Commun. math. Phys. 65, 91—96 (1979)

Communications in
Mathematical
Physics

Confinement of Static Quarks in Two Dimensional Lattice Gauge Theories

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Abstract. In two dimensional Higgs models on a lattice with Abelian or Nonabelian compact gauge group, fractional static charges are always confined.

1. Introduction

We consider lattice gauge theories [1] in two Euclidean space time dimensions with compact gauge group G with nontrivial center.[1] We admit multiplets of Bose fields $\phi(x)$, but no fermions. The Bose fields are required to transform trivially under an abelian (finite or Lie-) subgroup Γ contained in the center of G. Let C a closed path, $U[C]\in G$ the parallel transporter around C, and χ the character of a representation of G which is not trivial on Γ. We show that the Wilson loop integral satisfies

$$\langle \chi(U[C])\rangle \leqq \chi(\mathbb{1})\exp\{-\alpha\cdot(\text{area enclosed by } C)\} \tag{1.1}$$

with $\alpha > 0$. This indicates confinement of fractional static charges. Fractional means with nontrivial transformation law under Γ.

Example. $G = SU(2), \Gamma \simeq \mathbb{Z}_2, \phi$ any (reducible or irreducible) multiplet of scalar fields with integral "colour" isospins, $\chi(U) = \operatorname{tr} U$.

The proof of this result is elementary. It is essential that the Euclidean Lagrangean is real. Coupling constants are otherwise arbitrary.

For the special case of the Abelian Higgs-Villain model [9] a stronger result than ours is known [2] for small coupling constant β^{-1}.

2. An Example

For the sake of clarity we consider first a model with gauge group $G = SU(2)$, $\Gamma = \{\pm\mathbb{1}\} \simeq Z_2$, ϕ scalar fields transforming according to a one valued repre-

[1] By definition, the center of G consists of those elements γ of G which commute with all elements, i.e. $\gamma g = g\gamma$ for all g in G. Among the simply connected simple Lie groups, the requirement of nontrivial center excludes the exceptional groups G_2, F_4 and E_8

sentation of G/Γ. We consider the theory on a finite lattice $\Lambda \subset \mathbb{Z}^2$ with free boundary conditions. The character

$$\chi(U) = \operatorname{tr} U \tag{2.1}$$

is not trivial on Γ since $\frac{1}{2}\chi(\pm \mathbb{1}) = \pm 1$.

The (random) variables of the theory are $U(b) \in G$ associated with links b between nearest neighbours on the lattice, and $\phi(x)$ associated with vertices $x \in \Lambda$. If the direction of the link is reversed, $U(b) \to U(b)^{-1}$. For any directed path C consisting of links $b_1 \dots b_n$, the parallel transporter $U[C] = U(b_n) \dots U(b_1)$. We write $\dot{U}(b) = U(b)\Gamma \in G/\Gamma$ etc. Since the scalar fields are assumed to transform trivially under Γ, their interaction with the gauge field variables $U(b)$ involves only $\dot{U}(b)$.

The Euclidean Lagrangean will be taken to be of the form

$$L = \beta \sum_P \chi(U[P]) + L_1(\{\phi, \dot{U}\}). \tag{2.2}$$

The sum in the first term goes over all plaquettes P, i.e. closed paths of four links $b_1 \dots b_4$; the orientation of P is immaterial since $\chi(U) = \chi(U^{-1})$.

Let F any observable $=$ (μ-measurable) function of the random variables. Then

$$\langle F \rangle = Z^{-1} \int d\mu F; \; Z = \int d\mu \tag{2.3}$$
$$d\mu = \prod_b dU(b) \prod_x d\phi(x) e^L$$

dU is normalized Haar measure on G [3].

Let C a closed directed path, then it is the boundary $C = \partial \Xi$ of a region $\Xi \subset \Lambda$ with area $|\Xi|$. We study the expectation value of the Wilson loop integral,

$$\langle \chi(U[C]) \rangle = Z^{-1} \int \prod_b dU(b) \chi(U[C]) \exp \beta \sum_P \chi(U[P]) I(\{\dot{U}, \phi\}).$$
$$I(\{\dot{U}, \phi\}) = \int \prod_x d\phi(x) \exp L_1(\{\phi, \dot{U}\}). \tag{2.4}$$

The variables $U(b)$ take values in G and are integrated over G. Our method of proof consists in carrying out the sub-integration (rather: summation) over the subgroup Γ of G. If f is any integrable function on G, then

$$\int_G dU f(U) = \int_G dU \int_\Gamma d\gamma f(U\gamma) \tag{2.5}$$

because of invariance of Haar measure on G under the action of the group. Here

$$\int d\gamma(\dots) = \tfrac{1}{2} \sum_{\gamma \in \Gamma} (\dots)$$

is integration over Γ with normalized Haar measure on Γ.

