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Absence of Classical Lumps*

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Abstract. Solutions of the equations of classical Yang-Mills theory in four dimensional Minkowski space are studied. It is proved (Theorem 1) that there is no finite energy (nonsingular) solution of the Yang-Mills equations having the property that there exists ε, R, $t_0>0$ such that

$$E_R(t)=\int_{|\bar{x}|\leqq R}\theta_{00}(t,\bar{x})d^3\bar{x}\geqq\varepsilon \quad \text{for every} \quad t>t_0\,,$$

$\theta_{00}(\bar{x},t)$ being the energy density. Previously known theorems on the absence of finite energy nonsingular solutions that radiate no energy out to spatial infinity are particular cases of Theorem 1. The result stated in Theorem 1 is not restricted to the Yang-Mills equations. In fact, it extends to a large class of relativistic equations (Theorem 2).

I. Introduction

In a very interesting paper [1] Coleman has proved that there are no (nontrivial) finite energy nonsingular solutions of classical Yang-Mills theory in four dimensional Minkowski space that do not radiate energy out to spatial infinity. More precisely he has proved:

Theorem *(Coleman). The only finite energy nonsingular solution of the Yang-Mills equations in four dimensional Minkowski space satisfying*

$$\lim_{|\bar{x}|\to\infty}|\bar{x}|^{3/2+\delta}F^a_{\mu\nu}(t,\bar{x})=0\,,\qquad \delta>0 \tag{1}$$

uniformly in $|\bar{x}|$ *and* $t, t>0$, *is the vacuum solution.* $F^a_{\mu\nu}(t,\bar{x})$ *is the field strength,* $\mu\nu=0,1,2,3$ *are space-time indexes and* a *is an internal index.* $\bar{x}\in\mathbb{R}^3$, $t\in\mathbb{R}$, *and the* $F^a_{\mu\nu}(t,\bar{x})$ *are real valued functions in* $\mathbb{R}^4$.

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Coleman's theorem generalizes [1] previous results in the absence of finite energy nonsingular non-radiant solutions [1, 2].

Coleman's argument is remarkably simple (Coleman's proof is basically a refinement of an argument of Pagels [3]). (1) is a sufficient condition for the relevant surface integral to go to zero at spatial infinity uniformly in time.

I have been able to improve Coleman's argument. In this way I can get rid of the strong assumption (1) and I prove the following theorem:

Theorem 1. *There is no finite energy (nonsingular) solution of the Yang-Mills equations in four dimensional Minkowski space having the property that there exists* ε, R, $t_0>0$, *such that*

$$E_R(t)=\int_{|\bar{x}|\leqq R}\theta_{00}(t,\bar{x})d^3x\geqq\varepsilon \quad \textit{for any} \quad t>t_0\,. \tag{2}$$

$\theta_{00}(t,\bar{x})$ *being the energy density. From the definition of* $\theta_{00}(t,\bar{x})$ *and the relation* $F^a_{\mu\nu}=-F^a_{\nu\mu}$ *it follows that*

$$\int d^3x|F^a_{\mu\nu}|^2\leqq 2E \quad \textit{for any} \quad t\,. \tag{3}$$

This gives us all the uniformity in time that we need. E is the energy of the fields

$$E=\int d^3x\theta_{00}(t,\bar{x})\,.$$

The previously known results [1, 2] on the absence of finite energy, non-singular, non-radiant solutions are particular cases of Theorem 1.

The result stated in Theorem 1 is not restricted to the Yang-Mills equations. It extends to a large class of relativistic equations: Let $\theta_{\mu\nu}(t,\bar{x})$ be the energy momentum tensor associated with a relativistic equation. Assume

(α) $\theta_{00}(t,\bar{x})\geqq 0\,,\quad \theta^\mu_\mu=0\,,\quad \partial_\mu\theta^{\mu\nu}=0\,.$

(β) There is a constant $C>0$ such that

$$\int|\theta_{\mu\nu}(t,\bar{x})|d^3\bar{x}\leqq CE \quad \text{for any} \quad t\,,$$

where E is the energy integral:

$$E=\int\theta_{00}(t,\bar{x})d^3x\,.$$

Then

Theorem 2. *For any relativistic equation such that the associated energy momentum tensor satisfies* (α) *and* (β) *there is no finite energy (nonsingular) solution having the property that there exists* ε, R, $t_0>0$, *such that*

$$E_R(t)=\int_{|\bar{x}|\leqq R}\theta_{00}(t,\bar{x})d^3x\geqq\varepsilon \quad \textit{for any} \quad t>t_0\,.$$

II. The Proofs

Let $A^a_\mu(t,\bar{x})$, $t\in\mathbb{R}$, $\bar{x}\in\mathbb{R}^3$ be a set of vector fields. Define the field strength $F^a_{\mu\nu}(t,\bar{x})$ by

$$F^a_{\mu\nu}(t,\bar{x})=\partial_\mu A^a_\nu-\partial_\nu A^a_\mu+C^{abc}A^b_\mu A^c_\nu\,. \tag{4}$$

C^{abc} are the structure constants of a compact Lie group. We have

$$F^a_{\mu\nu}=-F^a_{\nu\mu}\,. \tag{5}$$

Summation over repeated indexes is understood (unless otherwise specified). The derivatives are taken in distribution sense and the $F^a_{\mu\nu}(t, \bar{x})$ are assumed to be real functions.

