int e^\eta \mathbf{x}\,d\mu_0=0$; this condition can always be satisfied by exploiting the invariance of $S[\eta]$ under the conformal transformations of S^2. The result is relevant for a recently proposed formulation of a theory of random surfaces.

1. Introduction

Let $ds^2=e^\eta ds_0^2$ denote a Riemannian metric on the two-dimensional sphere S^2, conformal to the standard metric $ds_0^2=d\theta^2+\sin^2\theta d\phi^2$. The points of S^2 will be parametrized, as usual, by a unit vector $\mathbf{x}$, by polar co-ordinates (θ,ϕ) or by a complex variable ξ, related to $\mathbf{x}$ by stereographic projection, i.e., $\xi=\cot\frac{\theta}{2}e^{i\phi}=(x_1+ix_2/1-x_3)$. The conformal factor e^η is assumed to be C^∞. Let $\Delta=e^{-\eta}\Delta_0$ be the Laplace–Beltrami operator associated to ds^2 and let $0=\lambda_0<\lambda_1\leqq\lambda_2\leqq\ldots\leqq\lambda_n\leqq\ldots\to\infty$ be the spectrum of $-\Delta$ (Δ_0 and $\{\lambda_n^0\}$ will denote the corresponding objects belonging to ds_0^2).

It was shown in Ref. [1] that the limit

$$\frac{\det\Delta}{\det\Delta_0}\equiv\lim_{n\to\infty}\prod_{k=1}^{n}\frac{\lambda_k}{\lambda_k^0}=e^{-S[\eta]} \tag{1}$$

exists provided that e^η is normalized, i.e.,

$$\int(e^\eta-1)d\mu_0=0, \tag{2}$$

* On leave from: Istituto di Fisica dell'Università di Parma, Sezione di Fisica Teorica, Parma, Italy

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where $d\mu_0 = \sin\theta\, d\theta \wedge d\phi$. A closed expression for $S[\eta]$ was obtained, namely

$$S[\eta] = \frac{1}{24\pi} \int_{S^2} \{\tfrac{1}{2}|\nabla_0 \eta|^2 + 2\eta\} d\mu_0, \tag{3}$$

where ∇_0 is the covariant gradient with respect to ds_0^2, i.e.

$$|\nabla_0 \eta|^2 = \left(\frac{\partial \eta}{\partial \theta}\right)^2 + (\sin\theta)^{-2}\left(\frac{\partial \eta}{\partial \phi}\right)^2 \tag{4}$$

The Euler–Lagrange equation for $S[\eta]$ under the constraint Eq. (2) has the simple geometrical meaning that the metric $e^{\eta} ds_0^2$ has constant curvature. It follows that the general solution, giving all the stationary points of $S[\eta]$ is the following:

$$\eta = \eta_g^{(0)}(\xi) = 2\ln \frac{1 + |\xi|^2}{|\alpha\xi + \beta|^2 + |\gamma\xi + \delta|^2} = -2\ln(\cosh\tau + \operatorname{sh}\tau\, \mathbf{n}\cdot\mathbf{x}), \tag{5}$$

where $g = \left(\begin{smallmatrix}\alpha & \beta \\ \gamma & \delta\end{smallmatrix}\right) \in SL(2, \mathbb{C})$, $\mathbf{n}$ is a unit vector and $\tau \in (0, +\infty)$. Here $S[\eta]$ vanishes at $\eta_g^{(0)}$ and its expansion around any of these stationary points has a positive semi-definite quadratic part, hence Eq. (5) gives indeed the local minima of $S[\eta]$. Since $S[\eta]$ is interpreted as the classical action of the field $\eta(\xi)$, it is important to know whether $\eta_g^{(0)}$ are merely local minima (metastable states) or whether they are indeed the absolute minima of $S[\eta]$. The problem is less trivial than it might appear at first sight, actually its solution requires some tools from non-linear analysis which are far from trivial.

The answer turns out to be very simple, however, as given by the following

Theorem. *$S[\eta]$ is positive semi-definite under the constraint $\int (e^{\eta} - 1) d\mu_0 = 0$ and $S[\eta] = 0$ implies $\eta = \eta_g^{(0)}$ for some $g \in SL(2, \mathbb{C})$.*

2. Proof of the Main Theorem

The proof of the theorem makes essential use of an "exponential" Sobolev inequality due to Aubin, combined with the invariance of $S[\eta]$ under conformal transformations.

