Commun. math. Phys. 62, 97–106 (1978)

Communications in
Mathematical
Physics

Pointwise Bounds for Schrödinger Eigenstates

Rene Carmona*

Département de Mathématique-Informatique, U.E.R. Marseille-Luminy, F-13288 Marseille Cedex 2, France

Abstract. In a different paper we constructed imaginary time Schrödinger operators $H_q = -\frac{1}{2}\Delta + V$ acting on $L^q(\mathbb{R}^n, dx)$. The negative part of typical potential function V was assumed to be in $L^\infty + L^q$ for some $p > \max\{1, n/2\}$. Our proofs were based on the evaluation of Kac's averages over Brownian motion paths. The present paper continues this study: using probabilistic techniques we prove pointwise upper bounds for L^q-Schrödinger eigenstates and pointwise lower bounds for the corresponding groundstate. The potential functions V are assumed to be neither smooth nor bounded below. Consequently, our results generalize Schnol's and Simon's ones. Moreover probabilistic proofs seem to be shorter and more informative than existing ones.

I. Introduction

The study of Schrödinger operator via probabilistic techniques originates in the famous work of Kac [14]. Using Wiener's measure in the case of heat equation, he made mathematically rigorous the heuristic prescriptions given by Feynman to solve Schrödinger equation. Stimulated by Kac's paper, something of an industry has developed, and the probabilistic approach is by now standard. To-day it is clear that Brownian motion averages are a good device to define and study the diffusion semigroup for fairly general potential functions V, and to prove the negative infinitesimal generator is a suitable extension of the formal differential operator $-\frac{1}{2}\Delta + V$. See for example [18, 10, 1, 11, 5, 16 and 4]. Let us note that in most of these papers V was assumed to be bounded below. Thanks to [2, Théorème 1] or [4, Theorem 2.1] this restrictive hypothesis can be weakened. Moreover nice spectral properties have been proved by means of similar techniques (see for example [9, 15, 2, and 4]).

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* Laboratoire de Mathématiques de Marseille associé au C.N.R.S L.A.225
New address (from October 1st 1978): Département de Mathématiques, U.E.R. Sciences, Université de Saint-Etienne, 23 rue du Docteur Michelon, F-42100 Saint-Etienne, France

0010-3616/78/0062/0097/$02.00

The present paper is written in the same spirit as the above mentioned ones. It deals with the asymptotic behavior of the various eigenstates.

For a complete review of all the results on Schrödinger operator that can be proved by path space integral methods we refer to [22], and we mention [7] and [8] for using stochastic processes to view the same problems from a slightly different point of view.

Although some of the results presented here are essentially known, our proofs are new and surprisingly short. While these results agree with those found using differential equation techniques, as desired, it is informative to probe their anatomy in the path space picture. The main novelty of the paper is the use of stochastic process techniques to study pointwise bounds for L^q-Schrödinger eigenstates. We generalize all the W.K.B. type estimates of [21] which were proved to hold for C^∞-potential functions V. Furthermore, when $V\to\infty$ at infinity we prove exponential fall-off of the eigenfunctions, even though V is not assumed to be bounded below as it is the case in [23] and [21]. The last section of the paper is devoted to the study of pointwise lower bounds for the groundstate eigenfunction. Our desire to get lower bounds was motivated by the study of hyper and super contractive estimates via logarithmic Sobolev inequalities (see [12, 6, 19 and 4]).

We would like to emphasize the fact that, even though the probabilistic approach is powerful in that it does not require any smoothness assumption on the potential functions, it provides us with constants in the rate of exponential fall off which are far from being optimal. It should be worthwhile to push the techniques in order to obtain better ones.

The relation of our results to certain problems in quantum field theory should not be unnoticed. Indeed, even though the discussion of this paper is confined solely to the study of quantum mechanical Hamiltonians, our work has been motivated, in large measure, by a potential applicability of the concepts and methods we use to fundamental problems in quantum field theory. We hope the present work serves to stimulate further research in this direction.

To keep the bibliography to a moderate length, we have adopted the convention that "see reference [x]" means "see reference [x] and the papers referred to there in ".

