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Two-Cluster Scattering of N Charged Particles

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Abstract. *We define geometrically two-cluster scattering states by their asymptotic space-time behavior.* We show that these subspaces coincide with the ranges of the two-cluster wave operators, or modified wave operators if both clusters are charged. In particular this proves asymptotic completeness and absence of a singular continuous spectrum of the Hamiltonian below the lowest three-body threshold.

Introduction

It is common belief that two-cluster scattering and two-particle (=potential-)scattering are closely related. Asymptotically it should not matter whether two particles or two bounded subsystems move apart from each other and become free. Indeed Combes [2] could prove asymptotic completeness below the three-body threshold where the asymptotic breakup into more than two clusters is energetically forbidden. Simon simplified [14] Combes' proof and extended it [15] to include more potentials. Simon's treatment allowed charged particles but at least one of the clusters had to be neutral[1]. Our method includes the case where both clusters are charged (which requires the use of modified wave operators), also more general short range interactions which may be velocity dependent are included.

To avoid the energy restriction we introduce two subspaces ($\mathbf{2}_{\text{in}}, \mathbf{2}_{\text{out}}$) describing the incoming or outgoing two-cluster scattering states. We characterize them "geometrically", i.e. by their behavior in space and time. In these states the particles within each of the clusters "stay together" asymptotically in the past or future whereas the clusters separate (we give the precise definition below). This is a *natural extension of Ruelle's geometric characterization of bound states and* scattering states [13].

We show that $\mathbf{2}_{\text{in/out}}$ coincides with the direct sum of the ranges of the corresponding two-cluster (modified) wave operators. Using compactness arguments we can decompose any state from $\mathbf{2}_{\text{out}}$ at a sufficiently late time into a finite

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1 See "Note Added in Proof"

sum of simple parts: each component is a bound state in each cluster and the clusters are arbitrarily far separated. These pieces can be further treated with the methods developed for potential scattering in [5] and [6]. We assume distinguishable particles but the changes necessary to accomodate identical particles are obvious.

We have not given the most general results. Combining this paper with the detailed analysis of Simon [16] it should not be hard to include more general Hamiltonians, Dirac-systems, highly singular repulsive potentials, absorptive interactions, exterior static electromagnetic fields etc.

For a general N-particle state the decomposition used here can be continued such that only finitely many bound states have to be considered within each cluster. This will be given in a separate paper.

N-Body Dynamics, Some Notation

We study a system of N distinguishable particles of masses m_i, each moving in ν-dimensional space. We separate off the conserved center of mass (CM) motion, the $n=\nu(N-1)$ dimensional configuration space is

$$X=\left\{x=(\mathbf{x}^1,\mathbf{x}^2,\dots,\mathbf{x}^N)|\mathbf{x}^i\in\mathbb{R}^\nu,\ \sum_{i=1}^N m_i\mathbf{x}^i=\mathbf{0}\right\}. \tag{1}$$

The very convenient N-particle notation used here is taken from Appendix 1 in [3], see there for details, some of our definitions differ by a factor 2.

The conjugate momenta p are elements of the dual space X^*

$$X^*=\left\{p=(\mathbf{p}_1,\mathbf{p}_2,\dots,\mathbf{p}_N)|\mathbf{p}_i\in\mathbb{R}^\nu,\ \sum_{i=1}^N \mathbf{p}_i=\mathbf{0}\right\} \tag{2}$$

with the bilinear form

$$(p,x)=\sum_{i=1}^N \mathbf{p}_i\cdot\mathbf{x}^i, \tag{3}$$

$\mathbf{p}_i\cdot\mathbf{x}^i$ the usual inner product in $\mathbb{R}^\nu$. The velocity $v=(\mathbf{v}^1,\dots,\mathbf{v}^N)=dx/dt\in X$. The relation $\mathbf{p}_i=m_i\mathbf{v}^i$ provides a natural identification of X with X^* and thus from (3) a scalar product on X

$$\langle x,y\rangle=\sum_{i=1}^N m_i\mathbf{x}^i\cdot\mathbf{y}^i. \tag{4}$$

dx denotes the volume element of X w.r.t. the metric (4), then $\mathscr{H}=L^2(X,dx)$ is the state space. The free Hamiltonian $\mathbb{H}^0$ on $\mathscr{H}$ is the selfadjoint operator

$$\mathbb{H}^0=\sum_{i=1}^N (2m_i)^{-1}(\mathbf{p}_i)^2. \tag{5}$$

Let $D=(C,C')$ be a two-cluster decomposition of $\{1,2,\dots,N\}=C\cup C'$, $C\cap C'=\emptyset$; $C,C'\neq\emptyset$. (In this paper D always denotes a non trivial two-cluster decomposition.)

$$m_C=\sum_{i\in C} m_i;\qquad \mathbf{x}_C=m_C^{-1}\sum_{i\in C} m_i\mathbf{x}^i \tag{6}$$

are the mass and the CM-coordinate of the cluster C resp.

$$(\Pi_C x)^i = \begin{cases} \mathbf{x}^i - \mathbf{x}_C & \text{if } \; i \in C \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

defines an orthogonal projection on X, equipped with the scalar product (4), moreover Π_C, $\Pi_{C'}$, and Π_D given by

$$\Pi_D = \mathbb{1} - \Pi_C - \Pi_{C'} \tag{8}$$

are pairwise orthogonal projections. They generate the splitting

$$X = X_D \oplus X_C \oplus X_{C'} \tag{9}$$

leading to the decomposition

$$\mathscr{H} = \mathscr{H}_D \otimes \mathscr{H}_C \otimes \mathscr{H}_{C'} = L^2(X_D) \otimes L^2(X_C) \otimes L^2(X_{C'}) . \tag{10}$$

$X_C = \Pi_C X$ is a $\nu(K-1)$ dimensional space if K particles are in the cluster. $z_C = \Pi_C x$ describes the internal configuration of the cluster in its CM frame.

$$|z_C|^2 = \langle x, \Pi_C x \rangle = \sum_{j \in C} m_j (\mathbf{z}^j)^2 . \tag{11}$$

The volume element on X_C corresponds to (11). $\mathscr{H}_C = \mathbb{C}$ if only one particle belongs to C.

X_D is ν-dimensional, it can be parametrized by the coordinate

$$\mathbf{y} = \mathbf{x}_{C'} - \mathbf{x}_C \tag{12}$$

and has Euclidean metric.

