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Hölder Continuity of Sample Paths in Euclidean Field Theory

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Abstract. Hölder continuity of sample paths of the stochastic process $\xi_t(f)=\varphi(\delta_t f)$ $(f\in\mathscr{S}(R^{d-1}))$ in Euclidean field theory is proved under some assumptions on correlation functions. These assumptions are fulfilled in $P(\varphi)_2$ and in theories in which the GHS inequality holds. The continuity index α is determined by the condition $\int d\sigma(p)|p_0|^{2\alpha}<\infty$, where $d\sigma(p)$ is the Fourier transform of the two-point function.

Introduction

Euclidean field theory is assumed to be defined by a Euclidean invariant measure $d\mu(\varphi)$ on $\mathscr{S}'(R^d)$. However, the support of the measure μ is usually concentrated on a much smaller subspace of $\mathscr{S}'(R^d)$. It is important to determine this support in order to know which operations are we allowed to perform on the generalized functions $\varphi\in\mathscr{S}'(R^d)$. The support properties of the free Euclidean measure were investigated in [1–3]. It has been shown that, if we smear out $\varphi(t,\mathbf{x})$ with respect to $d-1$ variables then $\xi_t(f)=\varphi(\delta_t f)$ as a function of t is Hölder continuous with continuity index $\alpha<\frac{1}{2}$ for φ from the support of the free Euclidean measure. Cannon [2] has given also some arguments that the continuity of paths being a local property should hold for $P(\varphi)_2$ measures which are absolutely continuous with respect to the free measure when restricted to bounded regions from R^2 [4]. A continuity of paths in $P(\varphi)_2$ has been shown by Albeverio and Hoegh-Krohn [5] to result from a theory of stochastic processes with values in infinite dimensional spaces.

In this paper we prove Hölder continuity of the sample paths $\xi_t(f;\omega)$ under some assumptions on the correlation functions. Estimates on the correlation functions lead then to bounds on $|\xi_t(f)-\xi_{t'}(f)|$. Applying Fröhlich estimates in $P(\varphi)_2$ [6] we get Hölder continuity of sample paths with continuity index arbitrarily close to 1/2. The GHS inequality [7] estimates the generating functional in terms of the two-point function. This inequality applies to φ^4 and

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some other models [8] in any dimension (cp. [9]). We show that even if such models are non-canonical (assuming that fixed time fields make sense) the sample functions are still Hölder continuous, but with continuity index smaller than in the canonical field theory.

Hölder Continuity of Sample Functions

We usually begin with a random field $\varphi(g)$ over the Schwartz space $\mathscr{S}(R^d)$ in order to define a probability measure on $\mathscr{S}'(R^d)$. This random field can be then extended to a larger test-function space $\mathscr{T}$ if only $g_n \to g$ in $\mathscr{T}$ implies $\varphi(g_n) \to \varphi(g)$ in measure [10]. We assume that a norm $\| \ \|$ on $\mathscr{T}$ exists such that

(a) $|E[e^{u\varphi(g)}]| \leqq e^{u^{2k}\|g\|^{2k}}$ for u sufficiently large (here k is chosen to be an integer) i.e. $E\left[\exp\left(\frac{u}{\|g\|}\varphi(g)\right)\right]$ is an entire analytic function of finite order.

In field theory the Källen-Lehmann spectral representation holds

$$S_2(x-y) = E[\varphi(x)\varphi(y)] = \int d\sigma(p) e^{ip(x-y)}$$
$$= \int dp e^{ip(x-y)} \int_0^\infty d\varrho(m) \frac{1}{p^2+m^2}. \tag{1}$$

We assume that

(b) there exist positive numbers $\varepsilon < \frac{1}{2}$ and $\alpha < \frac{1}{2}$ such that for $f \in \mathscr{S}(R^{d-1})$

$\int d\sigma(p) |\tilde{f}(\mathbf{p})|^2 |p_0|^{2q} < \infty$ if $-\varepsilon \leqq q \leqq \alpha$

(c) $\|f\|_\alpha \equiv \| |p_0|^\alpha \tilde{f} \| < \infty$ with $\tilde{f} \in \mathscr{S}(R^{d-1})$

(d) there exists an integer $r > \frac{1}{2\varepsilon}$ such that for certain C

$E[\varphi^{2r}(g)] \leqq C(E[\varphi^2(|g|)])^r$.

