On Generalizations of the KMS-Boundary Condition

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Abstract. We investigate the possibility to generalize the KMS-boundary condition for a thermodynamical system by following essentially the same procedure that for a finite system would amount to choosing a certain class of more general density functions on phase space (or density matrices) than the ones corresponding to the canonical or grand-canonical ensemble.

1. Introduction

Gibbs states of thermodynamical systems in classical physics are described by means of conditional probabilities that are related to the energy of interaction of the configurations of finite systems with the outside world by means of an exponential function.

In [1] states of classical systems are considered where the above mentioned exponential function is replaced by a set of more general (albeit invertible) functions. These states are called regular conditional equilibrium (CE) states. A subclass of these states can, under suitable conditions, be shown to be Gibbs states [1]. The CE states are to be viewed as states obtained as thermodynamical limits of microcanonical states for finite systems. On the other hand Gibbs states are thermodynamical limits of canonical states for finite systems with specified boundary conditions (or convex combinations thereof) (cf. [2]).

Once one can show as in [1] that, under suitable conditions, regular CE states are Gibbs states, then one has made statements about the "equivalence of ensembles." Loosely speaking one can say that the exponential function, representing the Gibbs character of the state, is the thermodynamical significant object.

For quantum systems, under suitable conditions on the way one performs the thermodynamic limit on the finite volume correlation functions and the dynamics, one obtains the well-known KMS-boundary condition [3]. The KMS-boundary condition as introduced in [3] reads:

$$\int_{-\infty}^{\infty} \omega(A\alpha_t(B)) \, f(t - i\beta) \, dt = \int_{-\infty}^{\infty} \omega(\alpha_t(B)A) \, f(t) \, dt \tag{1}$$

with

$$t \to \omega(A\alpha_t B) \quad \text{a continuous function on } \mathbb{R}^1.$$

Here $\{\alpha_t\}_{t \in \mathbb{R}^1}$ is a one-parameter group of automorphisms of a C^*-algebra $\mathfrak{U}$, f a function with its Fourier transform $\hat{f} \in D$ and ω a state on $\mathfrak{U}$.

By Fourier transforming (1) we obtain

$$\hat{F}_{AB}(\check{\hat{f}} e^{-\beta \varepsilon}) = \hat{F}_{BA}(\hat{f}), \tag{2}$$

where $\hat{F}_{AB}$ is the Fourier transform of the distribution determined by the function $t \to \omega(A\alpha_t(B))$, $\check{f}(x) = f(-x)$. From (2) one sees that the Gibbs-like nature of the thermodynamical KMS state ω is reflected in the appearance of $e^{-\beta \varepsilon}$ as a multiplier in the sense of distribution theory.

Regular CE states in [1] are obtained by replacing the exponential weight-function appearing in the conditional probabilities for configurations of finite systems by another invertible function. In this paper we intend to study to what extent something similar for thermodynamical systems as a whole, can be done. More precisely we are going to study the following equation:

$$\hat{F}_{AB}(\check{\hat{f}}_\phi) = \hat{F}_{BA}(\hat{f}), \tag{3}$$

or equivalently

$$\int_{-\infty}^{\infty} \omega(A\alpha_t B) f_\phi(t)\, dt = \int_{-\infty}^{\infty} \omega(\alpha_t(B)A) f(t)\, dt, \tag{4}$$

where $t \to \omega(A\alpha_t B)$ is continuous for all A and B, $\hat{f}_\phi = \hat{f} \cdot \phi$ and ϕ is a C^∞-function from $\mathbb{R}$ into $\mathbb{C}$ such that $\phi(x) - 1$ has no zeros except at $x = 0$. States satisfying condition (4) with a given ϕ are called ϕ-KMS states. If $\phi(x) = e^{\beta x}$ (3) and (4) are nothing but the KMS-boundary condition.

Equation (3) and (4) are also obtained as an intermediate step in deriving the KMS-boundary condition for a system with a strongly continuous dynamics $\Big($i.e. $\|\alpha_t(A) - A\| \underset{t \to 0}{\to} 0\Big)$ from stability properties of the system, sufficiently fast decrease properties of the time-correlation functions up to fourth order and the assumption that the spectrum of the generator of time translations fills $\mathbb{R}^1$. By making use of the cluster-properties of the two-point function it is then shown that ϕ is an exponential function [4, 5] (cf. also [6, 7]). The cluster-properties used in the proof restrict its applicability to a one-phase region since the obtained KMS state is extremal.

