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N Body Scattering in the Two-Cluster Region*

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Abstract. We extend Combes' result on completeness of N-body scattering at energies below the lowest 3-body threshold from potentials with $|x|^{-\nu-\varepsilon}$ falloff ($\nu \equiv$ number of dimensions for each particle) to central potentials with $|x|^{-1-\varepsilon}$. We also treat the scattering of electrons from neutral atoms in the two cluster region.

§1. Introduction

This paper is a contribution to our program [2, 8, 3] of using geometric ($\equiv$ configuration space) methods to study multiparticle non-relativistic quantum mechanical systems. Indeed, it can be viewed as an addendum to Section 3 of [8].

Consider the Hamiltonian of N particles in ν-dimensions, i.e.,

$$\tilde{H} = -\sum_{i=1}^{N} (2m_i)^{-1} \Delta_i + \sum_{i<j} V_{ij}(\vec{r}_i - \vec{r}_j)$$

on $L^2(R^{N\nu})$. As is traditional, we separate out the center of mass and consider the "reduced" operator, H on $L^2(R^{(N-1)\nu})$. Corresponding to any decomposition α of $\{1, \ldots, n\}$ into two non-trivial subsets $C_1^{(\alpha)}, C_2^{(\alpha)}$, H decomposes into $H(C_1^{(\alpha)}) + H(C_2^{(\alpha)}) + T_\alpha + I_\alpha$ where I_α is the interaction between clusters, i.e.,

$$\sum_{i \in C_1^{(\alpha)},\, j \in C_2^{(\alpha)}} V_{ij}$$

and T_α is the relative kinetic energy of the clusters.

One defines

$$\Sigma_{2,\alpha} = \inf(\sigma(H(C_1^{(\alpha)}) + H(C_2^{(\alpha)})))$$

and

$$\Sigma_{3,\alpha} = \inf(\sigma_{\text{ess}}(H(C_1^{(\alpha)}) + H(C_2^{(\alpha)})))$$

where the operator is viewed as acting on $L^2(R^{(N-2)\nu})$. Finally, $\Sigma_2 = \min(\Sigma_{2,\alpha})$,

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$\Sigma_3 = \min_\alpha(\Sigma_{3,\alpha})$. The HVZ theorem (see [5, 6] for general information on N-body systems) asserts that the essential spectrum of H is $[\Sigma_2, \infty]$ and implies that Σ_3 is the lowest energy for breakup into three clusters. In the energy range $[\Sigma_2, \Sigma_3]$, the only open scattering channels are two cluster channels. The problem is genuinely multichannel since excitations and rearrangements are allowed but as it is "essentially two body" one would hope that two-body methods, especially the elegant Kato-Birman theory (reviewed in [5], §XI.3) would be applicable.

That this is indeed the case was a discovery of Combes [1]. In §3 of [8], we showed how Combes' proof could be streamlined and made quite elementary. Both [1] and [8] require $V_{ij}(x)$ to (more-or-less) behave at $|x|^{-\nu-\varepsilon}$ at infinity ([1] only considers $\nu=3$) which is not surprising since this is well-known to be the restrictions needed for the applicability of the Kato-Birman theory in the two body case.

What is not so well-known is the remark of Kuroda [4] that for central potentials where one can make a partial wave expansion, the Kato-Birman theory applicable so long as V is $0(|x|^{-1-\varepsilon})$. Our goal in §2 below will be to combine this remark of Kuroda [4] with [1, 8] to prove completeness of the scattering in the energy range $[\Sigma_2, \Sigma_3]$ for *central* potentials V_{ij} with $0(|x|^{-1-\varepsilon})$ falloff. There is one irony in this situation: The study of non-central potentials in the two-body case is often defended on the basis that they will better mock up the N-body situation where a partial wave expansion is not available. At least in the two cluster region, this argument is discredited by our considerations in §2. To avoid the special value $l = -\frac{1}{2}$ which occurs in the $\nu=2$ partial wave expansion, we will suppose $\nu \geqslant 3$ in §2. It is probably quite easy to accommodate $\nu=2$.