Remarks

1. Due to the Euclidean invariance the norm $\| \ \|$ can be chosen Euclidean invariant, then $\|g_a\| = \|g\|$ where $g_a(x) = g(x-a)$.

2. $\int d\sigma(p) |\tilde{f}(\mathbf{p})|^2$ should be finite if the fixed time fields [10, 11] are to exist. If $m \geqq m_p > 0$ in the Källen-Lehmann representation (1) then $\int d\sigma(p) |\tilde{f}(\mathbf{p})|^2 |p_0|^{-2\varepsilon}$ is finite for any $\varepsilon < \frac{1}{2}$. If $\int d\varrho(m) < \infty$ then α can be arbitrarily close to 1/2.

3. From (a) it follows that for r sufficiently large $E[\varphi^{2r}(g)] \leqq C_r \|g\|^{2r}$. However, $\| \ \|$ need not be continuous with respect to the S_2 norm.

In Appendix we put together some known results in order to show that conditions (a)–(d) are fulfilled in $P(\varphi)_2$ (see [6, 12]) with α arbitrarily close to 1/2 and $2k$ being the degree of the polynomial $P(\varphi)$. If the GHS inequality holds (see [7–9]) then the condition (a) is fulfilled with $k=1$ and $\|g\|^2 = S_2(|g|, |g|)$. In such a case (d) is true for any r and the Hölder continuity is determined by the two-point function.

Under the assumptions on the two-point function we can define (see [10, 11]) $\xi_t(f) = \varphi(\delta_t f)$ where $\delta_t f(x) = \delta(t-x_0) f(\mathbf{x})$ with $f \in \mathscr{S}(R^{d-1})$. Further on we shall also use Fourier transforms. Then $\delta_t f(p) = e^{ip_0 t} \tilde{f}(\mathbf{p})$ and $\xi_t(f) = \tilde{\varphi}(e^{ip_0 t} \tilde{f})$. We are interested now whether (almost all) sample functions of the process $\xi_t(f)$ are continuous. In the language of Euclidean field theory this means that $\varphi(\delta_t f)$ is continuous in t for almost all φ from the support of the Euclidean measure. We will prove

Theorem 1. *If conditions (a)–(d) are fulfilled, then there exists a constant A and a random variable $B_T \geqq 2T$ finite almost certainly (moreover integrable) such that the inequality*

$$|\xi_t(f)-\xi_{t'}(f)| \leqq A\|f\|_\alpha |t-t'|^\alpha \left[\lg \frac{B_T}{|t-t'|}\right]^{1-1/2k} \tag{2}$$

holds for almost every sample function $\xi_{(\cdot)}(f;\omega)$ if $|t|, |t'| \leqq T$.

Before starting the proof let us define an auxiliary stochastic process [this can be done due to the assumption (b)]

$$\eta_t(f) = \tilde{\varphi}(e^{ip_0t}|p_0|^\alpha \tilde{f}). \tag{3}$$

The condition b) ensures continuity of $\eta_t(f)$ in probability and Lebesgue integrability of $E[|\eta_s(f)|]$ on a finite interval. Therefore, due to Doob's theorem [13, p. 61, Theorems 2.6 and 2.7] we can choose a measurable and Lebesgue integrable process, which is equivalent to $\eta_t(f)$ in order to define the integral

$$\zeta_t(f) = \int_0^t \eta_s(f) ds \tag{4}$$

as an integral over the sample functions of the process $\eta_s(f)$. We will introduce still another stochastic process as a sampoe integral over $\eta_s(\alpha > 0)$

$$\xi_t^n(f) = \frac{\Gamma(1-\alpha)}{\pi} \sin\frac{\alpha\pi}{2} \int_{-n}^{n} |s|^{\alpha-1} e^{-n^{-\gamma}|s|^\delta} \eta_{t-s}(f) ds \tag{5}$$

here $0 < \gamma < \delta < \varepsilon$ and $\frac{\Gamma(1-\alpha)}{\pi} \sin\frac{\alpha\pi}{2} |s|^{\alpha-1}$ is the Fourier transform of $|p_0|^{-\alpha}$. So we expect that as $n \to \infty \xi_t^n(f) \to \xi_t(f)$ [cp. Eq. (3)]. Now our proof of Theorem 1 goes through the following steps

(i) we show first that $\zeta_t(f)$ (4) has paths which are Hölder continuous with continuity index 1.

