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PCT, Spin and Statistics, and All That for Nonlocal Wightman Fields

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Abstract. It is shown how classical results of axiomatic field theory such as the PCT and spin-statistics theorem may be generalized to nonlocal tempered fields. The method is also applicable to the theory of essentially local non-localizable fields.

1. Introduction and Results

It is known [5] that there are nontrivial models fulfilling all the Wightman axioms (see [6], Chap. 3, the test space has to be modified), except local commutativity. On the other hand, despite tremendous effort nobody has been able to construct nontrivial models in 4-dimensional space-time fulfilling all the Wightman axioms including local commutativity. In this situation it is quite natural to ask why local commutativity is generally believed to be indispensable. Cogent physical arguments are not known—although the idea of macrocausality should be somehow incorporated into the axiomatic scheme, of course. Thus it seems to be its enormous predictive power that makes physicists believe in local commutativity. The C^*-algebraic as well as the Euclidean approach to field theory cannot be imagined without this axiom and the standard proofs of most of the classical results of axiomatic field theory rely on local commutativity.

This does not mean, however, that such important results as the PCT and spin-statistics theorem cannot be proved without local commutativity. Indeed, the purpose of this paper is to stress the contrary.

Let us consider a field theory with the field operators $\varphi_1(x), \dots, \varphi_N(x)$ fulfilling all the Wightman axioms except local commutativity. If g is a tempered test function such that

(A1): the Fourier transform $\tilde{g}$ is positive-valued and[1]

(A2): $g(x)$ as well as all its partial derivatives of arbitrary order vanish stronger than exponentially when $\|x\| \to \infty$

we call two fields $\varphi_j(x), \varphi_k(x)$ *g-asymptotically commuting* (*resp. anti-commuting*)

[1] The symbol $\|\cdot\|$ is used for the norm in finite-dim. Euclidean space as well as in the Hilbert space of states

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if for every vector Φ from the common invariant domain D of the field operators the $\mathcal{O}_M(\mathbb{R}^4)$-function

$$F(a) \equiv \| \int dx dy [\varphi_j(x)\varphi_k(y) \underset{(\text{resp.} +)}{-} \varphi_k(y)\varphi_j(x)] g(x-a) g(y) \Phi \|$$

has finite

$$\| F \|_{\lambda,\varepsilon} \equiv \sup_{\|\vec{a}\| > \lambda |a^0|} \exp(\varepsilon \| a \|) |F(a)|$$

for all $\lambda, \varepsilon > 1$.

At this point the following remark is in order: Bearing in mind a well-known result by Borchers and Pohlmeyer [1] one might expect g-asymptotical commutativity to imply local commutativity. However, this is not the case. Counter-examples were given in [3].

To keep things as simple as possible, assume $\varphi(x) \in \{\varphi_1(x), \ldots, \varphi_N(x)\}$ to be a neutral scalar field. Then we shall prove:

Theorem 1. *Let g be a tempered test function fulfilling* (*A*1) *and* (*A*2). *If for every $n > 1$ and for every $\hat{y} = (y_1, \ldots, y_n) \in (\mathbb{R}^4)^n$ the $\mathcal{O}_M(\mathbb{R}^4)$-function*[2]

$$F_{\hat{y}}(a) \equiv \int d\hat{x} [(\Psi_0, \varphi(x_1) \ldots \varphi(x_n) \Psi_0) - (\Psi_0, \varphi(x_n) \ldots \varphi(x_1) \Psi_0)]$$
$$\cdot g(x_1 - y_1 + 1a) \ldots g(x_n - y_n + na)$$

has finite $\| F_{\hat{y}} \|_{\lambda,\varepsilon}$ for all $\lambda, \varepsilon > 1$ then the field $\varphi(x)$ fulfills the PCT condition

$$(\Psi_0, \varphi(x_1) \ldots \varphi(x_n) \Psi_0) = (\Psi_0, \varphi(-x_n) \ldots \varphi(-x_1) \Psi_0)$$

and vice versa.

