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The Quasi-Classical Limit of Quantum Scattering Theory

Kenji Yajima*
Mathematical Research Institute, ETH-Zürich, Switzerland, and Department of Mathematics, University of Tokyo, Tokyo, Japan**

Abstract. We study the quasi-classical limit of the quantum mechanical scattering operator for non-relativistic simple scattering system. The connection between the quantum mechanical and classical mechanical scattering theories is obtained by considering the asymptotic behavior as $\hbar \to 0$ of the quantum mechanical scattering operator on the state $\exp(-ip\cdot a/\hbar)f(p)$ in the momentum representation.

Introduction

Let us consider the Schrödinger operator

$$H^h = -\frac{\hbar^2}{2m}\Delta + V(x), \quad \Delta = \frac{\partial^2}{\partial x_1^2} + \dots + \frac{\partial^2}{\partial x_1^2} \tag{0.1}$$

in the Hilbert space $\mathscr{H} = L^2(\mathbb{R}^n)$ and let $H_0^h = -\frac{\hbar^2}{2m}\Delta$. Here $\hbar = \frac{h}{2\pi}$ and h is the small positive parameter called Planck's constant. We assume the potential $V(x)$ to satisfy the following condition.

Assumption (A). (1) $V(x)$ is a real valued infinitely differentiable function on $\mathbb{R}^n$.
(2) For any multi-index α, there exist constants $m(\alpha) > |\alpha| + 1$ and $C_\alpha > 0$ such that

$$\left|\left(\frac{\partial}{\partial x}\right)^\alpha V(x)\right| \leqq C_\alpha(1 + |x|)^{-m(\alpha)}.$$

Under this condition H_0^h and H^h are self-adjoint operators with the domain $\mathscr{D}(H_0^h) = \mathscr{D}(H^h) = H^2(\mathbb{R}^n) =$ the Sobolev space of order 2. Furthermore it is well known that the wave operators $W_\pm^h$ defined as

$$W_\pm^h = \text{s} - \lim_{t \to \pm\infty} e^{itH^h/\hbar} e^{-itH_0^h/\hbar}$$

exist and are complete:

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** Current address

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$\mathscr{R}(W_+^h) = \mathscr{R}(W_-^h) =$ the spectrally absolutely continuous subspace of $\mathscr{H}$ w.r.t. H^h.

(See Agmon [1] and Kuroda [10] for more general results.) The scattering operator S^h for the system is defined as

$$S^h = (W_+^h)^* W_-^h$$

and is therefore a unitary operator on $\mathscr{H}$. We shall study the asymptotic behavior of the operator S^h as $h \to 0$ and its relation to the corresponding classical mechanical scattering theory for the Hamiltonian

$$H(x, \xi) = \xi^2/2m + V(x). \tag{0.2}$$

In the classical mechanical scattering theory, the following results are well-known under the Assumption (A): i) For any $(a, \eta) \in \mathbb{R}^n \times \mathbb{R}^n$ $(\eta \neq 0)$ there exists a unique solution $(x_-(t, a, \eta), \xi_-(t, a, \eta))$ of the equation of motion

$$\frac{dx}{dt} = \frac{\xi}{m}, \quad \frac{d\xi}{dt} = -\operatorname{grad} V(x) \tag{0.3}$$

such that

$$\begin{aligned} &\lim_{t \to -\infty} |x_-(t, a, \eta) - t\eta/m - a| = 0, \\ &\lim_{t \to -\infty} |\xi_-(t, a, \eta) - \eta| = 0; \end{aligned} \tag{0.4}$$

ii) there exists a closed null set (i.e., Lebesgue measure zero) $e \subset \mathbb{R}^n \times \mathbb{R}^n$ such that for any $(a, \eta) \notin e$, there exists $(a_+(a, \eta), \eta_+(a, \eta)) \in \mathbb{R}^n \times \mathbb{R}^n$ such that

$$\begin{aligned} &\lim_{t \to \infty} |x_-(t, a, \eta) - t\eta_+(a, \eta)/m - a_+(a, \eta)| = 0, \\ &\lim_{t \to \infty} |\xi_-(t, a, \eta) - \eta_+(a, \eta)| = 0; \end{aligned} \tag{0.5}$$

iii) the classical scattering operator S^{cl} defined on $\mathbb{R}^n \times \mathbb{R}^n \backslash e$ as $S^{cl}(a, \eta) = (a_+(a, \eta), \eta_+(a, \eta))$ is a C^1-canonical mapping.

We define as $e(a) = \{\eta \in \mathbb{R}^n; (a, \eta) \in e\} \cup \{0\}$. Obviously $e(a)$ is a null set for almost all $a \in \mathbb{R}^n$, i.e., $e(a)$ has Lebesgue measure zero in $\mathbb{R}^n$. (See Herbst [6], Simon [12] and Huniziker [9]). We write as $\dot{x}(t) = \dfrac{dx}{dt}$ and $\dot{\xi}(t) = \dfrac{d\xi}{dt}$.

It will be proved that for any $a \in \mathbb{R}^n$, $\eta_+(a, \eta)$ and $a_+(a, \eta)$ are smooth functions of $\eta \in \mathbb{R}^n \backslash e(a)$. We define as

$$e(a)^{ex} = \{\eta \in \mathbb{R}^n \backslash e(a) : \det[\partial\eta_+(a, \eta)/\partial\eta] = 0\} \cup e(a). \tag{0.6}$$

For any $f \in \mathscr{H}$, let us write as $f_a^h(\eta) = e^{-i(a \cdot \eta/\hbar)} f(\eta)$. Our main theorem can be stated as follows.

Theorem. *Let Assumption* (A) *be satisfied. Let* $a \in \mathbb{R}^n$ *and let* $e(a)^{ex}$ *be defined as* (0.6). *Then for any* $f \in \mathscr{H}$ *with* supp f *a compact subset of* $\mathbb{R}^n \backslash e(a)^{ex}$

$$\lim_{h\downarrow 0}\left\| (\hat{S}^h f_a^h)(\eta) - \sum_{\eta=\eta_+(a,\eta_j)} e^{i\operatorname{Ind}\gamma(\eta,\eta_j)\pi/2 + i(S(\eta,\eta_j)-\eta\cdot a_+(a,\eta_j))/\hbar} \cdot \left|\det[\partial\eta_+(a,\eta)/\partial\eta]|_{\eta=\eta_j}\right|^{-1/2} f(\eta_j(\eta))\right\| = 0, \tag{0.7}$$

where the summation is taken over all η_j's satisfying $\eta = \eta_+(a,\eta_j)$.
Here $\hat{S}^h$ is the scattering operator in momentum representation: $\hat{S}^h = \mathscr{F}^h S^h(\mathscr{F}^h)^{-1}$,

$$\mathscr{F}^h f(\eta) = h^{-n/2}\int e^{-ix\cdot\eta/\hbar} f(x)dx; \tag{0.8}$$

$\operatorname{Ind}\gamma(\eta,\eta_j)$ *is so-called Keller-Maslov-index of the orbit* $\{x_-(t,a,\eta_j), \xi_-(t,a,\eta_j)\}$; $S(\eta,\eta_j)$ *is defined as*

$$S(\eta,\eta_j) = \lim_{\substack{t\to\infty\\ s\to-\infty}} \left\{ \int_s^t L(x(u,a,\eta_j), \dot{x}(u,a,\eta_j))du + \frac{s}{2m}\eta_j^2 - \frac{t}{2m}\eta^2 \right\}, \tag{0.9}$$

where $L(x,\dot{x}) = \frac{m}{2}\dot{x}^2 - V(x)$ is the Lagrangian of the system.

The connection between the quantum mechanics (0.1) and the classical mechanics (0.2) has been discussed since the advent of the quantum mechanics and there are several mathematically rigorous arguments. The refined form of WKBJ-method by Maslov [11] and the rigorous form of Ehrenfest's theorem by Hepp [4] seem to be outstanding, among others. However, unfortunately, these works are mostly concerned with the dynamics in finite time and the connection of the scattering theories has not been discussed (see, however, Hepp's unpublished note [5]). In this paper we study the connection of these scattering theories using the WKBJ-method which seems to be fit best for the purpose. The main tools for proving the theorem are the approximate fundamental solution constructed by Fujiwara [3] and the L^2-boundedness theorem for certain kind of oscillatory integral operators by Asada-Fujiwara [2].

Before proceeding to the text we want to discuss here about a meaning of the theorem. For a smooth real function $S(\eta)$, we can consider the wave function $e^{-iS(\eta)/\hbar}f(\eta)$ to represent, asymptotically as $\hbar\to 0$, the ensemble of classical particles $\left(\frac{\partial S}{\partial\eta}(\eta), \eta\right)$ with momentum distribution $|f(\eta)|^2 d\eta$. Hence by taking $e^{-ia\cdot\eta/\hbar}f(\eta)$ as an initial state, we prepare it as to represent the ensemble of particles concentrated at the configuration $x = a$ with momentum distribution $|f(\eta)|^2 d\eta$. Thus the incoming particles are prepared as $x(t) \sim a + t\eta/m$, $\xi(t) \sim \eta$ at remote past. What the theorem says is that the final state $\hat{S}^h(e^{-ia\cdot\eta/\hbar}f)$ is asymptotically represented as the incoherent superposition of the wave functions

$$e^{i\operatorname{Ind}\gamma(\eta,\eta_j)\pi/2 + i(S(\eta,\eta_j)-\eta\cdot a_+(a,\eta_j))/\hbar}\left|\det\left(\frac{\partial\eta_+}{\partial\eta}(a,\eta_j)\right)\right|^{-1/2} f(\eta_j)$$

which represents the particles

$$\left(-\frac{\partial}{\partial\eta}(S(\eta,\eta_j) - \eta\cdot a_+(a,\eta_j)), \eta\right) = (a_+(a,\eta_j), \eta_+(a,\eta_j))$$

with the momentum distribution

$$\left|\det\left(\frac{\partial\eta_+}{\partial\eta_j}\right)(a,\eta_j)\right|^{-1} |f(\eta_j)|^2 d\eta = |f(\eta_j)|^2 d\eta_j .$$

Moreover, since $a_+(a,\eta_k) \neq a_+(a,\eta_j)$ for $j \neq k$, by the canonical property of S^{cl}, this incoherent superposition turns to the coherent one, as $h \to 0$. Thus we may say that the classical mechanical scattering can be represented as a limit of $h \to 0$ of the quantum scattering theory.

The plan of the paper is as follows. In Sect. 1 we shall study the asymptotic behavior as $h \to 0$ of the wave packet

$$e^{-itH_0^\epsilon/h}\tilde{f}_a^h, \ \tilde{f}_a^h(x) = h^{-n/2}\int e^{ix\cdot\eta/h} f_a^h(\eta)d\eta .$$

In Sects. 2 and 3, we shall prepare the materials from the classical mechanics which are necessary in the following sections. In Sect. 4 we shall study the asymptotic behavior of $W_-^h\tilde{f}_a^h$. In Sects. 5 and 6, we shall study $\hat{S}^h f_a^h$. In the last section, 7, some remarks will be given. Especially a result which is related to Dollard's cone scattering theory [13] will be obtained.

We list here the notation and the conventions used in the paper.

For domains Ω_1 and $\Omega_2, \Omega_1 \subset\subset \Omega_2$ means that Ω_1 is a precompact subset of Ω_2 and $\bar{\Omega}_1$, the closure of Ω_1, is contained in Ω_2. $L^2(\Omega)$ is the Hilbert space of all square integrable functions on Ω equipped with the inner product and the norm as

$$(f,g) = \int_\Omega f(x)\cdot\overline{g(x)}dx, \quad \|f\| = (f,f)^{1/2}.$$

If $\Omega = \mathbb{R}^n$, we simply write as $L^2(\mathbb{R}^n) = \mathscr{H}$. For $x = (x_1, \dots, x_n) \in \mathbb{R}^n, |x| = (\sum x_j^2)^{1/2}$ is the Euclidean norm. For the multi-index $(\alpha_1, \dots, \alpha_n) = \alpha, \alpha_j \in \mathbb{N}, |\alpha| = \sum \alpha_j$. $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$ and $(\partial/\partial x)^\alpha = (\partial/\partial x_1)^{\alpha_1} \dots (\partial/\partial x_n)^{\alpha_n} \cdot dx = dx_1 \dots dx_n$. $\mathscr{B}(\mathbb{R}^n)$ is the space of all C^∞-bounded functions with their derivatives. For vector-valued function $f(x), \frac{\partial f}{\partial x}$ stands for the matrix $\left(\frac{\partial f_j}{\partial x_k}\right)_{j,k}$. For matrix $A = (a_{ij}), |A| = \sup_i \left\{\sum_j a_{ij}^2\right\}^{1/2}$. For $h > 0, \mathscr{F}^h$ is the Fourier transform defined by (0.8) and if $h = 1$, we omit the index h. For the Fourier transform we write as $\hat{f}^h = \mathscr{F}^h f$, $\tilde{f}^h = \mathscr{F}^{h*} f = (\mathscr{F}^h)^{-1} f$. $f_a^h(\eta) = e^{-i\eta\cdot a/h} f(\eta)$ and $\tilde{f}_a^h = \mathscr{F}^{h*} f_a^h$.

π_1 and π_2 are the projections from $\mathbb{R}^n \times \mathbb{R}^n$ to $\mathbb{R}^n$: $\pi_1(x,p) = x$ and $\pi_2(x,p) = p$. For $a \in \mathbb{R}^n$ we define as $\pi_{2,a}^*(p) = (a,p)$ and $\pi_1^*(x) = (x,0)$. For an operator T in a Hilbert space $\mathscr{X}$, $\mathscr{D}(T), \mathscr{R}(T)$ stand for the domain and the range of the operator. The constants appearing in the formulas are distinguished in one context but in the other they are not distinguished and may be written by the same symbol if it is clear that which constants are meant. We sometimes omit the indices and suffices or other parameters if no confusions are feared.

1. Uniform Estimate for the Free Propagator

In Sect. 4 we shall study the asymptotic behavior of the wave operator W_-^h on the coherent state $\tilde{f}_a^h$. As a starting point we study here the asymptotic behavior of $\exp(-itH_0^h/\hbar)\tilde{f}_a^h$ as $h \to 0$. Actually we prove the following theorem, the essential feature of which is that the estimate is uniform with respect to time t outside $t = 0$.