(ii) using (i) and integrating by parts in (5) we express ξ_t^n by ζ_t and show that ξ_t^n fulfills the estimate (2) with constants A, B independent of n

(iii) we prove that $\xi_t^n(f) \to \bar{\xi}_t(f)$ almost certainly for every t, where $\bar{\xi}_t(f)$ is a stochastic process equivalent to $\xi_t(f)$.

The step (iii) together with the estimate (2) for ξ_t^n ends the proof of Theorem 1. We begin with

Lemma 2. *Assuming (a)–(d) there exists a constant $r_0 > 0$ such that the random function $\exp\left|\frac{r}{\|f\|_\alpha}\eta_s(f)\right|^\beta$ $\left(\text{where } \beta = \frac{2k}{2k-1}\right)$ is Lebesgue integrable over s for $0 \leqq r \leqq r_0$ and the random variable ($B_T \geqq 2T$)*

$$B_T = \int_{-T}^{T} ds \exp\left|\frac{r}{\|f\|_\alpha}\eta_s(f)\right|^\beta \tag{6}$$

is finite almost certainly. Moreover, $\lim_{T\to\infty} \frac{1}{T} B_T$ *exists almost certainly and* $E[B_T] \leqq BT$ *where B does not depend on f.*

Proof. It is sufficient to show that $\int_{-T}^{T} ds\, E\left[\exp\left|\frac{r}{\|f\|_\alpha}\eta_s(f)\right|^\beta\right]$ because then Doob's Theorem 2.7 ([13]) implies that the integral (6) makes sense and $E[B_T]$ is finite. In order to show that the expectation value is finite let us notice that

Lemma 2a. *There exist constants* a_k, b_k *such that*

$$\exp|x|^{\frac{2k}{2k-1}} \leqq b_k \int_{-\infty}^{\infty} dy \exp(-y^{2k}) \exp(a_k xy). \tag{7}$$

To prove (7) we use the identity

$$-y^{2k} + zy = -p(y-v)^{2k} + v^{2k} + v^{2k}[p(1-y/v)^{2k} - (y/v)^{2k} - 1 + 2ky/v] \tag{8}$$

where $v = \left(\frac{z}{2k}\right)^{\frac{1}{2k-1}}$. It is then easy to show that there exists p such that the last term in Eq. (8) is positive. Hence

$$\int dy \exp(-y^{2k} + zy) \geqq \int dy \exp[-p(y-v)^{2k}] \exp v^{2k}.$$

Now, using Eq. (7) we get

$$E\left[\exp\left|\frac{r}{\|f\|_\alpha}\eta_s(f)\right|^\beta\right] \leqq b_k \int_{-\infty}^{\infty} dy \exp(-y^{2k}) E\left[\exp\left(ya_k \frac{r}{\|f\|_\alpha}\eta_s(f)\right)\right] \tag{9}$$

and from the assumptions (a) and (c)

$$E\left[\exp\left(ya_k \frac{r}{\|f\|_\alpha}\eta_s(f)\right)\right] \leqq \exp[y^{2k}(a_k r)^{2k}] \quad \text{as} \quad y\to\infty.$$

Therefore for $a_k r < 1$ the integral in Eq. (9) is finite and bounded by a constant $\frac{1}{2}B$. This result is at the same time sufficient for the Birkhoff-Khinchine ergodic theorem [13] to hold, hence $\lim_{T\to\infty} \frac{1}{T} B_T$ exists almost certainly.

Remark. As defined by Eq. (6) B_T depends on f and we are able to show only that $E[B_T]$ is bounded by a constant independent of f, but not B_T itself. This obscures the f-dependence[1] of the right side of Eq. (2) (A does not depend on f). We could prove that B_T is bounded by a random variable independent of f if we could show e.g. that $e^{|\eta_s|}$ is Lebesgue integrable, where $|\eta_s|$ is the norm of the linear functional $\eta_s(f)$.

Lemma 3. *The (Hölder) inequality* (2) *holds for the process* $\zeta_t(f)$ *[Eq. (4)] with* $\alpha = 1$.