Theorem 2. *Let g be a tempered test function fulfilling* (*A*1) *and* (*A*2). *If $\varphi(x)$ is g-asymptotically anti-commuting with itself then $\varphi(x)\Psi_0 = 0$. If for every $j \in \{1, \ldots, N\}$ $\varphi(x)$ is g-asymptotically commuting or anti-commuting with $\varphi_j(x)$ then $\varphi(x)\Psi_0 = 0$ implies $\varphi(x) = 0$.*

2. Auxiliary Results

The proofs both of Theorem 1 and of Theorem 2 are based on the following

Lemma. *Let $\mathcal{N}$ be a connected complex neighborhood of $(\mathbb{R}^2 \setminus \overline{V_+^{(2)}}) \cup (\mathbb{R}^2 + iV_+^{(2)}) \cup (\mathbb{R}^2 - iV_+^{(2)})$, where $V_+^{(2)}$ denotes the 2-dimensional open forward light cone:*

$$V_+^{(2)} \equiv \{(x^0, x^1) \in \mathbb{R}^2 : x^0 > |x^1|\}.$$

Then the envelope of holomorphy of $\mathcal{N}$ is $\mathbb{C}^2$.

Proof. Let $F(z_0, z_1)$ be a (single-valued) holomorphic function on $\mathcal{N}$. Then we have to show that F is the restriction of an entire function. Without loss of generality we may assume $\mathcal{N}$ to have the property:

$$(z_0, z_1) \in \mathcal{N} \Rightarrow (z_0, \operatorname{Re}(z_1) + i\lambda \operatorname{Im}(z_1)) \in \mathcal{N} \quad \text{for all } \lambda \in (-1, +1). \tag{1}$$

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[2] As usual Ψ_0 denotes the vacuum state vector and (.,.) denotes the inner product of the Hilbert space of states

Choose a sequence $\varepsilon_1, \varepsilon_2, \ldots$ converging to zero such that[3]

$$\left|\left[-\frac{z+N+3N^4}{4}+\sqrt{\varepsilon_N+\left(\frac{z+N-3N^4}{4}\right)^2}\right]+\frac{z+N}{2}\right| < 1 \quad \text{for all } z\in\mathbb{C} \text{ with } |z|<3N \tag{2}$$

if N is big enough and define

$$z_{N,0}(w_0,w_1)\equiv -(w_0)^2-(w_1)^2+\frac{4\varepsilon_N}{3N^4+(w_0)^2+(w_1)^2}-N,$$
$$z_{N,1}(w_0,w_1)\equiv +(w_0)^2-(w_1)^2.$$

Both functions are holomorphic on $\mathbb{C}_N\times\mathbb{C}_N$, where $\mathbb{C}_N\equiv\{w\in\mathbb{C}:|\mathrm{Im}(w)|<N^2\}$. If N is big enough then

$$(w_0,w_1)\to(z_{N,0}(w_0,w_1),z_{N,1}(w_0,w_1))$$

maps $\mathbb{R}\times\mathbb{C}_N\cup\mathbb{C}_N\times\mathbb{R}$ into $\mathcal{N}$, hence, by the composition theorem

$$G_N(w_0,w_1)\equiv F(z_{N,0}(w_0,w_1),z_{N,1}(w_0,w_1)) \tag{3}$$

defines a holomorphic function on some suitable complex neighborhood of $\mathbb{R}\times\mathbb{C}_N\cup\mathbb{C}_N\times\mathbb{R}$. Via the Malgrange-Zerner theorem (see [4], Theorem 3) $G_N(w_0,w_1)$ may be analytically continued into the region

$$\mathcal{W}_N\equiv\{(w_0,w_1)\in\mathbb{C}^2:|\mathrm{Im}(w_0)|+|\mathrm{Im}(w_1)|<N^2\}.$$

Define

$$\mathcal{O}_N\equiv\left\{z\in\mathbb{C}:|\mathrm{Re}(z)|<\frac{N}{2},|z|<3N\right\}\cup\{z\in\mathbb{C}:|z|\in(2N,3N),$$
$$\mathrm{Re}(z)<0,\mathrm{Im}(z)<1\}.$$