Theorem 1.1. *For any $a \in \mathbb{R}^n$ and $t > 0$ let us define the operator $T_{0,a}^h(t)$ on $\mathscr{H}$ as*

$$T_{0,a}^h(t)f(x) = e^{im(x-a)^2/2t\hbar}(m/it)^{n/2}f(m(x-a)/t). \tag{1.1}$$

Then
1) *$T_{0,a}^h(t)$ is a unitary operator in $\mathscr{H}$ and is strongly continuous in $t > 0$;*
2) *for any $\delta > 0$,*

$$\lim_{h \to 0} \sup_{|t| \geqq \delta} \| e^{-itH_0^h/\hbar}\tilde{f}_a^h - T_{0,a}^h(t)f \| = 0. \tag{1.2}$$

Proof. Since $\exp(-itH_0^h/\hbar)\tilde{f}_a^h(x) = \exp(itH_0^h/\hbar)\tilde{f}_0^h(x-a)$ and $T_{0,a}^h(t)f(x) = T_{0,0}^h(t)f(x-a)$, it is sufficient to prove 1) and 2) for the case $a = 0$. In what follows we omit the index $a = 0$. Since the first statement is clear, we prove 2) only. Since $T_0^h(t)$ and $\exp(-itH_0^h/\hbar)\mathscr{F}^{h*}$ are unitary, the standard limiting procedure shows that it suffices to prove (1.2) for $f \in C_0^\infty(\mathbb{R}^n)$. By the Plancherel's theorem

$$\begin{aligned}
e^{-itH_0^h/\hbar}\tilde{f}^h(x) &= h^{-n/2}\int e^{(x\cdot p - p^2t/2m)/\hbar}f(p)dp \\
&= h^{-n/2}e^{imx^2/2t\hbar}\int e^{-it(p-mx/t)^2/2m\hbar}f(p)dp \\
&= h^{-n/2}e^{imx^2/2t\hbar}(m\hbar/it)^{n/2}\int e^{im\hbar y^2/2t - imx\cdot y/t}\hat{f}(y)dy \\
&= T_0^h(t)f + e^{imx^2/2t\hbar}(m/it)^{n/2}(2\pi)^{-n/2} \\
&\quad \cdot \int_0^1 \int_{\mathbb{R}^n} (im\hbar/2t)e^{-imx\cdot y/t}e^{im\hbar s y^2/2t}y^2\hat{f}(y)dyds.
\end{aligned}$$

Hence by Minkowski's inequality and the Parseval relation we get

$$\| e^{-itH_0^h/\hbar}\hat{f}^h - T_0^h(t)f \| \leqq (m\hbar/2t)\| y^2\hat{f}(y)\| \leqq (m\hbar/2t)\| f \|_2,$$

which implies the relation (1.2). (Q.E.D.)

2. Properties of Classical Orbit

In this section we study several properties of the motion of classical particle which will be needed in the following sections. We write the canonical transformations in the phase space $\mathbb{R}^n \times \mathbb{R}^n$ associated with the Hamiltonians $H(x,\xi) = \xi^2/2m + V(x)$ and $H_0(\xi) = \xi^2/2m$ as $U(t)$ and $U_0(t)$, that is,

$$U_0(t)(x_0, \xi_0) = (x_0 + t\xi_0/m, \xi_0) \tag{2.1}$$

and

$$U(t)(x_0, \xi_0) = (x_t, \xi_t), \tag{2.2}$$

where (x_t, ξ_t) is the solution of the Hamilton equation

$$\dot{x}(t) = \xi(t)/m, \quad \dot{\xi}(t) = -\operatorname{grad}_x V(x(t)),$$
$$x(0) = x_0, \quad \xi(0) = \xi_0. \tag{2.3}$$

In the followings we always assume Assumption (A). For $(a,\eta)\in\mathbb{R}^n\times\mathbb{R}^n$ we write as $U(t)U_0(s)(a,\eta) = (x_s(t,a,\eta), \xi_s(t,a,\eta))$.

Lemma 2.1. *Let $K \subset\subset \mathbb{R}^n\backslash\{0\}$. Then there exists a constant $R>0$ depending only on K such that for $s<0$ with sufficiently large $|s|$ and any multi-index α, there exists a constant C_α such that for any $\eta\in K$ and $0\leqq t\leqq |s+R|$*

$$\left|\left(\frac{\partial}{\partial\eta}\right)^\alpha x_s(t,a,\eta) - (s+t)m^{-1}\frac{\partial^\alpha\eta}{\partial\eta^\alpha} - \frac{\partial^\alpha a}{\partial\eta^\alpha}\right|$$
$$\leqq (2m)^{-1}|s+t|\left|\frac{\partial^\alpha\eta}{\partial\eta^\alpha}\right| + C_\alpha|s+t|^{-\varepsilon}, \tag{2.4}$$

$$\left|\left(\frac{\partial}{\partial\eta}\right)^\alpha \xi_s(t,a,\eta) - \frac{\partial^\alpha\eta}{\partial\eta^\alpha}\right| \leqq C_\alpha|s+t|^{-1-\varepsilon}. \tag{2.5}$$

Here $\varepsilon = \min(m(l)-l-1 : l\leqq|\alpha|)$ and $m(l) = \max(m(\alpha) : |\alpha| = l)$.

Proof. Since the case $a\neq 0$ can be treated in a similar way, we prove the case $a=0$ and we omit the variable a in the following expressions. For simplicity we also assume $m=1$. Since $V(x)$ satisfies the Assumption (A), $\left(\frac{\partial}{\partial\eta}\right)^\alpha x_s(t,\eta)$ and $\left(\frac{\partial}{\partial\eta}\right)^\alpha \xi_s(t,\eta)$ satisfy the integral equations

$$\left.\begin{aligned}
&\left(\frac{\partial}{\partial\eta}\right)^\alpha x_s(t,\eta) = (s+t)\frac{\partial^\alpha\eta}{\partial\eta^\alpha} + \int_0^t (t-u)\left(\frac{\partial}{\partial\eta}\right)^\alpha(-\operatorname{grad}_x V(x_s(u,\eta)))du \\
&\text{and} \\
&\left(\frac{\partial}{\partial\eta}\right)^\alpha \xi_s(t,\eta) = \frac{\partial^\alpha\eta}{\partial\eta^\alpha} + \int_0^t \left(\frac{\partial}{\partial\eta}\right)^\alpha(-\operatorname{grad}_x V(x_s(u,\eta)))du
\end{aligned}\right\} \tag{2.6}$$

Let us write $F(x) = -\operatorname{grad}_x V(x)$. By an elementary calculus we see that $\left(\frac{\partial}{\partial\eta}\right)^\alpha F(x_s(t,\eta))$ is the sum of $\frac{\partial F}{\partial x}\cdot\left(\frac{\partial}{\partial\eta}\right)^\alpha x_s(t,\eta)$ and the terms of the form

$$\frac{\partial^\beta F}{\partial x^\beta}(x_s(t,\eta))\cdot\left(\frac{\partial^{\sigma_1}x}{\partial\eta^{\sigma_1}}\right)^{\delta_1}\cdots\left(\frac{\partial^{\sigma_j}x}{\partial\eta^{\sigma_j}}\right)^{\delta_j},$$

where $|\beta|\geqq 2, \beta\leqq\alpha, \delta_1+\dots+\delta_j = |\beta|, \sigma_l<\alpha$ and $\sum|\sigma_l|\cdot\delta_l = |\alpha|$. Let us take as $\varepsilon = \min(m(0)-1, m(1)-2)$ and $R = \sup_{\eta\in K}\{(\max\{C_1, \frac{3}{2}nC_2\})^{1/(1+\varepsilon)}\varepsilon^{-1/(1+\varepsilon)} \cdot|\eta|^{-(3+\varepsilon)/(1+\varepsilon)}(1+|\eta|)\}$

where $C_j = \max_{|\alpha|=j} C_\alpha$ and C_α's are the constants appeared in Assumption (A). We first prove (2.4) for the cases $\alpha = 0$ and $|\alpha| = 1$ then prove the general cases by induction.

(a) The case $\alpha = 0$. Put

$$t^* = \sup\{0 \leqq t \leqq -s : |x_s(u,\eta) - (s+u)\eta| \leqq 2^{-1}|s+u||\eta| \text{ for } 0 < u < t\}.$$

It suffices to prove $t^* \geqq -s-R$. By Assumption (A) and the definition of t^* we get

$$\begin{aligned}|x_s(t^*,\eta) - (s+t^*)\eta| &\leqq C_1 \int_0^{t^*} (t^*-u)(1+|x_s(u,\eta)|)^{-2-\varepsilon} du \\ &\leqq C_1 \int_0^{t^*} (t^*-u)(2^{-1}|s+u||\eta|)^{-2-\varepsilon} du \\ &\leqq C_1 2^{2+\varepsilon}|\eta|^{-2-\varepsilon}\varepsilon^{-1}(-s-t^*)^{-\varepsilon}.\end{aligned}$$

Hence if $t^* < -s-R$, the right hand side is strictly smaller than $2^{-1}(-s-t^*)|\eta|$ which leads to a contradiction because $x_s(t,\eta)$ is continuous in t.

(b) The case $|\alpha| = 1$. By (2.6) we have the equation for $g_s(t,\eta) = \dfrac{\partial x}{\partial \eta}(t,\eta) - (s+t)e$, $e = \{\delta_{ij}\}_{ij}$,

$$g_s(t,\eta) = \int_0^t (t-u)\frac{\partial F}{\partial x}(x_s(u,\eta))(s+u)e\,du + \int_0^t (t-u)\frac{\partial F}{\partial x}(x_s(u,\eta))g_s(u,\eta)du.$$

For $0 \leqq t \leqq -s-R$, we get by the result for the case $\alpha = 0$ that

$$\begin{aligned}&\left|\int_0^t (t-u)\frac{\partial F}{\partial x}(x_s(u,\eta))(s+u)e\,du\right| \\ &\quad \leqq nC_2(2^{-1}|\eta|)^{-3-\varepsilon}\int_0^t (t-u)(-s-u)^{-2-\varepsilon}du \\ &\quad \leqq nC_2(2^{-1}|\eta|)^{-3-\varepsilon}\varepsilon^{-1}(-s-t)^{-\varepsilon}. \qquad (2.7)\end{aligned}$$

Let us define as

$$t^{**} = \sup\{0 \leqq t \leqq -s : |g_s(u,\eta)| \leqq 2^{-1}|s+u| \text{ for } 0 \leqq u \leqq t\}.$$

Again it suffices to prove $t^{**} \geqq -s-R$. By (2.7) we have

$$\begin{aligned}|g_s(t^{**},\eta)| &\leqq nC_2(2^{-1}|\eta|)^{-3-\varepsilon}\Big\{\varepsilon^{-1}(-s-t^{**})^{-\varepsilon} \\ &\qquad + (1/2)\int_0^{t^{**}} (t^{**}-u)(-s-u)^{-3-\varepsilon}(-s-u)du\Big\} \\ &\leqq C_2\frac{3n}{2}(2^{-1}|\eta|)^{-3-\varepsilon}\varepsilon^{-1}(-s-t^{**})^{-\varepsilon}. \qquad (2.8)\end{aligned}$$

Hence if $t^{**} < -s-R$, the right hand side of (2.8) is strictly smaller than $2^{-1}|s+t^{**}|$, which again leads to a contradiction.

(c) The general case. Let us assume that the statement is true for all $\alpha' < \alpha$, $2 \leqq |\alpha|$. $\left(\frac{\partial}{\partial \eta}\right)^{\alpha} x_s(t,\eta)$ satisfies the equation

$$\left(\frac{\partial}{\partial \eta}\right)^{\alpha} x_s(t,\eta) = \int_0^t (t-u)\frac{\partial F}{\partial x}(x_s(u,\eta))\left(\frac{\partial}{\partial \eta}\right)^{\alpha} x_s(u,\eta)du$$
$$+\int_0^t (t-u)\left\{\left(\frac{\partial}{\partial \eta}\right)^{\alpha} F(x_s(u,\eta)) - \frac{\partial F}{\partial x}(x_s(u,\eta))\left(\frac{\partial}{\partial \eta}\right)^{\alpha} x_s(u,\eta)\right\}du. \tag{2.9}$$

By the assumption of the induction and the remark preceding part (a), we can easily see that

$$\left|\left(\frac{\partial}{\partial \eta}\right)^{\alpha} F(x_s(u,\eta)) - \frac{\partial F}{\partial x}(x_s(u,\eta))\left(\frac{\partial}{\partial \eta}\right)^{\alpha} x_s(u,\eta)\right|$$
$$\leqq C|s+u|^{-2-\varepsilon} \quad \text{for } 0 \leqq u \leqq -s-R, \quad \eta \in K.$$

Hence using other constants M_1 and M_2, we get

$$\left|\left(\frac{\partial}{\partial \eta}\right)^{\alpha} x_s(t,\eta)\right| \leqq M_1|s+t|^{-\varepsilon} + M_2\int_0^t (t-u)|s+u|^{-2-\varepsilon}\left|\left(\frac{\partial}{\partial \eta}\right)^{\alpha} x_s(u,\eta)\right|du$$
$$\leqq M_1|s+t|^{-\varepsilon} + M_2\int_0^t |s+u|^{-1-\varepsilon}\left|\left(\frac{\partial}{\partial \eta}\right)^{\alpha} x_s(u,\eta)\right|du.$$

Therefore by Gronwall's inequality we get

$$\left|\left(\frac{\partial}{\partial \eta}\right)^{\alpha} x_s(t,\eta)\right| \leqq M_1(-s-t)^{-\varepsilon} + M_2\cdot M_1\int_0^t |s+u|^{-1-2\varepsilon}e^{C|t+s|^{-\varepsilon}}du$$
$$\leqq M_1|s+t|^{-\varepsilon} + M_3|s+t|^{-2\varepsilon}$$
$$\leqq M_4|s+t|^{-\varepsilon},$$

which proves the general case.

The second relation (2.5) is obvious by virtue of the estimate

$$\left|\int_0^t \left(\frac{\partial}{\partial \eta}\right)^{\alpha} F(x_s(u,\xi))du\right| \leqq M|s+t|^{-1-\varepsilon} \quad \text{for } 0 \leqq t \leqq -s-R. \qquad \text{(Q.E.D.)}$$

Corollary 2.2. *For sufficiently large* $-s$,

$$\pi_j U(t)U_0(s)\pi^*_{2,a},\ j=1,2 \text{ and } 0 \leqq t \leqq -s-R,$$

is C^∞*-diffeomorphism from* K *to its image.*

This is a well-known consequence of (2.4) and (2.5) (see also Hörmander [8]).