Let us dispose of the constraint [Eq. (2)] by introducing

$$\eta = \psi - \ln \int e^{\psi} \frac{d\mu_0}{4\pi} \tag{6}$$

(ψ is defined up to an additive constant, which we may fix by requiring $\int \psi d\mu_0 = 0$, but this will not be necessary). The unconstrained functional is now

$$S[\eta] = \tfrac{1}{3}\int \{\tfrac{1}{4}|\nabla_0 \psi|^2 + \psi\} \frac{d\mu_0}{4\pi} - \tfrac{1}{3}\ln \int e^{\psi} \frac{d\mu_0}{4\pi}, \tag{7}$$

which was introduced long ago in a purely geometrical context [2]. It was shown by Moser [3] that $S[\eta]$ is bounded from below by some absolute constant. A sharper version of the inequality may hold, however, under additional constraints on ψ such as a parity condition [4] $\psi(x) = \psi(-x)$. More generally, Aubin [5] proved that if ψ

satisfies

$$\int e^{\psi}\mathbf{x}\, d\mu_0 = 0, \tag{8}$$

then

$$\int e^{\psi}\frac{d\mu_0}{4\pi} \leqq C(\varepsilon)\exp\left\{(\tfrac{1}{8}+\varepsilon)\int|\nabla_0\psi|^2\frac{d\mu_0}{4\pi} + \int\psi\frac{d\mu_0}{4\pi}\right\} \tag{9}$$

for any $\varepsilon > 0$ and some constant $C(\varepsilon)$. Since the coefficient in the exponential is now $\frac{1}{8}+\varepsilon < \frac{1}{4}$, it follows that

$$3S[\eta] \geqq \left(\frac{1}{8}-\varepsilon\right)\int|\nabla_0\eta|^2\frac{d\mu_0}{4\pi} - \ln C(\varepsilon). \tag{10}$$

Under these circumstances it is known that *the infimum of S is actually attained at the solutions of Euler–Lagrange equation* (see Aubin [5] for details on this point and Berger [6] for the general theory).

At this point, provided η satisfies the additional constraint (8), one has the sharp inequality

$$\begin{cases} S[\eta] \geqq 0 \\ S[\eta] = 0 \Rightarrow \eta = 0. \end{cases} \tag{11}$$

In fact the Euler–Lagrange equation under the constraints (2) and (8) is

$$-\Delta_0\eta + 2 = \lambda e^{\eta} + \boldsymbol{\mu}\cdot\mathbf{x}e^{\eta}. \tag{12}$$

By integrating over S^2 one finds $\lambda = 2$. It is also known (Kazdan and Warner [7]) that the equation

$$\Delta_0\eta = 2 - (2 + \boldsymbol{\mu}\cdot\mathbf{x})e^{\eta} \tag{13}$$

does not admit any solution except for $\boldsymbol{\mu} \equiv 0$, in which case we are led back to the general solution Eq. (5). Only $\eta = 0$ satisfies the constraint (8).

Now we come to the crucial observation that allows us to apply Aubin's result in general:

Lemma. *The functional* $S[\eta]$ *is invariant under the transformations*

$$\eta \to (T_g\eta)(\xi) = \eta(g^{-1}\xi) + \chi(g^{-1},\xi), \tag{14}$$

where

$$g\xi = \frac{\alpha\xi+\beta}{\gamma\xi+\delta}, \quad g = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \in \mathrm{SL}(2,\mathbb{C}), \tag{15}$$

$$\chi(g,\xi) = 2\ln\frac{1+|\xi|^2}{|\alpha\xi+\beta|^2+|\gamma\xi+\delta|^2}. \tag{16}$$

A direct proof is not difficult, but it is rather cumbersome and not particularly enlightening. It is preferable to rely on the link between $S[\eta]$ and the Laplacian [Eq. (1)] and realize that SL(2, $\mathbb{C}$) is the largest connected group of conformal transformations of S^2 onto itself, Eq. (14) giving the transformation rule for η. The spectrum of the Laplacian is clearly the same for η and $T_g\eta$.

Now, without changing the value of $S[\eta]$, we can look for a $g\in \mathrm{SL}(2,\mathbb{C})$ such that Eq. (8) is satisfied by $T_g\eta$. If such a g exists then, by Eq. (11),

$$S[\eta]=S[T_g\eta]\geqq 0, \tag{17}$$

and $S[\eta]=0\Rightarrow T_g\eta=0$ for some g, which is the assertion of the theorem. So everything is reduced to the problem of finding a root of the equation

$$\int e^{(T_g\eta)(\xi)}\mathbf{x}(\xi)d\mu_0=0. \tag{18}$$

A simple topological argument will show that such a root actually exists, and the proof of the theorem will be complete. By inserting the definition of $T_g\eta$ and changing the integration variable to $g^{-1}\xi$, we get the equation

