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# Existence of a Homoclinic Point for the Hénon Map

Michał Misiurewicz[1] and Bolesław Szewc[2]

[1] Institute of Mathematics, Warsaw University, PL-00-901 Warsaw,

[2] Institute of Applied Mathematics and Statistics, SGGW-Agricultural University, PL-02-975 Warsaw, Poland

**Abstract.** We prove analytically that for the Hénon map of the plane into itself $(s, t) \mapsto (t+1-1.4a^2, 0.3s)$, there exists a transversal homoclinic point.

Curry in his paper [1] investigates numerically the map $T: \mathbb{R}^2 \to \mathbb{R}^2$:

$$T(s,t) = (t+1-as^2, bs), \qquad a=1.4, \qquad b=0.3$$

(defined first by Hénon [2]). His results suggest the existence of a transversal homoclinic point. However, he writes that his arguments cannot be considered as a rigorous proof. Moreover, he is sceptical about the possibility of obtaining such a proof with a computer.

For some values of $a$ and $b$, the existence of a transversal homoclinic point was proved analytically by Marotto [3], but it seems impossible that his methods could be applied to the case $a=1.4$, $b=0.3$.

In the present paper we show that the problem (for $a=1.4$, $b=0.3$) is less complicated than it looks. We obtain an analytical proof just by making appropriate estimates. The most complicated computations are of the type $1.42 \times 0.626 = 0.88892$, therefore a reader who wants to check all of them needs only a pen and paper (although a pocket calculator is of course better). We omit some computations in Step 1, but they are standard, and besides, they were already done by Hénon.

We introduce the new coordinates: $x = \dfrac{t}{0.3}$, $y = s$. In these coordinates the map has the form

$$f(x,y) = (y, 1-1.4y^2+0.3x).$$

It has one (minor) advantage: when we take an image, or an inverse image of a point then we introduce only one new number.

The map $f$ has a hyperbolic fixed point $P = (x_0, x_0)$, where

$$x_0 = \frac{-0.7+\sqrt{6.09}}{2.8}.$$

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**Theorem.** *There exists a transversal homoclinic point for P.*

We divide the proof into 9 steps.

*Step 1* (see [2]). Denote by $\Omega$ the quadrilateral $ABCD$, where $A=(1.4, -1.33)$, $B=\left(\frac{1.33}{3}, 1.32\right)$, $C=\left(\frac{-1.4}{3}, 1.245\right)$, $D=\left(\frac{-5}{3}, -1.06\right)$. We show that $f(\Omega)\subset\Omega$.

The line $AB$ is given by the equation

$$2.87y+7.95x=7.3129,$$

$BC$ – by $\quad 2.73y-0.225x=3.50385,$

$CD$ – by $\quad 3.6y-6.915x=7.709,$

$DA$ – by $\quad 9.2y+0.81x=-11.102.$

The images of the points $A$, $B$, $C$, $D$ are:

$$f(A)=(-1.33, -1.05646), \qquad f(B)=(1.32, -1.30636),$$

$$f(C)=(1.245, -1.310035), \qquad f(D)=(-1.06, -1.07304).$$

The image of the line $AB$ is a parabola, given by the equation

$$7.95y=-1.4x^2-0.861x+10.14387.$$

It is easy to check that this parabola lies below the lines $AB$, $BC$, and $CD$, and that $f(A)$, $f(B)$, $f(C)$, and $f(D)$ lie above the line $DA$. To complete the proof, it is enough to notice that the images of the sides $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$ of $\Omega$ are contained in parabolas "pointing upwards" and they lie below the parabola $f(AB)$ (this follows from the fact that the horizontal lines are mapped onto vertical ones).

*Step 2.* We make estimates for the fixed point and its eigenvectors.

We have

$$0.631<x_0<0.632.$$

It is easy to check that $P\in\operatorname{Int}\Omega$. From this it follows that the whole unstable manifold of $P$ is contained in $\Omega$.