The relative velocity of the centers of mass of the clusters is

$$\mathbf{v} = \mathbf{v}_{C'} - \mathbf{v}_C \tag{13}$$

and $\mathbf{p}$ is the conjugate momentum to $\mathbf{y}$ if

$$\mathbf{p} = \mu \mathbf{v} , \qquad \mu = m_C m_{C'} (m_C + m_{C'})^{-1} , \tag{14}$$

μ is the reduced mass of the two clusters which depends on the decomposition D.

Under the decomposition (10) the free Hamiltonian (5) splits into

$$\begin{aligned} \mathbb{H}^0 &= H_0 \otimes \mathbb{1} \otimes \mathbb{1} + \mathbb{1} \otimes \mathbb{H}^0_C \otimes \mathbb{1} + \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{H}^0_{C'} \\ &= H_0 + \mathbb{H}^0_C + \mathbb{H}^0_{C'} , \end{aligned} \tag{15}$$

where $\mathbb{H}^0_C$ is the kinetic energy of the particles in cluster C after separating off their CM-motion, $H_0 = (2\mu)^{-1} \mathbf{p}^2$ is the relative kinetic energy of the clusters.

The operators $F(\cdot)$ and $F_0(\cdot)$ are the same as in [6], $F(\cdot)$ denotes multiplication with the characteristic function of the operator and the region specified in the parantheses. $F_0(\cdot)$ is multiplication with the function obtained by convoluting the characteristic function of the specified region with a fixed function $\zeta \in \mathscr{S}(\mathbb{R}^\nu)$. It obeys $\int d^\nu x \zeta(\mathbf{x}) = 1$, and $\operatorname{supp} \tilde{\zeta}(\mathbf{p})$ is contained in a ball of radius $\left(\min_D \mu\right) a/2$; $0 < a \leqq 1$ a constant to be fixed later. We will need one more multiplication operator. Let $\psi \in \mathscr{D}(\mathbb{R})$ obey $0 \leqq \psi(\lambda) \leqq 1$, $\psi(\lambda) = 1(0)$ if $\lambda \leqq 1$ $(\geqq 2)$.

Define for $r>0$

$$E(|\mathbf{u}|\leqq r)=\psi(|\mathbf{u}|/r). \tag{16}$$

Then $\|E\|=1$, $E\in\mathscr{D}(\mathbb{R}^m)$

$$\lim_{r\to\infty}\|\nabla E\|=0, \qquad \lim_{r\to\infty}\|\Delta E\|=0 \tag{17}$$

no matter what the dimension m of the vector $\mathbf{u}$ is or which of the norms above is used. We use the shorthand

$$E(|\mathbf{u}|\geqq r):=\mathbb{1}-E(|\mathbf{u}|\leqq r).$$

The interaction is given by pair potentials W_{ij} and $\mathscr{W}_{ij}$ (also three- and more-body forces can easily be included). W_{ij} describe possibly singular and velocity dependent forces of short range given as multiplication or pseudo-differential operators. We assume that on $L^2(\mathbb{R}^\nu)$ they are form bounded relative to $(-\Delta+\mathbb{1})$ and obey

$$\|(-\Delta+\mathbb{1})^{-1/2}W_{ij}(-\Delta+\mathbb{1})^{-1}F(|\mathbf{x}|\geqq R)\|=:h(R)\in L^1([0,\infty),dR). \tag{18}$$

The long range forces (like Coulomb forces in $\mathbb{R}^3$) are given by $\mathscr{W}_{ij}$, the multiplication with the once continuously differentiable function $\mathscr{W}_{ij}(\mathbf{x})$ vanishing at infinity which fulfills

$$|\nabla\mathscr{W}_{ij}(\mathbf{x})|\leqq\text{const}(1+|\mathbf{x}|)^{-(1+\gamma)}, \qquad 1\geqq\gamma>(3\nu+2)/(3\nu+3). \tag{19}$$

According to a construction of Hörmander [7] any such function can be split into a smooth function with specific decay properties of its Fourier transform and a remainder of short range. If one decomposition of the interaction $\bar{W}_{ij}+\bar{\mathscr{W}}_{ij}$ obeys (18) and (19) then we can replace it by a better one $W_{ij}+\mathscr{W}_{ij}$. Thus without loss of generality we may assume that $\tilde{\mathscr{W}}_{ij}$ has the specific decay properties, for details see [6].

$$I_C=\sum_{i<j}W_{ij}+\mathscr{W}_{ij}(\mathbf{x}^i-\mathbf{x}^j), \qquad i,j\in C \tag{20}$$

is the interaction within the cluster C. Our next assumption is that for all clusters C (including that of all particles)

$$|(\Psi,I_C\Psi)|\leqq a(\Psi,\mathbb{H}_C^0\Psi)+b\|\Psi\|^2 \tag{21}$$

for all $\Psi\in\mathscr{Q}(\mathbb{H}_C^0)$, $a<1$.

$$H_C=\mathbb{H}_C^0+I_C \tag{22}$$

is a selfadjoint semibounded operator with $\mathscr{Q}(H_C)=\mathscr{Q}(\mathbb{H}_C^0)$ (Theorem X.17 in [11])

$$H_D=H_0+H_C+H_{C'}=H-I_D \tag{23}$$

is the Hamiltonian where the inter-cluster interactions

$$I_D=\sum_{i,j}W_{ij}+\mathscr{W}_{ij}(\mathbf{x}^i-\mathbf{x}^j), \qquad i\in C', \qquad j\in C \tag{24}$$

are removed, H the total Hamiltonian.

$$\exp(-iH_Dt)=\exp(-iH_0t)\otimes\exp(-iH_Ct)\otimes\exp(-iH_{C'}t) \tag{25}$$

preserves the decomposition (10).

The last assumption is technical, we have to exclude slowly decaying wave functions of bound states. Let C be a proper subset of $\{1, \ldots, N\}$. For any $\Phi \in \mathscr{H}_C$ with $H_C \Phi = E\Phi$ we require

$$\int_0^\infty dr \| F(|z_C| \geqq R^\alpha)\Phi \| < \infty , \tag{26}$$

for some $\alpha < \gamma \leqq 1$, or $\alpha = 1$ if no long range forces are present.

For big classes of interactions all bound states decay exponentially, but there is also an explicit example in one dimension of a bound state at the threshold which violates (26) [16]. We hope to remove this assumption eventually.

The intercluster long range force is given by

$$\mathscr{W}_D(\mathbf{y}) = \sum \mathscr{W}_{ij}(\mathbf{y}), \quad i \in C', \quad j \in C. \tag{27}$$

For Coulomb forces $\mathscr{W}_D \neq 0$ only if both clusters are charged. If $\mathscr{W}_D$ is short range it may be treated with the short range part of I_D and be disregarded below.