The following lemma is concerned with the existence of the wave operator in the classical mechanics.

Lemma 2.3. i) *For any* $a \in \mathbb{R}^n$ *and* $\eta \in K \subset\subset \mathbb{R}^n \backslash \{0\}$,

$$\lim_{s \to -\infty}\left(\left(\frac{\partial}{\partial \eta}\right)^{\alpha} x_s(-s-R,a,\eta),\ \left(\frac{\partial}{\partial \eta}\right)^{\alpha} \xi_s(-s-R,a,\eta)\right)$$

exists for any multi-index α. *Furthermore the convergence is uniform on K. Hence*

$$W^{cl}_{-,R} = \lim_{s \to -\infty} U(-s-R)U_0(s)$$

is a C^∞*-canonical transformation from* $\pi^*_{2,a}K$ *to its image.*
ii) $\pi_j W^{cl}_{-,R}\pi^*_{2,a} = \Omega^{R,a}_{j,-}(j=1,2)$ *is a* C^∞*-diffeomorphism from K to its image.*

Proof. i) is known for $|\alpha| \leqq 1$, see Simon [12] and Herbst [6]. For general $|\alpha| \geqq 2$, we can mimic their proof. ii) is an immediate consequence of i) and Collorary 2.2. (Q.E.D.)

Remark 2.4. Lemma 2.3 and its proof show that for any $(a,\eta)\in\mathbb{R}^n \times (\mathbb{R}^n\backslash\{0\})$ there exists a solution of (2.3) such that

$$\lim_{t\to-\infty} |x(t) - a - t\eta/m| = 0, \quad \lim_{t\to-\infty} |\xi(t) - \eta| = 0. \tag{2.10}$$

We write this solution as $(x_-(t,a,\eta), \xi_-(t,a,\eta))$.

The important theorem in the scattering theory of the classical mechanics is the one by Hunziker and Siegel on the completeness of the scattering states. In our context it may be stated as follows (see Hunziker [9] for the details and the proof).

Theorem 2.5. (*Hunziker-Siegel*) *Let Assumption* (*A*) *be satisfied. Then there exists a closed null set* $e \subset \mathbb{R}^n \times \mathbb{R}^n$ *such that the following statements hold: i) For any* $(a,\eta)\in\mathbb{R}^{2n}\backslash e$ *there exists* $(a_+(a,\eta), \eta_+(a,\eta))\in\mathbb{R}^{2n}$ *such that the solution* $(x(t),\xi(t)) = (x_-(t,a,\eta), \xi_-(t,a,\eta))$ *satisfies*

$$\lim_{t\to\infty} |x(t) - a_+ - \eta_+ t/m| = 0, \quad \lim_{t\to\infty} |\xi(t) - \eta_+| = 0. \tag{2.11}$$

ii) *The mapping* S^{cl} *defined as* $S^{cl}(a,\eta) = (a_+(a,\eta), \eta_+(a,\eta))$ *is a* C^1*-canonical mapping.*
Hence by Fubini's theorem, we get the following result:

Corollary 2.6. *For any* $a\in\mathbb{R}^n$, *let us define* $e(a) = \{\eta\in\mathbb{R}^n : (a,\eta)\in e\} \cup \{0\}$. *Then the statement of Theorem 2.5 holds for any* (a,η) *with* $\eta\in\mathbb{R}^n\backslash e(a)$ *and* $e(a)$ *is null set for almost all* $a\in\mathbb{R}^n$.

Remark. If $n=1$, we can easily prove that $e(a)$ is closed null set for all $a\in\mathbb{R}^n$. We believe that this is true even for $n \geqq 2$. However we could not prove it.

Now we study the asymptotic behavior as $t\to\infty$ of the solutions of (2.3) which satisfy Theorem 2.5. The study has been intensively done by Hörmander [8] and others [6], [12] in other contexts, however, we need a slightly different estimates.

Lemma 2.7. *Let* $a\in\mathbb{R}^n$ *and* $K \subset\subset \mathbb{R}^n\backslash e(a)$. *Then the following statements hold.*
i) $\eta_+(a,\eta)$ *and* $a_+(a,\eta)$ *are infinitely differentiable functions of* $\eta\in K$.
ii) *There exists sufficiently large* $t_0 > 0$ *such that for any* $\alpha > 0$, *there exists a constant* C_α *independent of* $\eta\in K$ *such that*

$$\left|\left(\frac{\partial}{\partial\eta}\right)^\alpha (x_-(t,a,\eta) - \eta_+(a,\eta)t/m - a_+(a,\eta))\right| \leqq C_\alpha t^{-\varepsilon}, \tag{2.12}$$

$$\left|\left(\frac{\partial}{\partial\eta}\right)^\alpha (\xi_-(t,a,\eta) - \eta_+(a,\eta))\right| \leqq C_\alpha t^{-1-\varepsilon} \tag{2.13}$$

for $t > t_0$.

Proof. We mimic the argument of Hunziker [9]. We set $m=1$. For $t>0$, let us write as $C_t=\{\eta\in\mathbb{R}^n\backslash e(a): \dot{x}_-(t,a,\eta)\cdot x_-(t,a,\eta)\geqq(t/2)|\eta|^2$ and $C_0(1+|x_-(t,a,\eta)|)^{-m(0)}+C_1(1+|x_-(t,a,\eta)|)^{-m(1)}|x_-(t,a,\eta)|\leqq 2^{-1}|\eta|^2\}$ (let us remark $|\eta_+|=|\eta|$ by the conservation of energy). Since for $x(t)=x_-(t,a,\eta)$,

$$2^{-1}(d/dt)x(t)^2=x(t)\cdot\dot{x}(t)$$

and

$$2^{-1}(d^2/dt^2)x(t)^2=|\eta|^2-\{2V(x(t))-F(x(t))\cdot x(t)\},$$

C_t is monotonically increasing in $t>0$ and by Theorem 2.5 $\bigcup_{t>0} C_t=\mathbb{R}^n\backslash e(a)$. Since $\bar{K}$ is compact, there exists a constant $t_0>0$ such that $C_{t_0}\supset\bar{K}$, which implies the existence of (another) constant t_0 such that

$$|x_-(t,a,\eta)|\geqq 2^{-1}|\eta|t,\quad \text{for } \eta\in K \text{ and } t>t_0. \tag{2.14}$$

Now let us note that $x_-(t,a,\eta)$ and $\xi_-(t,a,\eta)$ satisfy the integral equations (see Simon [12]):

$$x_-(t,a,\eta)=a_++\eta_+t-\int_t^\infty(t-u)F(x_-(u,a,\eta))du, \tag{2.15}$$

$$\xi_-(t,a,\eta)=\eta_+-\int_t^\infty F(x_-(u,a,\eta))du \tag{2.16}$$

as well as the equations

$$x_-(t,a,\eta)=a+t\eta+\int_{-\infty}^t (t-u)F(x_-(u,a,\eta))du, \tag{2.17}$$

$$\xi_-(t,a,\eta)=\eta+\int_{-\infty}^t F(x_-(u,a,\eta))du. \tag{2.18}$$

Hence

$$a_+(a,\eta)=a-\int_{-\infty}^\infty u\cdot F(x_-(u,a,\eta))du, \tag{2.19}$$

$$\eta_+(a,\eta)=\eta+\int_{-\infty}^\infty F(x_-(u,a,\eta))du. \tag{2.20}$$

Therefore Lemma 2.1 and 2.11 show the continuity of $a_+(a,\eta)$ and $\eta_+(a,\eta)$. Now differentiating (2.17) by $\left(\frac{\partial}{\partial\eta}\right)^\alpha$ and using the Gronwall's inequality, we can easily see that for any $0\leqq\alpha$, $\left|\left(\frac{\partial}{\partial\eta}\right)^\alpha x_-(t,a,\eta)\right|$ is majorized by $C_\alpha t$. This with Lemma 2.1 and with Lebesgue's dominated convergence theorem implies that

$$\left(\frac{\partial}{\partial\eta}\right)^\alpha\int_{-\infty}^\infty uF(x_-(u,a,\eta))du \text{ and } \left(\frac{\partial}{\partial\eta}\right)^\alpha\int_{-\infty}^\infty F(x_-(u,a,\eta))du$$

exists. This completes the proof for i).

For proving ii), it suffices to prove (2.13), since (2.12) can be obtained by integrating (2.13). Differentiating (2.16) and noting $\left|\left(\frac{\partial}{\partial\eta}\right)^\alpha x_-(t,a,\eta)\right| \leqq C_\alpha t$, we have

$$\left(\frac{\partial}{\partial\eta}\right)^\alpha (\xi_-(t,a,\eta)-\eta_+) = \int_t^\infty \left(\frac{\partial}{\partial\eta}\right)^\alpha F(x_-(u,a,\eta))du.$$

Since $\left|\left(\frac{\partial}{\partial\eta}\right)^\alpha F(x_-(u,a,\eta))\right| \leqq C|u|^{-2-\varepsilon}$ by virtue of the remark given in the proof of Lemma 2.1, inequality (2.13) follows obviously. (Q.E.D.)

Let us define the scattering operator S_a^{cl} with index $a\in\mathbb{R}^n$ as

$$S_a^{cl}\eta = \eta_+(a,\eta).$$

3. Properties of Action Integrals

In this section, we shall study several properties of the solutions of the Hamilton-Jacobi equation

$$\left(\frac{\partial S}{\partial t}\right)(t,x) + (2m)^{-1}(\text{grad}_x S(t,x))^2 + V(x(t,x)) = 0. \tag{3.1}$$

We first discuss them in the region $(-\infty, t)$ with $-t$ sufficiently large, using the estimates of Lemma 2.1.

Lemma 3.1. *Let $K \subset\subset \mathbb{R}^n\backslash\{0\}$ and let $R>0$ be the constant determined in Lemma 2.1. For $-s>R$ let us define the function $S_s(t,x)$ on the domain $\mathcal{Q}_{s,R}(K,a) = \bigcup_{0\leqq t\leqq -s-R} \{t\}\times \pi_1 U(t)U_0(s)\pi^*_{2,a}K \subset \mathbb{R}^1\times\mathbb{R}^n$ as*

$$S_s(t,x) = \frac{s}{2m}\eta_s(t,a,x)^2 + \int_0^t L(x_s(u,a,\eta_s(t,a,x)), \dot{x}_s(u,a,\eta_s(t,a,x))du, \tag{3.2}$$

where $L(x,\dot{x}) = m\dot{x}^2/2 - V(x)$ is the Lagrangian of the system and $\eta_s(t,a,x) = (\pi_1 U(t)U_0(s)\pi^*_{2,a})^{-1}x$. Then the following statements hold:

i) *$S_s(t,x)$ is an infinitely differentiable function of (t,x) on $\mathcal{Q}_{s,R}(K,a)$.*

ii) *$S_s(t,x)$ satisfies the Hamilton-Jacobi Eq. (3.1) with the initial condition*

$$S_s(0,x) = m(x-a)^2/2s. \tag{3.3}$$

iii) *Let us define as*

$$K_s(t,x) = -(2m)^{-1}\int_0^t (\Delta_x S_s)(u, x_s(u,\eta_s(t,a,x)))du, \tag{3.4}$$

then there exists a constant $C>0$ such that

$$\left|\frac{\partial}{\partial y}K(t,x)\right| \leqq C|s|^{-1}|s+t|^{-1-\varepsilon}, \tag{3.5}$$

$$\left|\frac{\partial^2}{\partial y^2}K(t,x)\right| \leqq C|s|^{-2}|s+t|^{-1-\varepsilon}, \tag{3.6}$$

where $y = s\eta_s(t,a,x)+a$.

iv) *For any multi-index* α, $\lim_{s\to-\infty}\left(\frac{\partial}{\partial x}\right)^\alpha S_s(-s-R, x_s(-s-R,a,\eta))$ *exists uniformly on* K, *and* $\lim_{s\to-\infty}\left(\frac{\partial}{\partial x}\right)^\alpha S_s(-s-R,x)$ *exists on* $\Omega_{1,-}^{R,a}(K)$ *uniformly. We write as*

$$S_{-\infty}^R(x)=\lim_{s\to-\infty} S_s(-s-R,x).$$

Proof. Since the case $a\neq 0$ can be treated similarly by changing the variable x to $x-a$, we assume $a=0$ in what follows and omit the variable a. We also assume $m=1$. Statement i) is obvious and ii) is well known (see Landau-Lifschitz [15]). Let us prove iii) and iv).

Proof of iii). Since as is well known, $(\partial S/\partial x)(u,x_s(u))=\xi(u,\eta_s(t,x))$, $(\Delta_x S)\cdot(u,x_s(u,\eta_s(t,x)))=\sum_{j=1}^n \partial\xi_j/\partial\bar{x}_j=\sum_{j,k=1}^n(\partial y_k/\partial\bar{x}_j)(\partial\xi_j/\partial y_k)$, where $y=s\eta_s(t,x)$ and $\bar{x}=x_s(u)=x_s(u,\eta_s(t,x))$. Then by elementary calculations, we get

$$\begin{aligned}&(\partial/\partial y_m)(\Delta_x S)(u,x_s(u,\eta_s(t,x)))\\ &\quad=\sum_{j,k,l}(\partial\bar{x}_l/\partial y_m)(\partial^2 y_k/\partial\bar{x}_j\partial\bar{x}_l)(\partial\xi_j/\partial y_k)+\sum_{j,k}(\partial y_k/\partial\bar{x}_j)(\partial^2\xi_j/\partial y_k\partial y_m),\end{aligned}\tag{3.7}$$

$$\begin{aligned}&(\partial^2/\partial y_m\partial y_n)(\Delta_x S)(u,x_s(u,\eta_s(t,x)))\\ &\quad=\sum_{j,k,l}(\partial^2\bar{x}_l/\partial y_m\partial y_n)(\partial^2 y_k/\partial\bar{x}_j\partial\bar{x}_l)(\partial\xi_j/\partial y_k)\\ &\qquad+\sum_{j,k,l,i}(\partial\bar{x}_l/\partial y_m)(\partial\bar{x}_i/\partial y_n)(\partial^3 y_k/\partial\bar{x}_l\partial\bar{x}_j\partial\bar{x}_i)(\partial\xi_j/\partial y_k)\\ &\qquad+\sum_{j,k,l}(\partial\bar{x}_l/\partial y_m)(\partial^2 y_k/\partial\bar{x}_l\partial\bar{x}_j)(\partial^2\xi_j/\partial y_k\partial y_n)\\ &\qquad+\sum_{j,k,l}(\partial^2 y_k/\partial\bar{x}_j\partial\bar{x}_l)(\partial\bar{x}_l/\partial y_n)(\partial^2\xi_j/\partial y_k\partial y_m)\\ &\qquad+\sum_{j,k}(\partial y_k/\partial\bar{x}_j)(\partial^3\xi_j/\partial y_k\partial y_m\partial y_n).\end{aligned}\tag{3.8}$$

Since $\bar{x}=x_s(u,y/s)$, Lemma 2.1 and simple calculations show that for $y=s\eta$ and $\bar{x}=x_s(u,\eta)$

$$\begin{aligned}&|(\partial y/\partial x)|\leqq C|s||s+u|^{-1},\\ &|(\partial^\alpha y/\partial x^\alpha)|\leqq C_\alpha|s||s+u|^{-1-\varepsilon-|\alpha|},\quad |\alpha|\geqq 2.\end{aligned}\tag{3.9}$$

Thus using (3.9) and Lemma 2.1, we get from (3.7) and (3.8) the estimates

$$|(\partial/\partial y_k)(\Delta_x S)(u,x(u))|\leqq C|s|^{-1}|s+u|^{-2-\varepsilon}$$

and

$$|(\partial^2/\partial y_n\partial y_m)(\Delta_x S)(u,x(u))|\leqq C|s|^{-2}|s+u|^{-2-\varepsilon},$$

from which the desired results (3.5) and (3.6) follow by integration.