The derivative of $f$ at a point $(x, y)$ is $Df(x, y)=\begin{pmatrix} 0 & 1 \\ 0.3 & -2.8y\end{pmatrix}$. The characteristic polynomial of $Df(P)$ is $\lambda^2+2.8x_0\lambda-0.3$. We have $1.76<2.8x_0<1.77$. Thus $2.07<\sqrt{(2.8x_0)^2+4\cdot 0.3}<2.1$, and we get the estimates for the eigenvalues of $Df(P)$:

$$\lambda_1<-1.915<-1.12, \qquad \tfrac{1}{8}<0.15<\lambda_2<0.17<\tfrac{1}{2}.$$

The corresponding eigenvectors are $\begin{pmatrix}1\\ \lambda_1\end{pmatrix}$ and $\begin{pmatrix}1\\ \lambda_2\end{pmatrix}$ respectively. We denote the stable and unstable manifolds of $P$ by $W^s$ and $W^u$ respectively.

*Step 3.* We investigate the unstable manifold $W^u$.

Denote: $\Phi=\{(x,y): y \geqq 0.4\}$, $\Gamma=\left\{\begin{pmatrix}a\\b\end{pmatrix}: -\frac{b}{a} \geqq 1.12\right\}$. Let $(x,y) \in \Phi$, $\begin{pmatrix}a\\b\end{pmatrix} \in \Gamma$, $\begin{pmatrix}a_1\\b_1\end{pmatrix}=Df(x,y)\begin{pmatrix}a\\b\end{pmatrix}$. We have $a_1=b$, $b_1=0.3a-2.8by$. Hence, $-\frac{b_1}{a_1}=-0.3\frac{a}{b}+2.8y$ $\geqq 2.8y \geqq 1.12$, and therefore $\begin{pmatrix}a_1\\b_1\end{pmatrix}$ belongs also to $\Gamma$. Besides, we get $|a_1| \geqq 1.12|a|$.

Notice that $P \in \operatorname{Int}\Phi$ and $\begin{pmatrix}1\\\lambda_1\end{pmatrix} \in \operatorname{Int}\Gamma$. Therefore a small piece of $W^u$, containing $P$, is contained in $\Phi$ and vectors tangent to it belong to $\Gamma$. Look at the images of this piece. As long as they are contained in $\Phi$, vectors tangent to them belong to $\Gamma$ and the absolute value of the difference of the first coordinates of their endpoints grows at least by a factor 1.12 at each step. Since the whole $W^u$ is contained in $\Omega$, this cannot continue forever. Hence, there exists a point $(c, 0.4)$ belonging to $W^u$. We take such a point closest (along $W^u$) to $P$. The point $(d_1, d_2)=f^2(c, 0.4)$ also belongs to $W^u$. We get

$$\begin{aligned} d_2 &= 1-1.4(1-1.4\cdot 0.4^2+0.3c)^2+0.3\cdot 0.4 \\ &= 1.12-1.4(0.776+0.3c)^2 < 1.4(1.676+0.3c)(0.124-0.3c). \end{aligned}$$

Since $\begin{pmatrix}x_0-c\\x_0-0.4\end{pmatrix} \in \Gamma$, we have $c > x_0 > 0.6$, and consequently $d_2 < 0$. Moreover, since $\lambda_1 < 0$, for all points $(x,y)$ lying on $W^u$ between $P$ and $(d_1, d_2)$ we have $\begin{pmatrix}x_0-x\\x_0-y\end{pmatrix} \in \Gamma$ and $y \leqq x_0$. Hence $x \geqq x_0$. Since $d_2 < 0$, among those points there is one with the second coordinate equal to 0. We denote it by $Q=(q,0)$, and the piece of $W^u$ between $Q$ and $P$ by $\Sigma^u$.

*Step 4.* We investigate the stable manifold $W^s$.