II. Notations—Prerequisites

In this paper $n\geqq 1$ is a fixed integer, $|\cdot|$ denotes the Euclidean norm of $\mathbb{R}^n$ and L^p and L^p_{loc} stand for the usual Lebesgue spaces with respect to Lebesgue's measure on $\mathbb{R}^n$.

Let us introduce now the standard path space realisation of Brownian motion: $\Omega=C(\mathbb{R}_+,\mathbb{R}^n)$ is the space of continuous functions from $\mathbb{R}_+$ into $\mathbb{R}^n$, for each $t\geqq 0$, X_t is the coordinate function:

$$X_t=\Omega\ni\omega\to x_t(\omega)=\omega(t)\,,$$

$\mathscr{F}$ is the smallest σ-field relative to which all the functions X_t, for $t\geqq 0$, are measurable and the collection of probability measures $\{W_x;x\in\mathbb{R}^n\}$ is defined as follows. W is the unique probability measure on $(\Omega,\mathscr{F})$ such that:

(i). $W\{X_0=0\}=1$,

(ii). $X_{t_1}-X_{t_0},\dots,X_{t_n}-X_{t_{n-1}}$ are independent, Gaussian, mean zero and their variances are $t_1-t_0,\dots,t_n-t_{n-1}$, whenever $0\leqq t_0<t_1<\dots<t_n$.

W is the so-called Wiener measure (see for example [13, Chapter 1] for several constructions). Now, for each $x \in \mathbb{R}^n$, the probability measure W_x is defined by:

$$W_x\{A\} = W\{\tau_{-x}(A)\} \qquad A \in \mathscr{F},$$

where τ_x is the path translation:

$$[\tau_x \omega](t) = x + \omega(t) \qquad x \in \mathbb{R}^n, \qquad \omega \in \Omega, \qquad t \geqq 0.$$

Expectation with respect to W_x is denoted by the symbol E_{W_x}.

Now, let us review briefly results of [4] which we use in the sequel. If V is a measurable function on $\mathbb{R}^n$ which admits a breakup $V = V_1 - V_2$ with $V_1 \in L^1_{\text{loc}}$, V_1 bounded below, $V_2 \geqq 0$ and $V_2 \in L^p$ for some $p > \max\{1, n/2\}$ then the formula:

$$[T_t f](x) = E_{W_x}\left\{ f(X_t) \exp\left[-\int_0^t V(X_s) ds \right] \right\} \qquad t \geqq 0, \qquad x \in \mathbb{R}^n \tag{2.1}$$

defines a strongly continuous semigroup $\{T_t; t \geqq 0\}$ on L^q for all $q \in [1, \infty[$. Among the smoothing properties of these semigroups let us mention that T_t is a bounded operator from L^q into L^r for all extended real numbers q and r in $[1, \infty]$ provided $q \leqq r$ and q is finite. Furthermore, if $t > 0$, T_t is a bounded operator on L^∞ and if $f \in L^q$ with q finite, $[T_t f](x)$ converges to zero when $|x|$ goes to ∞. Finally, if $V_1 \in L^{+n/2}_{\text{loc}}$, $T_t f$ is a continuous function whenever $t > 0$ and f is in any Lebesgue space [a measurable function g belongs to $L^{+\alpha}_{\text{loc}}$ if for any compact set K in $\mathbb{R}^n$ there is a real number $p(K) > \alpha$ such that $g \mathbb{1}_K \in L^{p(k)}$, where $\mathbb{1}_K$ denotes the indicator function of K]. The proofs of these results depend essentially on the crucial estimate:

$$\sup_{x \in \mathbb{R}^n} E_{W_x}\left\{ \exp\left[r \int_0^t V_2(X_s) ds \right] \right\} \leqq k_\varepsilon \exp[c(p)^{1/\varepsilon} \| V_2 \|_p^{1/\varepsilon} r^{1/\varepsilon} t] \tag{2.2}$$

which holds for all $t \geqq 0$, for some positive constant k_ε, with:

$$\varepsilon = 1 - n/2p \quad \text{and} \quad c(p) = (2\pi)^{-n/2p}(1 - p^{-1})^{(1-p^{-1})n/2} \tag{2.3}$$

and where $\| \cdot \|_p$ stands for the L^p-norm.