Proof of iv). We prove only the case $\alpha = 0$. Other cases can be proved similarly. By the definition of $S_s(t, x)$ and the law of conservation of energy, we have

$$\begin{aligned} S_s(t,x) &= t\{\eta_s(t,x)^2/2m + V(s\eta_s(t,x))\} \\ &\quad - 2\int_0^t V(x_s(u,\eta_s(t,x)))du + (s\eta_s(t,x))^2/2sm \\ &= (s+t)\eta_s(t,x)^2/2m + tV(s\eta_s(t,x)) - 2\int_0^t V(x_s(u,\eta_s(t,x)))du. \end{aligned}$$

Therefore

$$\begin{aligned} S_s(-s-R,x) = &-R\eta_s(-s-R,x)^2/2m + (-s-R)V(s\eta_s(-s-R,x)) \\ &-2\int_0^{-s-R} V(x_s(u,\eta_s(-s-R,x))du. \end{aligned}$$

Take K_1 such that $K \subset\subset K_1 \subset\subset \mathbb{R}^n\backslash\{0\}$. For $x \in \Omega^R_{1,-}(K)$, $\eta_s(-s-R,x) \in K_1$ for sufficiently large $-s$. $\lim_{s\to-\infty} \eta(-s-R,x)$ exists and $\lim_{s\to-\infty}(-s-R)V \times (s\eta_s(-s-R,x)) = 0$ uniformly on K, since $(\pi_1 U(-s-R)U_0(s)\pi^*_{2,a})^{-1}$ converges uniformly on $\Omega^R_{1,-}(K)$. Now Lemma 2.1 and Lebesgue's dominated convergence theorem imply the uniform convergence of the integral $\int_0^{s-R} V(x_s(u,\eta_s(-s-R,x))du$ as $s \to -\infty$. This completes the proof.

(Q.E.D.)

Remark 3.2. We remark here that for $x = \Omega^{R,a}_{1,-}(\eta)$,

$$S^R_{-\infty}(x) = \lim_{s\to-\infty}\left\{\int_s^{-R} L(x_-(u,a,\eta), \quad \dot{x}_-(u,a,\eta))du + \frac{\eta^2}{2m}s\right\}. \tag{3.10}$$

Now we turn to the study of the solution of (3.1) in the region (t, ∞) with $t > 0$ sufficiently large. We take $K \subset\subset \mathbb{R}^n\backslash e(a)^{ex}$ and take K_1 such that $K \subset\subset K_1 \subset\subset \mathbb{R}^n\backslash e(a)^{ex}$. Then obviously there exists a constant $\delta_{K_1} > 0$ such that

$$|\det(\partial\eta_+(a,\eta)/\partial\eta| \geqq \delta_{K_1}, \quad \eta \in K_1. \tag{3.11}$$

Therefore the mapping $S^{cl}_a\eta = \eta_+(a,\eta)$ is a locally uniform diffeomorphism on K_1 and for any $\eta_+ \in S^{cl}_a K$, there exists a small neighbourhood $U_+(\eta_+) \subset S^{cl}_a K_1$ such that the inverse image $(S^{cl}_a)^{-1}U_+(\eta_+)$ consists of a finite number of disjoint components $\{U_k(\eta_+)\}$ on each of which S^{cl}_a is diffeomorphism. Thanks to the property (2.12), the correspondence from $\eta_+(a,\eta)$ to $x_-(t,a,\eta)$ is diffeomorphism for sufficiently large $t > t_0, t_0$ is determined only by K. So that the mapping $x_-(t,a,\cdot)$ for $t > t_0$ is also diffeomorphic on $U_k(\eta_+)$. Let us take such $U_k(\eta_+)$ for each $\eta_+ \in S^{cl}_a K$. Obviously $\{U_k(\eta_+), \eta_+ \in S^{cl}_a K\}$ forms an open covering of K. Then a simple compactness argument shows there exists a finite number of relatively compact open covering $\{U_k\}$ of K such that $U_k \subset K_1$ and on each U_k, S^{cl}_a and the mapping $x_-(t,a,\cdot)(t > t_0)$ are diffeomorphisms. Let us take one of these U_k's and label it as U.

Lemma 3. 3. *Let Assumption (A) be satisfied. Let* $K \subset\subset K_1 \subset\subset \mathbb{R}\backslash e(a)^{ex}$. *Let us take* U *and* $t_0 > 0$ *as in the preceding remark and define the function* $S(t, x)$ *on the domain* $\mathscr{D}(t_0, U) = \bigcup_{t \geqq t_0} \{t\} \times \pi_1 U(t+R) W^{cl}_{-,R} \pi^*_{2,a} U$ *as*

$$S(t,x) = \int_{-R}^{t} L(x_-(u,a,\eta(t,a,x)), \quad \dot{x}_-(u,a,\eta(t,a,x))du$$

$$+ S^R_{-\infty}(x_-(-R,a,\eta(t,a,x)). \tag{3.12}$$

where $\eta(t,a,x) \in U$ *is determined by the relation* $x = x_-(t,a,\eta(t,a,x))$. *Then the following statements hold:*

i) $S(t,x)$ *is an infinitely differentiable function on* $\mathscr{D}(t_0, U)$.

ii) $S(t,x)$ *satisfies the Hamilton-Jacobi equation (3.1) on the domain* $\mathscr{D}(t_0, U)$.

iii) $(\partial S/\partial x)(t,x) = \xi_-(t,a,\eta(t,a,x))$.

iv) *Let us define the function* $K_{t_0}(t,x)$ *on the domain* $\mathscr{D}(t_0, U)$ *as*

$$K_{t_0}(t,x) = -(2m)^{-1} \int_{t_0}^{t} (\Delta_x S)(u, x_-(u,a,\eta(t,a,x)))du. \tag{3.13}$$

Then for any multi-index α *there exists a constant* $C_\alpha > 0$ *such that*

$$|(\partial/\partial y)^\alpha K_{t_0}(t,x)| \leqq C_\alpha \log t, \tag{3.14}$$

where $y = x_-(t_0, \eta(t,a,x))$.

Proof. Statements i), ii) and iii) can be easily checked. We prove iv) only. As in the proof of Lemma 3.1, we can easily get the following estimates from Lemma 2.7 and (3.11):

$$|(\partial/\partial x)^\alpha y(t,x)| \leqq C_\alpha t^{-|\alpha|}, \quad t \geqq t_0, \quad |\alpha| \geqq 1; \tag{3.15}$$

$$|(\partial/\partial y)^\alpha x(t, \eta(t_0,a,y))| \leqq C_\alpha t, \quad t \geqq t_0, \quad |\alpha| \geqq 1; \tag{3.16}$$

$$|(\partial/\partial y)^\alpha (\xi_-(t, \eta(t_0,a,y)))| \leqq C_\alpha, \quad t \geqq t_0, |\alpha| \geqq 1; \tag{3.17}$$

Since $(\partial S/\partial x)(t,x) = \xi_-(t)$, we can use the formulas (3.7) and (3.8), and after a simple calculation we get

$$|(\partial/\partial y)^\alpha (\Delta_x S)(y,a,x_-(u,a,\eta(t,a,x)))| \leqq C_\alpha u^{-1}, \tag{3.18}$$

for any $u \geqq t_0$ and multi-index α, from which the desired estimates follow by integration. (Q.E.D.)

4. The Asymptotic Behavior of the Wave Operator W^h_-

In this section we study the asymptotic behavior of the wave operator W^h_- on the coherent state $\tilde{f}^h_a$. The crucial step of the study is the following theorem.

Theorem 4.1. *Let Assumption (A) be satisfied. Let* $K \subset\subset \mathbb{R}^n \backslash \{0\}$ *and* R *be the constant determined in Lemma 2.1. For* $f \in \mathscr{H}$ *with* supp $f \subset K$, *let us define the function* $f_{a,s}(t,x)$ *for* $t \leqq -s-R$ *as*

$$f_{a,s}(t,x) = (m/is)^{n/2} f(\eta_s(t,a,x)) e^{K_s(t,x)}. \tag{4.1}$$

Then

$$\lim_{h\downarrow 0} \sup_{-s\geqq R} \sup_{0\leqq t\leqq -s-R} \| e^{-itH^h/\hbar} T^h_{0,a}(s)f - e^{iS_s(t,x)/\hbar} f_{a,s}(t,x) \| = 0. \tag{4.2}$$

Proof. For simplicity we assume $a=0$ and $m=1$. First of all we remark that by the well known result $\exp(2K(t,x))dx = dy$, for $y = s\eta_s(t,a,x)$ (see for example Maslov [11]), we easily see $\| f_s(t,x)\| = \| f \|$. Since $\exp(-itH^h/\hbar)T^h_0$ is also unitary, it suffices to prove (4.2) for $f \in C_0^\infty(K)$. Since, as can be easily checked, $f_s(t,x)$ satisfies the transport equation

$$\frac{\partial}{\partial t} f_s(t,x) + \sum_{k=1}^{n} \left(\frac{\partial S}{\partial x_k}\right)\left(\frac{\partial}{\partial x}\right) f_s(t,x) + 2^{-1}(\Delta_x S)(t,x) f_s(t,x) = 0, \tag{4.3}$$

we get

$$\left(i\hbar\frac{\partial}{\partial t} + \frac{\hbar^2}{2}\Delta - V(x)\right) e^{iS_s(t,x)/\hbar} f_s(t,x) = 2^{-1}\hbar^2 e^{iS_s(t,x)/\hbar}(\Delta_x f_s)(t,x). \tag{4.4}$$

Since $\exp(iS_s(0,x)/\hbar)\cdot f_s(0,x) = T^h_0(s)\cdot f$, we get by (4.4) and Duhamel's principle that

$$\begin{aligned} &\| e^{-itH^h/\hbar} T^h_0(s)f - e^{iS_s(t,x)/\hbar} f_s(t,x) \| \\ &\leqq (\hbar/2) \left\| \int_0^t e^{-i(t-u)H^h/\hbar} (e^{iS_s(u,x)/\hbar}(\Delta f_s)(u,x)) du \right\| \\ &\leqq (\hbar/2) \int_0^t \| (\Delta_x f_s)(u,x) \| du. \end{aligned} \tag{4.5}$$

In what follows we omit the index s. Let us write $s\eta_s(u,x) = y(u,x)$ and $g(y/s) = s^{-n/2} f(y/s)$.
Then

$$\begin{aligned} \Delta_x f(u,x) &= \Delta_x(e^{K(u,x)} g(y/s)) \\ &= \sum_k (\Delta_x y_k)\{(\partial K/\partial y_k) g(y/s) + s^{-1}(\partial g/\partial y_k)(y/s)\} e^{K(u,x)} \\ &\quad + \sum_{k,j,i} (\partial y_k/\partial x_j)(\partial y_i/\partial x_j)[\{\partial K/\partial y_k)(\partial K/\partial y_i) + (\partial^2 K/\partial y_k \partial y_i) g(y/s)\} \\ &\quad + s^{-1}\{(\partial g/\partial y_k)(y/s)\cdot(\partial K/\partial y_i) + (\partial g/\partial y_i)\partial K/\partial y_k)\} \\ &\quad + s^{-2}(\partial^2 g/\partial y_k \partial y_i)(y/s)] e^{K(u,x)}. \end{aligned} \tag{4.6}$$

Now remembering that $\| \exp(K(u,x)) s^{-n/2} h(y/s) \| = \| h \|$ for any $h \in L^2(K)$ by the preceding remark, we get by Lemma 3.1 and (3.9) that

$$\begin{aligned} \| \Delta_x f(u,x) \| &\leqq C|s\|s+u|^{-3-\varepsilon}\{|s|^{-1}|s+u|^{-1-\varepsilon}\| f \| + |s|^{-1}\| f \|_1\} \\ &\quad + |s|^2|s+u|^{-2}\{(|s|^{-2}|s+t|^{-2-\varepsilon} + |s|^{-2}|s+t|^{-1-\varepsilon}\| f \|_1 + s^{-2}\| f \|_2\} \\ &\leqq C|s+t|^{-2}\| f \|_2 . \end{aligned} \tag{4.7}$$

Therefore for $0 \leqq t \leqq |s+R|$,

$$(\hbar/2)\int_0^t \| \Delta f(u,x) \| du \leqq Ch \| f \|_2 \int_0^t |s+u|^{-2} du$$

$$\leqq Ch \| f \|_2 |s+t|^{-1} \leqq Ch \| f \|_2 R^{-1},$$

which proves the desired result. (Q.E.D.)

Now we can state and prove the main theorem in this section.

