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Remarks on the Modular Operator and Local Observables

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Abstract. In this paper we give a characterization of the modular group of a von Neumann algebra $\mathscr{R}$, with a cyclic and separating vector, which provides at the same time a necessary and sufficient condition so that two von Neumann algebras $\mathscr{R}_1$ and $\mathscr{R}_2$, such that $\mathscr{R}_1 \subseteq \mathscr{R}_2'$, are the mutual commutants, i.e. $\mathscr{R}_1 = \mathscr{R}_2'$.

An application is made to the duality property in Quantum Field Theory, and we give a sufficient condition for PCT invariance in a theory of local observables.

1. Introduction

It is known that if $\mathscr{R}$ is a von Neumann algebra with a cyclic and separating vector Ω, then the associated modular operator is characterized by the following conditions:

i) $\Delta = \Delta^*$, $\Delta > 0$;
ii) for each $t \in \mathbb{R}$ $\Delta^{it}\Omega = \Omega$;
iii) for each $t \in \mathbb{R}$ $\Delta^{it}\mathscr{R}\Delta^{-it} = \mathscr{R}$;
iv) the automorphism group $\sigma_t = \Delta^{it} \cdot \Delta^{-it}$, satisfies the KMS condition for the state $\omega_0 = (\Omega, \cdot\Omega)$.

Recall that $\Delta^{1/2}$ is the modulus in the polar decomposition of the *-operator $A\Omega \to A*\Omega$, $A \in \mathscr{R}$; the phase J is an antiunitary involution such that $J\Delta^{1/2}A\Omega = A*\Omega$, and $J\mathscr{R}J = \mathscr{R}'$. By these relations $\Delta^{1/2}\mathscr{R}^{\mathrm{sa}}\Omega = \mathscr{R}'^{\mathrm{sa}}\Omega$, where we denote with $\mathscr{R}^{\mathrm{sa}}$ the selfadjoint operators of $\mathscr{R}$ [8].

Conversely the KMS condition is easily implied by the condition

iv′) $\Delta^{1/2}\mathscr{R}^{\mathrm{sa}}\Omega \subset \mathscr{R}'^{\mathrm{sa}}\Omega$.

In this note we show that condition iv′) independently from Tomita-Takesaki theory, implies a commutation theorem, and at the same time characterizes the modular group, producing another proof of the uniqueness of the modular automorphisms.

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This is suggested by the proof of duality in field theory [1], and in analogy with that analysis can possibly be further generalized to *-algebras of unbounded operators.

2.

In this paper the Rieffel-van Daele commutation theorem is very important [6]; we will use the following simplified form of the theorem:

Theorem 1. *Let $\mathscr{R}_1$ and $\mathscr{R}_2$ be von Neumann algebras on the Hilbert space $\mathscr{H}$, with a cyclic and separating vector Ω. We assume that $\mathscr{R}_1$ and $\mathscr{R}_2$ commute. Then $\mathscr{R}_1 = \mathscr{R}_2'$ if and only if the following condition is satisfied: if $\xi \in \mathscr{H}$ and for each $A \in \mathscr{R}_1^{\mathrm{sa}}$, $B \in \mathscr{R}_2^{\mathrm{sa}}$ we have*

$$\operatorname{Im}(A\Omega, \xi) = 0, \qquad \operatorname{Re}(B\Omega, \xi) = 0$$

then $\xi = 0$.

We shall utilize the following lemma.

Lemma 2. *Let $\mathscr{R}$ be a von Neumann algebra with a cyclic and separating vector Ω, and let $U(t)$ be an unitary strongly continuous group such that $U(t)\Omega = \Omega$, and $U(t)\mathscr{R}U(t)^* = \mathscr{R}$ for every $t \in \mathbb{R}$.*

If we assume that the unbounded operator $U(i) = e^{-K}$ (where K is the generator of the group $U(t)$) satisfies the following condition

$$\mathscr{R}\Omega \subseteqq \mathscr{D}(U(i)) \quad \text{and} \quad U(i)\mathscr{R}^{\mathrm{sa}}\Omega \subseteqq \mathscr{R}^{\mathrm{sa}}\Omega$$

then $U(t) = I$ for each $t \in \mathbb{R}$.