In the method of [8], one had to show that an operator $AI_\alpha B$ was trace class and there (and also in §2, below) we accomplished this by looking at each $AV_{ij}B$. Doing this prevents the use of any kind of cancellations between the V_{ij}. Such cancellations are expected to be present in the scattering of an electron from a neutral system. We discuss this problem in §3 and prove completeness in the two cluster region by treating $AI_\alpha B$ all at once. This is the first completeness result for Coulomb systems with more than two particles.

§2. Short Range Potentials with Slow Decay

In this section, we will prove:

Theorem 1. *Let $\nu \geqslant 3$. Let each potential V_{ij} be central (spherically symmetric) and obey:*

$$(1+|x|^2)^{1/2+\varepsilon} V_{ij}(x) \in L^\infty + L^p$$

for $p = 2(\nu=3), > 2(\nu=4), \nu/2(\nu \geqslant 5)$. Then scattering is complete in the energy range $[\Sigma_2, \Sigma_3]$, i.e.,

$$\bigoplus \operatorname{Ran} \Omega_\alpha^\pm \supset \operatorname{Ran}(\mathrm{E}_{\mathrm{ac}} \mathrm{E}_{[\Sigma_2,\Sigma_3]})$$

where $\Omega_\alpha^\pm$ are the usual channel wave operators, and $E_\Omega(E_{\mathrm{ac}})$ are spectral projections for H.

Proof. We begin by recalling some notation and results from [8]. Let P_α be the projection onto all states of the form $\phi_1(\zeta_1)\phi_2(\zeta_2)\psi(\eta_\alpha)$ where ζ_i is the coordinate internal to cluster $C_i^{(\alpha)}, \eta_\alpha$ is the coordinate between the center of mass of the clusters and ϕ_1, ϕ_2 are eigenfunctions of $H(C_i^{(\alpha)})$ with energies E_i obeying $E_1 + E_2 < \Sigma_3$. By definition of Σ_3, we can (by replacing Σ_3 by $\Sigma_3 - \varepsilon$) suppose each P_α is "finite dimensional" in the ζ_i coordinates.

Let P be the projection onto the span of the P_α and let $Q = I - P$. Then, as in [8], $(P - P_\alpha)e^{-itH}E\varphi$ and $Q(e^{-itH}E\varphi) \to 0$ as $t \to \infty$ where $E = E_{\mathrm{ac}}E_{[\Sigma_2, \Sigma_3]}$. The proofs of these facts depended not at all on the $|x|^{-\nu-\varepsilon}$ falloff of V. Thus, as in [8], the proof is reduced to showing the existence of the limits $(H_\alpha = H - I_\alpha)$

$$\Omega^{\pm}(H_\alpha, H; P_\alpha) \equiv s - \lim_{t \to \mp\infty} e^{itH_\alpha} P_\alpha e^{-itH}.$$

Now let J be the total angular momentum of the particles about their center of mass. We use the symbols $Q_{J \leqslant j}$ (resp. $Q_{J=j}$) to denote the projection onto all vectors with $J^2 \leqslant j(j+1)$ (resp. $J^2 = j(j+1)$). We are supposing for notational simplicity that $\nu = 3$; otherwise more involved notation is needed to label the relevant spherical harmonics and representations. For each α, we can write $J = L_\alpha + S_\alpha$ where S_α is the angular momentum of the cluster about their centers of mass and L_α is the angular momentum of the two clusters about each other. Notice that J commutes with H, H_α, P_α and that L_α, S_α commutes with H_α, P_α. Moreover

$$P_\alpha Q_{S_\alpha \leqslant s} = P_\alpha \tag{1}$$

for some s, since P_α is "finite dimensional in the ζ_i coordinates".

Since J commutes with H, the existence of $\Omega^{\pm}(H_\alpha, H; P_\alpha)$ follows from the existence of $\Omega^{\pm}(H_\alpha, H; P_\alpha Q_{J \leqslant j})$ for each j. By Birman's form of the Kato-Birman theorem, we only need that (for $[H_\alpha, P_\alpha Q_{J \leqslant j}] = 0$)

$$C = H_\alpha(P_\alpha Q_{J \leqslant j}) - (P_\alpha Q_{J \leqslant j})H = -P_\alpha Q_{J \leqslant j} I_\alpha.$$

has the property that $E_I(H_\alpha)CE_I(H)$ is trace class for each bounded interval. (Mutual subordination is trivial.) As in the proof of Lemma 3.4 of [8], write

