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Brownian Motion in a Convex Ring and Quasi-Concavity

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Abstract. Let X be the Brownian motion in $\mathbb{R}^n$ and denote by τ_M the first hitting time of $M \subseteq \mathbb{R}^n$. Given convex sets $K \subseteq L \subseteq \mathbb{R}^n$ we prove that all the level sets

$$\{(x, t) \in \mathbb{R}^n \times [0, +\infty[\,; P_x[\tau_K \leqq t \wedge \tau_{L^c}] \geqq \lambda\}, \lambda \in \mathbb{R},$$

are convex.

1. Introduction

The background of the present paper is a very beautiful theorem of Gabriel [3, 4] and Lewis [5] stating that the equilibrium potential of a convex body in $\mathbb{R}^n$ relative to a surrounding convex body is quasi-concave. Below we will show the same property for the solution of the corresponding heat conduction problem with zero initial data. Here recall that a real-valued function f defined on a convex set is said to be quasi-concave if all the level sets $\{f \geqq \lambda\}$, $\lambda \in \mathbb{R}$, are convex.

Throughout, X denotes the Brownian motion in $\mathbb{R}^n$ and, for each $M \subseteq \mathbb{R}^n$, τ_M stands for the first hitting time of M, that is, $\tau_M = \inf\{t > 0; X(t) \in M\}$.

Theorem 1.1. *Suppose $K, L \subseteq \mathbb{R}^n$ are convex sets such that $K \subseteq L$. Then the function*

$$w(x, t) = P_x[\tau_K \leqq t \wedge \tau_{L^c}], \qquad (x, t) \in \mathbb{R}^n \times [0, +\infty[,$$

is quasi-concave.

Here, for short, L^c means $\mathbb{R}^n \backslash L$.

To prove Theorem 1.1 there is no loss of generality to assume that (i) K is a convex body in $\mathbb{R}^n$, (ii) L is the interior of a convex body in $\mathbb{R}^n$, and (iii) $d(K, L^c) > 0$. In what follows, we always assume (i)–(iii) are fulfilled. Then, in particular,

$$\begin{cases} \Delta w = 2w_t' & \text{in} \quad (L \backslash K) \times]0, +\infty[\\ w = 0 & \text{on} \quad \{(L \backslash K) \times \{0\}\} \cup \{\partial L \times [0, +\infty[\} \\ w = 1 & \text{on} \quad \partial K \times [0, +\infty[\end{cases}$$

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and, hence, $w(x,+\infty)$ is the equilibrium potential of K relative to L (see e.g. Friedman [2]).

The proof of Theorem 1.1 is divided into three steps. In the first step we use the isoperimetric inequality of Brownian motion to obtain a certain differential inequality, which is basic for the subsequent arguments. Theorem 1.1 then follows from Step 1 exploiting the same main line of reasoning as in the time-stationary case (Step 3). There is just one new difficulty, namely to handle the discontinuity points of w (Step 2).

2. Step 1: A Differential Inequality

Suppose $(x,t)\in(L\backslash K)\times]0,+\infty[$ and $y\in K$ are fixed.

We claim that

$$(y-x)\cdot\nabla_x w(x,t)-2tw_t'(x,t)\geqq d(y,K^c)(2\pi t)^{-1/2}\exp[-(\Phi^{-1}(w(x,t)))^2/2], \quad (2.1)$$

where $d(x,y)=|x-y|=((x-y)\cdot(x-y))^{1/2}$ is the usual metric on $\mathbb{R}^n$ and

$$\Phi(\lambda)=\int_{-\infty}^{\lambda}\exp(-s^2/2)ds/(2\pi)^{1/2}, \qquad -\infty\leqq\lambda\leqq+\infty.$$

Before the proof of (2.1) let us remark that the weaker differential inequality $(y-x)\cdot\nabla_x w(x,t)-tw_t'(x,t)\geqq 0$ is a corollary to Theorem 1.1.