The mapping $f^{-1}$ is given by the formula

$$f^{-1}(x,y)=\left(\frac{y-1+1.4x^2}{0.3}, x\right).$$

We have

$$Df^{-1}(x,y)=\begin{pmatrix}\frac{2.8}{0.3}x & \frac{1}{0.3}\\ 1 & 0\end{pmatrix}.$$

Denote:

$$\Psi=\{(x,y): 0.475 \leqq 2.8x \leqq 1.9\}, \qquad \Delta=\left\{\begin{pmatrix}a\\b\end{pmatrix}: 2 \leqq \frac{a}{b} \leqq 8\right\}.$$

Let $(x,y) \in \Phi$, $\begin{pmatrix}a\\b\end{pmatrix} \in \Delta$, $\begin{pmatrix}a_1\\b_1\end{pmatrix}=Df^{-1}(x,y)\begin{pmatrix}a\\b\end{pmatrix}$. We have

$$\frac{a_1}{b_1}=\frac{2.8}{0.3}x+\frac{1}{0.3}\frac{b}{a}$$

and therefore

$$2=\frac{0.475}{0.3}+\frac{1}{8\cdot 0.3}\leqq\frac{a_1}{b_1}\leqq\frac{1.9}{0.3}+\frac{1}{2\cdot 0.3}=8\,.$$

Thus, $\begin{pmatrix}a_1\\ b_1\end{pmatrix}\in\Delta$. Besides, we get $|b_1|\geqq 2|b|$.

Notice that $P\in\operatorname{Int}\Psi$ and $\begin{pmatrix}1\\ \lambda_2\end{pmatrix}\in\operatorname{Int}\Delta$. Therefore a small piece of $W^s$, containing $P$, is contained in $\Psi$ and vectors tangent to it belong to $\Delta$. Look at the images of this piece (under $f^{-n}$, $n=1,2,\ldots$). As long as they are contained in $\Psi$, vectors tangent to them belong to $\Delta$ and the absolute values of the differences between the first coordinates of their endpoints and $x_0$ grow at least by the factor 2 at each step. This time $\lambda_2>0$ and we may look separately at what happens to the right and to the left of $P$.

Hence, we get the points $R=\left(\frac{1.9}{2.8},\,r\right)$ and $S=\left(\frac{0.475}{2.8},\,s\right)$ on $W^s$, closest to $P$ (along $W^s$) to the right and to the left of $P$, respectively. We denote the pieces of $W^s$ between $S$ and $P$ and between $P$ and $R$ by $\Sigma_1^s$ and $\Sigma_2^s$ respectively. We denote also $\Sigma^s=\Sigma_1^s\cup\Sigma_2^s$. For all $(x,y)\in\Sigma^s$ we have $\begin{pmatrix}x-x_0\\ y-x_0\end{pmatrix}\in\Delta$. In particular it follows that $x-x_0$ and $y-x_0$ have the same sign.

*Step 5.* We show that $\Sigma^s$ divides $f(\Omega)$ into two parts and one of them is contained in the half-plane

$$\Pi=\{(x,y):2.8x\geqq 0.475\}\,.$$

The parabola $f(CD)$ is given by the equation $y=-1.4x^2+\frac{1.08}{6.915}x+\frac{4.6023}{6.915}$. If $x=\frac{0.475}{2.8}$ then, using the estimates $0.1<x<0.17$, we get

$$y>-1.4\cdot 0.17^2+\frac{0.108+4.6023}{6.915}>-0.04046+0.68>x_0\,.$$

Therefore $\Sigma_1^s$ intersects $f(CD)$.

The parabola $f(AB)$ is given by the equation

$$y=-1.4x^2-\frac{0.861}{7.95}x+\frac{10.14387}{7.95}\,.$$

If $x=\frac{1.9}{2.8}$ then, using the estimate $x>0.65$, we get

$$y<-1.4\cdot 0.65^2+\frac{-0.861\cdot 0.65+10.14387}{7.95}<-0.5915+1.21<x_0\,.$$

Therefore $\Sigma_2^s$ intersects $f(AB)$.

Those points of intersection are unique, because vectors tangent to $\Sigma^s$ lie in the first and the third quadrants and the tops of the parabolas $f(CD)$ and $f(AB)$ lie to the left of the line $x=\frac{0.475}{2.8}$. Hence, $\Sigma^s$ divides $f(\Omega)$ into two parts. Since $\Sigma^s$ is contained in $\Pi$, one of the parts also must be contained in $\Pi$. We denote this part by $\Lambda_1$ and the other one by $\Lambda_2$.