Let us call H_q the negative infinitesimal generator of the semigroup $\{T_t; t \geqq 0\}$ acting on L^q. Then it is proved in [4, Section IV] that H_q is a reasonable self-adjoint extension of the formal differential operator $-\frac{1}{2}\Delta + V$. In fact H_q is bounded below and H_2 coincides with the usual Schrödinger operator defined as sum of quadratic forms.

We end this section with two estimates on some sup-functionals of Brownian paths. First let us note that for each $a > 0$ and each $t > 0$, we have:

$$\begin{aligned} W_0\{\sup_{0 \leqq s \leqq t} |X_s| > a\} &\leqq 2 W_0\{|X_t| > a\} \\ &= 2(2\pi)^{-n/2} \sigma_{n-1} \int_{a/\sqrt{t}}^{\infty} r^{n-1} e^{-r^2/2} dr, \end{aligned}$$

where the first inequality is obtained via Levy's maximal inequality for sums of independent random variables and where σ_{n-1} denotes the area surface of the unit sphere in $\mathbb{R}^n$. Consequently there is a constant c (which depends on n) such that for any $t>0$ and any $a>0$ we have:

$$W_0\{\sup_{0\leqq s\leqq t}|X_s|>a\}\leqq c[(a/\sqrt{t})^{\max\{0,n-2\}}+1]e^{-a^2/2t}. \tag{2.4}$$

Second we prove, in the linear case (i.e. $n=1$) a lower bound which we will need in Section IV.

Lemma 2.1. *If $n=1$ and if a, α and t are positive real numbers which satisfy:*

$$\alpha<a/2 \quad \text{and} \quad a^2>t,$$

then for any $x\in[-(a-\alpha),a-\alpha]$ we have:

$$W_0\{\sup_{0\leqq s\leqq t}|X_s|\leqq a, X_t\in[x-\alpha,x+\alpha]\}\geqq\frac{1}{2}\frac{\alpha^2 a}{t\sqrt{2\pi t}}e^{-9a^2/8t} \tag{2.5}$$

Remark 2.1. We want to emphasize the fact that the constants in (2.5) are irrelevant; in fact for α such that $0<\alpha<a$ and a^2/t suitably bounded below (by a constant which depends on α), relation (2.5) holds provided 1/2 and 9/8 are replaced by suitable α-dependent positive constants.

Remark 2.2. The boundedness below assumption on $a/\sqrt{t}$ cannot be dropped without modifying the right hand side of (2.5) because the left hand side of (2.5) is bounded above by $W_0\{\sup_{0\leqq s\leqq t}|X_s|\leqq a\}$, and the latter converges to zero faster than any power of $a/\sqrt{t}$ (see for example [3, Lemma 2]).

Sketch of Proof. The proof is easy but requires lengthy computations. That is why we merely outline what a complete proof should be.

Since the left hand side of (2.5) is an even function of x, we prove (2.5) only for $x\geqq 0$. In this case a lower bound is clearly given by:

$$f(x)=W_0\{\sup_{0\leqq s\leqq t}|X_s|\leqq a, X_t\in[x,x+\alpha]\}.$$

The joint distribution of $\sup_{0\leqq s\leqq t}|X_s|$ and X_t is known (see for example [17, Corollary 5.3]). Thereby $f(x)$ can be computed explicitly. It is easy to check that f is differentiable and that its derivative is negative which implies:

$$f(x)\geqq f(a-\alpha) \qquad x\in[0,a-\alpha].$$

Now, in order to conclude, we write $f(a-\alpha)$ as the sum of an alternate series the first terms of which possess standard equivalents. □

III. Upper Bounds for Eigenstates

For each constant $a>0$ and each real valued function f on $\mathbb{R}^n$ we use the following notation:

$$f^{(a)}(x)=\inf\{f(y);|y-x|\leqq a\} \qquad x\in\mathbb{R}^n.$$

When a is a positive function $f^{a(x)}(x)$ will be shortened into $f^a(x)$.