The "modified free" Hamiltonian [4] $H_0'(t) = H_0 + \mathscr{W}_D(\mathbf{V}t)$ generates on $\mathscr{H}_D$ the time evolution

$$U_0(t,0) = \exp -i\left\{H_0 t + \int_0^t d\tau \mathscr{W}_D(\mathbf{V}\tau)\right\}. \tag{28}$$

On $\mathscr{H}$ the modified free time evolution corresponding to the cluster decomposition D is

$$U_D(t,0) = U_0(t,0) \otimes \exp(-iH_C t) \otimes \exp(-iH_{C'} t). \tag{29}$$

Denote by $P_{\text{cont}}(H)$, $P_p(H)$ the projection on the continuous resp. point – spectral subspace of a Hamiltonian H. Then the modified two-cluster wave operators are

$$\Omega_{\mp}^D = \underset{t \to \pm\infty}{\text{s-lim}} \exp(iHt) U_D(t,0) P_p(H_C) P_p(H_{C'}). \tag{30}$$

The existence of these wave operators is well known [12], it also follows as a byproduct of our completeness proof. For different D the ranges are orthogonal.

Finally we remark that $(\mathbb{H}_C^0 + i)^{1/2}$ and $(H_C + i)^{1/2}$ are bounded relative to each other, the same is true for $(\mathbb{H}^0 + i)^{1/2}$, $(H_D + i)^{1/2}$, and $(H + i)^{1/2}$. Consequently $F(|x| \leqq R)(H + i)^{-1/2}$ and $F(|z_C| \leqq R)(H_C + i)^{-1/2}$ are compact for any $R < \infty$.

Two-Cluster Scattering States, the Main Result

Ruelle [13] gave a geometric characterization of scattering states and bound states, the latter were those states where all particles stayed together uniform in time. He showed that these subspaces coincide with $P_{\text{cont}}(H)$ and $P_p(H)$ resp. Amrein and Georgescu [1] extended the result and found that the crucial assumption was the compactness of $F(|x| \leqq R)(H + i)^{-1}$ for all $R < \infty$.

Applying Ruelle's ideas to the subsystems we define the incoming (outgoing) two-cluster subspaces $\mathbf{2}_{\text{in}}$ ($\mathbf{2}_{\text{out}}$) ($A \vee B := A + B - AB$ for commuting operators A and B).

Definition. $\Psi \in \mathbf{2}_{\text{out}}$ if $\Psi \in P_{\text{cont}}(H)$ and for any $\varepsilon > 0$ there is an $r(\varepsilon) < \infty$ such that

$$\sup_{t \geqq 0} \left\| \left[\mathbb{1} - \bigvee_D F(|z_C| \leqq r(\varepsilon)) F(|z_{C'}| \leqq r(\varepsilon)) \right] \exp(-iHt)\Psi \right\| < \varepsilon . \tag{31}$$

$\Psi \in \mathbf{2}_{\text{in}}$ if the same holds for $t \leqq 0$.

Obviously $\mathbf{2}_{\text{in}}$ and $\mathbf{2}_{\text{out}}$ are closed linear subspaces, furthermore they are invariant under finite time translations: assume (31) holds for $t \geqq \tau$. For any Ψ the set $\{\exp(-iHt)\Psi | 0 \leqq t \leqq \tau\}$ is compact because $\exp(-iHt)$ is strongly continuous and thus all $\exp(-iHt)\Psi$ lie in an ε-neighbourhood of finitely many $\exp(-iHt_i)\Psi$. For a big enough $r'(\varepsilon) < \infty$

$$\sup_{0 \leqq t \leqq \tau} \| [\mathbb{1} - F(|x| \leqq r(\varepsilon))] \exp(-iHt)\Psi \| < \varepsilon \tag{32}$$

which implies (31) for $t \geqq 0$ with a possibly bigger $r(\varepsilon)$. It follows that the states from $\mathbf{2}_{\text{out}}$ with finite energy are dense in $\mathbf{2}_{\text{out}}$.

We do not assume that the decomposition becomes stable. In fact one could imagine for a potential as given by Pearson [10] that two clusters move away from each other, come back, scatter and rearrange, move away, and so on indefinitely. Such a state could well be in $\mathbf{2}_{\text{out}}$. But for the interactions we consider this is ruled out by our completeness result.

We will show:

Theorem. *Let the interactions obey conditions* (18), (19), (21), *and* (26) *then for two-cluster decompositions D*

a) $\mathbf{2}_{\text{in}} = \bigoplus_D \operatorname{Ran} \Omega_+^D$

$$\mathbf{2}_{\text{out}} = \bigoplus_D \operatorname{Ran} \Omega_-^D .$$

b) *If* Σ_3 *is the lowest three-cluster breakup threshold of* H *then* $P_{\text{cont}}(H) F(H \leqq \Sigma_3) \mathscr{H} \subset \mathbf{2}_{\text{in}} \cap \mathbf{2}_{\text{out}}$.

Corollary. *Under the assumptions of the theorem* $H \restriction \mathscr{D}(H) \cap \operatorname{span}(\mathbf{2}_{\text{in}}, \mathbf{2}_{\text{out}})$ *has no singular continuous spectrum.*

Behaviour of Far Separated Clusters for Finite Times

Fix a two-cluster decomposition D. For big enough R the operator

$$F_0(|\mathbf{y}| \geqq R) E(|z_C| \leqq r) E(|z_{C'}| \leqq r) \tag{33}$$

singles out the component where the clusters C and C' are far separated up to a rapidly decreasing tail. For $i \in C'$, $j \in C$

$$|\mathbf{x}^i - \mathbf{x}^j| = |\mathbf{y} + \mathbf{z}^i + \mathbf{z}^j| \geqq |\mathbf{y}| - 4r(\min m_i)^{-1/2} \tag{34}$$

We first show that for any given time interval $|t| \leqq \tau < \infty$ the clusters remain far separated if they were that initially. We choose the constant $\beta > 0$ such that the clusters are separated at least by βR in the range of

$$E(|\mathbf{y}| \geqq R/3) F(|z_C| \leqq \beta R) F(|z_{C'}| \leqq \beta R) . \tag{35}$$

Remember that $F_0(\cdot)$ depends on a constant $0<a\leqq 1$ to be chosen later.