Proof. Let $A \in \mathscr{R}'^{\mathrm{sa}}$ and $B \in \mathscr{R}^{\mathrm{sa}}$. Since $B\Omega \in \mathscr{D}(U(i)) \subseteqq \mathscr{D}(U(z))$, for $z \in \mathbb{C}$ and $\operatorname{Im} z \in [0, 1]$ we can define the function

$$f(z) = (A\Omega, U(z)B\Omega)$$

by the spectral theory of selfadjoint operators f is analytic for $\operatorname{Im} z \in (0, 1)$ and continuous for $\operatorname{Im} z \in [0, 1]$. On the real axis f is a real function, because $AU(t)BU(z)^*$ is selfadjoint; on the axis $\operatorname{Im} z = 1$ we have

$$f(t+i) = (A\Omega, U(t)U(i)B\Omega);$$

but there is $\hat{B} \in \mathscr{R}^{\mathrm{sa}}$ such that $U(i)B\Omega = \hat{B}\Omega$, hence $f(t+i)$ is real. By [10], Lemma 1.6, f is bounded on the strip. By the principle of analytic reflection we can extend f to an entire bounded function, and by Liouville theorem f is constant. We have proved

$$\forall t \in \mathbb{R} (A\Omega, U(t)B\Omega) = (A\Omega, B\Omega)$$

but Ω is cyclic and separating for $\mathscr{R}$, and we obtain $U(t) \equiv I$.

The following lemma is well-known [8]:

Lemma 3. *Let U be an unitary operator such that $U\Omega = \Omega$ and $U\mathscr{R}U^* = \mathscr{R}$, where Ω is a cyclic and separating vector for the von Neumann algebra $\mathscr{R}$. Then U commutes with the modular group associated to Ω.*

3.

The following gives a density condition, and at the same time a characterization of the modular group:

Theorem 4. *Let R_1 and R_2 be von Neumann algebras with a common cyclic vector Ω. We assume that R_1 and R_2 commute, and there is a strongly continuous unitary group $U(t)$ satisfying the following conditions:*

i) $U(t)\mathscr{R}_l U(t)^* = \mathscr{R}_l$ *for* $l=1,2, t\in\mathbb{R}$;

ii) $U(t)\Omega=\Omega$, $t\in\mathbb{R}$;

iii) $\mathscr{R}_1\Omega \subseteqq (U(i))$, $U(i)\mathscr{R}_1^{sa}\Omega \subseteqq \mathscr{R}_2^{sa}$.

Then $\mathscr{R}_1=\mathscr{R}_2'$ *and* $\Delta^{-it/2}=U(t)$.

Proof. We utilize the Rieffel-Van Daele commutation theorem. Let $\xi\in\mathscr{H}$, and suppose that for $A\in\mathscr{R}_1^{sa}$ and $B\in\mathscr{R}_2^{sa}$, $(\xi, A\Omega)$ is real, and $(\xi, B\Omega)$ is pure imaginary. We want to show that $\xi=0$.

We define on the strip $\bar{I}=\{z\in\mathbb{C}|\operatorname{Im} z\in(0,1)\}^-$ the function

$$f(z)=(\xi, U(z)A\Omega).$$

By [10], Lemma 1.6, f is analytic on I, continuous and bounded on $\bar{I}$. Moreover $f(t)=(\xi, U(t)A\Omega)=(\xi, U(t)AU(t)^*\Omega)$ by condition ii, hence f is real on the real axis by condition i.

On the axis $\operatorname{Im} z=1$, we have $f(t+i)=(\xi, U(t)U(i)A\Omega)$, but by condition iii) there is $\hat{A}\in\mathscr{R}_2^{sa}$ such that $U(i)A\Omega=\hat{A}\Omega$. Hence $f(t+i)=(\xi, U(t)\hat{A}\Omega)$ is pure imaginary because $U(t)\hat{A}U(t)^*\in\mathscr{R}_2^{sa}$.

Since f is real on the real axis and pure imaginary on the axis $\operatorname{Im} z=1$, by the principle of analytic reflection, we can extend it to an entire function. By Liouville Theorem f is constant; but f is real on the real axis, whereas it is purely imaginary on the axis $\operatorname{Im} z=1$, hence $f\equiv 0$.