Proof. Due to the Jensen inequality $F\left(\frac{1}{h}\int_t^{t+h} ds\,\eta_s\right) \leqq \frac{1}{h}\int_t^{t+h} F(\eta_s)ds$ for any convex function F and from Eq. (4) $\int_t^{t+h} ds\eta_s(f) = \zeta_{t+h}(f) - \zeta_t(f)$. Using this we get from

1 For Gaussian processes a natural f-independent formulation is discussed in [17]

Eq. (6) [with $F(x)=\exp|x|^{\beta}$]

$$|\zeta_{t+h}(f)-\zeta_t(f)|\leqq r^{-1}h\|f\|_\alpha\left(\lg\frac{B_T}{h}\right)^{1/\beta}. \tag{10}$$

Lemma 4. *The process* ξ_t^n *[Eq. (5)] fulfills the inequality* (2) *with* A *and* B_T *independent of* n.

Proof. We shall divide the integration in Eq. (5) into four parts $(-n,-1)$, $(-1,0)$, $(0,1)$ and $(1,n)$ and show that each piece of ξ_t^n fulfills the inequality (2) with constants independent of n. First, integrating by parts we get (with $\int\eta_{t-s'}ds'=-\zeta_{t-s}$)

$$\xi_t^{(1,n)}=\int_1^n s^{\alpha-1}e^{-n^{-\gamma}s^\delta}\eta_{t-s}ds=-n^{\alpha-1}e^{-n^{\delta-\gamma}}\zeta_{t-n}+e^{-n^{-\gamma}}\zeta_{t-1}$$
$$-(1-\alpha)\int_1^n s^{\alpha-2}e^{-n^{-\gamma}s^\delta}\zeta_{t-s}ds-n^{-\gamma}\delta\int_1^n s^{\alpha-2+\delta}e^{-n^{-\gamma}s^\delta}\zeta_{t-s}ds. \tag{11}$$

Then from Eq. (10) we have

$$|\xi_{t+h}^{(1,n)}(f)-\xi_t^{(1,n)}(f)|\leqq r^{-1}n^{\alpha-1}e^{-n^{\delta-\gamma}}h\|f\|_\alpha\left(\lg\frac{B_{T+n}}{h}\right)^{1/\beta}$$
$$+r^{-1}e^{-n^{-\gamma}}h\|f\|_\alpha\left(\lg\frac{B_T}{h}\right)^{1/\beta}+(1-\alpha)r^{-1}h\|f\|_\alpha\int_1^n s^{\alpha-2}e^{-n^{-\gamma}s^\delta}\left(\lg\frac{B_{T+s}}{h}\right)^{1/\beta}ds$$
$$+\delta r^{-1}n^{-\gamma}\|f\|_\alpha h\int_1^n s^{\alpha-2+\delta}e^{-n^{-\gamma}s^\delta}\left(\lg\frac{B_{T+s}}{h}\right)^{1/\beta}ds. \tag{12}$$

Here B_{T+s} depends on the range of the time variable of the process ζ_{t-s} as follows from the derivation of Eq. (10). Now, when $n\to\infty$ $\frac{B_{T+n}}{T+n}$ is convergent almost certainly due to Lemma 2. Hence there exists a random variable $\tilde{B}$ (with $E[\tilde{B}]<\infty$) such that $\frac{B_{T+s}}{T+s}\leqq\tilde{B}$ (because B_{T+s} is an increasing function of s). Then

$$\lg\frac{B_{T+s}}{h}\leqq\lg\frac{\tilde{B}}{h}+\lg(T+s)\leqq\frac{\lg(T+s)}{\lg(T+1)}\lg\frac{\tilde{B}(T+1)}{h}$$

and both integrals on the right sight of Eq. (12) are bounded by a constant. Therefore we have

$$|\xi_{t+h}^{(1,n)}(f)-\xi_t^{(1,n)}(f)|\leqq Ch\|f\|_\alpha\left(\lg\frac{\tilde{B}(T+1)}{h}\right)^{1/\beta} \tag{13}$$

where C can be chosen independent of n.