We will find, without further assumptions, that for quantum lattice systems, ϕ satisfying (3) and (4) is necessarily an exponential function on the spectrum of the generator of time-translations (the Liouville-operator). Generalizing to continuous quantum systems we find that for such systems with a sufficiently regular dynamics and a compactness assumption on the set of states ω that obey (3) and (4) ϕ is also an exponential function on the spectrum of the generator of time-translations (Theorem 1, Sect. 3). Hence under these conditions (3) and (4) admit nothing but KMS states as solutions.

We will furthermore show that in the classical limit (3) coincides with the classical KMS condition for some (possibly negative) temperature.

Since the generalized KMS condition as proposed by J. V. Pulé in [8] is a particular case of the condition formulated in (3) and (4) (see Sect. 4) all our conclusions apply also to the condition formulated in [8].

2. Invariance, Separating Character and the Modular Group of a ϕ-KMS State

We start with the most simple case: a finite Quantum spin system. Some of the features discovered here will hold true in general. Let $\mathfrak{U} = \mathbb{C}^{n \times n}$ the algebra of $n \times n$ complex matrices. The one parameter group of automorphisms α_t is given by $\alpha_t(A) = U_t A U_{-t}$ where U_t is unitary. The infinitesimal generator of U_t is denoted by H. We write $H = \sum_{l=1}^{n} \varepsilon_l \cdot E_l$ where E_l are orthogonal rank one projectors and ε_l the eigenvalues of H. A state ω is defined by a positive matrix ϱ with trace $\varrho = 1$. ω satisfies the ϕ-KMS equation if

$$\int_{-\infty}^{\infty} \mathrm{Tr}(\varrho A \alpha_t(B)) f_\phi(t)\, dt = \int_{-\infty}^{\infty} \mathrm{Tr}(\varrho \alpha_t(B) A) f(t)\, dt. \tag{5}$$

Lemma 1. ϱ determines a ϕ-KMS state if and only if $\varrho = \sum_{l=1}^{n} \varrho_l E_l$ and $\varrho_i/\varrho_j = \phi(\varepsilon_j - \varepsilon_i)$.

Proof. The integrals $\int_{-\infty}^{\infty} f(t) \alpha_t(B)\, dt$ and $\int_{-\infty}^{\infty} f_\phi(t) \alpha_t(B)\, dt$ are both well defined. One computes:

$$\int_{-\infty}^{\infty} f(t) \alpha_t(B)\, dt = \sum_{k=1}^{n} \sum_{l=1}^{n} \hat{f}(\varepsilon_l - \varepsilon_k) E_k B E_l$$

and

$$\int_{-\infty}^{\infty} f_\phi(t) \alpha_t(B)\, dt = \sum_{k=1}^{n} \sum_{l=1}^{n} \phi(\varepsilon_l - \varepsilon_k) \hat{f}(\varepsilon_l - \varepsilon_k) E_k B E_l.$$

If we choose $\hat{f} = 1$ on $\Sigma := \{\varepsilon_l - \varepsilon_k | l, k = 1, 2, \ldots, n\}$ we see that (5) implies

$$\mathrm{Tr}(\varrho A \Gamma(B)) = \mathrm{Tr}(\varrho B A), \tag{6}$$

where $\Gamma : \mathfrak{U} \to \mathfrak{U}$ is defined by:

$$\Gamma(X) = \sum_{k=1}^{n} \sum_{l=1}^{n} \phi(\varepsilon_l - \varepsilon_k) E_k X E_l.$$

(6) is equivalent with

$$\Gamma(X) \cdot \varrho = \varrho X \quad \text{for all} \quad X \in \mathfrak{U}$$

and from this one easily finds $\varrho = \sum_{l=1}^{n} \varrho_l E_l (\Gamma(E_l) = E_l)$ and $\varrho_i/\varrho_j = \phi(\varepsilon_j - \varepsilon_i)$. It is easy to see that this is also sufficient. □

Remark 1. It follows from this lemma that the density matrix ϱ is a function of the Hamiltonian and ω is therefore invariant. Furthermore ϱ is invertible and ω is therefore separating. Note that also $\phi(x) \in \mathbb{R}$ and $\phi(x) \cdot \phi(-x) = 1$ for $x \in \Sigma = \{\varepsilon_i - \varepsilon_j; i, j \in (1 \ldots n)\}$. These properties turn out to be true in general. We now consider the general case and prove that every ϕ-KMS state is invariant and separating.