*Step 6*. We show that $f^4(\Sigma^u)$ intersects $\Sigma^s$ at some point different from $P$.

Suppose that $f^4(\Sigma^u)$ intersects $\Sigma^s$ only at $P$. Then $f^i(\Sigma^u)$ for $i=0,1,2,3$ also intersects $\Sigma^s$ only at $P$. Since $\lambda_1<0$ and $W^u\subset\Lambda_1\cup\Lambda_2$, we get $f^i(\Sigma^u)\subset\Lambda_1$ for $i=0,2,4$ and $f^i(\Sigma^u)\subset\Lambda_2$ for $i=1,3$. In particular it follows that $f^4(Q)\in\Lambda_1\subset\Pi$.

The line $AB$ intersects the $x$-axis at the point with the first coordinate $\frac{7.3129}{7.95}<0.92$ and consequently we get $0.63<x_0<q<0.92$. For the points $f(Q)=(0,q_1)$, $f^2(Q)=(q_1,q_2)$, $f^3(Q)=(q_2,q_3)$ and $f^4(Q)=(q_3,q_4)$ we obtain consecutively the following estimates:

$$1.18<q_1<1.28\,,\qquad q_2<-0.94\,,\qquad q_3<0.15\,.$$

Since $2.8\cdot 0.15<0.475$, we get $f^4(Q)\notin\Pi$ – a contradiction.

Hence, $f^4(\Sigma^u)$ intersects $\Sigma^s$ at some point different from $P$. Therefore also $f^2(\Sigma^u)$ intersects $f^{-2}(\Sigma^s)$ at some point different from $P$. We denote this point by $H$.

*Step 7*. We estimate the vector tangent to $W^u$ at $H$.

By Step 3, $\Sigma^u\subset\{(x,y):x\geqq x_0,\ 0\leqq y\leqq x_0\}$, and the vectors tangent to $\Sigma^u$ belong to $\Gamma$. Let $(x,y)\in\Sigma^u$, $\begin{pmatrix}a\\b\end{pmatrix}\in\Gamma$. Denote $\begin{pmatrix}a_1\\b_1\end{pmatrix}=Df(x,y)\begin{pmatrix}a\\b\end{pmatrix}$, $\begin{pmatrix}a_2\\b_2\end{pmatrix}=Df^2(x,y)\begin{pmatrix}a\\b\end{pmatrix}$. We have $a_1=b$, $b_1=0.3a-2.8yb$, $a_2=b_1$, $b_2=0.3a_1-2.8(1-1.4y^2+0.3x)b_1$. Hence,

$$-\frac{b_1}{a_1}=2.8y-0.3\frac{a}{b},$$

$$-\frac{b_2}{a_2}=2.8(1-1.4y^2+0.3x)-0.3\frac{a_1}{b_1}.$$

We get

$$0<-\frac{b_1}{a_1}\leqq 2.8x_0-\frac{0.3}{1.12}<1.8-0.2=1.6\,,$$

$$-\frac{b_2}{a_2}>2.8(1-1.4x_0^2+0.3x_0)+\frac{0.3}{1.6}>2.8x_0+0.18>1.9\,.$$

Thus, for the vector $\begin{pmatrix}a_u\\b_u\end{pmatrix}$ tangent to $W^u$ at $H$, we get $-\frac{b_u}{a_u}>1.9$.

*Step 8*. We estimate the vector $\begin{pmatrix}a_s\\b_s\end{pmatrix}$ tangent to $W^s$ at $H$.