Consider now $\xi_t^{(0,1)}$. We get by integrating by parts (we have here $\int\eta_{t-s'}ds'=\zeta_t-\zeta_{t-s}$)

$$\xi_t^{(0,1)}=\int_0^1 s^{\alpha-1}e^{-n^{-\gamma}s^\delta}\eta_{t-s}ds=e^{-n^{-\gamma}}(\zeta_t-\zeta_{t-1})-\lim_{\varepsilon\to0}\varepsilon^{\alpha-1}(\zeta_t-\zeta_{t-\varepsilon})$$
$$+(1-\alpha)\int_0^1 s^{\alpha-2}e^{-n^{-\gamma}s^\delta}(\zeta_t-\zeta_{t-s})ds+n^{-\gamma}\int_0^1 s^{\alpha+\delta-2}e^{-n^{-\gamma}s^\delta}(\zeta_t-\zeta_{t-s})ds. \tag{14}$$

The second term on the right side of Eq. (14) vanishes owing to Eq. (10). The first term is Hölder continuous with index 1 due to Lemma 3. In order to show that $\xi_t^{(0,1)}$ is Hölder continuous with index α it is sufficient to show this property for the third term on the right sight of Eq. (14). Then the last term will be Hölder continuous with index $\alpha+\delta$. So let us consider

$$\begin{aligned}\left|\int_0^1 s^{\alpha-2}e^{-n^{-\gamma}s^\delta}(\zeta_{t+h}-\zeta_{t+h-s}-\zeta_t+\zeta_{t-s})ds\right| &\leqq \int_0^h s^{\alpha-2}|\zeta_{t+h}-\zeta_{t+h-s}|ds \\ &+\int_0^h s^{\alpha-2}|\zeta_{t-s}-\zeta_t|ds+\int_h^1 s^{\alpha-2}|\zeta_{t+h}-\zeta_t|ds+\int_h^1 s^{\alpha-2}|\zeta_{t+h-s}-\zeta_{t-s}|ds \\ &\leqq 2r^{-1}\|f\|_\alpha\int_0^h s^{\alpha-1}\left(\lg\frac{B_T}{s}\right)^{1/\beta}ds+2r^{-1}\|f\|_\alpha h\left(\lg\frac{B_T}{h}\right)^{1/\beta}\int_h^1 s^{\alpha-2}ds \\ &\leqq K\|f\|_\alpha h^\alpha\left(\lg\frac{B_T}{h}\right)^{1/\beta}. \end{aligned} \tag{15}$$

The integrals over $(-n,-1)$ and $(-1,0)$ correspond to the replacement $\eta_{t-s}\to\eta_{t+s}$. So we get the same bounds (13) (15) and finally the inequality (2).

Lemma 5. *For each t there exists a linear functional $\bar{\xi}_t$ on $\mathscr{S}(R^{d-1})$ such that $\lim_{n\to\infty}\xi_t^n(f)=\bar{\xi}_t(f)$ almost certainly.*

Proof. It is sufficient to show that (a consequence of the Borel-Cantelli Lemma [13]) there exists a sequence $\{a_n\}$, $a_n\geqq 0$ and $\sum_{n=1}^\infty a_n<\infty$ such that

$$\sum_{n=1}^\infty a_n^{-2r}E[(\xi_t^{n+1}(f)-\xi_t^n(f))^{2r}]<\infty. \tag{16}$$

From the definition (5) we have

$$\begin{aligned}\xi_t^{n+1}-\xi_t^n=K\int_{-n}^n |s|^{\alpha-1}(e^{-(n+1)^{-\gamma}|s|^\delta}-e^{-n^{-\gamma}|s|^\delta})\eta_{t-s}ds \\ +K\int_n^{n+1} s^{\alpha-1}e^{-(n+1)^{-\gamma}s^\delta}\eta_{t-s}ds+K\int_n^{n+1} s^{\alpha-1}e^{-(n+1)^{-\gamma}s^\delta}\eta_{t+s}ds.\end{aligned}$$

Clearly $|E[\eta_s\eta_{s'}]|\leqq E[\eta_0^2]$ hence

$$\begin{aligned}E\left[\left(\int_n^{n+1} s^{\alpha-1}e^{-(n+1)^{-\gamma}s^\delta}\eta_{t-s}ds\right)^2\right]&\leqq E[\eta_0^2]\left(\int_n^{n+1} s^{\alpha-1}e^{-(n+1)^{-\gamma}s^\delta}ds\right)^2 \\ &\leqq E[\eta_0^2]e^{-2(n+1)^{-\gamma}n^\delta}.\end{aligned}$$