Lemma 2. *If ω is a ϕ-KMS state then ω is invariant for α_t.*

Proof. For $B \in \mathfrak{U}$ we write

$$G_B(t) = \omega(\alpha_t(B)).$$

We consider G_B as a distribution on $Z(Z = \hat{D})$. If we take $A = \mathbb{1}$ in (4) we get

$$(G_B, f_\phi) = (G_B, f).$$

Hence

$$(\hat{G}_B, \hat{f}_\phi) = (\hat{G}_B, \hat{f}).$$

And therefore:

$$(\hat{G}_B, f(\phi - 1)) = 0 \quad \text{for all} \quad f \in D.$$

Since $\phi(x) - 1$ has no zeros except at the origin, it follows that $\hat{G}_B$ has its support in 0. It follows from a standard argument [9a] that ω is invariant for α_t. □

To proceed further it is convenient to write Eq. (4) in the GNS representation. Let $(\mathscr{H}, \pi, \Omega)$ the GNS triple associated to ω. Since ω is invariant α_t can be implemented by a strongly continuous group of unitaries U_t such that $U_t \Omega = \Omega$. The generator of U_t, i.e. the Liouville operator, will be denoted by L. Σ is the spectrum of L. In the representation ϕ-KMS reads:

$$\int_{-\infty}^{\infty} (\Omega, \pi(A) U_t \pi(B) \Omega) f_\phi(t) \, dt = \int_{-\infty}^{\infty} (\Omega, \pi(B) U_{-t} \pi(A) \Omega) f(t) \, dt.$$

This equation can be extended to the von Neumann algebra $\pi(\mathfrak{U})''$ (cf. [9b])

$$\int_{-\infty}^{\infty} (\Omega, A U_t B \Omega) f_\phi(t) \, dt = \int_{-\infty}^{\infty} (\Omega, B U_{-t} A \Omega) f(t) \, dt. \tag{7}$$

We can rewrite this equation by defining bounded operators U_ϕ and U on $\mathscr{H}$ by

$$\int_{-\infty}^{\infty} (x, U_t y) f_\phi(t) \, dt = (x, U_\phi y),$$

$$\int_{-\infty}^{\infty} (x, U_{-t} y) f(t) \, dt = (x, U y).$$

One computes easily:

$$U_\phi = (\hat{f} \phi)^{\vee}(L) \quad \text{and} \quad U = \hat{f}(L).$$

Now (7) can be rewritten as

$$(A^* \Omega, (\hat{f} \phi)^{\vee}(L) B \Omega) = (B^* \Omega, \hat{f}(L) A \Omega) \tag{8}$$

for all $A, B \in \pi(\mathfrak{U})''$.

Lemma 3. *If ω is ϕ-KMS then Ω is separating.*

Proof. Suppose $B\Omega = 0$, then for all $\hat{f} \in D$

$$(B^*\Omega, \hat{f}(L)A\Omega) = 0.$$

We now take $\hat{f}_n \in D$ such that $\hat{f}_n \to 1$ (equal to 1 on large intervals) then $\hat{f}_n(L) \to 1$ strongly and therefore

$$(B^*\Omega, A\Omega) = 0 \quad \text{for all} \quad A.$$

Hence $B^*\Omega = 0$ and therefore, by standard arguments [9b], $B = 0$. $\quad\square$

Lemma 4. *If ω is ϕ-KMS then:*
  i) $\phi(x) \in \mathbb{R}$, $x \in \Sigma$
  ii) $\phi(x) \cdot \phi(-x) = 1$, $x \in \Sigma$.

