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An Inequality on S Wave Bound States, with Correct Coupling Constant Dependence

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Abstract. We prove that the number of S wave bound states in a spherically symmetric potential $gV(r)$ is less than

$$g^{1/2}\left[\int_0^\infty r^2 V^-(r)dr \int_0^\infty V^-(r)dr\right]^{1/4}$$

where V^- is the attractive part of the potential, in units where $\hbar^2/2M=1$.

I. Introduction

It is well known that in the limit of large coupling constants the number of S wave bound states in a potential $V(r)$ behaves asymptotically like [1], [2]

$$n(g) \simeq g^{1/2}\frac{1}{\pi}\int_0^\infty [V^-(r)]^{1/2}dr \tag{1}$$

where $V^-(r)$ is the attractive part of $V(r)$, in units such that $\hbar^2/2M=1$. This asymptotic theorem holds under various sufficient conditions. One of them is that $V(r)$ should be piecewise monotonous [1] with a finite number of monotony intervals. Another [2] is that $\int_0 [V^-(r)]dr$ converges and that V decreases fast enough at infinity. However, it is clearly impossible to turn the asymptotic equality (1) into a strict bound because bound states can easily be produced by delta function potentials; however the integral of the square root of a delta function is zero, crudely speaking. One way out is to require *monotony* of the potential, which excludes delta functions. Then one gets the Calogero bound [3]

$$n < g^{1/2}\frac{2}{\pi}\int_0^\infty [V^-(r)]^{1/2}dr \tag{2}$$

V^- monotonous decreasing.

In other classical bounds monotony is not assumed but the coupling constant dependence is in g, like in the Bargmann bound [4]

$$n < g \int_0^\infty r V^-(r) dr \tag{3}$$

or worse [5].

Here we want to show that it is nevertheless possible to get a strict bound with a $g^{1/2}$ dependence.

II. A Bound for Finite Intervals

As always, we count the number of bound states by counting the number of zeros of the zero energy radial wave function. Let r_{p-1} and r_p be two successive zeros of the reduced zero energy wave function $u(r)$. We have

$$\begin{aligned} 0 = \int_{r_{p-1}}^{r_p} [u'^2 + gVu^2] dr &\geqq \int_{r_{p-1}}^{r_p} [u'^2 - gV^-(r)u^2] dr \\ &> \int_{r_{p-1}}^{r_p} u'^2 dr - (\operatorname{Sup} u^2) \int_{r_{p-1}}^{r_p} gV^-(r) dr . \end{aligned} \tag{4}$$

Now

$$|u(r)| = \left| \int_{r_{p-1}}^{r} u' dr \right| < (r - r_{p-1})^{1/2} \left[\int_{r_{p-1}}^{r} u'^2 dr \right]^{1/2} \tag{5}$$

$$|u(r)| = \left| \int_{r}^{r_p} u' dr \right| < (r_p - r)^{1/2} \left[\int_{r}^{r_{p-1}} u'^2 dr \right]^{1/2} \tag{6}$$

and taking the half sum of (5) and (6) and using then the Schwarz inequality, we get

$$|u(r) < \tfrac{1}{2} |r_p - r_{p-1}|^{1/2} \left[\int_{r_{p-1}}^{r_p} u'^2 dr \right]^{1/2} \tag{7}$$

$$\tfrac{1}{2} |r_p - r_{p-1}|^{1/2} \left[\int_{r_{p-1}}^{r_p} gV^-(r) dr \right]^{1/2} > 1 . \tag{8}$$

Suppose now that we have n bound states. We get n inequalities of the type (8) with $r_0 = 0$. Hence, for any $m \leqq n$:

$$m \leqq \tfrac{1}{2} \sum_{p=1}^{m} |r_p - r_{p-1}|^{1/2} \left[\int_{r_{p-1}}^{r_p} gV^-(r) dr \right]^{1/2} \tag{9}$$

or, using Schwarz inequality

$$m \leqq \frac{g^{1/2}}{2} (r_m)^{1/2} \left[\int_0^{r_m} V^-(r) dr \right]^{1/2} . \tag{10}$$