The Yang-Mills equations are

$$\partial^\mu F^a_{\mu\nu} + C^{abc} A^{\mu b} F^c_{\mu\nu} = 0 . \tag{6}$$

The signature of the metric tensor is $(+, -, -, -)$. The energy momentum tensor is given by

$$\theta^{\mu\nu} = F^{\mu\gamma a} F^{\nu a}_{\gamma} - \tfrac{1}{4} g^{\mu\nu} F^a_{\gamma\sigma} F^{a\sigma\gamma} . \tag{7}$$

$\mu, \nu, \gamma, \sigma = 0, 1, 2, 3$. We have $\theta^\mu_\mu = 0$, $\partial_\mu \theta^{\mu\nu} = 0$, and

$$\theta^{00} = \tfrac{1}{2}(E^a_i E^a_i + H^a_i H^a_i) \geqq 0 . \tag{8}$$

$i = 1, 2, 3$, where E^a_i and H^a_i are the analogues of the electric and magnetic fields $E^a_i = F^a_{i,0}$, $H^a_i = \frac{1}{2}\varepsilon_{ijk} F^a_{jk}$, $ijk = 1, 2, 3$. The energy is given by

$$E = \int \theta^{00}(t, \bar{x}) d^3x = \tfrac{1}{2} \int (E^a_i E^a_i + H^a_i H^a_i) d^3\bar{x} . \tag{9}$$

We assume $E < \infty$.

It follows from the fact that the $F^a_{\mu\nu}$ are antisymmetric in μ and ν, and from (8) that

$$\int |F^a_{\mu\nu}|^2 d^3x \leqq 2E < \infty \tag{10}$$

for any t. This fact gives us, for finite energy solutions, all the uniformity in time that we need to prove Theorem 1. In fact, for the Yang-Mills equations (6) we have $\theta^\mu_\mu = 0$, $\partial_\mu \theta^\mu_\nu = 0$, $\theta_{00} \geqq 0$. Hence (α) of Theorem 2 is satisfied. Moreover by (10) (β) is satisfied as well. Then Theorem 1 follows if we prove Theorem 2.

Proof of Theorem 2. Suppose that there exists a finite energy solution having the property that there exists ε, R, $t_0 > 0$ such that

$$E_R(t) = \int_{|x| \leqq R} d^3x \theta_{00}(t, \bar{x}) d^3x \geqq \varepsilon \quad \text{for any} \quad t > t_0 .$$

Let $\mu(r) \in \mathscr{L}^1(R)$ satisfy the following conditions:

(i) $\mu(r) \geqq 0, \quad \mu(r) = 0 \quad \text{for} \quad r < R.$

(ii) $\|r\mu(r)\|_1 < \infty .$

(iii) $\|\mu\|_1 > K/\varepsilon \|r\mu(r)\|_\infty ,$

where K is a constant defined below. We define

$$w(t) = \int dr \mu(r) \int_{|\bar{x}| \leqq r} d^3x x_i \theta^{0i} .$$

By (β)

$$|w(t)| \leqq 3CE \|r\mu(r)\|_1 . \tag{11}$$

On the other side by $\partial_\mu \theta^{\mu\nu} = 0$ and $\theta^\mu_\mu = 0$

$$\partial_t w(t) = \int \mu(r) E_r(t) dr - \int \mu(r) dr \int_{|\bar{x}| \leq r} d^3x \sum_{ij} \partial_i (x^j \theta^{ij}) .$$

Clearly

$$\int \mu(r)E_r(t)dr \geqq \varepsilon \|\mu(r)\|_1 .$$

Moreover by the theorem of the divergence and (β)

$$\int \mu(r)dr \int_{|\vec{x}| \leqq r} d^3x \sum_{ij} \partial_i(x^j\theta^{ij}) \leqq K\|r\mu(r)\|_\infty , \qquad K=9CE .$$

Then

$$\partial_t w(t) \geqq \varepsilon \|\mu(r)\|_1 - K\|r\mu(r)\|_\infty = \Delta > 0 .$$

Then for $t > t_0$

$$w(t) \geqq \Delta t - w(t_0) . \tag{12}$$

Obviously (11) and (12) cannot be satisfied simultaneously and we reach a contradiction. The theorem will be proved if we show a function in $\mathscr{L}^1(\mathbb{R}^3)$ satisfying (i), (ii), and (iii). Take for example

$$\mu(r) = \begin{cases} 0, & r < R \\ \frac{1}{r}, & R \leqq r \leqq R_1 < \infty \\ 0, & r > R_1 , \end{cases}$$

where R_1 is so large that $\ln R_1 - \ln R_0 > K/\varepsilon$. Q.E.D.

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