Theorem 4.2. *Let Assumption (A) be satisfied. Let $K \subset\subset \mathbb{R}^n \backslash \{0\}$ and $f \in \mathscr{H}$ have* $\operatorname{supp} f \subset K$. *Let us define as*

$$(W^a_{-,R} f)(x) = e^{iS^R_{-,x}(x)/\hbar - in\pi/4} [\det(\partial \Omega^{R,a}_{1,-}(\eta)/\partial(\eta)|_{\Omega^{R,a}_{1,-}(\eta)=x}]^{-(1/2)} f((\Omega^{R,a}_{1,-})^{-1} x).$$

Then

$$\lim_{h\ 0} \| e^{iRH^h/\hbar} W^h_- \tilde{f}^h_a - W^a_{-,R} f \| = 0 \tag{4.8}$$

Proof. From (4.2) we get

$$0 = \lim_{h \downarrow 0} \lim_{s \to -\infty} \| e^{i(s+R)H^h/\hbar} e^{-isH^h_0/\hbar} \tilde{f}^h_a - e^{iS_s(-s-R,x)/\hbar} f_{a,s}(-s-R, x) \|. \tag{4.9}$$

The first summand in the right of (4.9) converges to $\exp(iRH^h/\hbar) W^h_- \tilde{f}^h_a$ and the second to $W^a_{-,R} f$ as $s \to -\infty$ by virtue of Lemma 2.3, Lemma 3.1 iv) and the relation $\exp(2K(t,x))dx = dy$, $y = s\eta_s(t,a,x)$. This proves the theorem. (Q.E.D.)

5. Approximate Fundamental Solution and the Stationary Phase Method

Continuing the approximation scheme, we study here the asymptotic behavior of the solution $e^{-itH^h/\hbar} W^h_- \tilde{f}^h_a$ as $h \to 0$ for sufficiently large $t > 0$. The fundamental tools in this section and the next section are the approximate fundamental solution of the Schrödinger operator due to Fujiwara [3], the L^2-boundedness theorem for some oscillatory integral operators by Asada-Fujiwara [2] and the stationary phase method. We first review results of Fujiwara [3] and Asada-Fujiwara [2].

Theorem 5.1. (Fujiwara). *Let $V(x)$ be a real valued infinitely differentiable function on $\mathbb{R}^n$ satisfying the condition*

$$\text{(C)} \left| \left(\frac{\partial}{\partial x} \right)^\alpha V(x) \right| \leqq C_\alpha \ \textit{for} \ |\alpha| \geqq 2.$$

Then there exists a constant $\delta > 0$ such that the following statements hold:

i) *For any $t \in [-\delta, \delta]$ and $x, y \in \mathbb{R}^n$, there exists a unique solution of the equation of the classical motion (2.3) such that $x(0) = y$, $x(t) = x$.*

ii) *Let $S(t,x,y) = \int_0^t L(x(u,y), \dot{x}(u,y))du$ be the action integral along the classical orbit and let $\phi(t,x,y) = tS(t,x,y)$. Then*

$$\phi(t,x,y) \in C^\infty((-\delta,\delta) \times \mathbb{R}^n \times \mathbb{R}^n), \tag{5.1}$$

$$|(\partial/\partial x)^\alpha (\partial/\partial y)^\beta \phi(t,x,y)| \leqq C_{\alpha\beta} \quad \text{if } |\alpha| + |\beta| \geqq 2. \tag{5.2}$$

$$|\det(\partial^2 \phi/\partial x \partial y)(t,x,y)| \geqq 2^{-1}, \tag{5.3}$$

$$|(\partial/\partial x)^\alpha (\partial/\partial y)^\beta (\Delta_x \phi - nm)| \leqq C_{\alpha\beta} t, \tag{5.4}$$

where $(\partial^2\phi/\partial x \partial y)$ in (5.3) is the matrix $\{\partial^2 \phi/\partial x_j \partial y_k\}_{j,k}$.

iii) *Define as*

$$a(t,x,y) = e^{-in\pi/4}(m/2\pi|t|)^{n/2}$$
$$\cdot \exp\left(-\frac{1}{2}\int_0^t (m^{-1}\Delta_x S(u,x(u),y) - n/m)du \right).$$

Then the function $\left(\frac{|t|}{m}\right)^{n/2} a(t,x,y)$ *is an infinitely differentiable function on* $(-\delta,\delta) \times \mathbb{R}^n \times \mathbb{R}^n$ *satisfying the following conditions:*

$$|(\partial/\partial x)^\alpha(\partial/\partial y)^\beta a(t,x,y)| \leqq C_{\alpha\beta}|t|^{n/2}, \tag{5.6}$$

$$|(\partial/\partial x)^\alpha(\partial/\partial y)^\beta \Delta_x a(t,x,y)| \leqq C_{\alpha\beta}|t|^{n/2+1}. \tag{5.7}$$

iv) *Let* $E(t,h)$ *be the integral operator* $(t\in[-\delta,\delta])$ *defined as*

$$E(t,h)f(x) = h^{-n/2}\int_{\mathbb{R}^n} e^{iS(t,x,y)/\hbar}a(t,x,y)f(y)dy, \quad \text{for } f\in C_0^\infty(\mathbb{R}^n). \tag{5.8}$$

Then the operator $E(t,h)$ *can be extended to a bounded operator on* $\mathscr{H}$ *and statifies*

$$\|E(t,h)\| \leqq C, \quad -\delta \leqq t \leqq \delta; \tag{5.9}$$

$$s-\lim_{t\to 0} E(t,h) = I, I \text{ is the identity operator.} \tag{5.10}$$

v) *Let* $T>0$ *and* $0=t_0<t_1<\ldots<t_N=T$ *be an arbitrary subdivision of* $[0,T]$ *such that* $\delta(\Delta) = \max_j |t_{j+1}-t_j| < \delta$. *Then for* $0<h\leqq 1$,

$$\|E(t_N - t_{N-1},h)\ldots E(t_1-t_0,h) - e^{-itH^h/\hbar}\| \leqq C_{T,\Delta}h \tag{5.11}$$

and moreover

$$\lim_{\delta(\Delta)\to 0}\|E(t_N-t_{N-1},h)\ldots E(t_1-t_0) - e^{-itH^h/h}\| = 0. \tag{5.12}$$

For the proof and the details, see Fujiwara [3].

The following theorem on the boundedness of certain integral operators due to Asada-Fujiwara [2] also turns out to be quite useful.

Theorem 5.2. (Asada-Fujiwara) *Let* $\phi(x,\theta,y)$ *be a real-valued infinitely differentiable function on* $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n$ *(the case* $m=0$ *is not excluded) such that*

$$|(\partial/\partial x)^\alpha(\partial/\partial\theta)^\beta(\partial/\partial y)^\sigma\phi(x,\theta,y)| \leqq C_{\alpha\beta\sigma} \quad \text{if } |\alpha|+|\beta|+|\sigma| \geqq 2, \tag{5.13}$$

there exists a constant $\delta>0$ *such that*

$$\left|\det\begin{bmatrix}(\partial^2\phi/\partial x\partial y) & (\partial^2\phi/\partial x\partial\theta)\\ (\partial^2\phi/\partial\theta\partial y) & (\partial^2\phi/\partial\theta\partial\theta)\end{bmatrix}\right| \geqq \delta > 0, \tag{5.14}$$

and let $a(x,\theta,y)\in\mathscr{B}(\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^n)$. *Let us define the integral operator* $A(\gamma)$ *as*

$$A(\gamma)f(x) = \gamma^{n/2}\int e^{i\gamma\phi(x,\theta,y)}a(x,\theta,y)f(y)dyd\theta. \tag{5.15}$$

Then there exists a constant $C>0$ *determined by a finite number of* $C_{\alpha\beta\sigma}$*'s and* $\delta>0$ *such that*

$$\|A(\gamma)f\| \leqq C\|f\| \quad \textit{for } \gamma \geqq 1. \tag{5.16}$$

Furthermore if $a \equiv 0$ *in a neighbourhood of* C_ϕ *:*

$$C_\phi = \{(x, \theta, y) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n : \partial\phi/\partial\theta = 0\}, \tag{5.17}$$

then for any $M > 0$, *there exists a constant* $C_M > 0$ *such that*

$$\| A(\gamma) f \| \leqq C_M \gamma^{-M} \| f \|, \quad |\gamma| \geqq 1. \tag{5.18}$$

See Asada-Fujiwara [2] for the proof, the rigorous definition of the integral (5.15) and the details.

For $a \in \mathbb{R}^n$, let us take the domain $U \subset\subset \mathbb{R}^n \backslash e(a)^{ex}$ and $t_0 \in \mathbb{R}^1$ as in Lemma 3.3. Let $f \in \mathscr{H}$ have $\operatorname{supp} f \subset U$. We study the asymptotic behavior of $\exp(-i(t_0 + R)H^h/\hbar)W_-^h \tilde{f}_a^h$. We write as $T = t_0 + R$. Choose N sufficiently large such that $T/N = \tau < \delta$, where δ is the constant appeared in Theorem 5.1. By Theorem 5.1 and Theorem 4.2 we have

$$\begin{aligned} &\lim_{h \downarrow 0} \| e^{-it_0 H^h/\hbar} W_-^h \tilde{f}_a^h - e^{-iTH^h/\hbar} W_{-,a}^R f \| \\ &\quad = \lim_{h \downarrow 0} \| e^{-it_0 H^h/\hbar} W_-^h \tilde{f}_a^h - E(\tau, h)^N W_{-,a}^R f \| = 0. \end{aligned} \tag{5.19}$$

We write as

$$\begin{aligned} &W_{-,a}^R f(x) = e^{iS_{-\infty}^R(x)/\hbar} h_R(x), \\ &h_R(x) = \left| \det(\Omega_{1,-}^R(\eta)/\partial\eta)|_{x = \Omega_{1,-}^{R,a}(\eta)} \right|^{-1/2} e^{-in\pi/4} f((\Omega_{1,-}^{R,a})^{-1} x). \end{aligned} \tag{5.20}$$

Then

$$\begin{aligned} E(\tau, h)^N W_{-,a}^R f(x) &= h^{-Nn/2} \int_{\mathbb{R}^{Nn}} \exp\left(i\hbar^{-1} \sum_{j=1}^N S(\tau, x_j, x_{j-1}) \right) \\ &\prod_{j=1}^N a(\tau, x_j, x_{j-1}) \exp(i\hbar^{-1} S_{-\infty}^R(x_0)) h_R(x_0) dx_0 \ldots dx_{N-1}, \end{aligned} \tag{5.21}$$

where $x_0 = y, x_N = x$. Let us write $\theta = (x_{N-1}, \ldots, x_1)$ and $\phi_{R,t_0}(x, \theta, y) = \sum_{j=1}^N S(\tau, x_j, x_{j-1}) + S_{-\infty}^R(x_0)$. We want to apply the stationary phase method to the expression (5.21). For each $x \in \mathbb{R}^n$, the point of stationary phase (θ, y) is determined by the equation

$$d_{(\theta,y)} \phi(x, \theta, y) = 0, \tag{5.22}$$

which is equivalent to the system of equations

$$\begin{aligned} &\partial S/\partial x_{N-1}(\tau, x, x_{N-1}) + \partial S/\partial x_{N-1}(\tau, x_{N-1}, x_{N-2}) = 0, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\ &\partial S/\partial x_1(\tau, x_2, x_1) + \partial S/\partial x_1(\tau, x_1, x_0) = 0, \\ &\partial S/\partial x_0(\tau, x_1, x_0) + \partial S_{-\infty}^R(x_0)/\partial x_0 = 0. \end{aligned} \tag{5.23}$$

Lemma 5.3. *For each* $x \in \pi_1 U(T) W_{-,R}^{cl} \pi_{2,a}^*(U)$, *there exists a unique* $(\theta, y) = (x_{N-1}, \ldots, x_1, y) \in \mathbb{R}^{Nn}$ *such that* (θ, y) *satisfies the equation (5.22) and such that* $y \in \pi_1 W_{-,R}^{cl} \pi_{2,a}^*(U)$, *and for* $x \notin \pi_1 U(T) W_{-,R}^{cl} \pi_{2,a}^*(U)$ *there is no point of stationary phase such that* $y \in \pi_1 W_{-,R}^{cl} \pi_{2,a}^*(U)$.

Proof. Since (a) $-\partial S/\partial y(\tau,x,y)$ and $\partial S/\partial x(\tau,x,y)$ are the initial and final momenta of the particle which starts at the point y at time zero and reaches the point x at time τ; (b) $\partial S^R_{-\infty}(y)/\partial y = \xi_-(-R,a,\eta(-R,a,y))$; (c) the mapping $U \ni \eta \to x(t_0,a,\eta)$ is diffeomorphism; (d) for any fixed $y \in \mathbb{R}^n$ the mapping $\mathbb{R}^n \ni x \to \partial S/\partial y(\tau,x,y) \in \mathbb{R}^n$ is a global diffeomorphism by virtue of (5.2) and (5.3), we can conclude that for any $x_0 \in \pi_1 W^{cl}_{-,R}\pi^*_{2,a}(U)$, the solution of (5.23), considering $(x, x_{N-1}, \dots, x_1)$ as an unknown variable and $x_0 = y$ as a parameter, is determined uniquely as $x_1 = x_-(-R+\tau, a, \eta(-R,a,x_0)), \dots, x_{N-1} = x_-(-R+(N-1)\tau, a, \eta(-R,a,x_0))$ and $x = x_N = x_-(-R+N\tau, a, \eta(-R,a,x_0)) = x_-(t_0,a,\eta(t_0,a,x))$. By virtue of Lemma 3.3 the mapping $\Omega^{R,a}_{1,-}(U) \ni x_-(-R,a,\eta) \to x_-(t_0,a,\eta) \in \pi_1 U(T) W^{cl}_{-,R}\pi^*_{2,a}(U)$ is a diffeomorphism, this implies the lemma. (Q.E.D.)

Let us define $C_\phi(U)$ as

$$C_\phi(U) = \{(x,\theta,y) \in \mathbb{R}^n \times \mathbb{R}^{(N-1)n} \times \mathbb{R}^n;\ d_{(\theta,y)}\phi(x,\theta,y) = 0,\ y \in \Omega^R_{-,a}(U)\} \tag{5.24}$$

and the mapping $i\colon C_\phi(U) \to i(C_\phi(U)) \equiv \Lambda_\phi(U)$ as $i(x,\theta,y) = (x, \partial\phi/\partial x(x,\theta,y)) = (x, \xi_-(t_0,a,\eta(t_0,a,x)))$. Then we have the following lemma.