In the following, let Ω be the standard Fréchet space of all continuous functions of $[0,+\infty[$ into $\mathbb{R}^n$ and assume $X-X(0)$ is represented as the identity mapping on Ω. Stated otherwise, we choose the Wiener picture of Brownian motion. The isoperimetric inequality of Brownian motion may then be described as follows.

Suppose U denotes the class of all absolutely continuous $\omega\in\Omega$ such that $\omega(0)=0$ and

$$\int_0^{+\infty}|\omega'(t)|^2dt\leqq 1.$$

Then $\Phi^{-1}(P_x[X\in A+\varepsilon U])\geqq\Phi^{-1}(P_x[X\in A])+\varepsilon, \varepsilon>0$, for each Borel set $A\subseteqq\Omega$ (Borell [1]).

To prove (2.1) we may set $y=0$ and, of course, it suffices to treat the special case when 0 belongs to the interior of K. If $\bar{B}(0;r)$ denotes the closed ball in $\mathbb{R}^n$ of centre 0 and radius $r>0$, then $\omega([0,s])\subseteqq\bar{B}(0;s^{1/2})$ for each $(\omega,s)\in U\times]0,+\infty[$, and, hence, for any fixed $\varepsilon>0$,

$$\Phi^{-1}(P_x[\tau_{K+\bar{B}(0;\varepsilon)}\leqq t\wedge\tau_{(L+\bar{B}(0;\varepsilon))^c}])\geqq\Phi^{-1}(w(x,t))+\varepsilon\cdot t^{-1/2}.$$

We now define $\varrho=1/d(0;K^c)$ and have $M+\bar{B}(0;\varepsilon)\subseteqq(1+\varepsilon\varrho)M$, $M=K,L$, because K and L are convex. Thus

$$\Phi^{-1}(P_x[\tau_{(1+\varepsilon\varrho)K}\leqq t\wedge\tau_{((1+\varepsilon\varrho)L)^c}])\geqq\Phi^{-1}(w(x,t))+\varepsilon\cdot t^{-1/2},$$

and by scaling the time,

$$\Phi^{-1}(w(x/(1+\varepsilon\varrho),t/(1+\varepsilon\varrho)^2)\geqq\Phi^{-1}(w(x,t))+\varepsilon\cdot t^{-1/2}$$

which immediately proves (2.1).

3. Step 2: Analysis of the Points of Discontinuity of w

Assume K satisfies the following additional conditions (iv) K is strictly convex and (v) $K=K_0+\bar{B}(0;r_0)$, where K_0 is a convex body in $\mathbb{R}^n$ and $r_0>0$. Let $0<T<+\infty$ be fixed and set $D=\{(x,t)\in\mathbb{R}^n;0\leqq t\leqq T\}$, $u=w_{|D}$, and

$$\tilde{u}(\xi)=\sup\{u(\eta)\wedge u(\zeta);\xi\in[\eta,\zeta],\eta,\zeta\in D\},\quad \xi\in D.$$

respectively. Finally, suppose $\varepsilon\in]0,1[$ and let $\sup[\tilde{u}-u^\varepsilon]=Q>0$.

We claim there exist $\xi_*,\eta_*,\zeta_*\in(L\backslash K)\times]0,T]$ such that $\tilde{u}(\xi_*)-u^\varepsilon(\xi_*)=Q$, $\xi_*\in]\eta_*,\zeta_*[$, and $\tilde{u}(\xi_*)=u(\eta_*)=u(\zeta_*)$.