Next

$$E[(\int K(s)\eta_{t-s}(f)ds)^2]\leqq E[(\int |K(s)|\eta_{t-s}(|f|)ds)^2] \tag{17}$$

because the two-point function (1) is positive and η_t (3) can be defined as a limit in the mean of $\varphi(u_t f)$ with $-u_t$ positive (the Fourier transform of $-|p_0|^\alpha$ is positive). Using the inequality

$$e^{-(n+1)^{-\gamma}|s|^\delta}-e^{-n^{-\gamma}|s|^\delta}\leqq|s|^\delta]n^{-\gamma}-(n+1)^{-\gamma}[\leqq n^{-1-\gamma}|s|^\delta.$$

We get from Eq. (17)

$$E\left[\left(\int_{-n}^{n} |s|^{\alpha-1}(e^{-(n+1)^{-\gamma}|s|^{\delta}}-e^{-n^{-\gamma}|s|^{\delta}})\eta_{t-s}(f)ds\right)^{2}\right]$$
$$\leqq \gamma^2 n^{-2-2\gamma}\int_{-n}^{n} ds \int_{-n}^{n} ds'|s|^{\alpha+\delta-1}|s'|^{\alpha+\delta-1}E[\eta_s(|f|)\eta_{s'}(|f|)] \tag{18}$$

where

$$E[\eta_s(|f|)\eta_{s'}(|f|)]=\int d\sigma(p)|\tilde{|f|}(\mathbf{p})|^2|p_0|^{2\alpha}e^{ip_0(s-s')}.$$

We will show that the last integral in (18) is bounded by a constant. Because $\int d\sigma(p)|\tilde{f}(\mathbf{p})|^2<\infty$ Fubini theorem allows to interchange p and s integration. Now

$$\left|\int_{-n}^{n} ds\, e^{isp_0}|s|^{\alpha+\delta-1}\right| = 2|p_0|^{-\alpha-\delta}\left|\int_0^{n|p_0|} dx\, x^{\alpha+\delta-1}\cos x\right| \leqq A|p_0|^{-\alpha-\delta}$$

for $\alpha+\delta-1<0$. Hence (18) is bounded by $A^2\gamma^2 n^{-2-2\gamma}\int d\sigma(p)|\tilde{|f|}(\mathbf{p})|^2|p_0|^{-2\delta}$ which is finite if $\delta\leqq\varepsilon$ due to the assumption b). Therefore

$$E[(\xi_t^{n+1}(f)-\xi_t^n(f))^2]\leqq K_1e^{-2n^{\delta-\gamma}}+K_2n^{-2-2\gamma}. \tag{19}$$

Now, from the assumption d) through the same argument, which led us to Eq. (17) [i.e. applying assumption d) to $g=\int u_{t-s}fK(s)ds$, where $u_{t-s}\to\delta_{t-s}$] we get

$$E[(\int K(s)\eta_{t-s}(f)ds)^{2r}]\leqq C(E[(\int |K(s)|\eta_{t-s}(|f|)ds)^2])^r \tag{20}$$

and combining (19) and (20)

$$E[(\xi_t^{n+1}-\xi_t^n)^{2r}]\leqq \bar{K}_1e^{-2n^{\delta-\gamma}}+\bar{K}_2n^{-2r(1+\gamma)}.$$

Hence, choosing $a_n=n^{-1-\varkappa}$, $\gamma<\delta\leqq\varepsilon$, $r>\dfrac{1}{2\varepsilon}$ with $\varkappa$ and $\delta-\gamma$ sufficiently small we get the convergence of the series (16).