Lemma 5.4. *$C_\phi(U)$ is n-dimensional C^∞-manifold immersed in $\mathbb{R}^{(N+1)n}$ and is precompact.*

Proof. Since $C_\phi(U)$ is determined by the equation $d_{(\theta,y)}\phi(x,\theta,y) = 0$, for proving the first statement it suffices to prove that $d(\partial\phi/\partial\theta_1), \dots, d(\partial\phi/\partial\theta_{(N-1)n})$, $d(\partial\phi/\partial y_1), \dots, d(\partial\phi/\partial y_n)$ are linearly independent at each point $(x,\theta,y) \in C_\phi(U)$ as elements of $T^*\mathbb{R}^{n(N+1)}$, that is,

$$\operatorname{rank}\begin{bmatrix} \partial^2\phi/\partial x\partial y & \partial^2\phi/\partial y\partial y & \partial^2\phi/\partial y\partial\theta \\ \partial^2\phi/\partial x\partial\theta & \partial^2\phi/\partial y\partial\theta & \partial^2\phi/\partial\theta\partial\theta \end{bmatrix} = nN. \tag{5.25}$$

Since $S(\tau,x,y)$ and $S(\tau,x,y) + S^R_{-\infty}(y)$ satisfy the condition (5.3), the remark of Asada-Fujiwara [2] (at the end of Sect. 2 of [2]) shows

$$\det\begin{bmatrix} \partial^2\phi/\partial x\partial y & \partial^2\phi/\partial\theta\partial x \\ \partial^2\phi/\partial\theta\partial y & \partial^2\phi/\partial\theta\partial\theta \end{bmatrix} \neq 0, \tag{5.26}$$

at each point $(x,\theta,y) \in C_\phi(U)$, which obviously implies (5.25). The second statement is obvious. (Q.E.D.)

Lemma 5.5. *The mapping $i\colon C_\phi(U) \to \Lambda_\phi(U)$ is Lagrangian immersion. The projection*

$$\pi_1 : \Lambda_\phi(U) \ni (x, \partial\phi/\partial x) \to x \in \pi_1(\Lambda_\phi(U))$$

is a diffeomorphism.

Proof. The first statement is proved by Hörmander [7] (see also Asada-Fujiwara [2]). The second is obvious by Lemma 3.3. (Q.E.D.)

Lemma 5.6. *There exists a constant $\kappa > 0$ such that on $C_\phi(U)$,*

$$|\det \operatorname{Hess}_{(\theta,y)}(\phi)(x,\theta,y)| \geqq \kappa > 0,$$

$$\text{Hess}_{(\theta,y)}(\phi)=\begin{bmatrix}\partial^2\phi/\partial y\partial y & \partial^2\phi/\partial\theta\partial y\\ \partial^2\phi/\partial y\partial\theta & \partial^2\phi/\partial\theta\partial\theta\end{bmatrix}. \tag{5.27}$$

Proof. By virtue of Lemma 5.5, $dx_1, dx_2, \dots dx_n$ are linearly independent on $C_\phi(U)$. On the other hand on $C_\phi(U), d_{(\theta,y)}\phi=0$ and $d(\partial\phi/\partial\theta_1)\dots, d(\partial\phi/\partial\theta_{(N-1)n})$, $d(\partial\phi/\partial y_1)\dots, d(\partial\phi/\partial y_n)$ are linearly independent in $T^*\mathbb{R}^{N(n+1)}$. Therefore dx_1, $dx_2, \dots, dx_n, d(\partial\phi/\partial\theta_1), \dots, d(\partial\phi/\partial y_n)$ span the whole $T^*_{(x,\theta,y)}\mathbb{R}^{n(N+1)}$. This implies obviously that $\det \text{Hess}_{(\theta,y)}(\phi)\neq 0$ on each $(x,\theta,y)\in C_\phi(U)$. On the other hand the function $\text{Hess}_{(\theta,y)}(\phi)$ is continuous on $\overline{C_\phi(U)}$ and $\overline{C_\phi(U)}$ is compact. Combining these facts, we get easily the desired constant κ which satisfies (5.27). (Q.E.D.)

Now we can state the main theorem of this section.

Theorem 5.7. *Let Assumption (A) be satisfied. Let $f\in\mathscr{H}$ be such that* $\operatorname{supp} f\subset U$. *Then*

$$\lim_{h\downarrow 0}\sup_{t\geq t_0}\| e^{-itH^h/h}W_-^h\tilde{f}_a^h - e^{-n\pi i/4}|\det(\partial x(t,a,\eta)/\partial\eta)|^{-1/2} \cdot e^{i\text{Ind}_t(\gamma(x))\pi/2+iS(t,x)/h}f(\eta(t,a,x))\|=0. \tag{5.28}$$

Here $S(t,x)=\int_{-R}^{t} L(x_-(u,a,\eta(t,a,x)),\ \dot{x}_-(u,a,\eta(t,a,x)))du+S^R_{-\infty}(x(-R,a,\eta(t,a,x)))$ *and* $\text{Ind}_t\gamma(x)$ *is the so-called Keller-Maslov index of the path* $\gamma_t=\{(x_-(u,a,x),\ \xi_-(u,a,\eta(t,a,x))) : -\infty<u\leq t\}$ *and is defined explicitly in the proof.*

For proving the theorem we need the following lemmas:

Lemma 5.8. *Let $a(t,x,y)$ be the function defined in Theorem 5.1. Fix $y\in\mathbb{R}^n$ and consider $a(t,x,y)=a(t,x(\eta,y),y)$ as a function of t and $\eta=-\dfrac{\partial S}{\partial y}(t,x,y)$. Then $a(t,x(t,\eta,y),y)^2dx(t,\eta,y)=e^{-in\pi/2}d\eta$ for any $0<t<\delta$.*

Proof. We first prove that $a(t,x(t,\eta),y)^2dx(t,\eta)$ is t-independent n-form. To see this we differentiate this n-form by t and get

$$\frac{d}{dt}a(t,x(t,\eta,y),y)^2dx(t,\eta)=2\left((\partial a/\partial t)+\sum_{j=1}^{n}(\partial a/\partial x_j)(dx_j/dt)\right)a(t,x,y)dx + a(t,x,y)^2\sum_{j=1}^{n}dx_1\wedge\dots\wedge\frac{d}{dt}(dx_j)\wedge\dots\wedge dx_n.$$

Since $d/dt(dx_j)=d(dx_j/dt)=d(\partial S/\partial x_j(t,x,y))=\sum_{k=1}^{n}(\partial^2S/\partial x_j\partial x_k)dx_k$, the right hand side is equal to $a(t,x,y)dx$ multiplied by

$$2\left(\partial a/\partial t+\sum_{j=1}^{n}(\partial a/\partial x_j)(\partial S/\partial x_j)+2^{-1}(\Delta S)\cdot a\right)$$

which vanishes identically by the definition of $a(t,x,y)$. Since $\lim_{t\to 0}t^{-n}dx(t,\eta)=\lim_{t\to 0}d((x(t,\eta,y)-y)/t)=d\eta$ and the construction of $a(t,x,y)$ shows that $\lim_{t\to 0}t^{n/2}a(t,x,y)=e^{-in\pi/4}$, we get the desired result. (Q.E.D.)

Lemma 5.9. *For* $(x, \theta, y) \in C_\phi(U)$,

$$\det \operatorname{Hess}_{(\theta,y)}(\phi)(x,\theta,y)$$
$$= \det \operatorname{Hess}_{(\theta,y)}\left(\sum_{j=1}^{N} S(\tau, x_j, x_{j-1}) + S^R_{-\infty}(y)\right)$$
$$= \det(\partial x/\partial y)\cdot\det(\partial^2 S/\partial x_N \partial x_{N-1}) \ldots \det(\partial^2 S/\partial x \partial_2 x_1)\cdot\det(\partial^2 S/\partial x_1 \partial y).$$

Proof. Since $d_{(\theta,y)}\phi(x(y),\theta(y),y) = 0$, $y \in \Omega^{R,a}_{1,-}(U)$, we have for $j = 0, \ldots, N-1$,

$$(\partial/\partial y)(\partial\phi/\partial x_j(x(y),\theta(y),y)) = \sum_{k=0}^{N} \partial^2\phi/\partial x_j \partial x_k \cdot \partial x_k/\partial y$$
$$= 0, \text{ where } x_N = x, x_0 = y.$$

Hence for $1 \leqq j$,

$$\begin{aligned}&(\partial^2 S/\partial x_j \partial x_{j+1})(\tau, x_{j+1}(y), x_j(y))\cdot\partial x_{j+1}/\partial y\\ &\quad + (\partial^2 S/\partial x_j^2)(\tau, x_{j+1}(y), x_j(y))\cdot\partial x_j/\partial y\\ &\quad + (\partial^2 S/\partial x_j^2)(\tau, x_j(y), x_{j-1}(y))\cdot\partial x_j/\partial y\\ &\quad + (\partial^2 S/\partial x_j \partial x_{j-1})(\tau, x_j(y), x_{j-1}(y))\cdot\partial x_{j-1}/\partial y = 0\end{aligned} \tag{5.29}$$

and

$$\begin{aligned}&\partial^2 S/\partial x_1 \partial y(\tau, x_1(y), y)\cdot\partial x_1/\partial y + \partial^2 S/\partial y^2(\tau, x_1(y), y)\\ &\qquad + \partial^2 S^R_{-\infty}/\partial y^2(\tau, x_1(y), y) = 0.\end{aligned} \tag{5.30}$$

By definition

$$\det \operatorname{Hess}_{(\theta,y)}(\phi) =$$
$$\det\begin{bmatrix} \partial^2 S^R_{-\infty}/\partial y^2 + \partial^2 S/\partial y^2 & \partial^2 S/\partial y \partial x_1 & 0 & 0 & \ldots 0 \\ \partial^2 S/\partial x_1 \partial y & \partial^2 S/\partial x_1^2(x_1, y) + \partial S/\partial x_1^2(x_2, x_1) & \partial^2 S/\partial x_1 \partial x_2 \cdots 0 \\ 0 & \partial^2 S/\partial x_2 \partial x_1 & \partial^2 S/\partial x_2^2(x_2, x_1) + \partial^2 S/\partial x_2^2(x_3, x_2) \cdots 0 \\ 0 \quad 0 & * \quad * \quad * & \partial^2 S/\partial x_{N-1}^2(x, x_{N-1}) \end{bmatrix}. \tag{5.31}$$

Multiplying the $(j+1)$-th column by $\partial x_j/\partial y$ and adding those columns to the first we get by (5.29) and (5.30) that the right hand side of (5.31) is equal to the determinant of the following matrix:

$$\begin{bmatrix} 0 & \partial^2 S/\partial y \partial x_1 & 0 & & & \\ 0 & \partial^2 S/\partial x_1 \partial x_2 & 0 & & 0 & \\ 0 & * & * & * & 0 & \\ \cdots & \cdots & \cdots & \cdots & & \\ 0 & 0 & \cdots\cdots\cdots * & & * & \partial^2 S/\partial x_{N-1}\partial x_{N-2} \\ -\partial^2 S/\partial x \partial x_{N-1}\cdot\partial x/\partial y & & * \quad * \quad * \quad * & & & * \end{bmatrix}$$

which is equal to the desired quantity. (Q.E.D.)

Corollary 5.10

$$|\det \mathrm{Hess}(\phi)|^{-1} \prod_{j=1}^{N} |a(\tau, x_j(y), x_{j-1}(y))|^2 = |\det(\partial y/\partial x(y))|. \tag{5.32}$$

Proof. Since $\partial^2 S/\partial x_{j+1}\partial x_j = -\partial\xi_j(y)/\partial x_{j+1}$, (5.32) is the immediate consequence of Lemma 5.8 and Lemma 5.9. (Q.E.D.)

Proof of Theorem 5.7. We first prove

$$\lim_{h\downarrow 0} \| \exp(-it_0 H^h/\hbar) W_-^h \tilde{f}_a^h - e^{i\mathrm{Ind}(\gamma)\pi/2 + iS(t_0,x)/\hbar - in\pi/4} \cdot |\det(\partial x(t_0, a, \eta)/\partial\eta)|^{-1/2} f(\eta(t_0, a, x)) \| = 0. \tag{5.33}$$

By virtue of (5.19), it suffices to prove (5.33) replacing $\exp(-itH^h/\hbar)W_-^h \tilde{f}_a^h$ by $E(\tau, h)^N W_{-,a}^R f$. Let $\chi(x, \theta, y)$ be an infinitely differentiable function on $\mathbb{R}^n \times \mathbb{R}^{(N-1)n} \times \mathbb{R}^n$ with compact support such that $\chi(x, \theta, y) \equiv 1$ on a small neighbourhood of $C_\phi(U)$. Using the function $\chi(x, \theta, y)$, $E(\tau, h)^N W_{-,a}^R f$ is divided into two parts:

$$\begin{aligned} E(\tau, h)^N W_{-,a}^R f &= I_1(h)f + I_2(h)f \\ &= h^{-Nn/2} \int e^{i\hbar^{-1}\phi(x,\theta,y)} A(x, \theta, y)\chi(x, \theta, y) h_R(y) dy d\theta \\ &\quad + h^{-Nn/2} \int e^{i\hbar^{-1}\phi(x,\theta,y)} A(x, \theta, y)(1 - \chi(x, \theta, y)) h_R(y) dy d\theta, \end{aligned} \tag{5.34}$$

where $A(x, \theta, y) = \prod_{j=1}^{N} a(\tau, x_j, x_{j-1})$. We first treat $I_2(h)f$. Since $d_{(\theta,y)}\phi \neq 0$ on the support of $1 - \chi(x, \theta, y)$, the first order differential operator

$$L = (|\partial\phi/\partial\theta|^2 + |\partial\phi/\partial y|^2)^{-1}(\partial\phi/\partial\theta \cdot \partial/\partial\theta + \partial\phi/\partial y \cdot \partial/\partial y)$$

has a meaning on the support of $1 - \chi(x, \theta, y)$, and

$$-i\hbar \cdot L(e^{i\phi(x,\theta,y)/\hbar}) = e^{i\phi(x,\theta,y)/\hbar}$$

Therefore

$$\begin{aligned} &\int e^{i\hbar^{-1}\phi(x,\theta,y)} A(x, \theta, y)(1 - \chi(x, \theta, y) h_R(y) dy d\theta \\ &= -\int i\hbar \cdot L(e^{i\hbar^{-1}\phi(x,\theta,y)}) A(x, \theta, y)(1 - \chi(x, \theta, y) h_R(y) dy d\theta \\ &= -\hbar \int i e^{i\hbar^{-1}\phi(x,\theta,y)} L^*(A(x, \theta, y)(1 - \chi(x, \theta, y)) h_R(y)) dy d\theta \end{aligned} \tag{5.35}$$