Lemma 1. *For any* $0<a\leqq 1$, $r<\infty$, $\tau<\infty$, $E<\infty$

$$\lim_{R\to\infty}\sup_{|t|\leqq\tau}\|F(H_0\leqq E)F(H_C\leqq E)F(H_{C'}\leqq E)F_0(|\mathbf{y}|\geqq R)E(|z_C|\leqq r)E(|z_{C'}|\leqq r)$$
$$\cdot\exp(-iH_Dt)(H_0+i)\{\mathbb{1}-E(|\mathbf{y}|\geqq R/3)F(|z_C|\leqq\beta R)F(|z_{C'}|\leqq\beta R)\}\|=0, \qquad (36)$$

where $\beta^{-1}=3(1+4(\min m_i)^{-1/2})$.

Proof. The norm in (36) is bounded by

$$\|F(H_0\leqq E)F_0(|\mathbf{y}|\geqq R)(H_0+i)\|$$
$$\times\{\|F(H_C\leqq E)E(|z_C|\leqq r)\exp(-iH_Ct)F(|z_C|\geqq\beta R)\|+\|C\leftrightarrow C'\|\}$$
$$+\|F(H_0\leqq E)F_0(|\mathbf{y}|\geqq R)\exp(-iH_0t)(H_0+i)E(|\mathbf{y}|\leqq R/3)\|. \qquad (37)$$

$F(H_C\leqq E)E(|z_C|\leqq r)$ is compact on $\mathscr{H}_C$ for all $E<\infty$, $r<\infty$, thus the set

$$\{e^{-iH_Ct}E(|z_C|\leqq r)F(H_C\leqq E)\Phi|\,|t|\leqq\tau<\infty,\ \Phi\in\mathscr{H}_C,\ \|\Phi\|=1\}$$

is compact in $\mathscr{H}_C$ (compare the paragraph following the definition of $\mathbf{2}_{\text{out}}$). Since $s\text{-}\lim_{R\to\infty} F(|z_C|\leqq R)=\mathbb{1}$ on $\mathscr{H}_C$ we have for any $\beta>0$

$$\lim_{R\to\infty}\sup_{|t|\leqq\tau}\|F(H_C\leqq E)E(|z_C|\leqq r)\exp(-iH_Ct)F(|z_C|\geqq\beta R)\|=0 \qquad (38)$$

on $\mathscr{H}_C$ and on $\mathscr{H}$.

For the estimate of the free relative motion of the clusters we use formula (8) and Lemma 1′ in [6].

$$\|F(H_0\leqq E)F_0(|\mathbf{y}|\geqq R)\exp(-iH_0t)E(|\mathbf{y}|\leqq R/3)\|$$
$$\leqq\|F(|\mathbf{y}|\leqq 2R/3)F_0(|\mathbf{y}+\mathbf{v}t|\geqq R)\chi(\mathbf{v})\| \qquad (39)$$

if $\chi\in\mathscr{D}(\mathbb{R}^\nu)$, $\chi(\mathbf{v})=1$ on $\mu|\mathbf{v}|^2\leqq 2E$.

For $|t|\leqq\tau$ $S(t)$ is contained in a ball of radins $2R/3+\text{const}$, the distance d becomes arbitrarily big for big R. Thus by Lemma 1′

$$\lim_{R\to\infty}\sup_{|t|\leqq\tau}\|F(H_0\leqq E)F_0(|\mathbf{y}|\geqq R)\exp(-iH_0t)E(|\mathbf{y}|\leqq R/3)\|=0 \qquad (40)$$

on $\mathscr{H}_D$ and on $\mathscr{H}$. The same is true if $E(|\mathbf{y}|\leqq R/3)$ is replaced by $(H_0+i)E(|\mathbf{y}|\leqq R/3)(H_0+i)^{-1}$ because the difference of the two terms vanishes in norm as $R\to\infty$. Moreover

$$\|(H_0+i)E(|\mathbf{y}|\leqq R/3)(H_0+i)\exp(-iH_0t)F_0(|\mathbf{y}|\geqq R)F(H_0\leqq E)\|$$

is bounded uniformly in $R>1$ and t, this implies

$$\lim_{R\to\infty}\sup_{|t|\leqq\tau}\|F(H_0\leqq E)F_0(|\mathbf{y}|\geqq R)\exp(-iH_0t)(H_0+i)E(|\mathbf{y}|\leqq R/3)\|=0.$$

With (37) this completes the proof. □

As long as the clusters stay far apart the coupling between the clusters I_D (24) does not affect the motion.

Lemma 2. *For any* $0<a\leqq 1$, $r<\infty$, $\tau<\infty$

a) $$\lim_{R\to\infty}\sup_{|t|\leqq\tau}\|(H_D+i)^{-1/2}F_0(|\mathbf{y}|\geqq R)E(|z_C|\leqq r)E(|z_{C'}|\leqq r)\cdot\{\exp(-iHt)-\exp(-iH_Dt)\}(H+i)^{-1/2}\|=0. \tag{41}$$

b) *For any* φ *with* $\tilde{\varphi}\in L^1(\mathbb{R},dt)$

$$\lim_{R\to\infty}\|(H_D+i)^{-1/2}F_0(|\mathbf{y}|\geqq R)E(|z_C|\leqq r)E(|z_{C'}|\leqq r)\cdot\{\varphi(H)-\varphi(H_D)\}(H+i)^{-1/2}\|=0. \tag{42}$$

Proof. b) The norm in (42) is bounded by

$$\int dt|\tilde{\varphi}(t)|\|(H_D+i)^{-1/2}F_0(|\mathbf{y}|\geqq R)E(|z_C|\leqq r)E(|z_{C'}|\leqq r)\cdot\{\exp(-iHt)-\exp(-iH_Dt)\}(H+i)^{-1/2}\|.$$

By the dominated convergence theorem it is sufficient to show that for each finite t the norm in the integrand vanishes as $R\to\infty$, which follows from Part a).

a) The semiboundedness of H_0, H_C, and $H_{C'}$ implies

$$\lim_{E\to\infty}\|(H_D+i)^{-1/2}[\mathbb{1}-F(H_0\leqq E)F(H_C\leqq E)F(H_{C'}\leqq E)]\|=0. \tag{43}$$

Thus it is sufficient to estimate (41) for finite energy E in each factor space. Furthermore

$$\{\exp(-iHt)-\exp(-iH_Dt)\}=-i\int_0^t ds\exp(-iH_Ds)I_D\exp(-iH(t-s)), \tag{44}$$

thus (41) follows if for any $0<a\leqq 1$; r, τ, $E<\infty$

$$\lim_{R\to\infty}\tau\sup_{|s|\leqq\tau}\|F(H_0\leqq E)F(H_C\leqq E)F(H_{C'}\leqq E)F_0(|\mathbf{y}|\geqq R)\cdot E(|z_C|\leqq r)E(|z_{C'}|\leqq r)\exp(-iH_Ds)I_D(H+i)^{-1/2}\|=0. \tag{45}$$