Then, by (2), the functions

$$w_{N,0}(z_0,z_1)\equiv\sqrt{-\frac{z_0+N+3N^4}{4}+\sqrt{\varepsilon_N+\left(\frac{z_0+N-3N^4}{4}\right)^2}+\frac{z_1}{2}}$$
$$w_{N,1}(z_0,z_1)\equiv\sqrt{-\frac{z_0+N+3N^4}{4}+\sqrt{\varepsilon_N+\left(\frac{z_0+N-3N^4}{4}\right)^2}-\frac{z_1}{2}}$$

are single-valued and holomorphic on

$$\mathcal{N}_N\equiv\{(z_0,z_1)\in\mathbb{C}^2:z_0\in\mathcal{O}_N,|z_1|<N/4\}$$

and we have

$$(w_{N,0}(z_0,z_1),w_{N,1}(z_0,z_1))\in\mathcal{W}_N \text{ for } (z_0,z_1)\in\mathcal{N}_N$$

if N is big enough. Hence, by the composition theorem

$$F_N(z_0,z_1)\equiv G_N(w_{N,0}(z_0,z_1),w_{N,1}(z_0,z_1))$$

[3] By $\sqrt{z}$ we always denote the single-valued analytic continuation of the positive square root into the region $\mathbb{C}\backslash\{z=it:t\leqq 0\}$

defines a single-valued holomorphic function on $\mathcal{N}_N$. Since

$$z_{N,j}(w_{N,0}(z_0, z_1), w_{N,1}(z_0, z_1)) = z_j \text{ for } j = 0, 1$$

and since, by (2) again, we have

$$w_{N,j}(z_0, z_1) > 0 \text{ for } (z_0, z_1) \in \mathcal{R}_N \equiv (-3N, -2N) \times (-N/4, +N/4)$$

if N is big enough, (3) implies

$$F_N(z_0, z_1) = F(z_0, z_1) \text{ for } (z_0, z_1) \in \mathcal{R}_N \subset (\mathcal{N} \cap \mathcal{N}_N).$$

$\mathcal{N} \cap \mathcal{N}_N$ is easily seen to be connected, by (1). Therefore F_N gives rise to a single-valued analytic continuation of F into the region $\mathcal{N} \cup \mathcal{N}_N$. Since, by (1) again, all regions of the form

$$(\mathcal{N} \cup \mathcal{N}_{N_1} \cup \ldots \cup \mathcal{N}_{N_n}) \cap \mathcal{N}_{N_{n+1}}$$

are connected, all the F_N with N big enough may be used simultaneously for single-valued analytic continuation of F. This yields the required entire function. ■

Corollary. *Let the $\mathcal{O}_M(\mathbb{R}^4)$-function F have finite $\|F\|_{\lambda,\varepsilon}$ for all $\lambda, \varepsilon > 1$. If the support of its Fourier transform is contained in the closed forward light cone $\bar{V}_+$, then F is identically zero.*

Proof. Due to finiteness of $\|F\|_{\lambda,\varepsilon}$ for $\lambda, \varepsilon > 1$ the Fourier transform

$$\tilde{F}_{\pm}(x^0, x^1) = (2\pi)^{-1} \int da^0 da^1 F_{\pm}(a^0, a^1) e^{i(a^0 x^0 - a^1 x^1)}$$

of

$$F_{\pm}(a^0, a^1) \equiv \pm\, \theta(\pm a^0) F(a^0, a^1, 0, 0)$$

is the boundary value of some function $\hat{F}_{\pm}(z^0, z^1)$ holomorphic in the forward (backward) tube $\mathbb{R}^2 \pm i V_+^{(2)}$ (see [6], Theorem 2–9):

$$\begin{aligned} &\int dx^0 dx^1 \tilde{F}_{\pm}(x^0, x^1) \tilde{f}(x^0, x^1) \\ &= \lim_{\substack{(y^0, y^1) \to 0 \\ \pm (y^0, y^1) \in V'}} \int dx^0 dx^1 \hat{F}_{\pm}(x^0 + iy^0, x^1 + iy^1) \tilde{f}(x^0, x^1) \end{aligned}$$