Since $h_R(y)$ has a compact support and $\phi(t, x, y)$ satisfies (5.3) we easily see that

$$\begin{aligned} &L^*(A(x, \theta, y)(1 - \chi(x, \theta, y)) h_R(y)) \\ &= B_1(x, \theta, y) h_R(y) + B_2(x, \theta, y) \cdot \partial h_R/\partial y(y), \end{aligned}$$

where $B_1(x, \theta, y)$ and $B_2(x, \theta, y)$ are the functions in the space $\mathscr{B}(\mathbb{R}^n \times \mathbb{R}^{n(N-1)} \times \mathbb{R}^n)$. Therefore by using Theorem 5.1 we get the relation

$$\| I_2(h)f \| \leqq Ch \| f \|_1$$

with suitable constant C. (For the rigorous justification of the step from the 2-nd expression to the third in (5.35), the partial integration, we refer to Asada-Fujiwara [2]. Such a justification will be also necessary in calculations in Sect. 6, though we shall not mention it there.) For $I_1(h)f$ we use the stationary phase method. First of all we remark that supp $I_1(h)f$ is contained in the fixed compact set $\{x\in\mathbb{R}^n:(x,\theta,y)\in\operatorname{supp}\chi$ for some $(\theta,y)\in\mathbb{R}^{(n-1)N}\times\mathbb{R}^n\}$. There exists a constant $\kappa>0$ such that $|\det \mathrm{Hess}_{(\theta,y)}(\phi)(x,\theta,y)|\geqq\kappa$ on $C_\phi(U)$. Then the stationary phase method implies

$$\Big|I_1(h)f(x)-e^{\mathrm{Inert(Hess}\,(\phi))\pi i/2-in\pi/4}\cdot e^{iNn\pi/4}\prod_{j=1}^{N}a(\tau,x_j(x),x_{j-1}(x))e^{i\phi(x,\theta(x),y(x))/\hbar}$$

$$|\det \mathrm{Hess}_{(\theta,y)}\phi(x,\theta(x),y(x))|^{-1/2}h_R(y(x))\Big|\leqq C_f h, \tag{5.36}$$

where Inert (Hess(ϕ)) is the inertia of the Hessian of ϕ and C_f is the constant determined by a suitable Sobolev norm of h_R, hence by the one of f. As in the proof of Lemma 5.3, $x_j(x)=x(-R+j\tau,a,\eta(t,a,x)), j=0,1,\ldots,N-1$. Corollary 5.10 shows $\prod_{j=1}^{N}a(\tau,x_j(x),x_{j-1}(x))\cdot|\det(\mathrm{Hess}_{(\theta,y)}\phi(x,\theta(x),y(x))|^{-1/2}=e^{-iNn\pi/4}|\det(\partial y(x)/\partial x)|^{-1/2}$. By the construction of $S(\tau,x,y)$, it is obvious that $\phi(x,\theta(x),y(x))=\sum S(\tau,x_j(x),x_{j-1}(x))+S^R_{-\infty}(y(x))=S(t_0,x)$. Therefore writing as $\mathrm{Ind}_t(\gamma)=\mathrm{Inert(Hess}\,\phi(x,\theta(x),y(x))$, we get for $f\in C_0^\infty(U)$

$$\lim_{h\downarrow 0}\|I_1(h)f-e^{i\mathrm{Ind}(\gamma)\pi/2+iS(t,x)/\hbar-in\pi/4}|\det\partial y(x)/\partial x|^{1/2}h_R(y(x))\|=0. \tag{5.37}$$

(5.37) obviously implies (5.33) for $f\in C_0^\infty(U)$. Since the relevant operators in (5.33) are isometries, we can get (5.33) for general $f\in\mathscr{H}$ with supp $f\subset U$ by the standard argument.

Let us now prove the relation (5.28). Let $K_{t_0}(t,x)=K(t,x)$ be the function defined by (3.13). Then the definition of $K(t,x)$ implies that we can write as

$$\begin{aligned} Z_a^h(t)f(x)&\equiv|\det(\partial x(t,a,\eta)/\partial\eta)|^{-1/2}e^{-in\pi/4}e^{i\mathrm{Ind}_t(\gamma)\pi/2+iS(t,x)/\hbar}f(\eta(t,a,x))\\ &=e^{iS(t,x)/\hbar}e^{K(t,x)}\{|\det(\partial x(t_0,a,\eta)/\partial\eta)|^{-1/2}e^{-in\pi/4}\\ &\quad\cdot e^{i\mathrm{Ind}\,(\gamma)\pi/2}f(\eta(t_0,a,x(t_0,a,\eta(t,a,x))))\}\\ &\equiv e^{iS(t,x)/}\,e^{K(t,x)}g(y(t,x)), \end{aligned}\tag{5.38}$$

where $y=x(t_0,a,\eta(t,a,x))$. Then the similar argument used in the proof of Theorem 4.1 shows that

$$\begin{aligned}\|e^{-itH^h/\hbar}W^h_-\tilde f^h_a-Z^h_a(t)f\|&\leqq Ch\int_{t_0}^{t}\|\Delta_x(e^{K(u,x)}g(y(u,x)))\|du\\ &\quad+\|e^{-it_0H^h/\hbar}W^h_-\tilde f^h_a-Z^h_a(t_0)f\|.\end{aligned}\tag{5.39}$$

By (3.14) through (3.17) and the similar formula as (4.6), the first summand of the right hand side of (5.39) can be estimated as follows:

$$\begin{aligned}Ch\int_{t_0}^{t}\|\Delta_x(e^{K(u,x)}g(y(u,x))\|du&\leqq Ch\int_{t_0}^{t}u^{-2}\log(u)\|g\|_2du\\ &\leqq CMh\|f\|_2,\end{aligned}\tag{5.40}$$

where C and M are the constants depending only on t_0. From (5.39), (5.40) and the result of the first part of the proof of this theorem, we can get the statement of the theorem by the standard argument. (Q.E.D.)

6. The Quasi-Classical Limit of the Scattering Operator

Using the materials studied in the previous sections, we study here the asymptotic behavior of the scattering operator $S^h=(W_+^h)^*W_-^h$ on the coherent state $\tilde{f}_a^h$. We study it in the momentum space representation, that is, we study the asymptotic behavior of

$$\hat{S}^h(e^{-ip\cdot a/\hbar}f(p))=\mathscr{F}^hS^h\mathscr{F}^{h*}(e^{-ip\cdot a/\hbar}f(p)).$$

Under the Assumption (A), $\eta_+=\eta_+(\eta)$ is a smooth function of $\eta\in\mathbb{R}^n\backslash e(a)$. We define as

$$e(a)^{ex}=\{\eta\in\mathbb{R}^n\backslash e(a):\det\partial\eta_+(\eta)/\partial\eta=0\}\cup e(a) \tag{6.1}$$

Obviously the set $e(a)^{ex}$ is a closed subset of $\mathbb{R}^n$.

Theorem 6.1. *Let Assumption (A) be satisfied. Let* $K\subset\subset\mathbb{R}^n\backslash e(a)^{ex}$. *Then the following statements hold.*

1) *For any* $\eta_+\in S_a^{cl}(K)$, *there exists at most finite number of* $\eta_j=\eta_j(\eta_+)\in K$ *such that* $\eta_+=S_a^{cl}(\eta_j)$.

2) *Let* $f\in\mathscr{H}$ *be such that* $\operatorname{supp} f\subset K$. *Then*

$$\lim_{h\downarrow 0}\Big\| \hat{S}^h f_a^h(\eta_+)-\sum_j e^{i\operatorname{Ind}\gamma(\eta_+,\eta_j)\pi/2}\left|\det\partial\eta_+(\eta)/\partial\eta\right|^{-1/2}\Big|_{\eta=\eta_j(\eta_+)} \cdot e^{i(S(\eta_+,\eta_j)-\eta_+\cdot a+(\eta_j(\eta_+)))/\hbar}f(\eta_j(\eta_+))\Big\|=0, \tag{6.2}$$

where $\operatorname{Ind}\gamma(\eta_+,\eta_j)$ *is the Keller-Maslov's index of the orbit* $(x_-(t,a,\eta_j),\xi_-(t,a,\eta_j):-\infty<t<\infty)$,

$$S(\eta_+,\eta_j)=\lim_{\substack{t\to\infty\\ s\to-\infty}}\left\{\int_s^t L(x_-(u,a,\eta_j),\dot{x}_-(u,a,\eta_j))du+s\eta_j^2/2m-t\eta_+^2/2m\right\}. \tag{6.3}$$

Proof. We assume $m=1$. Let us take K_1 as $K\subset\subset K_1\subset\subset\mathbb{R}^n\backslash e(a)^{ex}$. Since K_1 is precompact there exists a constant $\delta_{K_1}>0$ such that (3.11) holds for every $\eta\in K_1$. Then by the remark following (3.11), the first statement of the theorem is obvious. As in that remark, let us take finite precompact subsets $\{U_j\}$ such that $\{U_j\}$ is a covering of K and S_a^{cl} is a diffeomorphism on each U_j. It is sufficient to prove the relation (6.2) for $f\in\mathscr{H}$ with $\operatorname{supp} f\subset U_j$, for some U_j. Moreover, since in this case the operators appearing in (6.2) are isometric, we may assume that $f\in C_0^\infty(U)$. By virtue of Theorem 5.7 and by the unitarity of the propagator $\exp(iH_0^h/\hbar)$ and the Fourier transform $\mathscr{F}^h$, we get

$$\lim_{h\downarrow 0}\sup_{t\geq t_0}\left\|\mathscr{F}^h e^{itH_0^h/\hbar}e^{-itH^h/\hbar}W_-^h\tilde{f}_a^h-\mathscr{F}^h e^{itH_0^h/\hbar}Z_a^h(t)f\right\|=0. \tag{6.4}$$

Here we used the notation in the proof of Theorem 5.7 for $Z_a^h(t)$. The second term of the left hand side of (6.4) can be written as

$$\mathscr{F}^h e^{itH_0^h/\hbar} Z_a^h(t) f = e^{itp^2/2\hbar} \mathscr{F}^h Z_a^h(t) f. \tag{6.5}$$

To the right hand side of (6.5) we apply the stationary phase method. Using the notation of Theorem 5.7 and its proof, let us write as

$$\begin{aligned} h(t,x) &= e^{i\mathrm{Ind}_t \gamma(x)\pi/2} \left|\det(\partial x(t,a,\eta)/\partial\eta)\right|^{-1/2} e^{-in\pi/4} f(\eta(t,a,x)) \\ &= e^{K(t,x)} g(y(x,t)). \end{aligned} \tag{6.6}$$

Then

$$\mathscr{F}^h Z_a^h(t) f(\eta) = h^{-n/2} \int e^{i\hbar^{-1}(S(t,x)-x\cdot\eta)} h(t,x)dx. \tag{6.7}$$

Making the change of variable x to xt and writing $v = \hbar t^{-1}$, $\phi(t,x,\eta) = (S(t,tx)t^{-1} - x\cdot\eta)$, and $h_t(x) = t^{n/2}h(t,tx)$, we have

$$F^h Z_a^h(t) f(\eta) = (2\pi v)^{-n/2} \int e^{iv^{-1}\phi(t,x,\eta)} h_t(x)dx. \tag{6.8}$$

For each $\eta \in \mathbb{R}^n$, the point of stationary phase is determined by the equation $d_x\phi(t,x,\eta) = 0$, that is,

$$\partial S/\partial x(t,tx) = \eta.$$

Since $\partial S/\partial x(t,tx) = \xi_-(t,a,\eta(t,a,tx))$ and the mappings $\eta \to \xi_-(t,a,\eta)$ and $\eta \to x_-(t,a,\eta)$ are both diffeomorphisms on U by Lemma 2.7 and Lemma 3.3, we can easily see that if $\eta \in \pi_2 U(t+R) W_{-,R}^{cl}(U)$, there exists a unique point of stationary phase $x = x_t(\eta) \equiv x_-(t,a,\eta)/t$ and otherwise there is no point of stationary phase. Let us write as

$$C_t(U) = \{(x_t(\eta), \eta) : \eta \in \pi_2 U(t+R) W_{-,R}^{cl}(U)\} \tag{6.9}$$

By Lemma 2.7 and Lemma 3.3, we can see that there is a precompact subset $V \subset \mathbb{R}^n \times \mathbb{R}^n$, $W \subset K \subset W_1 \subset K_1$ such that $\bar{V}$ is a neighbourhood of the diagonal set of $S_a^{cl}(W) \times S_a^{cl}(W)$ contained in $S_a^{cl}(W_1) \times S_a^{cl}(W_1)$ such that $\bigcup_{t \geqq t_0} C_t(U) \subset\subset V$.