The following assumption will be implicit through this section:
V is a real valued function which possesses a breakup $V=V_1-V_2$ with $V_1\in L^1_{\rm loc}$ and V_1 bounded below on one hand, and $V_2\geqq 0$ and $V_2\in L^p$ for some $p>\max\{1,n/2\}$ on

the other. $q \geqq 1$ *is finite and* ψ *is any* L^q*-Schrödinger eigenstate, namely we have* $H_q\psi = E\psi$ *for some real number* E.

In fact the upper bounds we prove are easy consequences of the following:

Lemma 3.1. *For each* $t>0$, *for each* $x \in \mathbb{R}^n$ *and for each* $a>0$ *we have:*

$$|\psi(x)| \leqq k_\varepsilon \|\psi\|_\infty \exp[(E + c(p)^{1/\varepsilon} \|V_2\|_p^{1/\varepsilon} \Gamma(\varepsilon)^{1/\varepsilon} 2^{-1+1/\varepsilon})t]$$
$$[\cdot \exp[-2tV_1^a(x)] + c[(a/\sqrt{t})^{\max\{0,n-2\}} + 1]$$
$$\cdot \exp[-2t \inf V_1 - a^2/2t]]^{1/2} \tag{3.1}$$

where the constants $c(p)$, ε, k_ε *and* c *are those of* (2.3), (2.2), *and* (2.4).

Proof. Since ψ is an eigenstate of H_q with eigenvalue E, then, it is a T_t-eigenstate with eigenvalue e^{-tE} and, consequently a bounded function. Thus:

$$|\psi(x)|^2 = e^{2tE} \left| E_{W_x}\left\{\psi(X_t) \exp\left[-\int_0^t V_1(X_s)\right] \exp\left[\int_0^t V_2(X_s)ds\right]\right\}\right|^2$$
$$\leqq \|\psi\|_\infty^2 e^{2tE} E_{W_x}\left\{\exp\left[-2\int_0^t V_1(X_2)ds\right]\right\} E_{W_x}\left\{\exp\left[2\int_0^t V_2(X_s)ds\right]\right\}. \tag{3.2}$$

Now:

$$E_{W_x}\left\{\exp\left[-2\int_0^t V_1(X_s)ds\right]\right\}$$
$$= E_{W_x}\left\{\exp\left[-2\int_0^t V_1(X_s)ds\right]; \sup_{0\leqq s\leqq t}|X_s - x| \leqq a\right\}$$
$$+ E_{W_x}\left\{\exp\left[-2\int_0^t V_1(X_s)ds\right]; \sup_{0\leqq s\leqq t}|X_s - x| > a\right\}$$
$$\leqq \exp[-2tV_1^a(x)] + \exp[-2t \inf V_1] W_0\{\sup_{0\leqq s\leqq t}|X_s| > a\} \tag{3.3}$$

and the conclusion follows from the conjunction of (3.2), (2.2), (3.3), and (2.4). □

Our first estimation was obtained in [21] for C^∞-potential functions V.

Proposition 3.1. *If we assume that:*

$$V_1(x) \geqq \gamma |x|^{2m}$$

outside a compact set for some positive constants γ *and* m, *then for each positive real number* δ *which satisfies* $\delta < \gamma^{1/2} m^m / 2(m+1)^{m+1}$ *there is a real number* $D(\delta)$ *for which:*

$$\forall x \in \mathbb{R}^n, \quad |\psi(x)| \leqq D(\delta) \exp[-\delta |x|^{m+1}]. \tag{3.4}$$

Proof. Since ψ is bounded, increasing D if necessary, it suffices to prove (3.4) for $|x|$ large enough. Now for $\beta > 0$ and $0 < \alpha < 1$ let us plug:

$$a = a(x) = \alpha |x| \quad \text{and} \quad t = t(x) = \beta \gamma^{-1/2} |x|^{-(m-1)}$$

into (3.1); then for each $\delta < \min\{2\beta(1-\alpha)^{2m}, \alpha^2/2\beta\}$ relation (3.4) is satisfied. Letting α and β vary independently concludes the proof. □