$$E_I(H_\alpha)CE_I(H) = ABDE \tag{2}$$

$$A = E_I(H_\alpha)P_\alpha(I + T_\alpha)^{1/2+\varepsilon}$$

$$B = (I + T_\alpha)^{-1/2-\varepsilon}(1 + |\eta_\alpha|^2)^{-1/2-\varepsilon} Q_{J \leqslant j} P_\alpha \tag{3}$$

$$D = P_\alpha(1 + |\eta_\alpha|^2)^{1/2+\varepsilon} I_\alpha(H_0 + 1)^{-1} \tag{4}$$

$$E = (H_0 + 1)E_I(H)$$

where we have used $P_\alpha^2 = P_\alpha$ and the fact that P_α and J commutes with P_α, T_α and $|\eta_\alpha|^2$. Now A and E are bounded as in [8].

The proof will be completed if we show that D is bounded and B is trace class. To handle D, decompose $I_\alpha = \Sigma V_{ij}$ and for a given term, write

$$|\eta_\alpha|^2 \leqslant a|x_i - x_j|^2 + b|\zeta|^2.$$

for a suitable internal coordinate ζ. Then using that

$$(1+|x_i-x_j|^2)^{+1/2+\varepsilon}V_{ij}(x)(H_0+1)^{-1}$$

is bounded by hypothesis and that

$$P_\alpha(1+|\zeta|^2)^{1/2+\varepsilon}$$

is bounded by the exponential falloff of bound states, we conclude that D is bounded.

To control B, use (1) and write

$$\begin{aligned} Q_{J\leqslant j}P_\alpha &= Q_{J\leqslant j}Q_{S\leqslant s}P_\alpha \\ &= Q_{L\leqslant j+s}P_\alpha Q_{J\leqslant j}Q_{S\leqslant s} \end{aligned}$$

where we have used the fact that if $J^2=j(j+1)$, $S^2=s(s+1)$, then L^2 can only have eigenvalue $l(l+1)$ with $l\leqslant j+s$. Thus, we only need that

$$(I+T_\alpha)^{-1/2-\varepsilon}(1+|\eta_\alpha|^2)^{-1/2-\varepsilon}Q_{L=l}P_\alpha \tag{5}$$

is trace class for each fixed l. Now, P_α is a sum of rank 1 operators in the ζ's, so we can suppose that P_α is rank 1 in the ζ's. But then the operator in (5) is a finite direct sum of operators unitarily equivalent to

$$(h_{0,l}+1)^{-1/2-\varepsilon}(1+|r|^2)^{-1/2-\varepsilon}$$

on $L^2(0,\infty,dr)$ where $h_{0,l}$ is the operator with angular momentum barrier, l, i.e.,

$$h_{0,l}=-\frac{d^2}{dr^2}+\frac{l(l+1)}{r^2}.$$

Let p^2 be the operator $-d^2/dr^2$ on all of $(-\infty,\infty)$. Then,

$$h_{0,l}+1)^{-1}\leqslant(h_{0,l=0}+1)^{-1}\leqslant(p^2+1)^{-1}.$$

(We remove a Dirichlet B.C. in the last step.)

By a standard theorem,

$$(h_{0,l}+1)^{-1/2-\varepsilon}\leqslant(p^2+1)^{-1/2-\varepsilon}.$$

for $\varepsilon<\frac{1}{2}$ (which can be assumed without loss). Thus, since

$$(1+r^2)^{-1/4-\varepsilon}(p^2+1)^{-1/2-\varepsilon}(1+r^2)^{-1/4-\varepsilon}$$

is known to be trace class on $L^2(R)$ (see e.g. [5]), we have that

$$(1+r^2)^{-1/4-\varepsilon}(h_{0,l}+1)^{-1/2-\varepsilon}(1+r^2)^{-1/4-\varepsilon}$$

is trace class. Using the fact that

$$(1+T_\alpha)^{-1/2-\varepsilon}(1+|\eta_\alpha|^2)^{-1/4-\varepsilon}(1+T_\alpha)^{1/2+\varepsilon}(1+|\eta_\alpha|^2)^{1/4+\varepsilon}$$

is bounded, we conclude that B is trace class. □

§3. Scattering from Neutral Coulomb Systems

In this section, we want to prove:

Theorem 2. *Let $N = m+2$ and $v = 3$. Suppose that $V_{ij}(x) = |x|^{-1}$ for $i \neq j \leqslant m+1$ and $V_{ij}(x) = -m|x|^{-1}$ for $i \leqslant m+1, j = m+2$. Then scattering is complete in the region $[\Sigma_2, \Sigma_3]$.*

Remarks. 1. We describe a system with a nucleus of charge m and $m+1$ electrons. Thus we are talking scattering of an electron off a neutral atom at energies too small to ionize a second electron.