In particular for every $A\in\mathscr{R}_1^{sa}$ we have $(\xi, A\Omega)=0$; then $\xi\in(R,\Omega)^{\perp}$ and we obtain $\xi=0$. By Theorem 1 $\mathscr{R}_1=\mathscr{R}_2'$. We know, by Lemma 3, that Δ^{is} and $U(t)$ commute for every $t,s\in\mathbb{R}$. Trivially the unitary group $V(t)=\Delta^{it/2}U(t)$ is a strongly continuous unitary group satisfying the following conditions:

i) $V(t)\Omega=\Omega$ $t\in\mathbb{R}$;

ii) $V(t)\mathscr{R}_1 V(t)^*=\mathscr{R}_1$ $t\in\mathbb{R}$;

iii) $\mathscr{R}_1\Omega\subseteqq\mathscr{D}(V(i))$, $V(i)\mathscr{R}_1^{sa}\Omega\subseteqq\mathscr{R}_1^{sa}\Omega$.

Hence $V(t)\equiv I$ by the Lemma 2, and $U(t)=\Delta^{-it/2}$.

Remark. Note that the implication $\mathscr{R}_1=\mathscr{R}_2'$ in the theorem is independent from Tomita-Takesaki theory. If $\mathscr{R}_1$ is a *-algebra (not a von Neumann algebra) fulfilling the assumption of Theorem 4, then we can conclude $\mathscr{R}_1''=\mathscr{R}_2'$, by the Rieffel-Van Daele commutation theorem [6].

4.

We want to apply the previous result to proof of the duality for the von Neumann algebras $\mathscr{R}(W_R)$ and $\mathscr{R}(W_L)$ associated with a hermitian scalar field φ, satisfying the

Bisognano-Wichmann condition [1]: for every real test functions $f, g \in \mathcal{S}(\mathbb{R}^4)$ such that supp f and supp g are space-like separated, the field operators $\varphi(f)$ and $\varphi(g)$ have selfadjoint closure whose spectral projectors commute.

Let $V(t)$ be the representation of the group of the velocity transformations whose action is described by the matrix

$$\Lambda(t)=\begin{pmatrix} \operatorname{ch} t & \operatorname{sh} t & 0 & 0 \\ \operatorname{sh} t & \operatorname{ch} t & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Furthermore define $J=\vartheta U(R_{e_1}(\pi))$, where ϑ is the PCT conjugation, and $R_{e_1}(\pi)$ is the rotation of π about the x_1 axis.

If $W_R=\{\chi\in\mathbb{R}^4 | \chi_1>|\chi_0|\}=-W_L$, we denote $\mathcal{R}(W_k)$ (for $k=L,R$) the von Neumann algebra generated by the spectral projector of the operators $\overline{\varphi(f)}$, where f is a real test function of $\mathcal{S}(\mathbb{R}^4)$ whose support is contained in W_k. By the condition of Bisognano and Wichmann $\mathcal{R}(W_R)\subseteq\mathcal{R}(W_L)'$; in [1] they prove that for each $A\in\mathcal{R}(W_R)$

$$A\Omega\in\mathcal{D}(V(i\pi)), \qquad V(i\pi)A\Omega=JA^*\Omega.$$

By the action of the PCT operator ϑ on the field we have $J\Omega=\Omega$ and $J\mathcal{R}(W_R)J=\mathcal{R}(W_L)$; hence by Theorem 4 $\mathcal{R}(W_R)=\mathcal{R}(W_L)'$. Trivially the operator $JV(i\pi)$ is the S operator of the Tomita-Takesaki theory (note that $\mathcal{R}(W_R)\Omega$ is a core for $V(i\pi)$, since it is invariant for the group $\{V(t)\}$). If $\mathcal{A}(\mathcal{O})$ is the local net generated by the field operators, we have for a double cone $\mathcal{O}$ contained in W_R

$$\mathcal{R}(W_R)=\Big\{\bigcup_{t\in\mathbb{R}} V(t)\mathcal{A}(\mathcal{O})V(t)^*\Big\}''=\Big\{\bigcup_{\mathcal{O}\subseteq W_R}\mathcal{A}(\mathcal{O})\Big\}''$$

because these von Neumann algebras have the same S operator [8]; we also obtain

$$(*)\quad \mathcal{A}(\mathcal{O}')^{-W}=\Big\{\bigcup_{\Lambda W_R\subseteq\mathcal{O}'} U(\Lambda)\mathcal{R}(\mathrm{W}_R)U(\Lambda)^*\Big\}'',$$

where $\mathcal{A}(\mathcal{O}')$ is the C^*-algebra generated by the von Neumann algebras $\mathcal{A}(\mathcal{O}_\alpha)$ with $\mathcal{O}_\alpha$ space-like separated with $\mathcal{O}$.