$$\int e^{\eta(\xi)}\mathbf{x}(g\xi)d\mu_0=0, \tag{19}$$

where g is the unknown. The function

$$\mathfrak{X}(g)=\int e^{\eta(\xi)}\mathbf{x}(g\xi)d\mu_0 \tag{20}$$

defines a continuous map $\mathfrak{X}:\mathrm{SL}(2,\mathbb{C})\to\mathbb{R}^3$ the image being contained in the unit ball $\|\mathfrak{X}\|<1$. For any $\lambda>1$ let $\mathfrak{B}_\lambda$ denote a sphere in $\mathrm{SL}(2,\mathbb{C})$ defined by

$$\mathfrak{B}_\lambda=\left\{g\in\mathrm{SL}(2,\mathbb{C})\,|\,g=u\begin{pmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{pmatrix}u^\dagger,\ u\in\mathrm{SU}(2)\right\}. \tag{21}$$

If λ is taken sufficiently large the image of $\mathfrak{B}_\lambda$ under the map $\mathfrak{X}$ is close to the sphere $\|\mathfrak{X}\|=1$; in fact,

$$\mathbf{x}\left(u\begin{pmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{pmatrix}u^\dagger\xi\right)=\mathscr{D}(u)\mathbf{x}\left(\begin{pmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{pmatrix}u^\dagger\xi\right), \tag{22}$$

$\mathscr{D}:\mathrm{SU}(2)\to 0(3)$ being the three-dimensional representation of SU(2); but

$$\lim_{\lambda\to+\infty}\mathbf{x}(\lambda^2(u^\dagger\xi))=\begin{pmatrix}0\\0\\1\end{pmatrix} \tag{23}$$

except for a set of measure zero ($u^\dagger\xi=0$) which does not contribute to the integral. Hence

$$\lim_{\lambda\to+\infty}\mathfrak{X}\left(u\begin{pmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{pmatrix}u^\dagger\right)=\mathscr{D}(u)\begin{pmatrix}0\\0\\1\end{pmatrix}. \tag{24}$$

This shows that for sufficiently large λ the map $\mathfrak{X}:\mathfrak{B}_\lambda\to\mathbb{R}^3-\{0\}$ is homotopically non-trivial. Since $\mathfrak{B}_\lambda$ is contractible (it shrinks to the identity as $\lambda\to 1$) this implies the existence of a root. [A similar argument holds in a much more general setting (Gluck [8]).]

3. Concluding Remarks

We have shown that the action functional introduced in [1] in the context of Polyakov's theory of random surfaces [9] is indeed bounded from below and attains its absolute minimum at the "classical solutions" Eq. (5). Let us recall that the symmetry of $S[\eta]$ under conformal transformations is a reflection of the fact that Polyakov's "gauge choice" $g_{ab} = \rho\delta_{ab}$ does not completely fix the gauge in the case of simply connected surfaces. Our result shows that the residual gauge freedom can be consistently eliminated by imposing the additional constraint $\int e^{\eta}\mathbf{x}d\mu_0 = 0$, which near $\eta = 0$ reduces to the condition that η be orthogonal to the zero modes. All these problems are peculiar of the simply connected surfaces. For surfaces with Euler characteristic $\chi \leqq 0$ there is no residual gauge freedom, no zero modes and the effective action is manifestly positive definite.

From a mathematical point of view, we have obtained the best constant in the Moser–Trudinger inequality, which now reads

$$\int_{S^2} e^{\psi}\frac{d\mu_0}{4\pi} \leqq \exp\left\{\frac{1}{4\pi}\int_{S^2}[\psi + \tfrac{1}{4}|\nabla_0\psi|^2]d\mu_0\right\}. \tag{25}$$

If ψ is independent of ϕ, this reduces to the elementary inequality

$$\int_0^1 e^{\psi(t)}dt \leqq \exp\left\{\int_0^1 \psi(t)\,dt + \frac{1}{4}\int_0^1 t(1-t)\psi'(t)^2\,dt\right\}, \tag{26}$$

the equality sign implying

$$\psi(t) = \ln\left[\frac{c_1}{(1+c_2t)^2}\right], (c_1 > 0, c_2 > -1). \tag{27}$$

The inequality (26) is "complementary" to the arithmetic-geometric-mean inequality [10].

Finally, the result of the theorem implies the following bound on the spectrum of Δ, which does not seem to have been noticed previously

$$\lim_{n\to\infty}\prod_{k=1}^{n}\frac{\lambda_k}{\lambda_k^0} = e^{-S[\eta]} \leqq 1, \tag{28}$$

the bound being saturated only by the standard metric (up to isometries).

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