We use this to rewrite the U-integrations in (2.4). Of course, $\dot{U}(b)$ remains invariant under the substitution $U(b) \to U(b)\gamma(b), \gamma(b) \in \Gamma$. Moreover, since $\gamma(b)$ are in the center of $G, U[C] \to U[C]\gamma[C]$ for any closed path C made of links $b_1 \dots b_n$, with $\gamma[C] = \gamma(b_n) \dots \gamma(b_1)$. Also

$$\chi(U[C]\gamma[C]) = \omega(\gamma[C]) \chi(U[C]),$$

ω the 1-dimensional unitary representation of Γ given by $\omega(\gamma) = \chi(\gamma)/\chi(\mathbb{1})$.

Proceeding in this way we obtain from (2.4)

$$\langle \chi(U[C]) \rangle = Z^{-1} \int \prod_b dU(b) \chi(U[C]) \int \prod_b d\gamma(b) \omega(\gamma[C]) I(\{\dot{U}, \phi\}) \cdot \exp \beta \sum_P \chi(U[P]\gamma[P]).$$

In two space time dimensions, with free boundary conditions, all the variables $\gamma[P]$ are independent [10]. So we may integrate over them in place of $\gamma(b)$'s. Moreover, since Γ is abelian and ω a representation of Γ.

$$\omega(\gamma[C]) = \prod_{P \subset \Xi} \omega(\gamma[P]).$$

The product is over all plaquettes in the area Ξ whose boundary is C, with the same orientation as C. This gives

$$\langle \chi(U[C]) \rangle = Z^{-1} \int \prod_b dU(b) \chi(U[C]) \int \prod_P d\gamma[P] \left\{ \prod_{P' \subset \Xi} \omega(\gamma[P']) \right\} \cdot I(\{\dot{U}, \phi\}) \exp \beta \sum_P \chi(U[P]\gamma[P]).$$

Now the γ-integrations may be performed. We define $\eta(U)$ for $U \in G$ by

$$\int_\Gamma d\gamma \omega(\gamma) e^{\beta \chi(U\gamma)} = \eta(U) \int_\Gamma d\gamma e^{\beta \chi(U\gamma)}. \tag{2.5'}$$

Explicitly

$$\eta(U) = \tanh \beta \chi(U)$$

for the model at hand. It has the property that $\eta(U\gamma) = \eta(U)\omega(\gamma)^{-1}$. Therefore $|\eta(U)|$ depends on U only through $\dot{U}$, since $|\omega(\gamma)| = 1$. Moreover, since $\chi(U)$ is real

$$|\eta(\dot{U})| \leqq e^{-\alpha} \quad \text{with} \quad \alpha > 0 \text{ independent of } U. \tag{2.6}$$

Explicitly

$$\alpha = -\ln \tanh 2\beta > 0 \tag{2.7a}$$

since $|\chi(U)| \leqq \chi(\mathbb{1}) = 2$ for all U. For the same reason $|\chi(U[C])| \leqq \chi(\mathbb{1})$.

As a consequence we obtain the inequality

$$|\langle \chi(U[C]) \rangle| \leqq \chi(\mathbb{1}) Z^{-1} \int \prod_b dU(b) \left\{ \prod_{P' \subset \Xi} |\eta(\dot{U}[P'])| \right\} I(\{\dot{U}, \phi\}) \int \prod_P d\gamma[P] \exp \beta \sum_P \chi(U[P]\gamma[P]).$$

In this step we have made essential use of the fact that the Lagrangean is real. Now the variable transformation $U(b) \to U(b)\gamma(b)$ can be undone again and the γ-integrations are then trivial to perform. As a result

$$|\langle \chi(U[C]) \rangle| \leqq \chi(\mathbb{1}) Z^{-1} \int d\mu \prod_{P' \subset \Xi} |\eta(\dot{U}[P'])|. \tag{2.8}$$

Due to inequality (2.6) this gives the final result

$$|\langle \chi(U[C])\rangle| \leqq \chi(\mathbb{1})e^{-\alpha|\Xi|} \tag{2.7b}$$

independent of the total volume Λ.

It is interesting to note that the integral in (2.8) is the partition function of a system with *real* Lagrangean $L+\sum_P \ln|\eta(\dot{U}[P])|$ enclosed in a heat bath with Lagrangean L. Thus the right hand side of (2.8) is the difference of free energies of two true statistical mechanical systems in Ξ immersed in the same heat bath.

3. The General Case

The general case is treated in exactly the same way. The Euclidean Lagrangean is assumed to be of the form

$$L = \beta \sum_P \mathscr{L}_0(U[P]) + L_1(\{\dot{U},\phi\}).$$

It is assumed that

i) $\mathscr{L}_0$ and L_1 are real, and $\mathscr{L}_0(U) = \mathscr{L}_0(U^{-1})$.

ii) $\mathscr{L}_0$ is a continuous function of $U \in G$.

iii) $\mathscr{L}_0(VUV^{-1}) = \mathscr{L}_0(U)$ (gauge invariance).