Let $\mathcal{A}$ be a net of local algebras [4]. One may define a dual net by setting $\mathcal{A}^d(\mathcal{O})=\mathcal{A}(\mathcal{O}')'$ [7]. The net $\mathcal{A}^d$ satisfies the properties:

i) $\mathcal{A}\subseteq\mathcal{A}^d$.

ii) $\mathcal{A}$ is local [i.e. $\mathcal{A}(\mathcal{O}_1)$ commute with $\mathcal{A}(\mathcal{O}_2)$ if $\mathcal{O}_1$ and $\mathcal{O}_2$ are space-like separated] iff $\mathcal{A}\subseteq\mathcal{A}^d$.

iii) $\mathcal{A}$ satisfies the duality iff $\mathcal{A}=\mathcal{A}^d$.

iv) If $\mathcal{A}_1$ and $\mathcal{A}_2$ are two net of local algebras, and $\mathcal{A}_1\subseteq\mathcal{A}_2$ then $\mathcal{A}_2^d\subseteq\mathcal{A}_1^d$.

We say that a net of local algebras satisfies assential duality if $\mathcal{A}^d=\mathcal{A}^{dd}$. Now if $\mathcal{A}$ is a local net, i.e. if $\mathcal{A}\subseteq\mathcal{A}^d$, then essential duality is fulfilled if and only if $\mathcal{A}^d$ is a local net [7].

In the Bisognano-Wichmann analysis, the condition $(*)$ implies that $\mathcal{A}^d$ is a local net; hence the local net generated by a hermitian scalar field satisfies essential duality.

The above discussion can be generalized to a theory of local observables. Let $\mathscr{A}$ be a local net of local algebras Poincaré covariant, i.e. there is an unitary strongly continuous representation $\mathscr{U}$ of the Poincaré group $\mathscr{P}$, such that

$$\forall \{\Lambda, a\} \in \mathscr{P} \qquad U(\Lambda, a)\mathscr{A}(\mathcal{O})U(\Lambda, a)^* = \mathscr{A}(\{\Lambda, a\}\mathcal{O})$$

for every double cone $\mathcal{O}$. Let Ω be the vacuum (it is Poincaré invariant). The following theorem give a sufficient condition so that the local net $\mathscr{A}$ satisfies essential duality, independently from the existence of a Wightman field generating the net, and from the PCT invariance of the theory.

Theorem 5. *Let $\mathscr{A}$ be a local net of local algebras Poincaré covariant, Ω the vacuum state. We call $\mathscr{R}(W_R)$ and $\mathscr{R}(W_L)$ the von Neumann algebras generated by the algebras $\mathscr{A}(\mathcal{O})$ with $\mathcal{O} \subseteqq W_R$ ($\mathcal{O} \subseteqq W_L$ respectively). Then $\mathscr{A}$ fulfills essential duality if for each double cone $\mathcal{O} \subseteqq W_R$*

$$(**) \quad \mathscr{A}(\mathcal{O})\Omega \subseteqq \mathscr{D}(U(i\pi)), \qquad U(i\pi)\mathscr{A}(\mathcal{O})^{\mathrm{sa}}\Omega \subseteqq \mathscr{R}(W_L)^{\mathrm{sa}}\Omega,$$

where $U(t) = U(\Lambda(t), 0)$ implements the pure Lorentz transformations along the x_1 axis.

Proof. By remark to Theorem 4 we have that $U(t)$ is the modular group of $\mathscr{R}(W_R)$ associated with the cyclic and separating vector Ω (note Ω is an analytic vector for the energy) [2]; moreover $\mathscr{R}(W_R) = \mathscr{R}(W_L)'$. By an argument similar to Bisognano-Wichmann proof, this implies essential duality.