The norm in (45) can be split

$$\begin{aligned}&\|F(H_0\leqq E)F(H_C\leqq E)F(H_{C'}\leqq E)F_0(|\mathbf{y}|\geqq R)E(|z_C|\leqq r)E(|z_{C'}|\leqq r)\\&\cdot\exp(-iH_Ds)(H_0+i)\{\mathbb{1}-E(|\mathbf{y}|\geqq R/3)F(|z_C|\leqq\beta R)F(|z_{C'}|\leqq\beta R)\}\|\\&\cdot\|(H_0+i)^{-1}I_D(H+i)^{-1/2}\|+\|F(H_0\leqq E)F_0(|\mathbf{y}|\geqq R)(H_0+i)\|\\&\cdot\|E(|\mathbf{y}|\geqq R/3)F(|z_C|\leqq\beta R)F(|z_{C'}|\leqq\beta R)(H_0+i)^{-1}I_D(H+i)^{-1/2}\|.\end{aligned} \tag{46}$$

By Lemma 1 the first factor in the first summand decays as $R\to\infty$ uniform in $|s|\leqq\tau$, the second factor is bounded. The first factor of the second summand is bounded uniform in R. For any of the pairs $i\in C', j\in C|\mathbf{x}^i-\mathbf{x}^j|\geqq\beta R$ in the range of $E(|\mathbf{y}|\geqq R/3)F(|z_C|\leqq\beta R)F(|z_{C'}|\leqq\beta R)$. With the translation invariance of $(H_0+i)^{-1}$ we obtain for the last factor in (46) the bound

$$\sum_{i,j}\|F(|\mathbf{u}|\geqq\beta R)[-(2\mu)^{-1}\Delta_u+i]^{-1}(W_{ij}+\mathscr{W}_{ij})(H+i)^{-1/2}\| \tag{47}$$

which vanishes as $R\to\infty$. □

The following easy consequence of Lemmas 1 and 2 shows the stability of a cluster decomposition for a finite time interval.

Lemma 3. *Let $\bar{D}=(\bar{C},\bar{C}')$ denote the two-cluster decompositions different from D. For any r, r', $\tau<\infty$, $0<a\leqq 1$*

$$\lim_{R\to\infty}\sup_{|t|\leqq\tau}\|(H_D+i)^{-1/2}F_0(|\mathbf{y}|\geqq R)E(|z_C|\leqq r)E(|z_{C'}|\leqq r)$$
$$\cdot\exp(-iHt)\bigvee_{\bar{D}\neq D}F(|z_{\bar{C}}|\leqq r')F(|z_{\bar{C}'}|\leqq r')\|=0. \tag{48}$$

Proof. For any $\bar{D}$ there is at least one pair of particles which are within one cluster of $\bar{D}$ but in different clusters for D. Thus for big enough R

$$E(|\mathbf{y}|\geqq R/3)F(|z_C|\leqq\beta R)F(|z_{C'}|\leqq\beta R)\bigvee_{\bar{D}\neq D}F(|z_{\bar{C}}|\leqq r')F(|z_{\bar{C}'}|\leqq r')=0. \tag{49}$$

The application of Lemma 2a), (43), and Lemma 1 completes the proof. □

Corollary. *Lemmas 1–3 remain true if for one or both clusters $E(|z_C|\leqq r)$ is replaced by $E(|z_C|\leqq r)P_{\mathrm{cont}}(H_C)$ or $E(|z_C|\leqq r)P_p(H_C)$.*

Proof. In the proofs of Lemmas 1–3 in $\mathscr{H}_C$ we have used nothing but compactness of $P(H_C\leqq E)E(|z_C|\leqq r)$. This is not changed by the additional spectral projection. □

If a state belongs to the continuous spectral subspace of the Hamiltonian of one (or both) of the clusters then that cluster will decay if it is far enough separated from the other.

Lemma 4. *For any $0<a\leqq 1$, r, $\varrho<\infty$, $\varepsilon>0$ there exists a $\tau(\varepsilon)$ such that*

$$\lim_{R\to\infty}\|(H_D+i)^{-1/2}F_0(|\mathbf{y}|\geqq R)E(|z_C|\leqq r)P_{\mathrm{cont}}(H_C)E(|z_{C'}|\leqq r)$$
$$\cdot\exp\{-iH\tau(\varepsilon)\}F(|z_C|\leqq\varrho)\|<\varepsilon. \tag{50}$$

Proof. Choose E big enough such that the norm in (43) is smaller than $\varepsilon/2$. $F(H_C\leqq E)E(|z_C|\leqq r)P_{\mathrm{cont}}(H_C)$ is a compact operator on $\mathscr{H}_C$. By the estimate of Amrein and Georgescu [1]

$$\|F(H_C\leqq E)E(|z_C|\leqq r)P_{\mathrm{cont}}(H_C)\exp(-iH_Ct)F(|z_C|\leqq\varrho)\| \tag{51}$$

tends to zero in the time mean. So we can find a $\tau(\varepsilon)$ such that (51) is smaller than $\varepsilon/2$. This proves (50) for H_D, and by Lemma 2a) for H. □

Asymptotic Decomposition of Two-Cluster States

Let for some $E<\infty$ $\Psi\in\mathbf{2}_{\mathrm{out}}\cap F(H\leqq E)\mathscr{H}$. We show first that it eventually splits into far separated two-cluster components. From here on D is no longer fixed but runs over the two-cluster decompositions.

The coordinate $\mathbf{y}$ which depends on D will be denoted by $\mathbf{y}_D$.