The last requirement assures that $\mathscr{L}_0(U[P])$ does not depend on a choice of initial point on P, and the first that it does not depend on the orientation of P. $\int d\gamma$ is again normalized Haar measure on Γ, and $\omega(\gamma) = \chi(\gamma)/\chi(\mathbb{1})$, χ the character involved in the loop. We admit free boundary conditions, or any other boundary conditions that involve only constraints on $\dot{U}(b)$ and $\phi(x)$. The Bose fields $\phi(x)$ may take values in a vector space V on which a reducible or irreducible unitary representation of G/Γ acts, or on a submanifold of V on which G/Γ can act, e.g. a sphere $(\phi(x), \phi(x)) = 1$, or they may be absent altogether.

$\eta(U)$ is defined by the analog of Eq. (2.5′)

$$\int d\gamma\omega(\gamma)e^{\beta\mathscr{L}_0(U\gamma)} = \eta(U)\int d\gamma e^{\beta\mathscr{L}_0(U\gamma)} \tag{3.1}$$

$|\eta(U)|$ is a function of $\dot{U}$ only for the same reason as before. It remains to show inequality (2.6).

Both integrals in (3.1) involve integration of a continuous function over a compact space Γ. Therefore the results are continuous functions of the variable U which takes values in the compact space G. In particular, therefore, $\int d\gamma \exp\beta\mathscr{L}_0(U\gamma)$ must assume its minimum in U which is positive since $\exp\beta\mathscr{L}_0(U\gamma)$ is positive and never zero. Thus also $\eta(U)$ is a continuous function of U. But $|\eta(U)| < 1$ for all U. Indeed, $|\eta(U)| \leqq 1$ holds because $|\omega(\gamma)| = 1$. Since $\exp\beta\mathscr{L}_0$ is positive, equality could only hold if $\omega(\gamma)$ were constant on Γ except on a set of Haar measure zero. By hypothesis this is not the case since ω is a non-trivial representation of Γ. Being continuous on G, $|\eta(U)|$ must assume its maximum which is therefore not 1. Consequently (2.6) holds with a positive α.

All other computations are literally the same as in Sect. 2. The resulting bound (2.7b) is valid for any finite volume Λ and is independent of Λ.

Remark. If $\mathscr{L}_0$ depends itself only on $\dot{U}$ one obtains $\eta(U)=0$ and therefore $\langle \chi(U[C])\rangle = \chi(\mathbb{1}) \lim_{\alpha\to\infty} \exp - \alpha|\Xi|$ ("superconfinement").

4. Concluding Remarks

(1) The same mechanism of quark confinement works in theories in more than 2 dimensions treated by high temperature expansion (small β) [1, 4]. Carrying out the sub-integrations over Γ already produces a factor $\beta^{|\Xi|}$ since all lower terms in the power series expansion in β are integrated to zero.

(2) In the case of a finite group Γ our bounds are not good enough to establish confinement in the continuum limit in which $\beta \to \infty$. For instance, expression (2.7a) tends to zero exponentially as $\beta \to \infty$. For the Abelian Higgs-Villain model [9], Israel and Nappi [2] have derived bounds which do not have this feature. Thus our bounds are not optimal. Comparison with the treatment of Callan, Dashen and Gross [5] of the Abelian Higgs model in continuous space time reveals a possible reason. We think of our lattice as superimposed on continuous space time. All the models of interest here admit topological excitations labelled by an element of Γ [6]. The excitations of the "classical vacuum" (pure gauge) of lowest action are called instantons, they have a definite size in two dimensions [7]. Our method consisted in taking into account the effect of topological excitations of size smaller than one lattice cell (so they become infinitely small when the lattice spacing goes to zero). Such an excitation supported inside the plaquette P takes $U[P] \to U[P]\gamma$ and, more generally, $U[C] \to U[C]\gamma^n$ for any path C which winds n times around its support. When one starts from a pure gauge, making such an excitation costs action of order β. It tends to infinity in the continuum limit. Excitations of larger size would be more favorable.
We hope to come back to these issues elsewhere, see also [8].

Acknowledgement. The author is indepted to J. P. Eckmann, M. Lüscher and V. Petkova for discussions, and to the Deutsche Forschungsgemeinschaft for financial support.

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Communicated by R. Haag

Received August 17, 1978

Erratum

Fritz, J., Dobrushin, R.L.: Non-Equilibrium Dynamics of Two-dimensional Infinite Particle Systems with a Singular Interaction. Commun. math. Phys. **57**, 67–81 (1977)

In the definition of f (beginning of Sect. 4, p. 72) we need $f(-5R/2)=1/9$, $f(-R/2)=8/9$ instead of 1/3 and 2/3. Proofs remain unchanged, only (34) turns into $Q_{32} \leqq 3R(1+d)S_3+(8+2d)S_4$.