Note that in the Bisognano-Wichmann situation condition $(**)$ takes the more specific form:

for each double cone $\mathcal{O} \subseteqq W_R$

$$(***) \quad \mathscr{A}(\mathcal{O})\Omega \subseteqq \mathscr{D}(U(i\pi)), \qquad U(i\pi)\mathscr{A}(\mathcal{O})^{\mathrm{sa}}\Omega \subseteqq \mathscr{A}(\mathcal{O}^j)^{\mathrm{sa}}\Omega$$

where $\mathcal{O}^j = \{\chi \in \mathbb{R}^4 | (-\chi_0, -\chi_1, \chi_2, \chi_3) \in \mathcal{O}\}$.

This more restrictive assumption implies in the general case that there is a PCT operator ϑ such that for every Poincaré transformation $\{\Lambda, a\}$ we have $\vartheta U(\Lambda, a)\vartheta = U(\Lambda, -a)$.

Theorem 6. *Let $\mathscr{A}$ be a local net of local algebras Poincaré covariant, Ω the vacuum. If condition $(***)$ is fulfilled the anti-unitary operator $\vartheta = U(R_{e_1}(\pi))J$ (J is the modular conjugation associated to the von Neumann algebra $\mathscr{R}(W_R)$ for the cyclic and separating vector Ω; $R_{e_1}(\pi)$ is the rotation of π about the axis determined by the unit vector $e_1 = (0,1,0,0)$) satisfies the following conditions:*

1. $\vartheta^2 = I$, $\vartheta\Omega = \Omega$;
2. *for each double cone* $\mathcal{O}$ $\vartheta\mathscr{A}(\mathcal{O})\vartheta = \mathscr{A}(-\mathcal{O})$;
3. *for each Poincaré transformation* $\{\Lambda, a\}, \vartheta U(\Lambda, a) = U(\Lambda, -a)\vartheta$.

In particular ϑ is independent from the wedge region which we use to define it.

Proof. By an argument of analytical extension J commutes with $U(R_{e_1}(\pi))$ and we have 1. We want to prove that for $a \in \mathbb{R}^4$ $U(I, -a) = \vartheta U(I, a)\vartheta$. As $\{U(I, a)\}$ is a group, it suffices to prove the equality

$$U(I,a^j)=JU(I,a)J\,,$$

where $a^j=(-a_0,-a_1,a_2,a_3)$, for $a\in W_R$.

If $z\in\mathbb{C}$ we define

$$\Lambda(z)=\begin{pmatrix}\mathrm{ch}\,z & \mathrm{sh}\,z & 0 & 0\\ \mathrm{sh}\,z & \mathrm{ch}\,z & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix};$$

this is an extension of the pure Lorentz transformations along the x_1 axis, and $z\to\Lambda(z)$ is an analytic entire function. By a simple computation if $\mathrm{Im}\,z\in(0,\pi)$ and $a\in W_R$, $\Lambda(z)a\in\mathscr{I}=\mathbb{R}^4+iV_+$.

By spectrum condition [2] there is an extension $T(\zeta)$ of the translations group on the tube $\bar{\mathscr{I}}$. The function $\zeta\in\bar{\mathscr{I}}\to T(\zeta)$ is analytic on $\mathscr{I}$, strongly continuous on $\bar{\mathscr{I}}$, and for $x\in\mathbb{R}^4$ we have $T(x)=U(I,x)$.

If $a\in W_R$ then $a+W_R\subseteq W_R$; hence if $A\in\mathscr{R}(W_R)$, $U(I,a)AU(I,-a)\in\mathscr{R}(W_R)$. In particular $A\Omega$, $U(I,a)A\Omega\in\mathscr{D}(U(i\pi))\subseteq\mathscr{D}(U(z))$ for $\mathrm{Im}\,z\in[0,\pi]$.