for all $\tilde{f} \in \mathcal{S}(\mathbb{R}^2)$ and every closed subcone V' of $\{0\} \cup V_+^{(2)}$. Since $\operatorname{supp} \tilde{F} \subset \bar{V}_+$ implies $\operatorname{supp}(\tilde{F}_+ - \tilde{F}_-) \subset \overline{V_+^{(2)}}$ the edge of the wedge theorem (see [6], Theorem 2–16) says that both $\tilde{F}_+$ and $\tilde{F}_-$ are restrictions of one function holomorphic in a complex neighborhood $\mathcal{N}$ of $(\mathbb{R}^2 \setminus \overline{V_+^{(2)}}) \cup (\mathbb{R}^2 + i V_+^{(2)}) \cup (\mathbb{R}^2 - i V_+^{(2)})$. Hence, by our lemma, $\hat{F}_+$ and $\hat{F}_-$ are restrictions of one entire Function $\hat{F}$. But then we have

$$\begin{aligned} &\int da^0 da^1 F(a^0, a^1, 0, 0) f(-a^0, -a^1) \\ &= \int dx^0 dx^1 (\tilde{F}_+ - \tilde{F}_-)(x^0, x^1) \tilde{f}(x^0, x^1) \\ &= \lim_{\substack{(y^0, y^1) \to 0 \\ (y^0, y^1) \in V'}} \int dx^0 dx^1 [\hat{F}(x^0 + iy^0, x^1 + iy^1) - \hat{F}(x^0 - iy^0, x^1 - iy^1)] \tilde{f}(x^0, x^1) \end{aligned}$$

$= 0$ for all $\tilde{f} \in \mathscr{D}(\mathbb{R}^2)$.

Hence $F(0) = 0$. ■

3. Proof of Theorem 1

From spectrum condition and $P_+^\uparrow$-invariance one may conclude in the standard way (see [6], proof of Theorem 4–6) that $(\Psi_0, \varphi(x_n) \dots \varphi(x_1)\Psi_0)$ is both regular and equal to $(\Psi_0, \varphi(-x_n) \dots \varphi(-x_1)\Psi_0)$ at Jost points. By (A2) this implies that the $\mathscr{O}_M(\mathbb{R}^4)$-function

$$F'_{\hat{y}}(a) \equiv \int d\hat{x}[(\Psi_0, \varphi(x_n) \dots \varphi(x_1)\Psi_0) - (\Psi_0, \varphi(-x_n) \dots \varphi(-x_1)\Psi_0)]$$
$$\cdot g(x_1 - y_1 + 1a) \dots g(x_n - y_n + na)$$

has finite $\| F'_{\hat{y}} \|_{\lambda,\varepsilon}$ for all $\lambda, \varepsilon > 1$. Hence, if we define $W_{\hat{y}}(a) \equiv F_{\hat{y}}(a) + F'_{\hat{y}}(a)$, also $\| W_{\hat{y}} \|_{\lambda,\varepsilon}$ is finite for every $\hat{y} \in \mathbb{R}^{4n}$ and for all $\lambda, \varepsilon > 1$. Now, since the spectrum condition and translation invariance imply $\operatorname{supp} \tilde{W}_{\hat{y}} \subset \bar{V}_+$ the corollary tells us that $W_{\hat{y}}(0) = 0$ for all $\hat{y} \in \mathbb{R}^{4n}$. Therefore

$$\int d\hat{x}[(\Psi_0, \varphi(x_1) \dots \varphi(x_n)\Psi_0) - (\Psi_0, \varphi(-x_n) \dots \varphi(-x_1)\Psi_0)] f(x)$$

$= 0$ for all f of the form

$$f(\hat{x}) = \int d\hat{y} g(x_1 - y_1) \dots g(x_n - y_n) h(\hat{y}), \; h \in \mathscr{S}(\mathbb{R}^{4n}).$$

Thanks to positivity of $\tilde{g}$ (see (A1)) the set of all f of this form is dense in $\mathscr{S}(\mathbb{R}^{4n})$. Hence the PCT condition is fulfilled. The "vice versa" statement of Theorem 1 follows by (A2) from the standard PCT theorem (see [6], Theorem 4–6).