To see this, first note that the function $\tilde{u}-u^\varepsilon$ is non-positive on $\{(K\cup L^c)\times[0,T]\}\cup\{(L\backslash K)\times\{0\}\}$ and choose for each $i\in\mathbb{N}$ a $\xi_i\in(L\backslash K)\times]0,T]$ with $q_i=\tilde{u}(\xi_i)-u^\varepsilon(\xi_i)>0$ and such that $q_i\to Q$ as $i\to+\infty$. Without any loss of generality we may assume the sequence $(\xi_i)_{i\in\mathbb{N}}$ converges to a point $\xi_*\in(\bar{L}\backslash\mathring{K})\times[0,T]$. Next choose $\eta_i,\zeta_i\in D$ satisfying $\varsigma_i\in[\eta_i,\zeta_i]$ and so that $0<u(\eta_i)\wedge u(\zeta_i)=\tilde{u}(\xi_i)-\delta_i$, where $0\leqq\delta_i\to0$ as $i\to+\infty$. If $\eta_i\in K\times[0,T]$, then by (2.1) the function $u(\zeta_i+\lambda(\eta_i-\zeta_i))$, $0\leqq\lambda\leqq1$, increases and a similar assertion is true if $\zeta_i\in K\times[0,T]$. In view of these facts it may be assumed that $\eta_i,\zeta_i\in(L\backslash K)\times]0,T]$. In the following $\hat{\eta}_i=(x(\hat{\eta}_i),t(\hat{\eta}_i))$ denotes the point in $K\times]0,T]$ which is closest to η_i and we let $H(x(\hat{\eta}_i))$ be the supporting hyperplane of K at $x(\hat{\eta}_i)$. Analogous conventions will be used below with $\hat{\eta}_i$ replaced by $\hat{\zeta}_i$ and $\hat{\xi}_i$, respectively. Then, to begin with,

$$u(\eta_i)\leqq P_{x(\eta_i)}[\tau_{H(x(\hat{\eta}_i))}\leqq t(\eta_i)],$$

that is $u(\eta_i)\leqq\Psi(d^2(x(\eta_i),K)/t(\eta_i))$, where

$$\Psi(\lambda)=\int_0^{1/\lambda}(2\pi s^3)^{-1/2}\exp(-1/(2s))ds,\qquad\lambda>0,$$

and, in a similar way, $u(\zeta_i)\leqq\Psi(d^2(x(\zeta_i),K)/t(\zeta_i))$. We now use that Ψ decreases and that the function $d^2(x,K)/t$, $(x,t)\in\mathbb{R}^n\times]0,T]$ is convex to obtain the inequality $u(\eta_i)\wedge u(\zeta_i)\leqq\Psi(\lambda_i)$, where $\lambda_i=d^2(x(\xi_i),K)/t(\xi_i)$. In particular,

$$q_i-\delta_i+u(\xi_i)\leqq\Psi(\lambda_i)\tag{3.1}$$

and, accordingly, the sequence $(\lambda_i)_{i\in\mathbb{N}}$ must be bounded. Now set $d(K,L^c)=R_0$ and choose $\bar{B}(y_i;r_0)\subseteqq K$ such that $x(\hat{\xi}_i)\in\bar{B}(y_i;r_0)$. Then

$$u(\xi_i)\geqq P_{x(\xi_i)}[\tau_{\bar{B}(y_i;r_0)}\leqq t(\xi_i)\wedge\tau_{B^c(y_i;r_0+R_0)}]$$

and introducing $\mu_i=r_0/t^{1/2}(\xi_i)$, this means that $u(\xi_i)$ does not fall below the probability

$$P_{(\lambda_i^{1/2},0,\dots,0)}[\tau_{\mu_i\bar{B}((-1,0,\dots,0);1)}\leqq1\wedge\tau_{(\mu_iB((-1,0,\dots,0);1+R_0/r_0))^c}].$$

Here, if $t(\xi_*)=0$, the same probability becomes arbitrarily close to $\Psi(\lambda_i)$ for large i, which contradicts (3.1). Thus $t(\xi_*)>0$.