Proposition 3.2. *If we assume that:*

$$\alpha = \liminf_{|x|\to\infty} V_1(x) > E, \tag{3.5}$$

then, there are positive constants D and δ for which:

$$\forall x \in \mathbb{R}^n, \quad |\psi(x)| \leqq D e^{-\delta|x|}. \tag{3.6}$$

Proof. Without altering relation (3.5) it is possible to choose the breakup $V = V_1 - V_2$ such that:

$$c(p)^{1/\varepsilon} \| V_2 \|_p^{1/\varepsilon} \Gamma(\varepsilon)^{1/\varepsilon} 2^{-1+1/\varepsilon} < (\alpha - E)/4.$$

As above it suffices to prove (3.6) for $|x|$ large enough. In fact we assume that $|x|$ is sufficiently large in order to have $V_1^a(x) > (\alpha + E)/2$ with $a(x) = |x|/2$. Finally we set $t(x) = \beta|x|$, and if $\beta > 0$ is small enough, the existence of δ follows from (3.1). □

The conclusion of the next proposition is known as Šnol's result (see [23] and [20] and it worths pointing out that the existing proofs require the boundedness below of the potential function V.

Proposition 3.3. *Let us assume:*

$$\lim_{|x|\to\infty} V_1(x) = \infty.$$

Then for each $\delta > 0$ there is a positive real number $D(\delta)$ for which:

$$\forall x \in \mathbb{R}^n, \quad |\psi(x)| \leqq D(\delta) e^{-\delta|x|}.$$

Proof. From (3.1) there are positive constants c_1 and c_2 such that:

$$|\psi(x)| \leqq c_1 \exp\left[(E + c(p)^{1/\varepsilon} \| V_2 \|_p^{1/\varepsilon} \Gamma(\varepsilon)^{1/\varepsilon} 2^{-1+1/\varepsilon}) t\right] \cdot \left[\exp[-2t V_1^a(x)] + c_2 \exp[-2t \inf V_1 - a^2/4t]\right]^{1/2}. \tag{3.7}$$

If $\delta > 0$ is fixed, let M be a positive number which satisfies:

$$M - E - c(p)^{1/\varepsilon} \| V_2 \|_p^{1/\varepsilon} \Gamma(\varepsilon)^{1/\varepsilon} 2^{-1+1/\varepsilon} > \delta. \tag{3.8}$$

Now, let $\beta > 0$ be small enough in order to have:

$$2\beta \inf V_1 + 1/(16\beta) > 2M,$$

and let $\alpha > 0$ be such that:

$$|x| > \alpha \Rightarrow V_1^a(x) > M/\beta,$$

where $a(x) = |x|/2$. Finally, if we set $t(x) = \beta|x|$ and if we plug all this in (3.7) we obtain:

$$|\psi(x)| \leqq c_1 \exp\left[(E + c(p)^{1/\varepsilon} \| V_2 \|_p^{1/\varepsilon} \Gamma(\varepsilon)^{1/\varepsilon} 2^{-1+1/\varepsilon}) \beta|x|\right](1 + c) e^{-Mx}$$

which, because $\beta < 1$ and by (3.7) concludes the proof. □

The last result of this section gives an explicit dependence of the decay of the eigenstates on the behavior of the potential function near infinity. These estimates are usually guessed via heuristic arguments of W.K.B. type.

Proposition 3.4. *Let us assume that:*

$$\lim_{|x|\to\infty} V_1(x) = \infty.$$

If moreover there is a positive function on $\mathbb{R}^n$, *say* a, *such that:*

$$V_1^a(x) \geqq \alpha V_1(x)$$

for some positive constant α *and for* x *outside a compact* K, *then there are positive constants* d *and* δ *such that:*

$$\forall x \notin K \qquad |\psi(x)| \leqq d \exp[-\delta a(x) \sqrt{V_1(x)}].$$

Proof. It suffices to plug $t(x) = a(x) V_1(x)^{-1/2}$ in (3.1). □

IV. Lower Bounds for the Groundstate Eigenfunction

This section is devoted to the proof of lower bounds on the fall off of the ground state eigenfunction of H_q. Our interest in this problem comes from our desire to obtain hyper and supercontractive estimates via logarithmic Sobolev inequalities (see [4, Section V]). *In addition to the hypotheses made in Section III on* V, E *and* ψ, *we assume in this section that* $V_1 \in L_{\mathrm{loc}}^{+n/2}$ *and* E *is the infimum of the spectrum of* H_q. Then, it was pointed out in [4, Remark 4.4] that a simple consequence of Feynman-Kac's formula is that ψ can be chosen everywhere positive and locally bounded away from zero.