4. Proof of Theorem 2

By (A2), since local commutativity is known not to imply any additional restriction on the 2-point function of a neutral scalar field, $\| F_{\hat{y}} \|_{\lambda,\varepsilon}$ is finite for all $\hat{y} \in \mathbb{R}^8$ and $\lambda, \varepsilon > 1$. Here $F_{\hat{y}}$ is defined as in Theorem 1. Similarly, by (A2) again, if $\varphi(x)$ is g-asymptotically anti-commuting with itself we see that $\| F_{\hat{y}}^+ \|_{\lambda,\varepsilon}$ is finite for all $\hat{y} \in \mathbb{R}^8$ and $\lambda, \varepsilon > 1$, where

$$F_{\hat{y}}^+(a) \equiv \int d\hat{x}[(\Psi_0, \varphi(x_1)\varphi(x_2)\Psi_0) + (\Psi_0, \varphi(x_2)\varphi(x_1)\Psi_0)]$$
$$\cdot g(x_1 - y_1 + a) g(x_2 - y_2 + 2a).$$

Thus for $W_{\hat{y}}^+(a) \equiv F_{\hat{y}}(a) + F_{\hat{y}}^+(a)$ we also have

$$\| W_{\hat{y}}^+ \|_{\lambda,\varepsilon} < \infty \text{ for all } \hat{y} \in \mathbb{R}^8 \text{ and } \lambda, \varepsilon > 1.$$

Since $\operatorname{supp} \tilde{W}_{\hat{y}}^+ \subset \bar{V}_+$ for all $\hat{y} \in \mathbb{R}^8$ the corollary implies $W_{\hat{y}}^+(0) = 0$. Using the same argument as in the proof of Theorem 1 we conclude that the 2-point function of $\varphi(x)$ vanishes. Because of hermiticity of the field this is equivalent to $\varphi(x)\Psi_0 = 0$.

Now assume $\varphi(x)\Psi_0 = 0$ and g-asymptotical commutativity or anti-commutativity of $\varphi(x)$ with all the $\varphi_j(x)$. Let $\Phi \in D$ and let $j_2, \dots, j_n \in \{1, \dots, N\}$. Then first prove that

$$G_{\hat{y}}(a) \equiv \int d\hat{x}(\Phi, \varphi(x_1)\varphi_{j_2}(x_2) \dots \varphi_{j_n}(x_n)\Psi_0)$$
$$\cdot g(x_1 - y_1)g(x_2 - y_2 + a) \dots g(x_n - y_n + a)$$

is identically zero by induction with respect to n:

We always have supp $\tilde{G}_{\hat{y}} \subset \bar{V}_+$ by the spectrum condition. For $n = 2$ we also have

$$\| G_{\hat{y}} \|_{\lambda, \varepsilon} < \infty \text{ for all } \hat{y} \in \mathbb{R}^8 \text{ and } \lambda, \varepsilon > 1$$

due to (A2), g-asymptotical commutativity (resp. anti-commutativity) and translation invariance, since because of $\varphi(x)\Psi_0 = 0$ we may write

$$G_{\hat{y}}(a) = \int d\hat{x}[(\Phi, \varphi(x_1)\varphi_{j_2}(x_2)\Psi_0) \underset{(\text{resp.} +)}{-} (\Phi, \varphi_{j_2}(x_2)\varphi(x_1)\Psi_0)]$$
$$\cdot g(x_1 - y_1)g(x_2 - y_2 + a).$$

Hence the corollary implies $G_{\hat{y}}(a) = 0$. The step from n to $n+1$ is done by the same reasoning.

From $G_{\hat{y}}(a) = 0$ it follows by the same argument as in the proof of Theorem 1 that $(\Phi, \varphi(x_1)\varphi_{j_2}(x_2) \dots \varphi_{j_n}(x_n)\Psi_0) = 0$. Since the Wightman axioms require D to be dense in the Hilbert space of states and Ψ_0 to be cyclic with respect to the fields this implies $\varphi(\mathrm{x}) = 0$.

5. Summary

We proved strengthened versions of the PCT and spin-statistics theorem for a neutral scalar field applicable to certain classes of *nonlocal* fields. It is evident how to generalize the results to general spin. In a similar way one may derive strengthened versions of other classical results like Haag's theorem, fôr instance. In combination with techniques developped in [3] the method is also applicable to the theory of essentially local non-localizable fields introduced in [2], which may be relevant to non-renormalizable field theories. Full details will be given elsewhere. The present paper was just to demonstrate the essential idea in a familiar framework.

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