By Step 4, $\Sigma^s\subset\Psi$ and vectors tangent to $\Sigma^s$ belong to $\Delta$. Therefore also vectors tangent to $f^{-1}(\Sigma^s)$ belong to $\Delta$. Denote $H=(x_2,y_2)$, $f(H)=(x_1,y_1)$,

$Df(H)\begin{pmatrix}a_s\\b_s\end{pmatrix}=\begin{pmatrix}a_1\\b_1\end{pmatrix}$. The vector $\begin{pmatrix}a_1\\b_1\end{pmatrix}$ is tangent to $f^{-1}(\Sigma^s)$ and therefore it belongs to $\Delta$. We consider two cases:

*Case 1.* $f(H)\in\Sigma^s$. Then $\begin{pmatrix}a_s\\b_s\end{pmatrix}\in\Delta$.

*Case 2.* $f(H)\notin\Sigma^s$. Since for $x\geqq x_0$ the first coordinate of the inverse images of a vector from the first quadrant grows at least by the factor $\frac{2.8}{0.3}x_0>5$ at each step, no point of $\Sigma_2^s\backslash\{P\}$ can belong to $W^u$. Therefore $f^2(H)\in\Sigma_1^s$. Hence, if $f^2(H)=(x,y)$ then $0<x<y\leqq x_0$. Using the formula for $f^{-1}$ and the inequality $x^2\geqq 0.8x-0.16$, we get (notice that $y_1=x$)

$$x_1\geqq\frac{x-1+1.4(0.8x-0.16)}{0.3}=\frac{2.12y_1-1.224}{0.3}.$$

Consequently, $y_1\leqq\frac{0.3x_1+1.224}{2.12}$. Therefore (notice that $y_2=x_1$):

$$x_2\leqq\frac{1}{2.12}x_1+\frac{1.224}{2.12\cdot 0.3}-\frac{1}{0.3}+\frac{1.4}{0.3}x_1^2=\frac{1.4}{0.3}x_1^2+\frac{1}{2.12}x_1-\frac{0.896}{0.626}.$$

Since $H\in\Omega$, $H$ lies to the right of the line $CD$. Therefore

$$x_2\geqq\frac{3.6y_2-7.709}{6.915}.$$

Since $y_2=x_1$, we get

$$1.4x_1^2+\left(\frac{0.3}{2.12}-\frac{0.3\cdot 3.6}{6.915}\right)x_1+\left(-\frac{0.896}{0.626}+\frac{7.709}{6.915}\right)\cdot 0.3\geqq 0.$$

The second coordinate of the point of intersection of the $y$-axis and $f(CD)$ is $\frac{4.6023}{6.915}>0.66>x_0$. Along with the estimates from Step 5, this proves that the set

$$\left\{(x,y):0\leqq x\leqq\frac{0.475}{2.8},\ y\leqq x_0\right\}$$

is disjoint from $f(\Omega)$. Thus, $x_1<0$. Hence we get

$$\begin{aligned}0&\leqq 1.4x_1^2+(0.14-0.16)x_1+(-1.42+1.12)\cdot 0.3\\&=1.4x_1^2-0.02x_1-0.09.\end{aligned}$$

Since $0.02^2+4\cdot 1.4\cdot 0.09>0.5$, we have

$$x_1\leqq\frac{0.02-0.7}{2.8}=-\frac{0.68}{2.8}.$$

Since $a_s = \frac{2.8}{0.3} x_1 a_1 + \frac{1}{0.3} b_1$, $b_s = a_1$ and $\frac{a_1}{b_1} \geqq 2$, we get

$$\frac{a_s}{b_s} \leqq \frac{-0.68+0.5}{0.3} = -0.6.$$

In both cases we obtain $-\frac{1}{2} \leqq -\frac{b_s}{a_s} \leqq \frac{1}{0.6} < 1.7$.

*Step 9*. From Steps 7 and 8 it follows that $\begin{pmatrix} a_u \\ b_u \end{pmatrix} \nparallel \begin{pmatrix} a_s \\ b_s \end{pmatrix}$ for every (non-zero) vectors $\begin{pmatrix} a_u \\ b_u \end{pmatrix}$ and $\begin{pmatrix} a_s \\ b_s \end{pmatrix}$ tangent to $W^u$ and $W^s$ respectively at $H$. Consequently, $W^u$ intersects $W^s$ at $H$ transversally.

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