Lemma 5. *For any R there is a $0\leqq T(R)<\infty$ such that with $\varepsilon, r(\varepsilon)$ as given in (31) for any $r\geqq r(\varepsilon)$*

$$\lim_{R\to\infty}\Big\|\exp(-iHT(R))\Psi-\sum_D F_0(|\mathbf{y}_D|\geqq R)E(|z_C|\leqq r)E(|z_{C'}|\leqq r)$$
$$\cdot\exp(-iHT(R))\Psi\|<\varepsilon. \tag{52}$$

Proof. Since $\Psi \in P_{\text{cont}}(H)\mathscr{H}$ there exists by Ruelle's theorem for any R a $T(R)$ such that

$$\lim_{R\to\infty} \|F(|x|\leqq 2R)\exp(-iHT(R))\Psi\| = 0. \tag{53}$$

Then for each D, $r(\varepsilon)\leqq r<\infty$

$$\lim_{R\to\infty} \|F_0(|\mathbf{y}_D|\leqq R)E(|z_C|\leqq r)E(|z_{C'}|\leqq r)\exp(-iHT(R))\Psi\| = 0.$$

Finally $\bigvee_D$ can be replaced by $\sum_D$ because the product of F_0EE for two different clusters vanishes as $R\to\infty$. $\square$

Lemma 6. *For every* D, $0<a\leqq 1$, $r<\infty$

a) $$\lim_{R\to\infty} \|F_0(|\mathbf{y}_D|\geqq R)E(|z_C|\leqq r)E(|z_{C'}|\leqq r)$$
$$\cdot P_{\text{cont}}(H_C)\vee P_{\text{cont}}(H_{C'})\exp(-iHT(R))\Psi\| = 0. \tag{54}$$

b) *For any* φ *with* $\tilde{\varphi}\in L^1(\mathbb{R}, dt)$

$$\lim_{R\to\infty} \|F_0(|\mathbf{y}_D|\geqq R)E(|z_C|\leqq r)E(|z_{C'}|\leqq r)P_p(H_C)P_p(H_{C'})$$
$$\cdot\{\varphi(H)-\varphi(H_D)\}\exp(-iHT(R))\Psi\| = 0. \tag{55}$$

Proof. a) For any $\varepsilon>0$ there is a $r(\varepsilon)$ according to (31). By Lemma 4 there is a $\tau(\varepsilon)$ such that

$$\lim_{R\to\infty} \|(H_D+i)^{-1/2}F_0(|\mathbf{y}_D|\geqq R)E(|z_C|\leqq r)P_{\text{cont}}(H_C)E(|z_{C'}|\leqq r)$$
$$\cdot\exp(-iH\tau(\varepsilon))F(|z_C|\leqq r(\varepsilon))\exp\{-iH(T(R)-\tau(\varepsilon))\}\Psi\| < \varepsilon. \tag{56}$$

(56) and the corresponding estimate for cluster C', the corollary of Lemma 3 and (31) at time $T(R)-\tau(\varepsilon)$ give for any ε

$$\lim_{R\to\infty} \|(H_D+i)^{-1/2}F_0(|\mathbf{y}_D|\geqq r)E(|z_C|\leqq r)E(|z_{C'}|\leqq r)$$
$$\cdot P_{\text{cont}}(H_C)\vee P_{\text{cont}}(H_{C'})\exp(-iHT(R))\Psi\| < 4\varepsilon. \tag{57}$$

Observe that the expression (57) with $\|(H_D+i)^{+1/2}\ldots\|$ is bounded uniform in R, $T(R)$, and $r\geqq 1$, this implies (54).

b) (55) follows from the Corollary of Lemma 2b) and the final argument in Part a) of this proof. $\square$

If $\varphi\in\mathscr{D}(\mathbb{R})$, $\varphi(\omega)=1$ for $|\omega|\leqq E$, $|\varphi(\omega)|\leqq 1$, one has $\varphi(H)\Psi=\Psi$ and $\|\varphi(H)\|=1$. For these φ and any $r\geqq r(\varepsilon)$ we have shown

$$\lim_{R\to\infty} \|\exp(-iHT(R))\Psi - \sum_D F_0(|\mathbf{y}_D|\geqq R)E(|z_C|\leqq r)P_p(H_C)$$
$$\cdot E(|z_{C'}|\leqq r)P_p(H_{C'})\varphi(H_D)\exp(-iHT(R))\Psi\| \leqq \varepsilon. \tag{58}$$

There is some E' such that the range of $\varphi(H_D)$ is contained in the range of $F(H_C\leqq E')\otimes F(H_{C'}\leqq E')$. Let $\{|\alpha_D\rangle | \alpha\in\mathbb{N}\}$ be the orthonormal set of all product eigenstates of H_C and $H_{C'}$ simultaneously, then $P_p(H_C)P_p(H_{C'})=\sum_\alpha |\alpha_D\rangle\langle\alpha_D|$:

Since $E(|z_C|\leqq 2r(\varepsilon))F(H_C\leqq E')E(|z_{C'}|\leqq 2r(\varepsilon))F(H_{C'}\leqq E')$ is compact on $\mathscr{H}_C\otimes\mathscr{H}_{C'}$ there is an $N(\varepsilon)<\infty$ such that

$$\sum_D\left\|E(|z_C|\leqq 2r(\varepsilon))E(|z_{C'}|\leqq 2r(\varepsilon))\sum_{\alpha>N(\varepsilon)}|\alpha_D\rangle\langle\alpha_D|F(H_C\leqq E')F(H_{C'}\leqq E')\right\|<\varepsilon. \tag{59}$$

We use the Corollary of Lemma 2b) again and insert (59) into (58) to obtain for $r(\varepsilon)\leqq r\leqq 2r(\varepsilon)$

$$\lim_{R\to\infty}\left\|\exp(-iHT(R))\Psi-\sum_D F_0(|\mathbf{y}_D|\geqq R)\right.$$
$$\left.\cdot E(|z_C|\leqq r)E(|z_{C'}|\leqq r)\sum_{\alpha=1}^{N(\varepsilon)}|\alpha_D\rangle\langle\alpha_D|\exp(-iHT(R))\Psi\right\|<2\varepsilon. \tag{60}$$

In a last step we will eliminate the E-factors by adding $[\mathbb{1}-E(|z_C|\leqq 2r(\varepsilon))E(|z_{C'}|\leqq 2r(\varepsilon))]\ldots$ to the approximation in (60) with $r=2r(\varepsilon)$. To see that this contribution is small observe that the ranges of $E(|z_C|\leqq r(\varepsilon))E(|z_{C'}|\leqq r(\varepsilon))$ and $[\mathbb{1}-E(|z_C|\leqq 2r(\varepsilon))E(|z_{C'}|\leqq 2r(\varepsilon))]$ are orthogonal and that

$$\lim_{R\to\infty}\left\|\sum_D F_0(|\mathbf{y}_D|\geqq R)\{E(|z_C|\leqq r(\varepsilon))E(|z_{C'}|\leqq r(\varepsilon))+\mathbb{1}\right.$$
$$\left.-E(|z_C|\leqq 2r(\varepsilon))E(|z_{C'}|\leqq 2r(\varepsilon))\}\sum_{\alpha=1}^{N(\varepsilon)}|\alpha_D\rangle\langle\alpha_D|\right\|\leqq 1 \tag{61}$$

because

$$\lim_{\varrho\to\infty}\left\|[\mathbb{1}-E(|z_C|\leqq\varrho)E(|z_{C'}|\leqq\varrho)]\sum_{\alpha=1}^{N(\varepsilon)}|\alpha_D\rangle\langle\alpha_D|\right\|=0. \tag{62}$$