From now on we assume without any loss of generality that the sequences $(\eta_i)_{i\in\mathbb{N}}$ and $(\zeta_i)_{i\in\mathbb{N}}$ both converge to the limits $\eta_*\in(\bar{L}\backslash\mathring{K})\times[0,T]$ and $\zeta_*\in(\bar{L}\backslash\mathring{K})\times[0,T]$, respectively. If η_* or $\zeta_*=\xi_*$, the continuity of u at ξ_* implies the contradiction $u(\xi_*)-u^{\xi}(\xi_*)\geqq Q$. Hence $\xi_*\in]\eta_*,\zeta_*[$. If $x(\eta_*)=x(\zeta_*)\in K$, then $t(\eta_*)$ or $t(\zeta_*)>t(\xi_*)$, and by using the continuity of u off $\partial K\times\{0\}$ we again obtain a contradiction. From these results and the strict convexity of K it follows that η_* or $\zeta_*\notin K\times[0,T]$. Assuming $\eta_*\notin K\times[0,T]$, we have $u(\eta_*)-u^{\varepsilon}(\xi_*)\geqq Q$ and, in particular, $\eta_*\in(L\backslash K)\times]0,T]$. If $\zeta_*\in K\times[0,T]$, then by (2.1) the function $u(\eta_*+\lambda(\zeta_*-\eta_*))$, $0\leqq\lambda\leqq 1$, increases and we get $u(\xi_*)-u^{\varepsilon}(\xi_*)\geqq Q$, which is absurd. Consequently, $\zeta_*\notin K\times[0,T]$ and, as above, $u(\zeta_*)-u^{\varepsilon}(\xi_*)\geqq Q$ and $\zeta_*\in(L\backslash K)\times]0,T]$. From these facts, $\tilde{u}(\xi_*)-u^{\varepsilon}(\xi_*)=Q$, and by eventually moving η_* or ζ_* closer to ξ_* we have $u(\eta_*)=u(\zeta_*)=\tilde{u}(\xi_*)$, which completes the proof of the claim at the beginning of this section.

4. Step 3: The Gabriel-Lewis Argument

To prove Theorem 1.1 there is no loss of generality to assume that the conditions (i)–(v) are fulfilled. Let u be as in the previous section. Of course, it is enough to show that the function u is quasi-concave.

Suppose contrary to this that u is not quasi-concave and choose an $\varepsilon\in]0,1[$ with $\sup[\tilde{u}-u^{\varepsilon}]>0$. Let η_*,ζ_*, and $\xi_*=\theta\eta_*+(1-\theta)\zeta_*$ be as in Step 2. This will lead us to a contradiction as follows.

First recall that $\nabla_x u\neq 0$ in $(L\backslash K)\times]0,T]$ by (2.1) and suppose $h\in\mathbb{R}^n=(\mathbb{R}^n\times\{0\})$ satisfies the inequality $h\cdot\nabla_x u(\eta_*)>0$. Then for all small $s>0$, $u(\eta_*+sh)>u(\eta_*)$ and, hence, $\tilde{u}(\xi_*+s\theta h)\geqq\tilde{u}(\xi_*)$ yielding $u(\xi_*+s\theta h)\geqq u(\xi_*)$ and $h\cdot\nabla_x u(\xi_*)\geqq 0$. From this follows that the vectors $\nabla_x u(\xi_*)$ and $\nabla_x u(\eta_*)$ are parallel and in the same way we conclude that the vectors $\nabla_x u(\xi_*)$ and $\nabla_x u(\zeta_*)$ are parallel.

Set $a=|\nabla_x u^{\varepsilon}(\xi_*)|$, $b=|\nabla_x u(\eta_*)|$, $c=|\nabla_x u(\zeta_*)|$, and $v=(\nabla_x u^{\varepsilon}(\xi_*))/a$, respectively. Suppose $h\in\mathbb{R}^n$ and $\kappa=h\cdot v\neq 0$. For each $s\in\mathbb{R}$ close to the origin there exists a unique $r=r(s)$ with $|r|$ minimal and such that $u(\eta_*+sh/b)=u(\zeta_*+rh/c)$. Writing $\xi_s=\xi_*+(\theta s/b+(1-\theta)r(s)/c)h$, we now have $u(\eta_*+sh/b)-u^{\varepsilon}(\xi_s)\leqq u(\eta_*)-u^{\varepsilon}(\xi_*)$. In particular,

$$\begin{cases} D_s(u(\eta_*+sh/b)-u^{\varepsilon}(\xi_s))_{|s=0}=0\\ D_s^2(u(\eta_*+sh/b)-u^{\varepsilon}(\xi_s))_{|s=0}\leqq 0\,. \end{cases} \tag{4.1}$$