So if we take a finite interval $0 < r < R$ with the boundary condition $u(R) = 0$, corresponding to an infinite wall at $r = R$ we see that the number of solutions with negative energy is less than

$$\tfrac{1}{2} g^{1/2} R^{1/2} \left[\int_0^R V^-(r) dr \right]^{1/2} \tag{11}$$

On the other hand, if we take a potential of finite range R, with exactly n bound states, the last one being at zero energy, we have $u'(R)=0$. We get $n-1$ inequalities of the type (8), but the last one is

$$|R-r_{n-1}|^{1/2}\left[\int_{r_{n-1}}^{R} gV^-(r)dr\right]^{1/2} \geqq 1. \tag{12}$$

Hence, combining (12) and (8), we get

$$n-\tfrac{1}{2}\leqq \frac{g^{1/2}R^{1/2}}{2}\left[\int_0^R V^-(r)dr\right]^{1/2}. \tag{13}$$

This inequality is saturated by n equally spaced delta function potentials, the last one having a strength which is a half of the others.

It is possible to get also another inequality by changing r to

$$z=1/r \tag{14}$$

defining

$$u(r)=w(z)/z$$

one gets the equation

$$\frac{d^2w}{dz^2}+Ww=0$$

with

$$W(z)=V(r)/z^4. \tag{15}$$

Equation (15) has again the form of the Schrödinger equation and therefore, between two successive nodes we get, returning to the original variables, in analogy to (8):

$$\frac{1}{2}\left|\frac{1}{r_{p-1}}-\frac{1}{r_p}\right|^{1/2}\left[\int_{r_{p-1}}^{r_p} gr^2V^-(r)dr\right]^{1/2}>1. \tag{16}$$

If one has n bound states, the last one being at zero energy $u(r)\to$const for $r\to\infty$ and hence $W(z)\to 0$ for $z\to\infty$. Combining inequalities of type (16) with the Schwarz inequality, we get

$$n-p<\tfrac{1}{2}\left(\frac{1}{r_p}\right)^{1/2}\left[\int_{r_p}^{\infty} gr^2V^-(r)dr\right]^{1/2} \tag{17}$$

III. A Bound for Potentials Extending from $r=0$ to $r=\infty$

The bounds (10) and (17) diverge when, respectively, $r_m\to\infty$ or $r_p\to 0$. To get an inequality which gives a finite result we have to combine the two inequalities. The number of bound states satisfies

$$n<\tfrac{1}{2}g^{1/2}\left[r_p^{1/2}\left(\int_0^{r_p} V^-(r)dr\right)^{1/2}+\left(\frac{1}{r_p}\right)^{1/2}\left(\int_{r_p}^{\infty} r^2V^-(r)dr\right)^{1/2}\right] \tag{18}$$

for any r_p where the zero energy wave function has a node. It would be tempting to minimize with respect to r_p, which would give the desired result. However, r_p takes only discrete values, and we must proceed more carefully. A first method will give a result which will be improved later on. Let us distinguish n even and n odd.

i) n even

Take $p=n/2$, (10) and (17) give

$$\frac{n}{2} \leqq \frac{1}{2} g^{1/2} r_p^{1/2} \left[\int_0^{r_p} V^-(r)dr \right]^{1/2}$$

$$\frac{n}{2} \leqq \frac{1}{2} g^{1/2} \left(\frac{1}{r_p}\right)^{1/2} \left[\int_{r_p}^{\infty} r^2 V^-(r)dr \right]^{1/2} .$$

Taking the product of the two inequalities, we get

$$n < g^{1/2} \left[\int_0^{r_p} V^-(r)dr \int_{r_p}^{\infty} r^2 V^-(r)dr \right]^{1/4} \tag{19}$$

$$< g^{1/2} \left[\operatorname*{Sup}_R \int_0^R V^-(r)dr \int_R^{\infty} r^2 V^-(r)dr \right]^{1/4} \tag{20}$$

$$< g^{1/2} \left[\int_0^{\infty} V^-(r)dr \int_0^{\infty} r^2 V^-(r)dr \right]^{1/4} . \tag{21}$$

ii) n odd

Take $p=(n-1)/2$ then exactly the same method gives

$$(n^2-1)^{1/2} < g^{1/2} \left[\int_0^{r_p} V^-(r)dr \int_{r_p}^{\infty} r^2 V^-(r)dr \right]^{1/4} \tag{22}$$

$$< g^{1/2} \left[\operatorname*{Sup}_R \int_0^R V^-(r)dr \int_R^{\infty} r^2 V^-(r)dr \right]^{1/4} \tag{23}$$

$$< g^{1/2} \left[\int_0^{\infty} V^-(r)dr \int_0^{\infty} r^2 V^-(r)dr \right]^{1/4} . \tag{24}$$