Let us take as infinitely differentiable function $\chi(x,\eta)$ on $\mathbb{R}^n \times \mathbb{R}^n$ with compact support such that $\chi(x,\eta) \equiv 1$ on V. Using the function $\chi(x,\eta)$ we divide $\mathscr{F}^h Z_a^h(t) f(\eta)$ into two parts:

$$\begin{aligned} \mathscr{F}^h Z_a^h(t) f(\eta) &= (2\pi v)^{-n/2} \int e^{iv^{-1}\phi(t,x,\eta)} \chi(x,\eta) h_t(x)dx \\ &\quad + (2\pi v)^{-n/2} \int e^{iv^{-1}\phi(t,x,\eta)} (1-\chi(x,\eta)) h_t(x)dx \\ &= I_1(t,v) f(\eta) + I_2(t,v) f(\eta). \end{aligned} \tag{6.10}$$

We should remark here that in each integral of (6.10), there exists a compact subset of $\mathbb{R}_x^n$ such that the support of function $h_t(x)$ is contained in it for $t \geqq t_0$ by Lemma 2.7. We first study $I_2(t,v)f(\eta)$. Define the first order differential operator L_t by

$$L_t = \left(\sum_{j=1}^n (\partial\phi/\partial x_j)^2 \right)^{-1} \sum_{j=1}^n (\partial\phi/\partial x_j) \cdot \partial/\partial x_j$$

which satisfies

$$-ivL_t(e^{iv^{-1}\phi(t,x,\eta)}) = e^{iv^{-1}\phi(t,x,\eta)} \tag{6.11}$$

and

$$L_t^* = -L_t - \left\{\left(\sum_{j=1}^n (\partial\phi/\partial x_j)^2\right)^{-1} \Delta\phi - 2\left(\sum_{j=1}^n (\partial\phi/\partial x_j)^2\right)^{-2} \sum_{j,k=1}^n (\partial\phi/\partial x_j)(\partial\phi/\partial x_k)(\partial^2\phi/\partial x_k\partial x_j)\right\} \tag{6.12}$$

Hence writing the second term of the r.h.s of (6.12) as $G(t,x,p)$,

$$\begin{aligned}(2\pi)^{n/2}I_2(t,v)f(\eta) &= -iv\cdot v^{-n/2}\int L_t(r^{iv^{-1}\phi(t,x,\eta)})(1-\chi(x,\eta))h_t(x)dx \\ &= -iv\cdot v^{-n/2}\int e^{iv^{-1}\phi(t,x,\eta)}L_t^*((1-\chi(x,\eta))h_t(x))dx \\ &= iv\cdot v^{-n/2}\int e^{iv^{-1}\phi(t,x,\eta)}(1-\chi(x,\eta))L_t(h_t(x))dx \\ &\quad + iv\cdot v^{-n/2}\int e^{iv^{-1}\phi(t,x,\eta)}\{L_t(1-\chi(x,\eta))+G(t,x,\eta)\}h_t(x)dx \\ &= II_1 + II_2. \end{aligned} \tag{6.13}$$

By Lemma 2.7 and Lemma 3.3 we can easily see that

$$L_t(1-\chi(x,\eta))+G(t,x,\eta)\in\mathscr{B}(\mathbb{R}^n\times\mathbb{R}^n),$$

uniformly in $t\geqq t_0$, and $\phi(t,x,\eta)$ obviously satisfies (5.13) uniformly in $t\geqq t_0$ (taking the number of variable $\theta=m=0$). Hence by Theorem 5.2, we get

$$\|II_2\| \leqq Cv\|h_t(x)\| \leqq Cv\|h(t,x)\| \leqq Cv\|f\|. \tag{6.14}$$

To estimate II_1, we first note that, using the terminology (5.38),

$$\begin{aligned}\partial/\partial x_j(h_t(x)) &= \partial/\partial x_j(e^{K(t,tx)}g(y(t,tx)))t^{n/2} \\ &= t(\partial K/\partial x_j)e^{K(t,tx)}g(y(t,tx))t^{n/2} \\ &\quad + e^{K(t,tx)}(\partial g/\partial y)(y(t,tx))\cdot t(\partial y/\partial x)t^{n/2}.\end{aligned}$$

Since $(1-\chi(x,\eta))(\sum(\partial\phi/\partial x_j)^2)^{-1}\partial\phi/\partial x_j\in\mathscr{B}(\mathbb{R}^n\times\mathbb{R}^n)$ uniformly in $t;|\partial K/\partial x_j| = t|\sum(\partial K/\partial y_k)\cdot(\partial y_k/\partial x_j)| \leqq Ct\cdot t^{-1}\cdot\log t = C\log t$ and $t|\partial y/\partial x|\leqq C$ by Lemma 2.7 and Lemma 3.3, we get again by Theorem 5.2 that

$$\begin{aligned}\|II_1\| &\leqq Cv(\log t\|h_t(x)\| + \|e^{K(t,x)}(\partial g/\partial y)(y(t,x))\|) \\ &\leqq C\hbar/t(\log t\|f\| + \|f\|_1)\end{aligned} \tag{6.15}$$

Summing up the relations (6.14) and (6.15), we get

$$\|I_2(t,y)f\| \leqq Cht^{-1}\log t\|f\|_1. \tag{6.16}$$

Now we turn to the study of $I_1(t,v)f(\eta)$. Since $C_t(U)$ converges to $C_\infty(U)$, we may assume that for any $t\geqq t_0$ and $\eta\in\pi_2V$, there exists $x_t(\eta)\in S_a^{cl}(W_1)$ such that $0=d_x\phi(t,x_t(\eta),\eta)$. For $(x_t(\eta),\eta)$ the function $\phi(t,x,\eta)$ can be written as

$$\phi(t,x,\eta) = \phi(t,x_t(\eta),\eta) + (x-x_t(\eta))\cdot B(t,x,\eta)(x-x_t(\eta)). \tag{6.17}$$

where $B(t,x,\eta)=\int_0^1(1-u)\partial^2\phi/\partial x^2(t,ux+x_t(\eta)(1-u),\eta)du$. Since $\mathrm{Hess}_x\phi(t,x,\eta)=t(\mathrm{Hess}_x S)(t,tx)=t(\partial\xi_-/\partial x)(t,a,\eta(t,a,tx))$

$$=\partial\xi_-(t,a,\eta)/\partial\eta\big|_{\eta=\eta(t,a,xt)}t(\partial x_-(t,a,\eta)/\partial\eta\big|_{\eta=\eta(t,a,tx)})^{-1},$$

Lemma 2.7 and Lemma 3.3 imply that the family $\{\mathrm{Hess}_x\phi(t,\dots,)\}_{t\geqq t_0}$ is equicontinuous and are non-singular on $\bar{V}$. Therefore by Morse's lemma, implicit function theorem and a simple compactness argument, it follows that there exists a finite number of subsets V_k of $\mathbb{R}^n$ such that each V_k is precompact open set, $\cup V_k\supset\bar{V}$ and on each V_k there exists a change of variable x to $y=y(x,t,\eta)$ such that $\{y(x,t,\eta)\}_{t\geqq t_0}$ is a bounded set of $\mathscr{B}(\mathbb{R}^n\times\mathbb{R}^n)$ with its inverse, $y(x_t(\eta),t,\eta)=0$, $\det\partial x/\partial y(x_t(\eta),t,\eta)=1$ and $B(t,x,\eta)=2^{-1}\sum d_j(t,\eta)y^2$, where $d_j(t,\eta)$'s are the eigenvalues of the matrix $\mathrm{Hess}_x\phi(t,x_t(\eta),\eta)$. Let $\{\tilde{\omega}_k(x,\eta)\}\subset C_0^\infty(\mathbb{R}^n\times\mathbb{R}^n)$ be the partition of unity on V subordinate to the covering $\{V_k\},\omega_k(x,\eta)=\tilde{\omega}_k(x,\eta)\chi(x,\eta)$. Using the function $\omega_j(x,\eta)$ we divide $I_1(t,v)f(\eta)$ as

$$\begin{aligned}I_1(t,v)f(\eta)&=\sum_k I_{1,k}(t,v)f(\eta)\\&=\sum_k(2\pi v)^{-n/2}\int e^{iv^{-1}\phi(t,x,\eta)}\omega_k(x,\eta)h(t,tx)t^{n/2}dx\\&=\sum_k(2\pi v)^{-n/2}\int e^{iv^{-1}\phi(t,x_t(\eta),\eta)+iv^{-1}2^{-1}\sum d_jy_j^2}\\&\qquad\cdot t^{n/2}\omega_k(x(y,\eta,t),\eta)h(t,tx(y,\eta,t))\cdot\det(\partial x/\partial y)dy\\&=\sum_k e^{iv^{-1}\phi(t,x_t(\eta),\eta)}(2v\pi)^{-n/2}\int e^{iv^{-1}2^{-1}\sum d_jy_j^2}h_k(t,y,\eta)dy,\end{aligned}\tag{6.18}$$

where $h_k(t,y,\eta)=t^{n/2}\omega_k(x(y,\eta,t),\eta)h(t,tx(y,\eta,t))\det(\partial x/\partial y)$.

Since $(\mathrm{Hess}_x\phi(t,x,\eta))^{-1}$ is uniformly bounded in $(t,x,\eta)\in\{t\geqq t_0\}\times\bar{V},y(t,x,\eta)$ forms a bounded set of $\mathscr{B}(\mathbb{R}^n\times\mathbb{R}^n)$ for $t\geqq t_0$ with its inverse function. Furthermore there exists constants C_1 and C_2 such that $C_1\leqq|d_j(t,\eta)|\leqq C_2$ on the support of $h_R(t,y,\eta)$. Therefore by a similar argument used in the proof of Theorem 1.1 we get

$$\begin{aligned}&\|I_{1,k}(t,v)f(\eta)-e^{iv^{-1}\phi(t,x_t(\eta),\eta)+in\pi/4}t^{n/2}|\det(\mathrm{Hess}_x\phi(t,x_t(\eta),\eta))|^{-n/2}\\&\quad\cdot\omega_k(x_t(\eta),\eta)h(t,x_t(\eta)t)\|\\&\leqq Cv\left\|\int_0^1 ds\int(vs)^{-n/2}e^{i(vs)^{-1}\sum d_jy_j^2}\sum_l d_l(t,\eta)(\partial^2/\partial y_l^2)h_k(t,y,\eta)dy\right\|\\&\leqq Cv\sum_l\int_0^1 ds\left\|(vs)^{-n/2}\int e^{i(vs)^{-1}\sum d_jy_j^2}(\partial^2/\partial y_l^2)h_k(t,y,\eta)dy\right\|.\end{aligned}\tag{6.19}$$

Here we used that $\det(\mathrm{Hess}_x\phi(t,x_t(\eta),\eta))>0$ which follows from Lemma 2.7.

After changing back variable y to x, the integral in the third of (6.19) can be written in the form

$$\sum_{|\alpha|\leqq 2}(vs)^{-n/2}\int e^{i(vs)^{-1}\phi(t,x,y)}a_\alpha(t,x,\eta)(\partial/\partial x)^\alpha h(t,tx)t^{n/2}dx,$$

where the functions $a_\alpha(t,x,\eta)$ are contained in a bounded set of $\mathscr{B}(\mathbb{R}^n\times\mathbb{R}^n)$ for

$t > t_0$. Since $(\partial/\partial x)^\alpha h(t, tx)t^{n/2} = t^{n/2}\{(\partial/\partial x)^\alpha(e^{K(t,tx)}g(y(t, tx)))\}$ and $|(\partial/\partial x)^\alpha K(t, tx)| \leqq C \log t$, we get $\|(\partial/\partial x)^\alpha g(y(t, tx))\cdot \exp(K(t, tx))\| \leqq C\|g\|_2 \leqq C\|f\|_2$ and $\|(\partial/\partial x)^\alpha h(t, x)t^{n/2}\| \leqq C \log t$. Therefore again by Theorem 5.2, we get that the right hand side of (6.19) can be estimated by $Cht^{-1}(\log t)$. Combining these estimates with the estimate for $I_2(t, v)f$ we finally get the following estimate:

$$\limsup_{h\downarrow 0\ t\geqq t_0} \| \mathscr{F}^h e^{itH_0^h/\hbar} e^{-itH^h/\hbar} W_-^h \tilde{f}_a^h(\eta_+) - e^{i\,\mathrm{Ind}_t\gamma(t,\eta_+)} e^{it\eta_+^2/2\hbar} e^{i(S(t,x_t(\eta_+)t) - tx_t(\eta_+)\cdot\eta_+)/\hbar} \cdot |\det \partial\eta_+(t, a, \eta)/\partial\eta|_{\eta=\eta(t,a,\eta_+)}|^{-1/2} f(\eta(t, a, \eta_+))\| = 0. \tag{6.20}$$

Since $tx_t(\eta_+) = x_-(t, a, \eta(t, a, \eta_+))$, $tx_t(\eta_+)\eta_+ - t\eta_+^2/2 - a_+(a, \eta_+)\eta_+ = (x_-(t, a, \eta(t, a, \eta_+)) - t\eta_+ - a_+(a, \eta_+))\eta_+$ converges to zero as $t \to \infty$, we get the desired relation, taking the limit $t \to \infty$ in (6.20). (Q.E.D.)

7. Concluding Remarks

Remark 1. We first want to explain here the technical reason why we took $e^{-i\eta\cdot a/\hbar}f(\eta)$ as the initial state instead of $e^{ix\cdot\xi/\hbar}f(x)$ which, in the limit $h \to 0$, gives the ensemble of the classical particles with the fixed momentum ξ and the configuration distribution density $|f(x)|^2$. Indeed the fixed momentum scattering would be more suitable in the classical mechanics because of its popularity. However, in spite of the fact that as $h \to 0$ the state $e^{ix\cdot\xi/\hbar}f(x)$ is propagated by the free motion $\exp(itH_0^h/\hbar)$ as the ensemble of the classical particles with the fixed momentum ξ in *finite* time region, it is *not uniform* in $t > \delta > 0$. We can see it from the following remarkable fact: for any $f \in \mathscr{H}$,

$$\lim_{t\to\pm\infty} \| e^{itH_0}f(x) - (m/it)^{n/2} e^{imx^2/2t}\hat{f}(mx/t)\| = 0. \tag{7.1}$$

The relation (7.1) implies that we cannot prepare the ensemble of the fixed momentum incoming classical particles from the quantum mechanics, at least in L^2-theoretical framework.

Remark 2. In the theorem we have the finite sum over the incoming momentums with the same outgoing one. These summands are asymptotically orthogonal each other as $h \to 0$. This can be easily seen as follows. Since the classical mechanical scattering operator is a canonical mapping on $\mathbb{R}^n \times \mathbb{R}^n \backslash e$ we have $a_+(\eta_j) = \left(-\frac{\partial}{\partial\eta_+}\right)(S(\eta_+, \eta_j) - a_+(\eta_j)\cdot\eta_+) \neq \left(-\frac{\partial}{\partial\eta_+}\right)(S(\eta_+, \eta_k) - a_+(\eta_k)\cdot\eta_+) = a_+(\eta_k)$ if $\eta_k \neq \eta_j$. Therefore by the stationary phase method we can see that they are asymptotically orthogonal each other as $h \to 0$. Hence if $f \in \mathscr{H}$ has its support $\mathrm{supp}\, f \subset\subset \mathbb{R}^n \backslash e(a)^{ex}$,

$$\lim_{h\to 0} \int_D |S^h(e^{-i\eta\cdot a/h}f(\cdot))(\eta)|^2 d\eta = \int_{(S_a^{cl})^{-1}D} |f(\eta)|^2 d\eta, \tag{7.2}$$

and (7.2) is still true for $f \in \mathscr{H}$ with $\mathrm{supp}\, f \subset \mathbb{R}^n \backslash e(a)^{ex}$ (standard argument).
Thanks to this relation we may say that $\sum_j |\partial\eta_+(\eta_j)/\partial\eta_j|^{-1}$ gives the "differential cross section" associated with this classical scattering process.

Remark 3. If $f \in H^2(\mathbb{R}^n)$ and $\operatorname{supp} f \subset\subset \mathbb{R}^n \backslash e(a)^{ex}$, the theorem can be improved so that the norm before taking limit $h \to 0$ is estimated by a constant times $h \| f \|_2$. This is an obvious consequence of our calculus.

Remark 4. A more general situation where the initial state is $e^{-iS(\eta)/\hbar} f(\eta)$ with general $S \in C^\infty(\mathbb{R}^n)$ and $V(x)$ is a long range potential will be discussed in the forthcoming paper [14].

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