Lemma 6. *$\bar{\xi}_t(\mathbf{x})$ and $\varphi(t,\mathbf{x})$ are equivalent random fields, i.e. the random variables $\int dt\bar{\xi}_t(g_t)$ and $\varphi(g)$ have the same probability distribution for $g_t(x)=g(t,\mathbf{x})$ with $g\in\mathscr{S}(R^d)$.*

Proof. Because ξ_t^n are uniformly Hölder continuous the convergence $\xi_t^n(f)$ to $\bar{\xi}_t(f)$ is uniform on any finite interval. Hence, for each sample function the integral $\int_{-R}^{R}\bar{\xi}_t(g_t)dt$ exists and is equal to $\lim_{n\to\infty}\int_{-R}^{R}\xi_t^n(g_t)dt$. Moreover one can show that the limit $R\to\infty$ exists almost certainly for $g\in\mathscr{S}(R^d)$ (see [14] Sect. 5.4 for the proof). So $\int\bar{\xi}_t(g_t)dt$ makes sense and we will show that its characteristic function coincides with that of $\varphi(g)$. In fact, we have

$$|E[\exp(i\varphi(g))]-E[\exp(i\int\bar{\xi}_t(g_t)dt)]$$
$$=\lim_{R\to\infty}\lim_{n\to\infty}\left|E\left[\exp(i\varphi(g))\left(1-\exp\left(-i\varphi(g)+i\int_{-R}^{R}\xi_t^n(g_t)dt\right)\right)\right]\right|$$
$$\leqq\lim_{R\to\infty}\lim_{n\to\infty}\left(E\left[\left(\varphi(g)-\int_{-R}^{R}\xi_t^n(g_t)dt\right)^2\right]\right)^{1/2}=0$$

where in the last step we show first [similarly as in Eq. (19)] that $E[(\varphi(g(t,\cdot)) - \xi_t^n(g_t))^2] \to 0$ and then apply Lebesgue dominated convergence theorem. Lemma 6 concludes the proof of Theorem 1 as $\bar{\xi}_t(f)$ fulfills the inequality (2).

III. Some Additional Remarks

First, let us note that the result formulated in Theorem 1 cannot be essentially improved under given model-dependent assumptions. The integrability of B_T implies integrability of expressions like $\exp|t-s|^{-\varepsilon}|\xi_t(f)-\xi_s(f)|^\beta$ and this needs an assumptions on the growth of $E[\exp u\varphi(g)]$ as $u\to\infty$. Next integrability of B_T implies that the correlation functions of $\xi_{t_i}-\xi_{t_{i+1}}$ are bounded by their covariances. This looks similar to the condition d) (although d) itself says nothing about the short time behaviour because one cannot make sense of $|\delta_t-\delta_s|$). Finally it will be shown below (Proposition 7) that the continuity index α (if $<\frac{1}{2}$) cannot be larger than the value determined by b).

On the other hand if we restrict ourselves to a single path then the logarithmic term can be included into a constant and only the continuity index characterizes the path. In this sense paths in $P(\varphi)_2$ and in the free theory are the same both having continuity index α arbitrarily close to $\frac{1}{2}$. However, dealing with a set of paths we should be able to distinguish (even when restricted to a bounded region of space-time) between the paths of free and interacting theory on the basis of their concentration on the (ξ,t) plane.

The finiteness of the integral $\int d\sigma(p)|\tilde{f}(\mathbf{p})|^2|p_0|^{2\alpha}$ appearing in the assumption (b) is related to the behaviour of $E[(\xi_t(f)-\xi_s(f))^2]$ for $t-s\to 0$. This can be seen from the formula

$$\int_\varepsilon^1 dt\, t^{-1-2\alpha}E[(\xi_t(f)-\xi_0(f))^2]=2\int d\sigma(p)|\tilde{f}(\mathbf{p})|^2\int_\varepsilon^1 dt\, t^{-1-2\alpha}(1-e^{ip_0t}) \tag{21}$$

as it can be shown that the limit $\varepsilon\to 0$ exists if and only if the integral $\int d\sigma(p)|\tilde{f}(\mathbf{p})|^2|p_0|^{2\alpha}$ is finite. Moreover, if $\int d\sigma(p)|\tilde{f}(\mathbf{p})|^2<\infty$ and $m>0$ in the Källen-Lehmann representation (1) then $E[(\xi_t-\xi_0)^2]$ is differentiable on $[\varepsilon,1]$ (because $S_2\sim e^{-m|t|}$ for large m and $t\neq 0$) and the derivative is integrable on $[0,1]$. Hence integrating by parts on the left side of Eq. (21) we get that for $2\alpha<1$ the limit $\varepsilon\to 0$ of (21) exists if and only if the limit $\varepsilon^{-2\alpha}E[(\xi_\varepsilon-\xi_0)^2]$ exists. Therefore if this limit does not exist then for δ sufficiently small

$$E[(\xi_t(f)-\xi_0(f))^2]\geqq A_\delta t^{2\alpha} \quad \text{if} \quad 0\leqq t<\delta. \tag{22}$$