Proof. (8) implies

$$(A^*\Omega, (\hat{f}\phi)\check{\,}(L)A^*\Omega) \in \mathbb{R}$$

for all A when $\hat{f}$ is real. Hence $(\hat{f}\phi)(\lambda) \in \mathbb{R}$ for $\lambda \in \Sigma$ and therefore i) holds. To prove ii) we apply (8) twice to get

$$(B^*\Omega, \hat{f}(L)A\Omega) = (B^*\Omega, (\hat{f}\phi\check{\phi})(L)A\Omega).$$

So

$$\hat{f}(1 - \phi\check{\phi})(L) = 0 \quad \text{for all} \quad \hat{f} \in D,$$

and therefore $\phi(x) \cdot \phi(-x) = 1$ be the spectral mapping theorem. $\quad\square$

Since Ω is separating one can define the modular operator Δ corresponding to it. Δ turns out to be a function of L.

Lemma 5. $\Delta^2 = \check{\phi}(L)$.

Proof. The reasoning is the same as the one followed in [4]. $\quad\square$

3. The ϕ-KMS Condition on Quasi-Local Algebras

In this section we consider thermodynamical systems which have the following properties:
  a) The C^*-algebra $\mathfrak{U}$ is quasi-local (cf. [10]).
  b) The dynamics $(\alpha_t)_{t \in \mathbb{R}}$ is strongly continuous on a $\sigma(\mathfrak{U}, N)$ dense C^*-subalgebra $\mathfrak{U}_0 \subset \mathfrak{U}$. N is the set of locally normal states (cf. [11]).
  c) The set of ϕ-KMS states, K_ϕ, is compact in the weak * topology of $\mathfrak{U}^*$.
  d) There exists an amenable abelian group G, like space translations, which acts on $\mathfrak{U}$ by the automorphisms $(\tau_x)_{x \in G}$ commuting with α_t. This group $(\tau_x)_{x \in G}$ acts (weakly) asymptotically abelian.
  (In particular quantum lattice systems have these properties.)

Lemma 6. *Let $\omega \in K_\phi$ and suppose ω is G-invariant. Then ω is KMS.*

Proof. Let $I_G(\mathfrak{U})$ denote the set of τ_x-invariant states on $\mathfrak{U}$. We will argue that ω has a decomposition in extremal invariant ϕ-KMS states, i.e.

$$\omega = \int_{\mathrm{Ext}(I_G(\mathfrak{U}) \cap K_\phi)} \omega_\gamma \, d\mu(\gamma) \tag{9}$$

with ω_γ extremal invariant.

First note that $K_\phi \cap I_G(\mathfrak{U})$ is a simplex. Due to assumption c) we find that $K_\phi \cap I_G(\mathfrak{U})$ is metrizable and $\mathrm{Ext}(K_\phi \cap I_G(\mathfrak{U}))$ is a G_δ. (These results can be found in [12].) Therefore the decomposition (9) can be made. Since $(\tau_x)_{x \in G}$ is weakly asymptotically abelian (i.e. $\omega([A, \tau_x(B)]) \to 0$, $x \to \infty$) we can apply Theorem 2.3 of [13] and Lemma 4.7' of [12] to conclude that $I_G(\mathfrak{U})$ is a face of $K_\phi \cap I_G(\mathfrak{U})$ and therefore the decomposition of ω is into extremal invariant states. Hence ω_γ is weakly clustering for $(\tau_x)_{x \in G}$, i.e.

$$\eta_x(\omega_\gamma(A\tau_x(B)) - \omega_\gamma(A)\,\omega_\gamma(B)) = 0 \, ,$$

where η_x is a mean over G. Let $(\mathscr{H}_\gamma, \pi_\gamma, \Omega_\gamma)$ be the GNS triple for ω_γ. We know from Lemma 4 that

$$\Delta_\gamma^2 = \check{\phi}(L_\gamma) \, ,$$

where L_γ is the Liouville operator of α_t in this representation. We have the following groups of automorphisms of the von Neumann algebra $\pi_\gamma(\mathfrak{U})''$.

1) Time evolution $\alpha_t^\gamma(A) = U_t^\gamma A U_{-t}^\gamma$, $U_t^\gamma = \exp it L_\gamma$.
2) The modular group $A \to \Delta_\gamma^{is} A \Delta_\gamma^{-is}$.
3) The translation group $\tau_x(A) = T_x A T_{-x}$.