Then a simple calculation shows

$$\lim_{R\to\infty}\left\|\sum_D F_0(|\mathbf{y}_D|\geqq R)[\mathbb{1}-E(|z_C|\leqq 2r(\varepsilon))E(|z_{C'}|\leqq 2r(\varepsilon))]\right.$$
$$\left.\cdot\sum_{\alpha=1}^{N(\varepsilon)}|\alpha_D\rangle\langle\alpha_D|\exp(-iHT(R))\Psi\right\|<2\sqrt{\varepsilon}. \tag{63}$$

We sum up the results of this section:

Lemma 7. *Let $\Psi\in\mathbf{2}_{\text{out}}\cap F(H\leqq E)\mathscr{H}$ for some $E<\infty$. For any $0<a\leqq 1$, $\varepsilon>0$ there exists an $N(\varepsilon)<\infty$ and for any R a $0\leqq T(R)<\infty$ such that*

a) $$\lim_{R\to\infty}\left\|\exp(-iHT(R))\Psi-\sum_D F_0(|\mathbf{y}_D|\geqq R)\right.$$
$$\left.\sum_{\alpha=1}^{N(\varepsilon)}|\alpha_D\rangle\langle\alpha_D|\exp(-iHT(R))\Psi\right\|<2(\varepsilon+\sqrt{\varepsilon}). \tag{64}$$

b) *For any φ with $\tilde{\varphi}\in L^1(\mathbb{R},dt)$ and any D*

$$\lim_{R\to\infty}\left\|F_0(|\mathbf{y}_D|\geqq R)\sum_{\alpha=1}^{N(\varepsilon)}|\alpha_D\rangle\langle\alpha_D|\{\varphi(H)-\varphi(H_D)\}\exp(-iHT(R))\Psi\right\|=0.$$

The second statement follows from (62) and an argument like that in the proof of Lemma 6b).

Note that the number of channels $N(\varepsilon)$ depends on E and $r(\varepsilon)$ but is otherwise independent of Ψ. In particular it is independent of R and $T(R)$.

From the properties of the interaction we have used so far only compactness and that the forces decay at infinity, no matter how slowly. The specific decay assumption will be used only in the next section.

Reduction to Potential Scattering

From now on keep $\varepsilon>0$ and $N(\varepsilon)<\infty$ fixed. Denote by ε_{D_α} the eigenvalue of $|\alpha_D\rangle\langle\alpha_D|$ for $H_C+H_{C'}$ then for any φ

$$\varphi(H_D)|\alpha_D\rangle\langle\alpha_D|=\varphi(H_0+\varepsilon_{D_\alpha})|\alpha_D\rangle\langle\alpha_D|\,. \tag{65}$$

For Ψ and $\varepsilon>0$ given, there is a $\varphi\in\mathscr{D}(\mathbb{R})$ (which will also be kept fixed) such that

$$|\varphi(\omega)|\leqq 1\,,\qquad \|\Psi-\varphi(H)\Psi\|<\varepsilon \tag{66}$$

and $\varphi(\omega)=0$ in an open neighbourhood of each of the finitely many points $\omega=\varepsilon_{D_\alpha}$. Then there is a sufficiently small $0<a\leqq 1$ such that

$$\varphi(\omega)=0 \quad \text{on} \quad \varepsilon_{D_\alpha}\leqq\omega\leqq\varepsilon_{D_\alpha}+\left(\max_D \mu\right)(17a)^2/2 \tag{67}$$

for all D and all $\alpha\leqq N(\varepsilon)$. If in $\varphi(H)\Psi$ the clusters are far enough separated then by Lemma 7b) the relative velocity of the clusters in bounded below by $17a$.

From here on the remainder of the proof is almost identical to the potential scattering situation, we refer to [6] for all details (or [5] if no long range inter-cluster force is present). The position and velocity of the particle becomes in this context the relative position $\mathbf{y}$ or the relative velocity $\mathbf{v}$ of the clusters. With respect to these quantities the phase space decomposition is made, the intermediate and final modified free time evolution are analogous with $\mathscr{W}_D$ (27) used as long range potential, and the same estimates of the space-time behavior (in $\mathbf{y}$ and t) apply. The only difference is that the potential decays with the distance of the particles $|\mathbf{x}^i-\mathbf{x}^j|$ instead of the distance of the clusters $|\mathbf{y}|$. We take care of this with an additional splitting of the cluster states.

In the case of one particle the splitting was made time dependent by projecting to those parts of space where either the potential is small or the state, in both cases the time integral over the future (or past) had to be small. In the two-cluster case the same is true for the following splitting with $\alpha<\gamma\leqq 1$ if long range potentials are present or $\alpha=1$ otherwise; $\beta=(\min m_i)^{1/2}/3$:

$$F(|\mathbf{y}|\geqq n+at)F(|z_C|\leqq\beta(n+at)^\alpha)F(|z_{C'}|\leqq\beta(n+at)^\alpha) \tag{68}$$

and the orthogonal complement

$$F(|\mathbf{y}|\leqq n+at)\vee F(|z_C|\geqq\beta(n+at)^\alpha)\vee F(|z_{C'}|\geqq\beta(n+at)^\alpha)\,. \tag{69}$$

The time integral of the part of the state with $|\mathbf{y}|\leqq n+at$ is small due to the estimate of the last two terms of (44) in [6] if we choose $R=65n$. The other two contributions to (69) are integrable and small as $n\to\infty$ due to assumption (26).

It remains to estimate the short range inter-cluster forces, this corresponds to $\int_0^\infty h(n+at)dt$ in the third term of (44) in [6]. One contribution comes from the long range forces which have been approximated by $\mathscr{W}_D$, but the difference is small in the range of the projector (68):

$$\begin{aligned}&|[\mathscr{W}_{ij}(\mathbf{x}^i-\mathbf{x}^j)-\mathscr{W}_{ij}(\mathbf{y})]F(|\mathbf{y}|\geqq n+at)F(|z_C|\leqq\beta(n+at)^\alpha)F(|z_{C'}|\leqq\beta(n+at)^\alpha)|\\&\quad\leqq\sup_{0\leqq\lambda\leqq 1}(|\mathbf{z}^i|+|\mathbf{z}^j|)|\nabla\mathscr{W}_{ij}(\mathbf{y}+\lambda(\mathbf{z}^i+\mathbf{z}^j))|F(|\mathbf{y}|\geqq n+at)\\&\quad\cdot F(|\mathbf{z}^i|\leqq(n+at)^\alpha)F(|\mathbf{z}^j|\leqq(n+at)^\alpha)\leqq\mathrm{const}(n+at)^{-1-\gamma+\alpha},\end{aligned}\tag{70}$$

where γ was given in (19). When $\alpha<\gamma$ the time integral vanishes as $n\to\infty$.