Lemma 4.1. *For each* $x = (x^1, \ldots, x^n) \in \mathbb{R}^n \setminus \{0\}$ *and each positive real numbers* $\alpha_1, \ldots, \alpha_n$, $a_1, \ldots, a_n$, $b_1, \ldots, b_n$ *and* t *which satisfy:*

$$a_j^2 > t, \quad \alpha_j < a_j/2, \quad \text{and} \quad l([-a_j, a_j] \cap [-x^j - b_j, -x^j + b_j]) > \alpha_j, \quad j = 1, \ldots, n$$

where $l(I)$ *denotes the length of the interval* I, *then we have:*

$$\begin{aligned} -\operatorname{Log}\psi(x) \leqq & -Et - \operatorname{Log}\varepsilon(b) + n \operatorname{Log}(2t\sqrt{2\pi t}) - \sum_{j=1}^{n} \operatorname{Log}(\alpha_j^2 a_j) \\ & + \frac{9}{8t} \sum_{j=1}^{n} a_j^2 + t \sup\{V_1(y); |y^j - x^j| < a_j, j = 1, \ldots, n\} \end{aligned} \tag{4.1}$$

where $\varepsilon(b)$ *is a positive constant depending only on* $b = (b_1, \ldots, b_n)$, *and where we used a superscript* j *to denote the* j*-th coordinate of the elements of* $\mathbb{R}^n$.

Proof. If we set:

$$\varepsilon(b) = \inf\{\psi(y); |y^j| \leqq b_j \quad j = 1, \ldots, n\}$$

and:

$$A = \{|X_t^j| \leqq b_j, \sup_{0 \leqq s \leqq t} |X_s^j - x^j| \leqq a_j \quad j = 1, \ldots, n\}$$

then we have:

$$\begin{aligned} \psi(x) &= e^{tE} E_{W_x}\left\{\psi(X_t) \exp\left[-\int_0^t V_1(X_s) ds\right] \exp\left[\int_0^t V_2(X_s) ds\right]\right\} \\ &\geqq e^{tE} E_{W_x}\left\{\psi(X_t) \exp\left[-\int_0^t V_1(X_s) ds\right]\right\} \\ &\geqq e^{tE} E_{W_x}\left\{\psi(X_t) \exp\left[-\int_0^t V_1(X_s) ds\right]; A\right\}, \end{aligned}$$

where $E_P\{\Phi; A\}$ means expectation, with respect to probability P, over paths lying in A,

$$\geqq e^{tE}\varepsilon(b)\exp[-t\sup\{V_1(y); |y^j - x^j| \leqq a_j,\ j=1,\dots,n\}]W_x\{A\}. \tag{4.2}$$

Now, by definition of the probability measures W_x we have:

$$\begin{aligned} W_x\{A\} &= W_0\{|x^j + X_t^j| \leqq b_j, \sup_{0\leqq s\leqq t}|X_s^j| \leqq a_j, j=1,\dots,n\} \\ &= \prod_{j=1}^n W_0\{|x^j+X_t^j| \leqq b_j, \sup_{0\leqq s\leqq t}|X_s^j| \leqq a_j\} \\ &\geqq (2t\sqrt{2\pi t})^{-n}\left(\prod_{j=1}^n \alpha_j^2 a_j\right)\exp\left[-9\left(\sum_{j=1}^n a_j^2\right)\Big/8t\right] \end{aligned} \tag{4.3}$$

where we used, first the fact that, with respect to W_0, the $\{X_t^j; t \geqq 0\}$ are n independent one dimensional Brownian motion processes starting from the origin, and second, Lemma 2.1.