Moreover, introducing

$$u(\eta_*+sh/b)=u(\eta_*)+\kappa s+Bs^2+o(s^2)\quad\text{as}\quad s\to 0$$

and

$$u(\zeta_*+sh/c)=u(\zeta_*)+\kappa s+Cs^2+o(s^2)\quad\text{as}\quad s\to 0,$$

it follows that

$$r(s)=s+\kappa^{-1}(B-C)s^2+o(s^2)\quad\text{as}\quad s\to 0.$$

By now setting $\lambda=\theta/b+(1-\theta)/c$ and

$$u^{\varepsilon}(\xi_*+sh)=u^{\varepsilon}(\xi_*)+\kappa as+As^2+o(s^2)\quad\text{as}\quad s\to 0,$$

the above yields

$$u^{\varepsilon}(\xi_s)=u^{\varepsilon}(\xi_*)+\lambda\kappa as+[\lambda^2A+(1-\theta)(a/c)(B-C)]s^2+o(s^2)$$

as $s\to 0$. Thus from (4.1), $a=\lambda^{-1}$ and

$$B-[\lambda^2 A+(1-\theta)(a/c)(B-C)]\leqq 0.$$

To simplify the last inequality we define $\mu=\theta/(b\lambda)<1$ and so we have $\mu B+(1-\mu)C-\lambda^2 A\leqq 0$, that is

$$\sum_{\substack{1\leqq i\leqq n\\ 1\leqq j\leqq n}}[(\mu/b^2)u''_{x_ix_j}(\eta_*)+((1-\mu)/c^2)u''_{x_ix_j}(\zeta_*)-\lambda^2(u^\varepsilon)''_{x_ix_j}(\xi_*)]h_ih_j\leqq 0.$$

Of course, the same estimate is true for all $h\in\mathbb{R}^n$ and, accordingly,

$$(\mu/b^2)\Delta u(\eta_*)+((1-\mu)/c^2)\Delta u(\zeta_*)-\lambda^2\Delta u^\varepsilon(\xi_*)\leqq 0.$$

Since $\Delta u^\varepsilon(\xi_*)=2(u^\varepsilon)'_t(\xi_*)+\varepsilon(\varepsilon-1)u^{\varepsilon-2}(\xi_*)|\nabla_x u(\xi_*)|^2$ and $|\nabla_x u(\xi_*)|>0$, necessarily

$$\mu\beta/b^2+(1-\mu)\gamma/c^2-\lambda^2\alpha<0 \tag{4.2}$$

with $\alpha=(u^\varepsilon)'_t(\xi_*)$, and where $\beta>u'_t(\eta_*)$ and $\gamma>u'_t(\zeta_*)$ are sufficiently small. But then

$$1-\alpha(\theta/\beta+(1-\theta)/\gamma)\geqq 0, \tag{4.3}$$

as the derivative

$$D_s[u(\eta_*+(0,s/\beta))\wedge u(\zeta_*+(0,s/\gamma))-u^\varepsilon(\xi_*+(0,s(\theta/\beta+(1-\theta)/\gamma)))]_{|s=0^-}$$

is non-negative. It is readily seen that (4.2) and (4.3) are non-consistent. In fact, by (4.3) the left-hand side of (4.2) does not fall below

$$\begin{aligned}&\mu\beta/b^2+(1-\mu)\gamma/c^2-\lambda^2/(\theta/\beta+(1-\theta)/\gamma)\\&\quad=\lambda^{-1}(\theta/\beta+(1-\theta)/\gamma)^{-1}[(\theta\beta/b^3+(1-\theta)\gamma/c^3)(\theta/\beta+(1-\theta)/\gamma)-\lambda^3]\end{aligned}$$

where, by the Hölder inequality,

$$(\theta\beta/b^3+(1-\theta)\gamma/c^3)(\theta/\beta+(1-\theta)/\gamma)\geqq(\theta/b^{3/2}+(1-\theta)/c^{3/2})^2\geqq\lambda^3.$$

From these estimates we have that the left-hand side member of (4.2) is non-negative, which is a contradiction.

This completes the proof of Theorem 1.1.

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