We define the 2-parameter group

$$W_{t,s}(A) = U_t^\gamma \Delta_\gamma^{is} A \Delta_\gamma^{-is} U_{-t}^\gamma, \; A \in \pi_\gamma(\mathfrak{U})'' \, .$$

There exists a C^*-algebra $C_0 \subset \pi_\gamma(\mathfrak{U})''$ such that

  i) C_0 is $\sigma(\pi_\gamma(\mathfrak{U})'', \pi_\gamma(\mathfrak{U})''_*)$ dense in $\pi_\gamma(\mathfrak{U})''$,
  ii) $W_{t,s}$ is strongly continuous on C_0.

To see this define $\tilde{\mathfrak{U}}_0 = \{A_f \mid \hat{f} \in D, \, A \in \mathfrak{U}_0\}$ and $A_f = \int_{-\infty}^{\infty} \alpha_t(A) f(t) \, dt$.

Take C_0 the C^*-closure of $\pi_\gamma(\tilde{\mathfrak{U}}_0)$. Every $A \in \pi_\gamma(\tilde{\mathfrak{U}}_0)$ is anylytic for both α_t^γ and the modular group, therefore $W_{t,s}$ is strongly continuous on $\pi_\gamma(\tilde{\mathfrak{U}}_0)$, hence on C_0, proving ii). Since $\mathfrak{U}_0$ is $\sigma(\mathfrak{U}, N)$ dense in $\mathfrak{U}$ and $\pi_\gamma(\mathfrak{U})''_* \subset N$ (cf. [12]), it follows that i) is true. The group $W_{t,s}$ commutes with τ_x. We now apply Theorem 3.4 of [14] to conclude that the spectrum of $W_{t,s}$ is a subgroup of $\mathbb{R}^2$.

On the other hand the spectrum of $W_{t,s}$ is equal to

$$\{(\lambda, \tfrac{1}{2} \log \check{\phi}(\lambda)) \mid \lambda \in \Sigma_\gamma\} \, ,$$

where Σ_γ is the spectrum of L_γ. Hence there exists $\beta_\gamma \in \mathbb{R}$ such that

$$\check{\phi}(\lambda) = e^{\beta_\gamma \cdot \lambda}, \quad \lambda \in \Sigma_\gamma \, .$$

However the set Σ_γ is independent of γ. This was proved for KMS states in [10]. Theorem A, the same proof applies in our case and therefore:

$$\check{\phi}(\lambda) = e^{\beta \lambda} \quad \text{on } \Sigma_\omega \, .$$

So

$$\Delta_\omega^2 = e^{\beta L_\omega},$$

and therefore ω is a KMS state for α_t.   □

Theorem 1. *Under the assumptions* a)–d) *every* $\omega \in K_\phi$ *is KMS.*

Proof. Let $\omega \in K_\phi$ and let η be a mean over G. It follows from Lemma 3 in [15] that there exists a linear map $M_\eta : \mathfrak{U}^* \to \mathfrak{U}^*$ which projects on the G-invariant states. Therefore $\tilde\omega := M_\eta \omega$ is G invariant. $M_\eta \omega$ is in the weak * closure of the convex hull of the set $\{\omega \circ \tau_x | x \in G\}$ and since K_ϕ is compact $\tilde\omega$ is also a ϕ-KMS state. It follows from the proceding lemma that

$$\check\phi(\lambda) = e^{\beta \lambda} \quad \text{on } \Sigma_{\tilde\omega},$$

but since $\Sigma_{\tilde\omega} = \Sigma_\omega =$ spectrum of α_t, we find

$$\Delta_\omega^2 = e^{\beta L_\omega},$$

and therefore ω is KMS for α_t.   □

Remark. $\beta = 0$ is excluded for a quasi-local algebra that contains infinite von Neumann algebras.

4. Pulé's g-KMS Condition and the Classical Limit

In this paragraph $\mathfrak{U}$ is a C^*-algebra of observables, α_t a strongly continuous group of * automorphisms. We shall show that every g-KMS state in the sense of [8] is also ϕ-KMS. We start from the following equation of [8]:

$$\omega(\tau_{tA}(B) - B) = -i \int_0^t F f_{A, \tau_s A(B)} \, ds. \tag{10}$$

Here $\tau_{sA}(B) = \exp(isA) B \exp(-isA)$ for $A, B \in \mathfrak{U}$. $f_{A,B}(x) = \omega(\alpha_{-ix}(A)B)$ for A an analytic element for α_t. Furthermore F is a distribution in S' with compact support and $\hat F$ an entire analytic function with no zeros on the imaginary axis except at zero. We take the derivative, with respect to t, at both sides of (10) and put $t = 0$:

$$\omega(BA) = (T, f_{AB}), \tag{11}$$

where $T = F + \delta$. For B an entire analytic element for α_t with $\omega(A\alpha_z B) \in Z$ there exists a function $\xi_{AB} \in D$ such that:

$$\omega(A\alpha_z(B)) = \int_{-\infty}^{\infty} e^{-izt} \xi_{AB}(t) \, dt.$$

Now (11) can be rewritten into:

$$\omega(BA) = (T, f_{AB}) = (T, \mathscr{L}\xi_{AB}) = (\mathscr{L}T, \xi_{AB}) = \int_{-\infty}^{\infty} \hat T(ix) \xi_{AB}(x) \, dx, \tag{12}$$

where $\mathscr{L}f$ denotes the two-sided Laplace transform of f. Clearly ξ_{AB} satisfies

$$\xi_{A\alpha_s(B)}(t) = e^{-ist} \xi_{AB}(t).$$

Combining this with (12) we get for $\hat{f} \in D$:

$$\int_{-\infty}^{\infty} \omega(\alpha_s(B)A) f(s) \, ds = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{T}(ix) \, \xi_{A\alpha_s(B)}(x) f(s) \, dx \, ds$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-isx} f(s) \, \hat{T}(ix) \, \xi_{AB}(x) \, dx \, ds$$

$$= \int_{-\infty}^{\infty} \hat{T}(ix) \, \xi_{AB}(x) \, \hat{f}(x) \, dx.$$

Define $\phi(x) = \hat{T}(ix)$ then the latter equation is equivalent to

$$\int_{-\infty}^{\infty} \omega(\alpha_s(B)A) f(s) \, ds = \int_{-\infty}^{\infty} \xi_{AB}(x) \, \hat{f}(x) \, \phi(x) \, dx$$

$$= \int_{-\infty}^{\infty} \omega(A\alpha_s(B)) f_\phi(s) \, ds,$$

which is the ϕ-KMS condition. This equation can be extended to all $A, B \in \mathfrak{U}$ because in [8] α_t is strongly continuous, and the elements B, as we have chosen them, form a dense set in $\mathfrak{U}$.

The Classical Limit

The generalized KMS condition has no a priori meaning for classical systems. However if we take the classical limit, i.e. $\hbar \to 0$ in such a way that

$$\frac{[A, B]}{i\hbar} \to \{A, B\}$$

and

$$\frac{\delta^{(n)}(A)}{\hbar^n} \to \{A, H\}^{(n)},$$

where $\{A, B\}$ denotes the Poisson bracket and $\{A, H\}^{(n+1)} = \{\{A, H\}^{(n)}, H\}$ then it turns out that the generalized KMS equation yields the classical KMS condition. An easy way to see this is to start from (11)

$$\omega(BA) = T\omega(\alpha_{-ix}(A)B)$$

or

$$\omega(BA) - \omega(AB) = T\omega(\alpha_{-ix}(A)B - AB). \tag{13}$$

We divide both sides by $i\hbar$ and let $\hbar \to 0$. Take A analytic for α_t then:

$$\alpha_{-ix}(A) = \sum_{n=0}^{\infty} \frac{(-ix)^n}{n!} \delta^{(n)}(A).$$

The right hand side of (13) tends to

$$\omega(\{B, A\}),$$

whereas the left hand side tends to

$$- (T, x)\, \omega(\{A, H\}\, B)\,.$$

So this gives us

$$\omega(\{A, B\}) = (T, (x))\, \omega(\{A, H\}\, B)\,,$$

which is the classical KMS condition with inverse temperature [16]

$$\beta = (T, x)\,.$$

With $\phi(x) = \hat{T}(ix)$ one easily computes

$$\beta = \phi'(0)\,.$$

Remark 2. For a finite classical system one would now expect $\varrho(x) = e^{-\beta H(x)}$ as the only solutions of the ϕ-KMS equation. This is indeed the case. For a finite classical system ϕ-KMS should be (in view of Lemma 1)

$$\frac{\varrho(x)}{\varrho(y)} = \phi(H(x) - H(y))\,.$$

It is easy to see that this implies $\phi(x) = e^{\beta x}$ on the range of H.

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