Finally the conjunction of (4.2) and (4.3) gives the desired conclusion. □

The following result was proved in [21] under more restrictive conditions on the potential function V. Namely V was assumed to be infinitely differentiable. In fact Lemma 4.1 seems to be useful to estimate the fall-off of the groundstate eigenfunction under mild regularity assumptions on V.

Proposition 4.1. *If furthermore we assume that:*

$$V_1(x) \leqq \gamma|x|^{2m} \tag{4.4}$$

outside a compact set for some positive constants γ and m, then we have:

$$\forall x \in \mathbb{R}^n \qquad \psi(x) \geqq D\exp[-\delta|x|^{m+1}) \tag{4.5}$$

for some positive constants D and δ.

Proof. Since ψ is locally bounded away from zero, enlarging D if necessary it suffices to prove (4.5) for $|x|$ large. Now, for $|x|>1$ let us set:

$$t = |x|^{-(m-1)}, \quad a_j = 1 + |x^j|, \quad \alpha_j = 1/2 \quad \text{and} \quad b_j = 1 \quad \text{for} \quad j=1,\dots,n.$$

The assumptions of Lemma 4.1 are clearly satisfied. Moreover, by (4.4) we have:

$$\sup\{V_1(y); |y^j - x^j| < a_j, \quad j=1,\dots,n\} \leqq \gamma 2^{4m}|x|^{2m}$$

provided x is assumed to satisfy $|x| > (m/8)^{1/2}$. Pluging all that in (4.1) we obtain:

$$\begin{aligned} -\operatorname{Log}\psi(x) \leqq & -E|x|^{-(m-1)} - \operatorname{Log}\varepsilon(b) + n\operatorname{Log}(2\sqrt{2\pi}) + 2n\operatorname{Log}2 \\ & - \sum_{j=1}^n \operatorname{Log}(1+|x^j|) + \frac{9n}{4}|x|^{m-1} + \frac{9}{4}|x|^{m+1} + \gamma 2^{4m}|x|^{m+1}, \end{aligned}$$

which implies (4.5) with any $\delta > \gamma 2^{4m} + 9/4$. □

For similar reasons we have:

Proposition 4.2. *Let us assume that there is a positive function t which is defined and bounded outside a compact set and which satisfies:*

(i) $|x|^2 \leqq t(x)[c_1 V_1(x) + c_2]$ *for some positive constants c_1 and c_2,*

(ii) $t(x)\sup\{V_1(y); |y^j - x^j| \leqq \max\{t^{1/2}, |x^j|\} j=1,\dots,n\} \leqq c_3 V_1(x) + c_4$ *for some positive constants c_3 and c_4.*

Then, there are positive constants d_1 and d_2 such that for each $x \in \mathbb{R}^n$ we have:

$$-\operatorname{Log}\psi(x) \leqq d_1 V_1(x) + d_2. \tag{4.6}$$

Example 4.1. If we assume that:

$$\forall x \in \mathbb{R}^n, \quad a'|x|^{\alpha'} e^{\varepsilon b|x|^\beta} + c' \leqq V_1(x) \leqq a|x|^\alpha e^{b|x|^\beta} + c$$

for some positive constants a', α', a, α, ε, b, and β which satisfy:

$$0 < \beta < 1 \quad \text{and} \quad 4^{\beta/2} - 1 < \varepsilon < 1,$$

and real constants c' and c, then, (4.6) holds.

Example 4.2. Let $U_1, \ldots, U_n$ be bounded below measurable functions on $\mathbb{R}$ such that:

$$\forall x \in \mathbb{R}^n, \quad \sum_{j=1}^n a_j U_j(x^j) + c'_1 \leqq V_1(x) \leqq \sum_{j=1}^n b_j U_j(x^j) + c'_2$$

for some constants c'_1 and c'_2 and positive ones $a_1, \ldots, a_n$, $b_1, \ldots, b_n$. Furthermore let us assume that for each j we have:

$$\sup\{U_j(x^j); |y^j - x^j| \leqq \max\{1, |x^j|\}\} \leqq c^j_3 U_j(x^j) + c^j_4$$

outside a compact subset of $\mathbb{R}$ and for some constants c^j_3 and c^j_4. If moreover we assume

$$|x|^2 \leqq c_1 V_1(x) + c_2$$

outside a compact subset of $\mathbb{R}^n$ then (4.6) holds.