Here we see already that the case of n odd should be improved. Indeed, for $n=1$ the Bargmann bound gives [4]

$$1 < g^{1/2} \left[\int_0^{\infty} r V^-(r)dr \right]^{1/2} < g^{1/2} \left[\int_0^{\infty} V^-(r)dr \int_0^{\infty} r^2 V^-(r)dr \right]^{1/4} . \tag{25}$$

To improve the case of n odd, we must take into account the loss from inequality (23) to inequality (24):

$$(n^2-1)^2 < g^2 \int_0^{\infty} V^-(r)dr \int_0^{\infty} r^2 V^-(r)dr$$

$$- g^2 \int_0^{r_p} V^-(r)dr \int_0^{r_p} r^2 V^-(r)dr$$

$$- g^2 \int_{r_p}^{\infty} V^-(r)dr \int_{r_p}^{\infty} r^2 V^-(r)dr$$

with $p=(n-1)/2$.

For $n=3$,

$$g^2 \int_0^{r_p} V^- dr \int_0^{r_p} r^2 V^- dr > 1$$

from (25), and

$$g^2 \int_{r_p}^{\infty} V^- dr \int_{r_p}^{\infty} r^2 V^- dr > 16$$

from (21). (The interval of application of these inequalities need not be $0-\infty$.) Hence we get

$$64+16+1=81=3^4<g^2 \int_0^{\infty} V^-(r)dr \int_0^{\infty} r^2 V^-(r)dr \tag{26}$$

For $n \geqq 5$ we have

$$\begin{aligned} &g^2 \int_0^{r_p} V^- dr \int_0^{r_p} r^2 V^- dr + g^2 \int_{r_p}^{\infty} V^- dr \int_{r_p}^{\infty} r^2 V^- dr \\ &\geqq \mathrm{Inf}\left\{\left[\left(\frac{n-1}{2}\right)^2 - 1\right]^2 + \left(\frac{n+1}{2}\right)^4, \left[\left(\frac{n+1}{2}\right)^2 - 1\right]^2 + \left(\frac{n-1}{2}\right)^4\right\} \\ &= \left(\frac{n-1}{2}\right)^2 \left[\left(\frac{n+3}{2}\right)^2 + \left(\frac{n-1}{2}\right)^2\right] \geqq 2n^2+4n+1 . \end{aligned}$$

Hence we get

$$n^4 < n^4+4n+2 < g^2 \int_0^{\infty} V^-(r)dr \int_0^{\infty} r^2 V^-(r)dr . \tag{27}$$

In summary, in all cases we have

$$n < g^{1/2} \left[\int_0^{\infty} V^-(r)dr \int_0^{\infty} r^2 V^-(r)dr\right]^{1/4} . \tag{28}$$

In fact it is clear that this is not an optimal result and that for large n the coefficients in (28) can be improved. However, the improvement is not considerable: asymptotically, for large n, one gets

$$n \lesssim \left(\frac{8}{9}\right)^{1/4} g^{1/2} \left[\int_0^{\infty} V^-(r)dr \int_0^{\infty} r^2 V^-(r)dr\right]^{1/4} . \tag{29}$$

IV. Discussion

The bounds we have obtained, such as (28) or (13), are somewhat reminiscent of the bound obtained in the three-dimensional case [6]

$$N < \frac{g^{3/2}}{2\pi} [\int V^- d^3x \int (V^-)^2 d^3x]^{1/2} . \tag{30}$$

The difference is that (30) is not the best possible result in the three-dimensional case [7]. In fact, the best possible result is

$$N < \text{Const.}\, g^{3/2} \int |V^-(x)|^{3/2} d^3x\,. \tag{31}$$

In the one-dimensional case, on the other hand, there is no hope to obtain a bound in the form of a single integral, with the correct coupling constant dependence except if the potential is monotonous. Let us also notice that the present bound is in all circumstances better than the one proposed long ago by Calogero [3]

$$n < \tfrac{1}{2} + \frac{2g}{\pi} \left[\int V^-(r)dr \int r^2 V^-(r)dr\right]^{1/2}. \tag{32}$$

The same product of integrals appears in (28) and (32) but it is not raised to the same power.

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