2. Since atoms have an infinity of bound states [9, 7], as Σ_3 is approached, an infinity of channels open. But at no energy that we consider, is the problem genuinely infinite channel.

3. One can allow more than one positive change so long as one cluster is neutral in each allowed two cluster state in the energy range considered. Also, one can add *central* short range potentials as in §2.

Proof. We follow the proof of Theorem 1 including the decomposition (2). A and E are bounded as before and B is trace class. All that is different is the proof that D is bounded. Let $f(x) = |x|^{-1}$ (if $|x| \geqslant 1$) and $= 1$ (if $|x| \leqslant 1$). Let $g(x) = |x|^{-1} - f(x)$. Let $V_{ij} = f_{ij} + g_{ij}$ and let $I_\alpha = I_{\alpha,1} + I_{\alpha,2}$ corresponding to this decomposition. $I_{\alpha,2}$ is a sum of finite range pieces, so its contribution is bounded as in §2 and [8]. Restricting to a "rank 1" P_α and to specific coordinates, we find that it suffices to show that the following operator is bounded:

$$|\eta(x_1, \ldots, x_m)> \; <\eta(x_1, \ldots, x_m)|(1 + |\eta_{m+1}|^2)^{1/2+\varepsilon} \sum_{i=1}^{m} [f(x_{m+1} - x_m) - f(x_{m+1})]$$

where we have passed to atomic coordinates $x_j = r_j - r_{m+2}$ and we use Dirac notation for P_α.

Now, $|(\nabla f)(x)| \leqslant |x|_c^{-2}$ where $|x|_c = |x|$ (if $|x| \geqslant 1$) and $= 1$ (if $|x| \leqslant 1$). Thus

$$|f(x-y) - f(x)| \leqslant y[\min(x, x-y)]_c^{-2}.$$

It follows that we need only prove that

$$|\eta> \; <\eta| x_m (1 + |\eta_{m+1}|^2)^{1/2-\varepsilon} [\min(x_{m+1}, x_m - x_{m+1})]_c^{-2}$$

is bounded. This is easy; for

$$\begin{aligned}
&(1+a^2)[\min(a, a-b)]_c^{-2} \\
&\quad \leqslant \max[(1+a^2)a_c^{-2}, (1+a^2)(a-b)_c^{-2}] \\
&\quad \leqslant \max[2, 3+2b^2] \leqslant 3 + 2b^2
\end{aligned}$$

where we used $a^2 \leqslant 2(a-b)^2 + 2b^2$. Thus taking $\varepsilon = \frac{1}{2}$ (without loss) and using $\eta_{m+1}^2 \leqslant 2x_{m+1}^2 + 2|\zeta|^2$, we need only that

$$|\eta> \; <\eta| x_m (3 + 2x_m^2)$$

is bounded. This is immediate from the exponential falloff of η in the $x_j (j \leqslant m)$ variables. □

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References

1. Combes, J. M.: Nuovo Cimento **64**A, 111 (1969)
2. Deift, P., Simon, B.: Commun. Pure Appl. Math., to appear
3. Deift, P., Hunziker, W., Simon, B., Vock, E.: ETH Preprint, in preparation
4. Kuroda, S.: J. Math. Phys. **3**, 933 (1962)
5. Reed, M., Simon, B.: Methods of modern mathematical physics, Vol. III. Scattering theory. New York: Academic Press 1978
6. Reed, M., Simon, B.: Methods of modern mathematical physics, Vol. IV. Analysis of Operators. New York: Academic Press 1977
7. Simon, B.: Helv. Phys. Acta **43**, 607 (1970)
8. Simon, B.: Commun. math. Phys. **55**, 259 (1977)
9. Zhislin, G. M.: Tr. Mosk. Math. Obsc. **9**, 82 (1960)

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