Acknowledgements. We would like to express our gratitude to Professor L. Gross for numerous enlightening discussions on the subject of this paper, and to Professor B. Simon for reading the first draft of this paper.

References

1. Babbit, D.W.: The Wiener integral and perturbation theory of the Schrödinger operator. Bull. Am. Math. Soc. **70**, 254—259 (1964)
2. Berthier, A.M., Gaveau, B.: Critère de convergence des fonctionnelles de Kac et application en mécanique quantique et en géométrie. J. Funct. Anal. (to appear)
3. Carmona, R.: Potentials on abstract Wiener space. J. Funct. Anal. **26**, 215—231 (1977)
4. Carmona, R.: Regularity properties of Schrödinger and Dirichlet semigroups. J. Funct. Anal. (to appear)
5. Deift, P., Simon, B.: On the decoupling of finite singularities from the question of asymptotic completeness in two-body quantum systems. J. Funct. Anal. **23**, 218—238 (1976)
6. Eckmann, J.P.: Hypercontractivity for anharmonic oscillators. J. Funct. Anal. **16**, 388—404 (1974)
7. Ezawa, H., Klauder, J.R., Shepp, L.A.: A path space picture for Feynman-Kac averages. Ann. Phys. **88**, 588—620 (1974)
8. Ezawa, H., Klauder, J.R., Shepp, L.A.: Vestigial effects of singular potentials in diffusion theory and quantum mechanics. J. Math. Phys. **16**, 783—799 (1975)
9. Faris, W.G., Simon, B.: Degenerate and non degenerate ground states for Schrödinger operators. Duke Math. J. **42**, 559—567 (1975)
10. Feldman, J.: On the Schrödinger and heat equations for non negative potentials. Trans. Am. Math. Soc. **108**, 251—264 (1963)

11. Ginibre, J.: Some applications of functional integration in statistical mechanics. In: Statistical mechanics and quantum field theory (eds. C. De Witt, R. Stora) pp. 327—427. New York: Gordon and Breach 1970
12. Gross, L.: Logarithmic Sobolev inequalities. Am. J. Math. **97**, 1061—1083 (1976)
13. Ito, K., McKean, H.P.: Diffusion processes and their sample paths, 2nd printing. Berlin-Heidelberg-New York: Springer 1974
14. Kac, M.: On some connection between probability theory and differential and integral equations. Proc. 2nd Berk. Symp. on Math. Stat. and Prob. pp. 199—215 (1951)
15. Lieb, E.: Bounds on the eigenvalues of the Laplace and Schrödinger operators Bull. Am. Math. Soc. **82**, 751—753 (1976)
16. McKean, H.P.: $-\Delta$ plus a bad potential. J. Math. Phys. **18**, 1277—1279 (1977)
17. Parthasarathy, K.R.: Probability measures on metric spaces. New York: Academic Press 1967
18. Ray, D.: On the spectra of second order differential operators. Trans. Am. Math. Soc. **77**, 299—321 (1954)
19. Rosen, J.S.: Sobolev inequalities for weight spaces and supercontractivity. Trans. Am. Math. Soc. **222**, 367—376 (1976)
20. Simon, B.: Pointwise bounds on eigenfunctions and wave packets in N-body quantum systems. II. Proc. Am. Math. Soc. **45**, 454—456 (1974)
21. Simon, B.: Pointwise bounds on eigenfunctions and wave packets in N-body quantum systems. III. Trans. Am. Math. Soc. **208**, 317—329 (1975)
22. Simon, B.: Functional integration and quantum physics. New York: Academic press (to appear)
23. Šnol', J.E.: On the behavior of the eigenfunctions of Schrödinger's equation Mat. Sb. **46**, 273—286 (1957); Erratum 259 (Russian)

Communicated by J. Glimm

Received March 22, 1978