Next, one can show that a non-canonical short distance behaviour $\sim|x-y|^{2-d-\eta}$ of the two-point function is in agreement with Eq. (22) only if $\alpha<\frac{1}{2}-\frac{1}{2}\eta$. So in a non-canonical field theory (if such exists at all) we have to take into account paths with worse continuity properties. We formulate this obvious consequence of Eq. (22) as

Proposition 7. *If for a positive* $\alpha_0<\frac{1}{2}\int d\sigma(p)|\tilde{f}(\mathbf{p})|^2|p_0|^{2\alpha_0}$ *is infinite then the set of paths which are Hölder continuous with index* α_0 *has measure less than* 1[2].

2 For Gaussian random fields one can show [1, 14] that such paths have measure zero. Let us note that integrability of the random variable B_T appearing in the inequality (2) is not assumed here

Appendix. Estimates on Correlation Functions

We put together here some results showing that our assumptions a)–d) are fulfilled in a class of models. In $P(\varphi)_2$ the folowing estimates on correlation functions have been established ([6]; see also [16])

$$E[\varphi(g_1)\dots\varphi(g_n)] \leqq K^n(n!)^{1-\frac{1}{2k}}\|g_1\| \dots \|g_n\| \tag{A.1}$$

with

$$\|g\| = \|(-\Delta+m_0^2)^{-1}g\|_1 + \|(-\Delta+m_0^2)^{-1}g\|_{2k} + \|(-\Delta+m_0^2)^{-1/2}g\|_2 \tag{A.2}$$

where $\| \ \|_p$ denotes L_p-norm and $2k$ is the degree of the polynomial $P(\varphi)$. From Eq. (A.1) the condition a) follows and $\||p_0|^\alpha \tilde{f}\| < \infty$ if $\alpha < \frac{1}{2}$. Then Glimm and Jaffe [12] proved that for every n

$$E[\varphi(g_1)\dots\varphi(g_n)] \leqq L^n n! \|(-\Delta+m_0^2)^{-1/2}g_1\|_2 \dots \|(-\Delta+m_0^2)^{-1/2}g_n\|_2 . \tag{A.3}$$

This estimate shows that the assumptions b) and d) are fulfilled in $P(\varphi)_2$. Next, one can easily show [15] that if (GHS inequality [7])

$$E_\mu^T[\varphi^3(|g|)] = \lim_{\lambda\to 0} \frac{d^3}{d\lambda^3} \ln E[\exp(\lambda\varphi(|g|))] \leqq 0 \tag{A.4}$$

in a theory with an interaction $V(\varphi) - \mu\varphi (\mu \geqq 0)$ then when $\mu \to 0$ (assuming $E_0[\varphi] = 0$)

$$E[\exp(u\varphi(g))] \leqq \exp(u^2 E[\varphi^2(|g|)]) . \tag{A.5}$$

Hence a)–d) are fulfilled with $\|g\|^2 = E[\varphi^2(|g|)]$ if only the tow-point function fulfills b). There is a class of lattice models (see [9]) including φ^4 and exponential interactions in which the GHS inequality holds. This inequality allows also to show the existence of the continuum limit [9], but one does not know whether this limit is non-trivial in $d > 3$ dimensions.

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Note Added in Proof. Repeating the procedure of proving Theorem 1 to each component of the vector $x \in R^d$ of the random field $\varphi_g(x) \equiv \varphi(g(\cdot - x))$ we could show

$$|\varphi_g(x) - \varphi_g(x')| \leqq C|x-x'|^\alpha + D|x-x'|^\alpha \left| lg \frac{1}{|x-x'|} \right|^{1-1/2k}$$

where C is a random variable and D is a constant, assuming

$$\int d\sigma(p)|\tilde{g}(p)|^2 |p_\mu|^\alpha < \infty \quad \text{and} \quad \|\tilde{g}(p)|p_\mu|^\alpha\| < \infty\,, \quad \alpha \leqq 1.$$