For the singular short range part any $\alpha\leqq 1$ may be used. Then in the range of the projector (68) $|\mathbf{x}^i-\mathbf{x}^j|\geqq(n+at)/3$.

$$\begin{aligned}&\|(H+i)^{-1/2}W_{ij}(H_0+i)^{-1}F(|\mathbf{y}|\geqq n+at)F(|z_C|\leqq\beta(n+at))F(|z_{C'}|\leqq\beta(n+at))\|\\&\quad\leqq\|(H+i)^{-1/2}W_{ij}(H_0+i)^{-1}F(|\mathbf{x}^i-\mathbf{x}^j|\geqq(n+at)/3)\|\leqq\mathrm{const}\,h((n+at)/3).\end{aligned}\tag{71}$$

We have used the translation invariance of $(H_0+i)^{-1}$ on $\mathbb{R}^\nu$: Since only finitely many potentials are involved this shows that the full time evolution is arbitrarily well approximated by the intermediate modified free one.

The remainder of the proof is exactly as in the potential scattering case, one shows that the in-component disappears in the future and that the asymptotic time evolution $U_D(t,0)$ (29) is a good approximation of the total one for late enough times. Thus we have shown that

$$\mathbf{2}_{\text{out}}\subset\bigoplus_D\operatorname{Ran}\Omega_-^D\,.$$

For $\mathbf{2}_{\text{in}}$ analogously. The opposite inclusion is given in the next section.

Subspaces of $\mathbf{2}_{\text{in/out}}$

It remains to show that $\operatorname{Ran}\Omega_-^D$ is contained in $\mathbf{2}_{\text{out}}$ for all D. For $\Psi\in\operatorname{Ran}\Omega_-^D$ there is a $\Phi=P_p(H_C)P_p(H_{C'})\Phi$ such that for any ε

$$\|\exp(-iHt)\Psi-\exp(-iH_Dt)\Phi\|<\varepsilon\quad\text{for}\quad t\geqq T(\varepsilon)\,.$$

Then there is an $M(\varepsilon)$ such that $\left\|\Phi-\sum_{\alpha=1}^{M(\varepsilon)}|\alpha_D\rangle\langle\alpha_D|\Phi\right\|<\varepsilon$ and there is an $r(3\varepsilon)$ such that

$$\left\|[\mathbb{1}-F(|z_C|\leqq r(3\varepsilon))F(|z_{C'}|\leqq r(3\varepsilon))]\sum_{\alpha=1}^{M(\varepsilon)}|\alpha_D\rangle\langle\alpha_D|\right\|<\varepsilon\,.\tag{72}$$

This shows for $t\geqq T(\varepsilon)$ that

$$\|[\mathbb{1}-F(|z_C|\leqq r(3\varepsilon))F(|z_{C'}|\leqq r(3\varepsilon))]\exp(-iHt)\Psi\|<3\varepsilon\,.$$

The analogous argument for $\mathbf{2}_{\text{in}}$ and the time invariance of the spaces completes the proof of part a) of the theorem.

Choose now $\Psi \in P_{\text{cont}}(H)F(H \leqq \Sigma_3)\mathscr{H}$. We will show that it belongs to both $\mathbf{2}_{\text{in}}$ and $\mathbf{2}_{\text{out}}$. For any ε there is a $\delta > 0$ such that

$$\|F(H \leqq \Sigma_3 - 2\delta)\Psi - \Psi\| < \varepsilon \quad \text{and a} \quad \varphi \in \mathscr{D}(\mathbb{R}) \quad \text{with} \quad 0 \leqq \varphi(\omega) \leqq 1,$$

$\varphi(\omega) = 1(0)$ if $\omega \in \sigma(H)$ and $\omega \leqq \Sigma_3 - 2\delta$ $(\omega \geqq \Sigma_3 - \delta)$.

For any two-cluster decomposition $D\varphi(H_D)$ restricts the energy within each cluster to an interval strictly below the continuum limit. Thus by the HVZ-theorem [14] there is a finite number M such that for all D

$$\varphi(H_D) = \varphi(H_D) \sum_{\alpha=1}^{M} |\alpha_D\rangle\langle\alpha_D| = \sum_{\alpha=1}^{M} \varphi(H_0 + \varepsilon_{D_\alpha})|\alpha_D\rangle\langle\alpha_D|.$$

Furthermore one knows for a very wide class of interactions that these $|\alpha_D\rangle$ have exponential decay such that condition (26) is automatically fulfilled (see [3] and the references given therein, also [16] for a two body result).

By the estimate (72) the extension of the clusters for states in the range of $\varphi(H_D)$ is uniformly bounded.

An easy construction shows that one can split the configuration space of the particles into several pieces: either all particles have a certain minimal distance, or they are grouped in clusters where the distance between the clusters is large compared to their diameters, or all particles are inside a big enough ball. For the well clustered states one shows as in Lemma 2 that $\varphi(H)$ differs little from $\varphi(H_D)$ for the corresponding decomposition. (The details of this construction will be given in a forthcoming paper.) For any decomposition $\bar{D}$ into three or more clusters $\varphi(H_{\bar{D}}) = 0$. Thus up to a small error the state can be decomposed into a piece where all particles are in some finite region or the remaining components lie in the ranges of the $\varphi(H_D)$'s. The error depends on δ only but is otherwise independent of the state, so it is in particular time independent for $\exp(-iHt)\varphi(H)\Psi$. This shows that $\Psi \in \mathbf{2}_{\text{in}} \cap \mathbf{2}_{\text{out}}$.

Our decomposition into well clustered states is convenient but not necessary. One can make instead a decomposition into two-cluster states where the separation of all particles in one cluster is at least d from all particles in the other cluster, and a remainder where all particles are inside a ball of radins $(N-1)d$. Then Hunziker shows [9] by a different method that with increasing d $[\varphi(H) - \varphi(H_D)]$ becomes small no matter how big the extension of the clusters is.

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Note Added in Proof

B. Davies kindly informed us that he recently extended Simon's application of the Kato-Birman trace class theory [15] to treat centrally symmetric long range forces like Coulomb interactions even when both clusters are charged.