If $a\in W_R$ we define on the strip $G=\{z\in\mathbb{C}|\mathrm{Im}\,z\in(0,\pi)\}$

$$f(z)=U(z)U(I,a)A\Omega-T(\Lambda(z)a)U(z)A\Omega$$

this function is well-defined, analytic on G and continuous on $\bar{G}$. One easily see that for every $t\in\mathbb{R}$ $f(t)=0$, and by the principle of analytical continuation $f\equiv 0$. In particular $f(i\pi)=0$, i.e.

$$U(i\pi)U(I,a)A\Omega=T(\Lambda(i\pi)a)U(i\pi)A\Omega\,;$$

but $\Lambda(i\pi)a=a^j$ and by Tomita-Takesaki theory we have

$$JU(I,a)A^*\Omega=U(I,a^j)JA^*\Omega\,;$$

Ω is cyclic for $\mathscr{R}(W_R)$ and 4) is proved. (Note that we have used condition (**) only).

If $\mathscr{O}\subseteq W_R$ is a double cone, then

$$\vartheta\mathscr{A}(\mathscr{O})\Omega\subseteq\mathscr{A}(-\mathscr{O})\Omega$$

because of the action of $U(R_{e_1}(\pi))$ and condition (***).

But $\vartheta\mathscr{R}(W_R)\vartheta=\mathscr{R}(-W_R)=\mathscr{R}(W_L)$, hence $\vartheta\mathscr{A}(\mathscr{O})\vartheta\subseteq\mathscr{R}(W_L)$. Now if $A\in\mathscr{A}(\mathscr{O})$ $\vartheta A\vartheta\in\mathscr{R}(W_L)$, but $\vartheta A\vartheta\Omega=\vartheta A\Omega=A'\Omega$ where $A'\in\mathscr{A}(-\mathscr{O})$ and so $\vartheta A\vartheta\in\mathscr{A}(-\mathscr{O})$, as Ω is a separating vector for $\mathscr{R}(W_L)$. Hence $\vartheta\mathscr{A}(\mathscr{O})\vartheta\subseteq\mathscr{A}(-\mathscr{O})$. If $\mathscr{O}$ is a double cone there is $a\in W_R$ such that $a+\mathscr{O}\subseteq W_R$; then

$$\begin{aligned}T(-a)\vartheta\mathscr{A}(\mathscr{O})\vartheta T(a)&=\vartheta T(a)\mathscr{A}(\mathscr{O})T(-a)\vartheta=\vartheta\mathscr{A}(\mathscr{O}+a)\vartheta\\&\subseteq\mathscr{A}(-\mathscr{O}-a)=T(-a)\mathscr{A}(-\mathscr{O})T(a)\,.\end{aligned}$$

This implies 2.

We want to prove that for every $\Lambda\in L_+^\uparrow$ ϑ and $U(\Lambda)$ commute. If R is a rotation, the von Neumann algebra associated with the wedge region RW_R is $\mathscr{R}(RW_R)=U(R)\mathscr{R}(W_R)U(R)^*$, the modular group of $\mathscr{R}(RW_R)$ is $U_R(t)=U(R)U(t)U(R)^*$, and the modular conjugation is $J_R = U(R)JU(R)^*$. If e is the unit vector Re_1 [where $e_1 = (0, 1, 0, 0)$], we denote $R_e(\pi)$ the rotation of π about the axis determined by e. Trivially $R_e(\pi) = RR_{e_1}(\pi)R^{-1}$. If we denote $\vartheta_R = U(R_e(\pi))J_R$, ϑ_R satisfies assumption 2. Since the local net is covariant ϑ_R commutes with $U_R(t)$.

Let $V_R = \vartheta_R$; then for each double cone $\mathcal{O}$ $V_R\mathcal{A}(\mathcal{O})V_R^* = \mathcal{A}(\mathcal{O})$ because ϑ and ϑ_R satisfy 2; moreover $V_R\Omega = \Omega$. By Lemma 3, V_R commutes with the modular group $U_R(t)$ (each wedge region is invariant for V_R). Because ϑ_R commutes with $U_R(t)$, we have proved that ϑ commutes with $U_R(t)$. Hence for every pure Lorentz transformation Λ ϑ commutes with $U(\Lambda)$.

The group generated by the pure Lorentz transformations is a normal subgroup of $L_+^\uparrow$, and $L_+^\uparrow$ has only trivial normal subgroup; so $L_+^\uparrow$ is the group generated by the pure Lorentz transformations, and ϑ commutes with $U(L_+^\uparrow)$.

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